Proof of the lemma

$$\boldsymbol{(n-p)\frac{\widehat{\sigma^2}}{\sigma^2} \sim \chi^2_{n-p}}$$

where $\widehat{\sigma^2}$ is an unbiased estimator of σ^2

$$\widehat{\sigma^2} = \frac{RSS}{n-p} = \frac{1}{n-p}\sum_{i=1}^{n} \mathrm{res}_i{}^2$$

Proof

First we have another lemma below.

Lemma (the proof of this lemma is not shown here).

if A is a symmetric and idempotent (i.e.,. $A = A^T, AA = A$) $n \times n$ matrix, and $Z \sim N(0, I_N)$ is a random vector of n independent standard normal variables.

Then $Z^T AZ \sim \chi^2_r$, where $r = Trace(A)$.

Then consider

$$(n-p)\frac{\widehat{\sigma^2}}{\sigma^2} = \frac{\sum_{i=1}^{n} \mathrm{res}^2}{\sigma^2} = \frac{\left\|Y - X\hat{\beta}\right\|_2^2}{\sigma^2}$$

In the numerator,

$$\begin{aligned} Y - X\hat{\beta} = Y - X(X^TX)^{-1}X^TY &= (X\beta + \varepsilon) - X(X^TX)^{-1}X^T(X\beta + \varepsilon) \\ &= X\beta + \varepsilon - X(X^TX)^{-1}X^TX\beta - X(X^TX)^{-1}X^T\varepsilon \\ &= X\beta + \varepsilon - X\beta - X(X^TX)^{-1}X^T\varepsilon = (I_n - H)\varepsilon \\ &\quad \text{where } H = X(X^TX)^{-1}X^T \end{aligned}$$

So,

$$\begin{aligned} \left\|Y - X\hat{\beta}\right\|_2^2 &= [(I_n - H)\varepsilon]^T[(I_n - H)\varepsilon] = \varepsilon^T(I_n - H)^T(I_n - H)\varepsilon = \varepsilon^T\left[\left(I_n{}^T - H^T\right)(I_n - H)\right]\varepsilon \\ &= \varepsilon^T\left(I_n{}^T I_n - I_n{}^T H - H^T I_n + H^T H\right)\varepsilon = \varepsilon^T(I_n - H - H + H)\varepsilon = \varepsilon^T(I_n - H)\varepsilon \end{aligned}$$

So

$$\frac{\left\|Y - X\hat{\beta}\right\|_2^2}{\sigma^2} = \frac{\varepsilon^T(I_n - H)\varepsilon}{\sigma^2} = \frac{\varepsilon^T}{\sigma}(I_n - H)\frac{\varepsilon}{\sigma}$$

Let

$$Z = \frac{\varepsilon}{\sigma}$$

Since $\varepsilon \sim N(0, \sigma^2)$, we have $Z \sim N(0,1)$

Let $A = I_n - H$, we have

- $A^T = I_n{}^T - H^T = I_n - H = A$, i.e., A is a symmetric; and
- $AA = (I_n - H)(I_n - H) = I_n - H - H + HH = I_n - H = A$. I.e. A is idempotent.

According to the lemma mentioned at the beginning,

$$Z^T AZ \sim \chi_r^2$$

where $r = \mathrm{Trace}(A) = \mathrm{Trace}(I_n - H) = n - p$

That is,

$$\boldsymbol{(n-p)\frac{\widehat{\sigma^2}}{\sigma^2} \sim \chi^2_{n-p}}$$

QED