Degree of freedom in Person's Chi-squared test for a Poisson distribution

Consider the goodness of fit hypothesis test

H_0: $X \sim$ **Poisson distribution (assuming four values are observed in the sample)**

Categories	Counts	Observations
1	O_1	$x_{1,1}, x_{1,2}, \ldots, x_{1,O_1} = 1$
2	O_2	$x_{2,1}, x_{2,2}, \ldots, x_{2,O_2} = 2$
3	O_3	$x_{3,1}, x_{3,2}, \ldots, x_{3,O_3} = 3$
4	O_4	$x_{4,1}, x_{4,2}, \ldots, x_{4,O_4} = 4$

For this null hypothesis, we first need to get an estimate of λ

$$\hat{\lambda}_{MLE} = \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\left(\sum_{i=1}^{O_1} x_{1,i} + \sum_{i=1}^{O_2} x_{2,i} + \sum_{i=1}^{O_3} x_{3,i} + \sum_{i=1}^{O_4} x_{4,i}\right)$$

$$= \frac{1}{n}(O_1 + 2O_2 + 3O_3 + 4O_4)$$

Hence, there are two constrains in this case

(1) $\frac{1}{n}(O_1 + 2O_2 + 3O_3 + 4O_4) = \hat{\lambda}_{MLE} = \hat{\lambda}$

(2) $O_1 + O_2 + O_3 + O_4 = n$

Two of O_1, O_2, O_3, O_4 need to be fixed to satisfy both of these two constraints.

Now consider a counterexample, assuming O_1, O_2, O_3 are free to vary, and let's try to solve O_4:
From (1)

$$O_4 = \frac{n\hat{\lambda} - (O_1 + 2O_2 + 3O_3)}{4}$$

From (2)

$$O_4 = n - (O_1 + O_2 + O_3)$$

Note the equation below does not necessarily hold.

$$\frac{n\hat{\lambda} - (O_1 + 2O_2 + 3O_3)}{4} = n - (O_1 + O_2 + O_3)$$

Assuming O_1, O_2 are free to vary, and let's try to solve O_3 and O_4:

(1) $\frac{1}{n}(O_1 + 2O_2 + 3O_3 + 4O_4) = \hat{\lambda}$

(2) $O_1 + O_2 + O_3 + O_4 = n$

Solving this equation group yields the solution of O_3 and O_4

$$O_3 = n(4 - \hat{\lambda}) - 3O_1 - 2O_2$$
$$O_4 = n(\hat{\lambda} - 3) + 2O_1 + O_2$$

This means given random variables O_1 and O_2 and constants related to n and $\hat{\lambda}$, we can solve the equations to find the unique combination of O_3 and O_4 values to make that the two constraints (1) and (2) satisfied. Therefore, the reduction in degrees of freedom is 2.

(Note that it is not guaranteed that in the solution O_3 and O_4 are both positive. This depends on the values of $O_1, O_2, n, \hat{\lambda}$.)