Bayes estimate of Bernoulli(p) with Beta prior distribution

Consider the random variable X

$$X \sim \mathrm{Ber(p)}$$

Prior distribution: $p \sim \mathrm{Beta}(\alpha, \beta)$,
i.e.,

$$\pi(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}$$

where $B(\alpha, \beta)$ is the beta function, here it works as a normalizing constant.

The posterior distribution can be expressed as

$$\pi(p|X_1, \dots, X_n) \propto L(X_1, \dots, X_n|p)\pi(p) = \left[\prod_{i=1}^{n} p^{x_i}(1-p)^{(1-x_i)}\right]\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}$$
$$\propto p^{\alpha-1+\sum_{i=1}^{n} x_i} \times (1-p)^{\beta-1+\sum_{i=1}^{n}(1-x_i)} = p^{\alpha'-1}(1-p)^{\beta'-1}$$

where

$$\alpha' = \alpha + \sum_{i=1}^{n} x_i$$

$$\beta' = \beta + \sum_{i=1}^{n} (1-x_i) = \beta + \mathrm{n} - \sum_{i=1}^{n} x_i$$

Notice that, as we simplify our expression, we drop any constant terms that do not depend on p.
The next step is to identify what distribution this is. We can use the following statement:
General property: only one probability distribution is proportional to a functional form.

$$\pi(p|X_1, \dots, X_n) \propto p^{\alpha'-1}(1-p)^{\beta'-1} \propto \frac{p^{\alpha'-1}(1-p)^{\beta'-1}}{B(\alpha', \beta')}$$

So the posterior distribution is

$$\mathrm{Beta}(\alpha', \beta') = \mathrm{Beta}\left(\alpha + \sum_{i=1}^{n} x_i\,, \beta + \mathrm{n} - \sum_{i=1}^{n} x_i\right)$$

In this cases, the posterior distribution $\pi(p|X_1, \dots, X_n)$ is in the same probability distribution family as the prior probability distribution $\pi(p)$. The prior and posterior are then called **conjugate distributions**, and the prior is called a **conjugate prior**.
https://en.wikipedia.org/wiki/Conjugate_prior