Abstraction-Based Synthesis of Approximate Opacity for Cyber-Physical Systems

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***Abstract*— Opacity is an important information-flow security property which characterizes the plausible deniability of certain "secret behaviors" in dynamical systems. In this paper, we study the problem of synthesizing controllers enforcing opacity over discrete-time control systems with continuous state sets. It is known that, for systems with uncountable number of states, the opacity-enforcing controller synthesis problem is *undecidable* in general. To address this undecidability issue, in this paper, we develop an *abstraction-based* approach to tackle the controller synthesis problem. Specifically, we adopt a notion of approximate opacity which is suitable for continuous-space control systems. We propose a notion of *approximate initial-state opacity preserving alternating simulation relation* which characterizes the closeness of two systems in terms of opacity preservation. We show that, based on this new notion of system relation, one can synthesize an opacity-enforcing controller for the abstract system which is finite and then refine it back to enforce opacity over the original control system. Finally, we present a method for constructing opacity-preserving finite abstractions for discrete-time control systems under some stability properties. Our results are illustrated on a two-room temperature control problem.**

I. Introduction

With the advancements of cyber-physical systems (CPS) such as autonomous vehicles, smart manufacturing, and smart cities, information security and privacy issues have become increasingly important for design considerations due to large information exchanges in real-time [23]. For dynamical systems, an important aspect of security is to analyze what crucial information can be released through its *information flow*. In this work, we consider an important class of information-flow security properties called *opacity* [10]. Roughly speaking, opacity captures the system's plausible deniability for its "secret" behavior by requiring that its secret and non-secret behaviors should be indistinguishable for an intruder (passive eavesdropper) with respect to the released information.

In the last decades, a wide range of results on the analysis of opacity have been developed in the context of discrete event systems (DES). Depending on the secret requirements and the information structure of the system, different notions of opacity were proposed in the literature [21], [26]. The recent results in [4], [26] show that many of these opacity notions are transformable to each other. Among the various notions, initial-state opacity requires that the intruder can never determine for sure that the system was initiated from a secret state. To check opacity for DES models, the basic idea in the existing approaches is to construct *observers* which can be used to make appropriate initial-state or current-state estimations [21], [27]. When the original system is not opaque, many different approaches have also been proposed to enforce opacity for DES modeled as finite-state automata or Petri nets [5], [17]. Among them, one known approach is to use the supervisory control theory, where a controller is imposed to restrict the behavior of the system such that the closed-loop system under control is opaque [6], [30].

The aforementioned results on the verification or synthesis of opacity mainly deal with DES with discrete-state sets and event-triggered dynamics. However, many real-world CPS are hybrid involving continuous state-sets and time-driven dynamics. More recently, notions of opacity have been further extended from DES to general CPS with continuous state-sets; see, e.g., [3], [12], [16], [18]. Particularly, in a recent result [28], notions of *approximate opacity* have been proposed, which generalize the opacity concepts from DES to metric systems. Compared with notions of opacity in DES literature, approximate opacity takes into account the imprecise measurements which are typical in real-world applications, and thus are more suitable for CPS with continuous dynamics.

Related Work. Since the state-sets for continuous systems are uncountable, the verification or synthesis are *undecidable* in general. To address this undecidability issue, a promising approach is to use *abstraction-based techniques* [19], [24]. In this context, one needs to build a *finite abstraction (a.k.a. symbolic model)* of the original concrete system such that these two systems have certain *relations* under which the analysis or synthesis results over the finite abstractions can be refined and carried over to the original systems. Abstraction-based techniques have been developed only recently to tackle security properties including opacity ones [9], [14], [28], [29]. For the purpose of verifying approximate opacity for general control systems, opacity-preserving simulation relations together with the corresponding abstraction algorithms are first developed in [28], [29]. These works are then extended to analyze opacity for large-scale interconnected control (or switched) systems in [11], [14]. However, all

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these works, including [11], [14], [28], [29], are only dealing with verification rather than controller synthesis of opacity. In the context of synthesizing opacity-enforcing controllers, the results in [9] provide a notion of opacity-preserving alternating simulation relations that allows controller refinement with respect to opacity. However, the results therein are only applicable for systems with finite state sets and under the assumption of precise observations, which cannot be used for general CPS. More recently, the results in [16] propose a two-layer framework to enforce approximate opacity by combining barrier certificates with standard abstraction-based approaches. An abstraction-based controller is first designed for safety properties via standard alternating simulation relations without considering opacity. On top of that, approximate opacity is ensured by eliminating control inputs that violate opacity conditions using a control barrier function. However, this result appears to be conservative in the sense that the controller enforcing opacity is obtained under some feasibility assumptions and only limited to systems with finite input and output sets.

Our contribution. In this work, we propose a novel systematic approach for synthesizing controllers that enforce approximate initial-state opacity for CPS with continuous state-sets. To this end, we first propose a new system relation called *approximate initial-state opacity-preserving (AInitSOP) alternating simulation relation*. We show that this new system relation preserves approximate initial-state opacity between the abstract and the concrete system in terms of controller synthesis. In particular, one can synthesize opacity-enforcing controllers directly by applying existing synthesis algorithms to the finite abstractions that simulate the concrete systems with AInitSOP alternating simulation relation. We further propose an effective approach to construct finite abstractions which preserve the proposed system relation for a class of discrete-time control systems under some stability assumptions. To the best of our knowledge, this paper is the first to provide directly a controller synthesis approach to enforce opacity for continuous-space control systems using abstraction-based techniques.

The remaining of this paper is organized as follows. In Section II, we introduce some basic preliminaries including a notion of approximate opacity. The abstraction-based controller synthesis framework is presented in Section III. In Section IV, we present a new notion of approximate opacity-preserving alternating simulation relations and discuss the properties of this relation. An abstraction algorithm is provided in Section V to build the opacity-preserving finite abstractions for control systems. In Section VI, we illustrate the proposed scheme via a two-room temperature control example. Finally, we conclude the paper in Section VII.

II. Preliminaries

A. Notation

We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of non-negative integers and real numbers, respectively. They are annotated with subscripts to restrict them in the usual way, e.g., $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the infinity norm of x. A set $B \subseteq \mathbb{R}^m$ is called a *box* if $B = \prod_{i=1}^{m}[c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, m\}$. For any set $A = \bigcup_{j=1}^{M} A_j$ of the form of finite uion of boxes, where $A_j = \prod_{i=1}^{n}[c_i^j, d_i^j]$, we define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, where $span(A_j) = \min\{|d_i^j - c_i^j| \mid i = 1, \ldots, m\}$. For any $\mu \leq span(A)$, define $[A]_\mu = \bigcup_{j=1}^{M}[A_j]_\mu$, where $[A_j]_\mu = [\mathbb{R}^m]_\mu \cap A_j$ and $[\mathbb{R}^m]_\mu = \{a \in \mathbb{R}^m \mid a_i = k_i\mu, k_i \in \mathbb{Z}, i = 1, \ldots, m\}$. We denote the different classes of comparison functions by $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$, where $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \gamma$ is continuous, strictly increasing and $\gamma(0) = 0\}$; $\mathcal{K}_\infty = \{\gamma \in \mathcal{K} : \lim_{r\to\infty} \gamma(r) = \infty\}$; $\mathcal{KL} = \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} :$ for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty\}$.

B. System

In this paper, we employ a notion of "system" introduced in [24] as the underlying model of CPS describing both continuous-space and finite control systems, which is modeled by the 6-tuple

$$T = (X, X_0, U, \longrightarrow, Y, H),$$

where X is a (possibly infinite) set of states, $X_0 \subseteq X$ is the set of initial states, U is a (possibly infinite) set of inputs, $\longrightarrow \subseteq X \times U \times X$ is a transition relation, Y is a (possibly infinite) set of outputs, and $H : X \to Y$ is an output mapping. For the sake of simplicity, we also denote a transition $(x, u, x') \in \longrightarrow$ by $x \xrightarrow{u} x'$, where we say that x' is a u-successor, or simply successor, of x. For each state $x \in X$, we denote by $U(x)$ the set of all inputs defined at x, i.e., $U(x) = \{u \in U : \exists x' \in X \text{ s.t. } x \xrightarrow{u} x'\}$, and by $U_u^{post}(x)$ the set of u-successors of state x. A system T is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_{\geq 0}$;
- *finite* (or *symbolic*), if X and U are finite sets;
- *deterministic*, if for any state $x \in X$ and any input $u \in U$, $|U_u^{post}(x)| \leq 1$ and *nondeterministic* otherwise.

A finite state run is an internal behavior of a system S generated from an initial state $x_0 \in X_0$ under an input sequence $u_1 \cdots u_n$, which is a sequence of transitions $x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} x_n$ such that $x_i \xrightarrow{u_{i+1}} x_{i+1}$ for all $0 \leq i \leq n-1$. Note that the state run generated under the same input sequence may not be unique as the system can be non-deterministic in general. The corresponding output run (external behavior) is a sequence of outputs $H(x_0)H(x_1)\cdots H(x_n)$.

Let $T_a = (X_a, X_{a0}, U_a, \xrightarrow[a]{}, Y, H_a)$ and $T_b = (X_b, X_{b0}, U_b, \xrightarrow[b]{}, Y, H_b)$ be two metric systems with the same output set and metric $\mathbf{d}$. Let $\mathcal{I} \subseteq X_a \times X_b \times U_a \times U_b$ be an *ε-approximate interconnection relation* [24] such that $\forall (x_a, x_b) \in \pi_X(\mathcal{I}) : \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$, where $\pi_X(\cdot)$ denotes the projection to $X_a \times X_b$. The composition of T_a

and T_b with the interconnection relation $\mathcal{I}$ is a new system

$$T_a \times_{\mathcal{I}}^{\varepsilon} T_b = (X_{ab}, X_{ab0}, U_{ab}, \underset{ab}{\longrightarrow}, Y, H_{ab}),$$

where $X_{ab} = \pi_X(\mathcal{I})$, $X_{ab0} = X_{ab} \cap (X_{a0} \times X_{b0})$, $U_{ab} = U_a \times U_b$, $H_{ab}((x_a, x_b)) = \frac{1}{2}(H_a(x_a) + H_b(x_b))$ and $(x_a, x_b) \xrightarrow[ab]{(u_a,u_b)} (x'_a, x'_b)$ if (i) $x_a \xrightarrow[a]{u_a} x'_a$; (ii) $x_b \xrightarrow[b]{u_b} x'_b$; and (iii) $(x_a, x_b, u_a, u_b) \in \mathcal{I}$. The subscript $\mathcal{I}$ will be dropped when it is clear from the context.

C. Approximate Opacity

In this paper, we consider internal behaviors as the information available to the system, i.e., state information, while external behaviors are considered as the information available to the outside of the system (for example, an intruder). The information of the system is released by the output mapping $H : X \to Y$. Besides, the system model and its dynamics are also known by the outside intruders.

In many realistic CPS applications, the system might have some "secret" that does not want to be revealed to the outside world via the external behavior. For this, we adopt a state-based formulation of secret. Specifically, we assume that $S \subseteq X$ is a set of *secret states*, and hereafter, we write a system in the form of $T = (X, X_0, S, U, \longrightarrow, Y, H)$ by incorporating the secret state set. The notion of opacity captures the plausible deniability of the system's secret under the information leakage. Note that for metric systems whose outputs are physical signals, due to the imperfect measurement precision of outside observers (which is the case for almost all physical systems), it is very difficult to distinguish two observations if their distance is very small. Therefore, in this paper, we adopt a type of opacity called δ-approximate initial-state opacity [28] which quantifies the measurement precision of the intruder, and thus is more applicable to metric systems. The formal definition of δ-approximate initial-state opacity is recalled from [28] as follows.

Definition 1: Consider a system $T = (X, X_0, S, U, \longrightarrow, Y, H)$. We say that T is *δ-approximate initial-state opaque* if for any $x_0 \in X_0 \cap S$ and any finite state run $x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} x_n$, there exist $x'_0 \in X_0 \setminus S$ and a finite state run $x'_0 \xrightarrow{u'_1} x'_1 \xrightarrow{u'_2} \cdots \xrightarrow{u'_n} x'_n$ such that

$$\max_{i \in \{0,\ldots,n\}} \mathbf{d}(H(x_i), H(x'_i)) \leq \delta.$$

Intuitively, approximate initial-state opacity requires that an intruder with imperfect measurement precision captured by parameter δ can never know that the system was initiated from a secret state. Note that when $\delta = 0$, the above notion of approximate initial-state opacity boils down to the standard notion of initial-state opacity [22] (referred to as *exact* initial-state opacity in the sequel) which is widely adopted in DES literature. We use the the following example to illustrate this notion.

Example 1: Consider system $T_1 = (X_1, X_{1,0}, S_1, U_1, \longrightarrow, Y_1, H_1)$ as shown in Figure 1, where $X_1 = \{A, B, C, D\}$, $X_{1,0} = \{A, D\}$, $S_1 = \{D\}$, $U_1 = \{u\}$, $Y_1 = \{1.1, 2.9, 3.1\} \subseteq \mathbb{R}$ equipped with metric $\mathbf{d}$ defined by $\mathbf{d}(y_1, y_2) = |y_1 - y_2|$, $\forall y_1, y_2 \in Y_1$. We mark all secret states by red, and the output of each state is specified by a value associated to it. First, one can check that T_1 is not 0-approximate/exact initial-state opaque since we know immediately that the system is at secret state when a finite path $D \xrightarrow{u} C$ which generates output path [1.1][2.9] is observed. Next, consider an intruder with measurement precision $\delta = 0.2$. One can observe that T_1 is not 0.2-approximate initial-state opaque due to existence of a self-loop behavior at state C. For example, consider a secret-starting finite path $D \xrightarrow{u} C \xrightarrow{u} C$ which generates output path [1.1][2.9][2.9]. The intruder can infer for sure that the system started from a secret state since there is no path which started from a non-secret state $x_0 \notin S_1$ generating an equivalent output path which is close to [1.1][2.9][2.9] up to precision $\delta = 0.2$. However, once the self-loop is removed from state C, we can readily check that the new system is 0.2-approximate initial-state opaque since for every path starting from a secret state, there always exists a path that starts from a non-secret state with δ-close observations. ◇

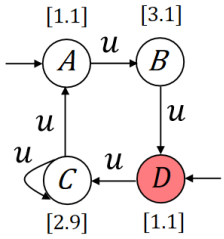


Fig. 1. Example to illustrate δ-approximate initial-state opacity on system T_1.

III. Abstraction-Based Controller Synthesis

In this section, we discuss how to leverage abstraction-based technique to synthesize controllers that enforce opacity of systems as defined in Subsection II-B.

A. Feedback Composition

When a system T does not satisfy some desired property, e.g., opacity, we can synthesize a controller for T such that the closed-loop system meets the specification. There are several (equivalent) definitions for controllers in the literature. In this paper, we adopt the definition in [24], in which a controller is considered also as a system that is composable to the original one through *approximate alternating simulation relation* defined as follows.

Definition 2: (Approximate Alternating Simulation Relation) Let $T_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y, H_a)$ and $T_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y, H_b)$ be two systems with the same output set. A relation $R \subseteq X_a \times X_b$ is said to be an approximate alternating simulation relation from T_a to T_b if the following conditions hold:

1) $\forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0} : (x_{a0}, x_{b0}) \in R$;
2) $\forall (x_a, x_b) \in R : \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3) $\forall (x_a, x_b) \in R, \forall u_a \in U_a(x_a), \exists u_b \in U_b(x_b)$ such that $\forall x_b \xrightarrow{u_b} x'_b, \exists x_a \xrightarrow{u_a} x'_a : (x'_a, x'_b) \in R$.

We say that T_a is ε-approximate alternatingly simulated by T_b (or T_b ε-approximate alternatingly simulates T_a, denoted by $T_a \preceq^{\varepsilon}_{AS} T_b$, if there exists an ε-approximate alternating simulation relation from T_a to T_b.

An alternating simulation relation $R \subseteq X_a \times X_b$ from T_a to T_b can also be extended to an interconnection relation $R^e \subseteq X_a \times X_b \times U_a \times U_b$ defined by: $(x_a, x_b, u_a, u_b) \in R^e$ if

(i) $(x_a, x_b) \in R$;
(ii) $u_a \in U_a(x_a), u_b \in U_b(x_b)$; and
(iii) $\forall x_b \xrightarrow{u_b} x'_b, \exists x_a \xrightarrow{u_a} x'_a : (x'_a, x'_b) \in R$.

Intuitively, R^e explicitly specifies which inputs we need to choose in order to maintain the alternating simulation relation.

The detailed control mechanism of (approximate) alternating simulation relation is explained in abstraction-based literatures for finite systems, e.g., [9], [24]. We recall this mechanism succinctly as follows. Consider two systems T_a and T_b under the above defined (approximate) alternating simulation relation, i.e., $T_a \preceq^{\varepsilon}_{AS} T_b$. Then, T_a can be a controller that offers an input $u_a \in U_a(x_a)$; this input is then transferred to T_b as a matching input $u_b \in U_b(x_b)$ via the interconnection relation R^e. Due to the non-determinism, T_b may go to any successor of u_b. Once T_b measures the successor state, T_a will update its state by matching the successor in T_b, and then offer a new input, and so forth. Note that controller T_a can also be non-deterministic as $u_a \in U_a(x_a)$ may not be unique. The above discussion is summarized by the following definition.

Definition 3: (Approximate Feedback Composition) A system T_c is said to be *ε-approximately feedback composable* with a system T_1 if there exists an ε-approximate alternating simulation relation R from T_c to T_1. When T_c is *ε-approximately feedback composable* with T_1, the feedback composition of T_c and T_1 is given by

$$T_c \times^{\varepsilon}_{\mathcal{F}} T_1 = (X_c \times X_1, X_{c0} \times X_{1,0}, U_c \times U_1, \xrightarrow[\mathcal{F}]{}, Y, H_{c1}),$$

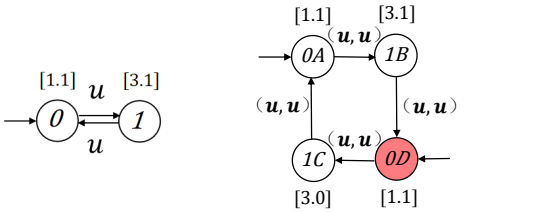
where the interconnection relation $\mathcal{F} = R^e$ is an extended ε-approximate alternating simulation relation as in Definition 2. For the sake of simplicity, the subscript $\mathcal{F}$ will be dropped when it is clear from the context.

Based on the above definition, we refer to T_c as a *controller* for system T if it is approximately feedback composable with T.

B. Opacity-Enforcing Control Problem

In this paper, in contrast to the existing results on verification of opacity [13], [28], our main goal is to tackle the *opacity-enforcing control problem* which requires to synthesize a controller T_c for system T such that it enforces approximate initial-state opacity on the composed system $T_c \times^{\varepsilon}_{\mathcal{F}} T$. More specifically, we say that T_c *enforces δ-approximate initial-state opacity* for T if for any $(x_{c0}, x_0) \in X_{c0} \times (X_0 \cap S)$ and any sequence

$$(x_{c0}, x_0) \xrightarrow[\mathcal{F}]{(u_{c1}, u_1)} (x_{c1}, x_1) \xrightarrow[\mathcal{F}]{(u_{c2}, u_2)} \cdots \xrightarrow[\mathcal{F}]{(u_{cn}, u_n)} (x_{cn}, x_n),$$



(a) Controller T_c (b) Closed-loop system $T_c \times^{\varepsilon_{c1}}_{R_{c1}} T_1$

Fig. 2. Example to illustrate the opacity-enforcing control problem.

there exist $x'_0 \in X_0 \setminus S$ and a sequence

$$x'_0 \xrightarrow{u'_1} x'_1 \xrightarrow{u'_2} \cdots \xrightarrow{u'_n} x'_n$$

such that $\max_{i \in \{0,\ldots,n\}} \mathbf{d}(H(x_i), H(x'_i)) \leq \delta$. Note that in this paper, we require that $T_c \times^{\varepsilon}_{\mathcal{F}} T$ is non-blocking, i.e., $\forall x \in X : U(x) \neq \emptyset$, which is a conventional assumption in symbolic control. Below, we illustrate the above-mentioned opacity-enforcing control problem on a simple finite system.

Example 2: Let us still consider system T_1 shown in Figure 1. To illustrate the opacity-enforcing control problem on T_1, we assume that there exists a controller $T_c = (X_c, X_{c0}, U_c, \xrightarrow[c]{}, Y_c, H_c)$ shown in Figure 2 (a), where $X_c = \{0, 1\}$, $X_{c0} = \{0\}$, $U_c = \{u\}$, $Y_c = \{1.1, 3.1\} \subseteq \mathbb{R}$ equipped with metric $\mathbf{d}$ defined by $\mathbf{d}(y_1, y_2) = |y_1 - y_2|$, $\forall y_1, y_2 \in Y_c$. The output of each state is specified by a value associated to it. One can readily check that $T_c \preceq^{\varepsilon_{c1}}_{AS} T_1$ with $\varepsilon_{c1} = 0.2$ through the approximate alternating simulation relation $R_{c1} = \{(0, A), (1, B), (1, C), (0, D)\}$. By Definition 3, the closed-loop system $T_c \times^{\varepsilon_{c1}}_{R_{c1}} T_1 = (X_c \times X_1, X_{c0} \times X_{1,0}, U_c \times U_1, \xrightarrow[\mathcal{R}_{c1}]{}, Y, H_{12})$ is shown in Figure 2(b). We can readily verify that for any path that started from a secret initial state in $(x_{c0}, x_0) \in X_{c0} \times (X_{1,0} \cap S_1)$ in the closed-loop system $T_c \times^{\varepsilon_{c1}}_{R_{c1}} T_1$, there exists an output-equivalent path initiated from a non-secret state $x \in X_{1,0} \setminus S_1$ in system T_1. For example, for $(0, D) \in X_{c0} \times (X_{1,0} \cap S_1)$ and a finite path $(0, D) \xrightarrow[\mathcal{R}_{1c}]{(u,u)} (1, C)$, there exist $A \in X_{1,0} \setminus S_1$ and a finite path $A \xrightarrow{u} B$ such that $|(H_{c1}(0, D) - H_1(A)| = 0 \leq 0.2$ and $|H_{c1}(1, C) - H_1(B)| = 0.1 \leq 0.2$. Therefore, we can claim that T_c is a controller that enforces 0.2-approximate initial-state opacity over T_1. $\diamond$

Note that parameters δ and ε in this paper specify two different types of precision. Parameter δ is used to specify the measurement precision of outside intruder under which we can guarantee the approximate opacity of a single system, while the parameters ε in the definition of approximate alternating simulation relation is used to describe the "distance" between two systems.

Note that the opacity-enforcing control problem is known to be undecidable for continuous-space systems. To this end, a promising approach is to leverage abstraction-based approaches as a bridge for the purpose of controller synthesis [24]. In this context, one first needs to build a finite abstraction of the concrete continuous-space control system, then synthesize a discrete controller based on the finite abstraction, and finally, refine the synthesized discrete

controller back as a hybrid controller to the original concrete system. The key to abstraction-based approach is to find appropriate relations between concrete systems and their finite abstractions such that properties of interest can be preserved under controller refinement. The abstraction-based controller refinement scheme is formalized in the following subsection.

C. Abstraction and Controller Refinement

Although approximate alternating simulation relations have shown to be useful [24] for controller refinement of properties such as ω-regular properties, unfortunately, they *do not preserve* security properties including opacity [1], [29]; check [29] for some counterexamples. Therefore, we introduce a new notion of *opacity-preserving* approximate alternating simulation relation, so that it can be applied to the abstraction-based opacity-enforcing control problem for continuous-space control systems.

Here, we propose a notion of so-called *approximate initial-state opacity preserving* (AInitSOP) alternating simulation relation. Specifically, this new notion of system relation from T_1 to T_2 is desired to satisfy the following requirements: (i) it is still an alternating simulation relation; (ii) enforcing opacity for T_1 implies the enforcement of opacity for T_2 after the controller refinement. The proposed notion of AInitSOP alternating simulation relation is introduced in the following definition.

Definition 4: (Approximate Initial-State Opacity Preserving Alternating Simulation Relation) Let T_1, T_2 be two systems, where $T_i = (X_i, X_{i,0}, S_i, U_i, \underset{i}{\longrightarrow}, Y, H_i), i = 1, 2$. A relation $R \subseteq X_1 \times X_2$ is said to be an ε-approximate initial-state opacity preserving (AInitSOP) alternating simulation relation from T_1 to T_2 if

1) a) $\forall x_{1,0} \in X_{1,0}, \exists x_{2,0} \in X_{2,0} : (x_{1,0}, x_{2,0}) \in R$;
   b) $\forall x_{1,0} \in X_{1,0} \setminus S_1, \exists x_{2,0} \in X_{2,0} \setminus S_2 : (x_{1,0}, x_{2,0}) \in R$;
2) $\forall (x_1, x_2) \in R : \mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon$
3) $\forall (x_1, x_2) \in R$, we have
   a) $\forall u_1 \in U_1(x_1), \exists u_2 \in U_2(x_2), \forall x_2 \xrightarrow{u_2} x_2', \exists x_1 \xrightarrow{u_1} x_1'$ such that $(x_1', x_2') \in R$;
   b) $\forall x_1 \xrightarrow{u_1} x_1', \exists x_2 \xrightarrow{u_2} x_2'$ such that $(x_1', x_2') \in R$.

We say that T_1 is AInitSOP alternatingly simulated by T_2 (or T_2 AInitSOP alternatingly simulates T_1), denoted by $T_1 \preceq^{\varepsilon}_{AIAS} T_2$, if there exists an AInitSOP alternating simulation relation from T_1 to T_2.

If $T_1 \preceq^{\varepsilon}_{AIAS} T_2$, we say that T_1 is an *abstraction* of T_2. In the sequel, we denote the original system by T_2 and the abstract system by T_1.

Note that an AInitSOP alternating simulation relation is still an alternating simulation relation, which makes the controller refinement procedure still possible. Next, we present the first main result of our paper which shows how to use the above-defined AInitSOP alternating simulation relation for the purpose of opacity-enforcing controller synthesis.

Theorem 1: Consider two systems T_1 and T_2, where $T_i = (X_i, X_{i,0}, S_i, U_i, \underset{i}{\longrightarrow}, Y, H_i), i = 1, 2$, and suppose that $T_1 \preceq^{\varepsilon_{12}}_{AIAS} T_2$. Then for any controller T_c that enforces δ-approximate initial-state opacity for the abstract system T_1 with $T_c \preceq^{\varepsilon_{c1}}_{AS} T_1$, the refined controller $T_{ref} = T_c \times^{\varepsilon_{c1}}_{\mathcal{F}_{c1}} T_1$ also enforces $\mathbf{max}\{(\frac{1}{2}\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12} + \delta), (\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12})\}$-approximate initial-state opacity for the original system T_2. [1]

In essence, the role of AInitSOP alternating simulation relation is to build a "bridge" between the original system and the controller of the abstract system. Based on this theorem, one can design a controller that enforces opacity of the finite abstract system, and then refine the controller back to enforce opacity of the original control system.

Note that in symbolic control, the controllers synthesized for abstract systems (with finite state set) are often precise, i.e., $\varepsilon_{c1} = 0$. In this case, we get a more succinct result as presented in the following corollary.

Corollary 1: Consider two systems T_1 and T_2, where $T_i = (X_i, X_{i,0}, S_i, U_i, \underset{i}{\longrightarrow}, Y, H_i), i = 1, 2$, and suppose that $T_1 \preceq^{\varepsilon_{12}}_{AIAS} T_2$. Then for any controller T_c that enforces δ-approximate initial-state opacity for the abstract system T_1 where $T_c \preceq^{0}_{AS} T_1$, the refined controller $T_{ref} = T_c \times^{\varepsilon_{c1}}_{\mathcal{F}_{c1}} T_1$ enforces $(\frac{3}{2}\varepsilon_{12} + \delta)$-approximate initial-state opacity over the original system T_2.

IV. Approximate Opacity-Preserving Finite Abstractions of Control Systems

In the previous section, we introduced a notion of approximate initial-state opacity preserving alternating simulation relations and discussed how it can be used to solve opacity-enforcing problem in an abstraction-based framework. Naturally, the next question for us is how to construct an opacity-preserving finite abstraction for a concrete control system so that it can be used for the sake of opacity-enforcing controller synthesis.

In general, the way to construct finite abstractions is system-dependent, and not all systems admit finite abstractions. Next, we show that a class of discrete-time control systems admits opacity-preserving finite abstractions under certain stability assumptions.

A. Discrete-time Control Systems

In this section, we consider a class of discrete-time control systems of the following form.

Definition 5: A discrete-time control system Σ is defined by the tuple $\Sigma = (\mathbb{X}, \mathbb{S}, \mathbb{U}, f, \mathbb{Y}, h)$, where $\mathbb{X}$, $\mathbb{U}$, and $\mathbb{Y}$ are the state, input, and output sets, respectively, and are subsets of normed vector spaces with appropriate dimensions. Set $\mathbb{S} \subseteq \mathbb{X}$ is a set of secret states. The map $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is called the transition function, and $h : \mathbb{X} \to \mathbb{Y}$ is the output map and assumed to satisfy the following Lipschitz condition: $\|h(x) - h(x')\| \leq \alpha(\|x - x'\|)$ for some $\alpha \in \mathcal{K}_\infty$ and all $x, x' \in \mathbb{X}$. The discrete-time control system Σ is

[1] All the detailed proofs of this paper are omitted due to the page limit and they are provided in XXXXXXXXXXXXXXXXXXXXXXXXXX.

described by difference equations of the form

$$\Sigma : \begin{cases} \xi(k+1) = f(\xi(k), \upsilon(k)), \\ \zeta(k) = h(\xi(k)), \end{cases} \tag{1}$$

where $\xi : \mathbb{N} \to \mathbb{X}$, $\zeta : \mathbb{N} \to \mathbb{Y}$, and $\upsilon : \mathbb{N} \to \mathbb{U}$ are the state, output, and input signals, respectively.

We denote by $\xi_{x\upsilon}(k)$ the point reached at time k under the input signal υ from initial condition $x = \xi_{x\upsilon}(0)$. Similarly, let $\zeta_{x\upsilon}(k)$ denote the output corresponding to state $\xi_{x\upsilon}(k)$, i.e. $\zeta_{x\upsilon}(k) = h(\xi_{x\upsilon}(k))$. Note that we implicitly assumed that set $\mathbb{X}$ is positively invariant[2] in the above definition.

B. Construction of Finite Abstractions

Next, we present how to construct finite abstractions for a class of discrete-time control systems. Specifically, the finite abstraction is built under the assumption that the concrete discrete-time control system is *incrementally input-to-state stable* as defined in [25] and recalled below.

Definition 6: System $\Sigma = (\mathbb{X}, \mathbb{S}, \mathbb{U}, f, \mathbb{Y}, h)$ is called incrementally input-to-state stable (δ-ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\forall x, x' \in \mathbb{X}$ and $\forall \upsilon, \upsilon' \in \mathbb{N} \to \mathbb{U}$, the following inequality holds for any $k \in \mathbb{N}$:

$$\|\xi_{x\upsilon}(k) - \xi_{x'\upsilon'}(k)\| \leq \beta(\|x - x'\|, k) + \gamma(\|\upsilon - \upsilon'\|). \tag{2}$$

Next, in order to construct approximate initial-state opacity preserving finite abstractions for a control system $\Sigma = (\mathbb{X}, \mathbb{S}, \mathbb{U}, f, \mathbb{Y}, h)$ in Definition 5, we define an associated metric system $T(\Sigma) = (X, X_0, X_S, U, \longrightarrow, Y, H)$, where $X = \mathbb{X}$, $X_0 = \mathbb{X}$, $X_S = \mathbb{S}$, $U = \mathbb{U}$, $Y = \mathbb{Y}$, $H = h$, and $x \xrightarrow{u} x'$ if and only if $x' = f(x, u)$. In the sequel, we will use $T(\Sigma)$ to denote the concrete control systems interchangeably.

Next, we introduce a symbolic system for the control system $\Sigma = (\mathbb{X}, \mathbb{S}, \mathbb{U}, f, \mathbb{Y}, h)$. To do so, in the rest of the paper, we assume that sets $\mathbb{X}$, $\mathbb{S}$ and $\mathbb{U}$ are of the form of finite union of boxes. Consider a concrete control system Σ and a tuple $\mathsf{q} = (\eta, \mu)$ of parameters, where $0 < \eta \leq \min\{span(\mathbb{S}), span(\mathbb{X} \setminus \mathbb{S})\}$ is the state set quantization, and $0 < \mu \leq span(\mathbb{U})$ is the input set quantization. Now let us introduce the symbolic system

$$T_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, X_{\mathsf{q}S}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \tag{3}$$

where $X_{\mathsf{q}} = X_{\mathsf{q}0} = [\mathbb{X}]_\eta$, $X_{\mathsf{q}S} = [\mathbb{S}]_\eta$, $U_{\mathsf{q}} = [\mathbb{U}]_\mu$, $Y_{\mathsf{q}} = \{h(x_{\mathsf{q}}) \mid x_{\mathsf{q}} \in X_{\mathsf{q}}\}$, $H_{\mathsf{q}}(x_{\mathsf{q}}) = h(x_{\mathsf{q}})$, $\forall x_{\mathsf{q}} \in X_{\mathsf{q}}$, and

- $x_{\mathsf{q}} \underset{\mathsf{q}}{\xrightarrow{u_{\mathsf{q}}}} x'_{\mathsf{q}}$ if and only if $\|x'_{\mathsf{q}} - f(x_{\mathsf{q}}, u_{\mathsf{q}})\| \leq \frac{1}{2}\eta$.

Now, we are ready to present the main result of this section, which shows that under some condition over the quantization parameters η and μ, the finite abstraction $T_{\mathsf{q}}(\Sigma)$ constructed in (3) indeed simulates our concrete control system $T(\Sigma)$ through the proposed approximate initial-state opacity preserving alternating simulation relation as in Definition 4.

[2]Set $\mathbb{X}$ is called positively invariant under (1) if $\xi_{x\upsilon}(k) \in \mathbb{X}$ for any $k \in \mathbb{N}$, any $x \in \mathbb{X}$, and any $\upsilon : \mathbb{N} \to \mathbb{U}$.

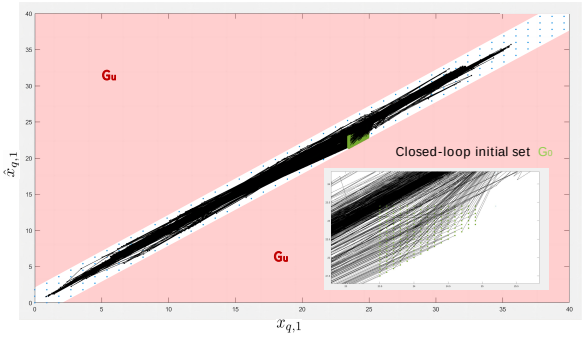


Fig. 3. Trajectories of the augmented closed-loop abstract system projected on the first-room coordinate starting from initial region G_0 (represented by the green area) under symbolic control. The black lines denote the state trajectories of the augmented abstract system. The red regions constitute the unsafe set G_u.

Theorem 2: Let $\Sigma = (\mathbb{X}, \mathbb{S}, \mathbb{U}, f, \mathbb{Y}, h)$ be a δ-ISS control system. For any desired precision $\varepsilon > 0$, and any tuple $\mathsf{q} = (\eta, \mu)$ of parameters satisfying

$$\beta\left(\alpha^{-1}(\varepsilon), 1\right) + \frac{1}{2}\eta \leq \alpha^{-1}(\varepsilon), \tag{4}$$

we have $T_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{AIAS} T(\Sigma)$.

Intuitively, this theorem shows that under certain conditions over the quantization parameter η, one can construct a finite abstraction as in (3) which is related to the original control system through the proposed AInitSOP alternating simulation relation. Let us recall that such an abstraction is a crucial bridge to the opacity-enforcing controller synthesis of continuous-space control systems. To be specific, one can first design symbolic controllers for the finite abstractions, and then leverage the results proposed in Theorem 1 to refine controllers to hybrid ones that enforce opacity on the original systems. Note that the design of symbolic controllers for finite abstractions is out of the scope of this paper. However, since the abstractions are finite, one can readily utilize the existing works and computational tools in the DES literature (e.g., [7]) to design controllers that enforce opacity on the abstractions.

We should mention that one can always find quantization parameters η such that (4) holds as long as $\beta\left(\alpha^{-1}(\varepsilon), 1\right) \leq \alpha^{-1}(\varepsilon)$. This inequality can be ensured by regarding the discrete-time control system as a sampled-data version of an original continuous-time system with large-enough sampling time (see more details in [28, Remark VI.8]). Note that one can also resort to incremental Lyapunov functions to prove Theorem 2 similar to the results in [8]. In particular, one can use the level sets of the Lyapunov functions as the underlying relation. By doing this, for a given ε, one can always find a quantization parameter η such that the corresponding inequality similar to (4) is always satisfied. For the sake of simple presentations, we decided not to follow this proof procedures.

V. Case Study

In this section, we demonstrate the proposed abstraction-based controller synthesis approach on a two-room temper-

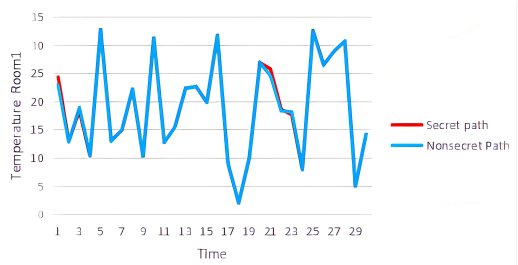


Fig. 4. First-room temperature trajectories initiated from different initial states (one from a secret state $x = [24.3; 20]$ and the other one from a non-secret state $x' = [22.9; 20]$.

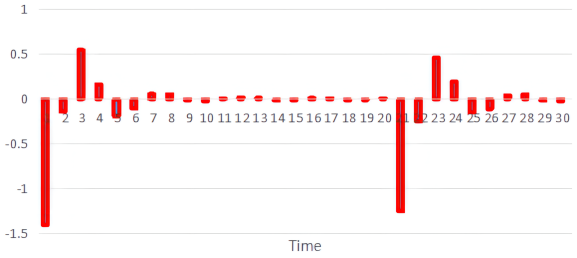


Fig. 5. Distance between the output trajectories corresponding to the two state trajectories depicted in Figure 4.

ature control problem, where each room is equipped with a heater. This model is borrowed from [15]. The temperature evolution of two rooms is:

$$\Sigma : \begin{cases} \boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + \alpha_h x_h \boldsymbol{u}(k) + \alpha_e \boldsymbol{x}_e, \\ \boldsymbol{y}(k) = h(\boldsymbol{x}(k)), \end{cases} \tag{5}$$

where $\boldsymbol{x}(k)$=[$\boldsymbol{x}_1(k)$;$\boldsymbol{x}_2(k)$], where $\boldsymbol{x}_i(k)$, $i \in \{1; 2\}$, represents the temperature of each room at time k, $\boldsymbol{u}(k)$=[$\boldsymbol{u}_1(k)$; $\boldsymbol{u}_2(k)$], where $\boldsymbol{u}_i(k) \in [0, 1]$, $\forall i \in [1; 2]$, represents the ratio of the heater valve being open in room i, $A \in \mathbb{R}^{2\times 2}$ is a heat exchange matrix for this model with elements $\{A\}_{11} = \{A\}_{22} = \alpha$, $\{A\}_{12} = 1 - 2\alpha - \alpha_e - \alpha_h c_1$, $\{A\}_{21} = 1 - 2\alpha - \alpha_e - \alpha_h c_2$. The parameters $\alpha = 0.1$, $\alpha_h = 0.5$, $\alpha_e = 0.1$, $c_1 = 0.4$ and $c_2 = 3$ are heat exchange coefficients of this model, $\boldsymbol{x}_e = [x_{e1}; x_{e2}] = [5\,°C; 5\,°C]$ represents the environment temperature and $x_h = 50\,°C$ represents the heater temperature. The output of this system is assumed to be the temperature of the second room, i.e., $\boldsymbol{y}(k) = h(\boldsymbol{x}(k)) = \boldsymbol{x}_2(k)$. In this example, our region of interest is as follows: $X = [0, 40] \times [0, 40]$, $X_0 = [20, 25] \times [20]$, $X_s = [23.5, 25] \times [20]$.

It is assumed that the secret of the system is whether the initial temperature of room 1 is higher than $23.5\,°C$,

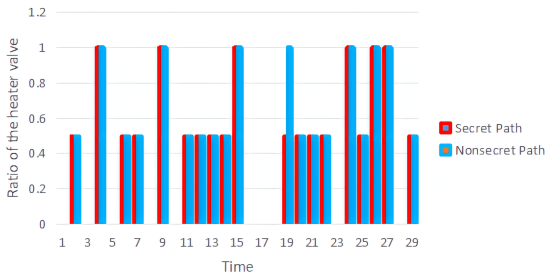


Fig. 6. The input runs corresponding to the state trajectories in Figure 4.

as this could mean that there are sensitive devices running or people are gathering in this room. We also assume that there is a malicious intruder wants to reason about the initial temperature of the first room by knowing the dynamics of the system and the output of the model. It is worth mentioning that due to the imperfect precision, the intruder cannot accurately obtain the output values of the system. Correspondingly, the measurement precision of the intruder is assumed to be $\delta_1 = 3.5$. Note that this can be captured as an δ_1-approximate initial-state opacity property of the system. Now, we apply our proposed abstraction-based framework to synthesize a controller to enforce approximate initial-state opacity on Σ. To do this, let us first build a finite abstraction of Σ using the approach presented in Subsection IV-B with a desired precision $\varepsilon = 1$. One can readily check that the system Σ is incrementally input-to-state stable. Hence, by leveraging Theorem 2 and based on inequality (4), we simply choose the state quantization parameter to be $\eta = 0.9$ and the input quantization parameter $\mu = 0.5$. Then, following the approach presented in Subsection IV-B, we can obtain a finite abstraction $T_{\mathsf{q}}(\Sigma)$ such that $T_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{AIAS} T(\Sigma)$ holds.

Next, we proceed with the opacity-enforcing controller synthesis by leveraging the result in Corollary 1. Specifically, in order to enforce the original system Σ to be 3.5-approximate initial-state opaque, we can design a 2.0-approximate initial-state opacity-enforcing controller for the abstract system T_{q}, and then refine it back to a controller that enforces 3.5-approximate initial-state opacity on the original system Σ.

As mentioned earlier, the design of symbolic controllers that enforces opacity of the finite abstractions is out of the scope of this paper. However, for the sake of completeness of the example, we briefly discuss our symbolic controller design process with the help of `SCOTS` [20] together with the ideas proposed in liu2020verification. In order to utilize `SCOTS` to design an opacity-enforcing symbolic controller, we resort to an approach developed in liu2020verification which essentially converts the opacity property of a single control system to a safety property of an augmented system which can be seen as the product of a control system and itself. We refer interested readers to liu2020verification for more details on the translation of opacity property to a safety one. Here, we briefly recall some of the notations that are used in this example: Given a single system T_{q}, an augmented system is defined as $T_{\mathsf{q}} \times T_{\mathsf{q}} = (X_q \times X_q, X_{q0} \times X_{q0}, X_{qS} \times X_{qS}, U_q \times U_q, f_q \times f_q, Y_q \times Y_q, H_q \times H_q)$. We use $G = X_q \times X_q$ to denote the augmented symbolic state set. Recall that a safety property essentially requires that any trajectory starting from a certain initial region should never reach an unsafe region. In this example, the initial and unsafe region for the obtained safety property is as follows: the initial region is G_0=$\{(x_q, \hat{x}_q) \in (X_{q0} \cap X_{qS}) \times (X_{q0} \setminus X_{qS}) |\ \|H(x_q) - H(\hat{x}_q)\| \leq \delta_2\}$, the unsafe region is G_u=$\{(x_q, \hat{x}_q) \in X_q \times X_q) |\ \|H(x_q) - H(\hat{x}_q)\| > \delta_2\}$, where $\delta_2 = 2$. Then, the safety controller synthesis problem is solved using `SCOTS`. In Figure 3, we show the state trajectories of the augmented closed-loop abstract system projected

on the first-room coordinate under the controller provided by SCOTS. It can be readily seen that the safety property is satisfied on the augmented system, which implies that the individual closed-loop abstract system is 2.0-approximate initial-state opaque.

So far, we have obtained a controller that enforces opacity on the abstract system with the closed-loop system denoted by $T_{ref} = T_c \times^0_{\mathcal{F}_{c1}} T_q(\Sigma)$. Then, according to Corollary 1, let T_{ref} be the refined controller for the original system. We have the guarantee that the closed-loop control system $T_{ref} \times^1_{\mathcal{F}_{12}} T(\Sigma)$ is δ_1-approximate initial-state opaque, where δ_1=($\frac{3}{2}\varepsilon + \delta_2$)=3.5, and the AInitSOP alternating simulation relation is as follows: $R_{12} = \{(x_{q,1}, x_1) \in X_{q,1} \times X_1 | \|x_1 - x_{\mathrm{q},1}\| \leq 1.0 \wedge (x_c, x_{q,1}) \in R_{c1}\}$. Figure 4 shows the simulation results of our implementation, which illustrates δ_1-approximate initial-state opacity of the closed-loop control system. In particular, two trajectories are depicted in this figure, where one is initiated from a secret state $[24.3; 20]$ while the other started from a non-secret state $x' = [22.9; 20]$. The distance between the corresponding output trajectories of these two state runs is depicted in Figure 5. The input runs of the trajectories are shown in Figure 6.

VI. Conclusion

In this work, we developed an abstraction-based approach for synthesizing controllers that enforce approximate initial-state opacity for continuous-space control systems. To this end, we proposed a notion of approximate initial-state opacity-preserving alternating simulation relation, which can be used to capture the distance between a concrete control system and its finite abstraction. Under this system relation, an opacity-enforcing controller designed for the finite abstraction can be refined back to the original control system. We further showed that under an incremental input-to-state stability assumption, a finite abstraction can be readily computed for a control system through the proposed system relation. Finally, we used a two-room temperature control example to illustrate our proposed abstraction-based controller synthesis framework. For future work, we plan to extend this work to more notions of opacity such as current-state opacity and infinite-step opacity. It would be also interesting to study how to relax the stability assumption on the control systems when building finite abstractions.

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VII. Appendix

In the sequel, we recall some definitions and results from [24], which are needed to prove Theorem 1.

Definition 7: [24] *(Approximate Simulation Relation)* Let $T_a = (X_a, X_{a0}, U_a, \xrightarrow[a]{}, Y, H_a)$ and $T_b = (X_b, X_{b0}, U_b, \xrightarrow[b]{}, Y, H_b)$ be two metric systems with the same output set. A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from T_a to T_b, if the following conditions are satisfied:

1) $\forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0} : (x_{a0}, x_{b0}) \in R$;
2) $\forall (x_a, x_b) \in R : \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3) $\forall (x_a, x_b) \in R, \forall x_a \xrightarrow[a]{u_a} x'_a, \exists x_b \xrightarrow[b]{u_b} x'_b : (x'_a, x'_b) \in R$.

We say that T_a is ε-approximately simulated by T_b (or T_b ε-approximately simulates T_a), denoted by $T_a \preceq^{\varepsilon}_S T_b$, if there exists an ε-approximate simulation relation from T_a to T_b.

The following result, recalled from [24], shows the usefulness of (approximate) alternating simulation relations as in Definition 2 in terms of controller refinement between two systems.

Proposition 1: [24] Consider systems T_1, T_2 and T_c with the same output set. Suppose T_c is ε_{c1}-approximate feedback composable with T_1 under ε_{c1}-approximate alternating simulation relation $R_{c1} \subseteq X_c \times X_1$. If there exists an ε_{12}-approximate alternating simulation relation $R_{12} \subseteq X_1 \times X_2$ from T_1 to T_2, then $T_c \times^{\varepsilon_{c1}}_{\mathcal{F}_{c1}} T_1$ is feedback composable with T_2 under $(\varepsilon_{c1}+\varepsilon_{12})$-approximate alternating simulation relation defined by:

$$R_{(c1)2} = \left\{ \begin{array}{c|c} ((x_c, x_1), x_2) & (x_c, x_1) \in R_{c1} \\ \in (X_c \times X_1) \times X_2 & \text{and } (x_1, x_2) \in R_{12} \end{array} \right\}. \tag{6}$$

Proposition 1 essentially says that, if $T_1 \preceq^{\varepsilon_{12}}_{AS} T_2$, then any controller T_c designed for T_1 can be *refined* to a controller for T_2. We denote by $T_{ref} = T_c \times^{\varepsilon_{c1}}_{\mathcal{F}_{c1}} T_1$ the refined controller with the interconnection relation defined in Equation (6). In particular, the refined controller has the following property.

Lemma 1: [2], [24] Let T_1, T_2 be two systems such that $T_1 \preceq^{\varepsilon_{12}}_{AS} T_2$. Suppose that T_c is a controller of T_1, i.e., $T_c \preceq^{\varepsilon_{c1}}_{AS} T_1$. Then for the refined controller $T_{ref} = T_c \times^{\varepsilon_{c1}}_{\mathcal{F}_{c1}} T_1$, we have

$$T_{ref} \times^{\varepsilon_{c1}+\varepsilon_{12}}_{\mathcal{F}} T_2 \preceq^{\frac{1}{2}(\varepsilon_{c1}+\varepsilon_{12})}_{S} T_c \times^{\varepsilon_{c1}}_{\mathcal{F}} T_1, \tag{7}$$

where $\preceq_S$ denotes the standard approximate simulation relation as in Definition 7.

Proof of Theorem1:

Proof: Let $R_{12} \subseteq X_1 \times X_2$ be an ε_{12}-approximate InitSOP alternating simulation relation from T_1 to T_2 and $R_{c1} \subseteq X_c \times X_1$ be an ε_{c1}-approximate alternating simulation relation from T_c to T_1 (since T_c is feedback composable with T_1). By Proposition 1, we also know T_{ref} is feedback composable with T_2 under the corresponding approximate alternating simulation relation $R_{(c1)2}$ defined in (6). In the following, we show that T_{ref} enforces approximate initial-state opacity for T_2, where $T_{ref} = T_c \times T_1$.

First, let us consider an arbitrary initial state $((x_{c,0}, x_{1,0}), x_{2,0})$ in $T_{ref} \times T_2$, where $x_{2,0} \in X_{2,0} \cap S_2$, and an arbitrary sequence

$$\begin{aligned} &((x_{c,0}, x_{1,0}), x_{2,0}) \xrightarrow{((u_{c,1}, u_{1,1}), u_{2,1})} ((x_{c,1}, x_{1,1}), x_{2,1}) \\ &\xrightarrow{((u_{c,2}, u_{1,2}), u_{2,2})} \cdots \xrightarrow{((u_{c,n}, u_{1,n}), u_{2,n})} ((x_{c,n}, x_{1,n}), x_{2,n}), \end{aligned} \tag{8}$$

where $((x_{c,i}, x_{1,i}), x_{2,i}) \in R_{(c1)2}$, $i \in [0, n]$. According to Equation (6), for $((x_{c,0}, x_{1,0}), x_{2,0}) \in R_{(c1)2}$, we have $(x_{c,0}, x_{1,0}) \in R_{c1}$ and $(x_{1,0}, x_{2,0}) \in R_{12}$. Next, we prove our theorem by considering the following two cases for $x_{1,0}$ in $((x_{c,0}, x_{1,0}), x_{2,0}) \in T_{ref} \times T_2$, i.e.,

Case1: $x_{1,0} \in X_{1,0} \cap S_1$ *such that* $(x_{c,0}, x_{1,0}) \in R_{c1}$;
Case2: $x_{1,0} \in X_{1,0} \setminus S_1$ *such that* $(x_{c,0}, x_{1,0}) \in R_{c1}$.

Consider *Case1*, we know that $x_{2,0} \in X_{2,0} \cap S_2$ and $x_{1,0} \in X_{1,0} \cap S_1$ in $((x_{c,0}, x_{1,0}), x_{2,0}) \in R_{(c1)2}$. From Lemma 1, we have $T_{ref} \times T_2 \preceq^{\frac{1}{2}(\varepsilon_{c1}+\varepsilon_{12})}_{S} T_c \times T_1$, for each state $((x_{c,i}, x_{1,i}), x_{2,i})$ in sequence (8), there exists a matching state $(x'_{c,i}, x'_{1,i})$, at least we have $(x'_{c,i}, x'_{1,i}) = (x_{c,i}, x_{1,i})$. Therefore, there exists an initial state $(x'_{c,0}, x'_{1,0}) \in R_{c1}$ in $T_c \times T_1$, where $x'_{1,0} \in X_{1,0} \cap S_1$ and an arbitrary sequence $(x'_{c,0}, x'_{1,0}) \xrightarrow{(u'_{c,1}, u'_{1,1})} (x'_{c,1}, x'_{1,1}) \xrightarrow{(u'_{c,2}, u'_{1,2})} \cdots \xrightarrow{(u'_{c,n}, u'_{1,n})} (x'_{c,n}, x'_{1,n})$ such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H(((x_{c,i}, x_{1,i}), x_{2,i})), H((x'_{c,i}, x'_{1,i}))) \leq \frac{1}{2}(\varepsilon_{c1} + \varepsilon_{12}). \tag{9}$$

Since T_c enforces δ-approximate initial-state opacity for T_1, we know that there exist a non-secret initial-state $x''_{1,0} \in X_{1,0} \setminus S_1$ in T_1 and a sequence $x''_{1,0} \xrightarrow{u''_{1,1}} x''_{1,1} \xrightarrow{u''_{1,2}} \cdots \xrightarrow{u''_{1,n}} x''_{1,n}$ such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H((x'_{c,i}, x'_{1,i})), H(x''_{1,i})) \leq \delta. \tag{10}$$

By the conditions 1)-b) and 3)-b) in Definition 4, we know that there exist a non-secret initial-state $x''_{2,0} \in X_{2,0} \setminus S_2$, and a sequence $x''_{2,0} \xrightarrow{u''_{2,1}} x''_{2,1} \xrightarrow{u''_{2,2}} \cdots \xrightarrow{u''_{2,n}} x''_{2,n}$, such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H(x''_{1,i}), H(x''_{2,i})) \leq \varepsilon_{12}. \tag{11}$$

By combining the inequalities (9), (10) and (11), and using the triangle inequality, we know that T_{ref} enforces $(\frac{1}{2}\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12} + \delta)$-approximate initial-state opacity for T_2 in *Case1*.

In *Case2*, we know that $x_{2,0} \in X_{2,0} \cap S_2$ and $x_{1,0} \in X_{1,0} \setminus S_1$ in the initial state $((x_{c,0}, x_{1,0}), x_{2,0})$ of sequence (8). Similarly, since $T_{ref} \times T_2 \preceq^{\frac{1}{2}(\varepsilon_{c1}+\varepsilon_{12})}_{S} T_c \times T_1$, for each state $((x_{c,i}, x_{1,i}), x_{2,i})$ in sequence (8), there exists a matching state $(x'''_{c,i}, x'''_{1,i})$, at least $(x'''_{c,i}, x'''_{1,i}) = (x_{c,i}, x_{1,i})$. Therefore, we know that there exists an initial state $(x'''_{c,0}, x'''_{1,0}) \in R_{c1}$ in $T_c \times T_1$, where $x'''_{1,0} \in X_{1,0} \setminus S_1$ and an arbitrary sequence $(x'''_{c,0}, x'''_{1,0}) \xrightarrow{(u'''_{c,1}, u'''_{1,1})} (x'''_{c,1}, x'''_{1,1}) \xrightarrow{(u'''_{c,2}, u'''_{1,2})} \cdots \xrightarrow{(u'''_{c,n}, u'''_{1,n})} (x'''_{c,n}, x'''_{1,n})$ such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H(((x_{c,i}, x_{1,i}), x_{2,n})), H((x'''_{c,i}, x'''_{1,i}))) \leq \frac{1}{2}(\varepsilon_{c1} + \varepsilon_{12}). \tag{12}$$

Since $T_c \times T_1 \preceq_S^{\frac{1}{2}\varepsilon_{c1}} T_1$, similarly, we know that there exist a non-secret initial-state $x'''_{1,0} \in X_{1,0} \setminus S_1$ in T_1 and a sequence $x'''_{1,0} \xrightarrow{u'''_{1,1}} x'''_{1,1} \xrightarrow{u'''_{1,2}} \cdots \xrightarrow{u'''_{1,n}} x'''_{1,n}$ such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H((x'''_{c,i}, x'''_{1,i})), H(x'''_{1,i})) \leq \frac{1}{2}\varepsilon_{c1}. \tag{13}$$

By the conditions 1)-b) and 3)-b) in Definition 4, we know that there exist a non-secret initial-state $x''''_{2,0} \in X_{2,0} \setminus S_2$, and a sequence $x''''_{2,0} \xrightarrow{u''''_{2,1}} x''''_{2,1} \xrightarrow{u''''_{2,2}} \cdots \xrightarrow{u''''_{2,n}} x''''_{2,n}$ such that $\forall i = 0, \ldots, n$,

$$\mathbf{d}(H(x'''_{1,i}), H(x''''_{2,i})) \leq \varepsilon_{12}. \tag{14}$$

By combining inequalities (12), (13), and (14), and using the triangle inequality, we know that T_{ref} enforces $(\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12})$-approximate initial-state opacity for T_2 in *Case2*.

Hence, summarizing the proof of *Case1* and *Case2*, we conclude that T_{ref} enforces $\mathbf{max}\{(\frac{1}{2}\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12} + \delta), (\varepsilon_{c1} + \frac{3}{2}\varepsilon_{12})\}$-approximate initial-state opacity for T_2. ■

Proof of Theorem2:

Proof: Consider the relation $R \subseteq X_{\mathsf{q}} \times X$ defined by $(x_{\mathsf{q}}, x) \in R$ if and only if $\|x - x_{\mathsf{q}}\| \leq \alpha^{-1}(\varepsilon)$. According to the construction of $T_{\mathsf{q}}(\Sigma)$, we have for any $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$, there exists $x_0 = x_{\mathsf{q}0} \in X_0$ such that $\|x_0 - x_{\mathsf{q}0}\| = 0 \leq \alpha^{-1}(\varepsilon)$. Hence, $(x_{\mathsf{q}0}, x_0) \in R$ and condition 1)-a) in Definition 4 is satisfied.

For every $x_{\mathsf{q}0} \in X_{\mathsf{q}0} \setminus X_{\mathsf{q}S}$, by choosing $x_0 = x_{\mathsf{q}0}$ which is also inside set $X_0 \setminus X_S$, one gets $(x_{\mathsf{q}0}, x_0) \in R$ and, hence, condition 1)-b) in Definition 4 holds as well.

Now consider any $(x_{\mathsf{q}}, x) \in R$. Condition 2) in Definition 4 is satisfied by the definition of R and the Lipschitz assumption on the output map:

$$\|H(x) - H_{\mathsf{q}}(x_{\mathsf{q}})\| = \|h(x) - h(x_{\mathsf{q}})\| \leq \alpha(\|x - x_{\mathsf{q}}\|) \leq \varepsilon.$$

Next, we proceed with showing condition 3) in Definition 4. Consider any pair of states $(x_{\mathsf{q}}, x) \in R$. For any input $u_q \in U_q$, let $u = u_q$. Consider the unique transition $x \xrightarrow{u} x' = f(x, u)$ in $T(\Sigma)$. Be leveraging the δ-ISS assumption on Σ, we get that the distance between x' and $f(x_{\mathsf{q}}, u_{\mathsf{q}})$ is bounded as:

$$\begin{aligned}\|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}})\| \leq& \beta\left(\|x - x_{\mathsf{q}}\|, 1\right) + \gamma\left(\|u - u_{\mathsf{q}}\|\right) \\ \leq& \beta\left(\alpha^{-1}(\varepsilon), 1\right).\end{aligned} \tag{15}$$

Moreover, by the structure of the symbolic system, we have that there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|f(x_{\mathsf{q}}, u_{\mathsf{q}}) - x'_{\mathsf{q}}\| \leq \frac{1}{2}\eta, \tag{16}$$

which, by the definition of $T_{\mathsf{q}}(\Sigma)$, implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $T_{\mathsf{q}}(\Sigma)$. Combining the inequalities (4), (15), (16), and the triangle inequality, we obtain:

$$\begin{aligned}\|x' - x'_{\mathsf{q}}\| &\leq \|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}}) + f(x_{\mathsf{q}}, u_{\mathsf{q}}) - x'_{\mathsf{q}}\| \\ &\leq \|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}})\| + \|f(x_{\mathsf{q}}, u_{\mathsf{q}}) - x'_{\mathsf{q}}\| \\ &\leq \beta\left(\alpha^{-1}(\varepsilon), 1\right) + \frac{1}{2}\eta \leq \alpha^{-1}(\varepsilon).\end{aligned}$$

Hence, we get $(x'_{\mathsf{q}}, x') \in R$ and condition 3)-a) in Definition 4 holds.

Similarly, we can show that condition 3)-b) in Definition 4 holds. Consider any $(x_{\mathsf{q}}, x) \in R$ and any $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Choose the input $u = u_{\mathsf{q}}$ and consider the unique $x' = f(x, u)$ in $T(\Sigma)$. Using δ-ISS assumption for Σ, we can bound the distance between x' and $f(x_{\mathsf{q}}, u_{\mathsf{q}})$ as:

$$\|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}})\| \leq \beta\left(\|x - x_{\mathsf{q}}\|, 1\right) \leq \beta\left(\alpha^{-1}(\varepsilon), 1\right). \tag{17}$$

Using the definition of $T_{\mathsf{q}}(\Sigma)$, the inequalities (4), (17), and the triangle inequality, we obtain:

$$\begin{aligned}\|x' - x'_{\mathsf{q}}\| \leq& \|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}}) + f(x_{\mathsf{q}}, u_{\mathsf{q}}) - x'_{\mathsf{q}}\| \\ \leq& \|x' - f(x_{\mathsf{q}}, u_{\mathsf{q}})\| + \|f(x_{\mathsf{q}}, u_{\mathsf{q}}) - x'_{\mathsf{q}}\| \\ \leq& \beta\left(\alpha^{-1}(\varepsilon), 1\right) + \frac{1}{2}\eta \leq \alpha^{-1}(\varepsilon).\end{aligned}$$

This implies that $(x'_{\mathsf{q}}, x') \in R$ and condition 3)-b) in Definition 4 holds as well. Therefore, we can conclude that $T_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{AIAS} T(\Sigma)$. ■