

Jeffreys prior for Ber(p) distribution

Jeffreys prior is a non-informative prior distribution for a parameter space; its density function is proportional to the square root of the determinant of the Fisher information matrix:

$$\pi_J(p) \propto \sqrt{\det [I(p)]}$$

For a Ber(p) distribution:

$$L_1(p) = p^x(1-p)^{1-x}$$

$$l_1(p) = \ln[p^x(1-p)^{1-x}] = x \ln(p) + (1-x) \log(1-p)$$

There are two ways to obtain the fisher information, $\text{Var}[l'_1(p)]$ or $-E[l''_1(p)]$.

Given that

$$l'_1(p) = \frac{x}{p} - \frac{1-x}{1-p}$$

and

$$l''_1(p) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$$

We can get

$$\begin{aligned} \text{Var}[l'_1(p)] &= \text{Var}\left[\frac{x}{p} - \frac{1-x}{1-p}\right] = \text{Var}\left(\frac{x}{p} - \frac{1}{1-p} + \frac{x}{1-p}\right) = \text{Var}\left(\frac{x}{p} + \frac{x}{1-p} + \text{constant}\right) \\ &= \text{Var}\left[\frac{1}{p(1-p)}x\right] = \left[\frac{1}{p(1-p)}\right]^2 \text{Var}(x) = \frac{p(1-p)}{[p(1-p)]^2} = \frac{1}{p(1-p)} \end{aligned}$$

and

$$\begin{aligned} -E[l''_1(p)] &= -E\left[-\frac{x}{p^2} - \frac{1-x}{(1-p)^2}\right] = -\left[-\frac{E(x)}{p^2} - \frac{1-E(x)}{(1-p)^2}\right] = \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} \\ &= \frac{1}{p(1-p)} \end{aligned}$$

The two method leads to the same result:

$$\text{Fisher Information} = \frac{1}{p(1-p)}$$

and

$$\sqrt{\det [I(p)]} = \sqrt{\frac{1}{p(1-p)}} = p^{-0.5}(1-p)^{-0.5} = p^{0.5-1}(1-p)^{0.5-1}$$

So,

$$\pi_J(p) \propto \text{Beta}(0.5, 0.5)$$