Wald test statistic for Normal distribution

The general form of W statistic is

$$W = \frac{\hat{\theta} - \theta}{\sqrt{var(\hat{\theta})}}$$

For a given sample, we genially plug in the estimated $var(\hat{\theta})$ to the denominator, i.e.,

$$W = \frac{\hat{\theta} - \theta}{\sqrt{\widehat{var}(\hat{\theta})}}$$

In the following hypothesis test, $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown.

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The Maximum likelihood estimator of μ is

$$\hat{\mu} = \overline{\mathrm{X}}_{\mathrm{n}}$$

The sample variance of $\hat{\mu}$ is

$$\widehat{var}(\hat{\mu}) = \widehat{var}(\bar{X}_n) = \widehat{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\widehat{var}\left(\sum_{i=1}^{n} X_i\right) = \frac{n}{n^2}\widehat{Var}(X_i)$$
$$= \frac{\hat{\sigma}^2}{n}$$

Next find an estimator for $\hat{\sigma}^2$.
Consider

$$\hat{\sigma}^2 = s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}}_{\mathrm{n}})^2$$

It can be proved (below) that s_n^2 is an unbiased estimator of σ^2, i.e., $E(s_n^2) = \sigma^2$.

Proof

$$E(s_n^2) = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}}_{\mathrm{n}})^2\right] = \frac{1}{n-1}E\left[\sum_{i=1}^{n} {X_i}^2 + n(\overline{\mathrm{X}}_{\mathrm{n}})^2 - 2\sum_{i=1}^{n} X_i\overline{\mathrm{X}}_{\mathrm{n}}\right]$$
$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n} {X_i}^2 - n(\overline{\mathrm{X}}_{\mathrm{n}})^2\right] = \frac{1}{n-1}\left[nE\left({X_i}^2\right) - nE\left({\overline{\mathrm{X}}_{\mathrm{n}}}^2\right)\right]$$
$$= \frac{n}{n-1}\left[E\left({X_i}^2\right) - E\left({\overline{\mathrm{X}}_{\mathrm{n}}}^2\right)\right]$$
$$= \frac{n}{n-1}\left([E(X_i)^2 + Var(X_i)] - [E(\overline{\mathrm{X}}_{\mathrm{n}})^2 + Var(\overline{\mathrm{X}}_{\mathrm{n}})]\right)$$
$$= \frac{n}{n-1}\left([\mu^2 + \sigma^2] - \left[\mu^2 + \frac{1}{n}Var(X_i)\right]\right) = \frac{n}{n-1} \times \left(\frac{n-1}{n}\sigma^2\right) = \sigma^2$$

QED

Plugging $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}}_{\mathrm{n}})^2$ into $\widehat{var}(\hat{\mu})$ yields

$$\widehat{var}(\hat{\mu}) = \frac{\hat{\sigma}^2}{n} = \frac{s_n^2}{n}$$

So, the W statistic is

$$\frac{\hat{\mu} - \mu}{\sqrt{\widehat{var}(\hat{\mu})}} = \frac{\hat{\mu} - \mu}{\sqrt{\frac{s_n^2}{n}}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{s_n^2}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}}_{\mathrm{n}})^2}}$$