Bayes estimate of $\boldsymbol{Poisson(\lambda)}$ with Exponential prior distribution

For a random variable $X \sim \text{Poisson}(\lambda)$, the Bayes estimate of λ with an exponential prior distribution

$$\lambda \sim \text{Exp(a)}$$

i.e.,

$$\pi(\lambda) = \ ae^{-a\lambda}$$

The posterior distribution can be expressed as

$$\pi(\lambda|X_1, \dots, X_n) \propto L(X_1, \dots, X_n|\lambda)\pi(\lambda) = \left(\prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}\right) ae^{-a\lambda} = \frac{e^{-n\lambda} \times \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} ae^{-a\lambda} \propto \ e^{-(n+a)\lambda} \times \lambda^{\sum_{i=1}^{n} x_i}$$

The next step is to identify what distribution this is. We can prove that this is a Gamma distribution. The PDF of $\text{Gamma}(\alpha, \beta)$ is given by https://en.wikipedia.org/wiki/Gamma_distribution

$$f(x, \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \propto x^{\alpha-1} e^{-\beta x}$$

where $\Gamma(\alpha)$ is the gamma function.

To find α and β, use $e^{-(n+a)\lambda} \times \lambda^{\sum_{i=1}^{n} x_i} = x^{\alpha-1} e^{-\beta x}$

Consider

(1) $e^{-(n+a)\lambda} = e^{-\beta x}$

(2) $\lambda^{\sum_{i=1}^{n} x_i} = x^{\alpha-1}$

From (2), we have $x = \lambda$ and $\alpha = \sum_{i=1}^{n} x_i + 1$

From (1), we have $\beta x = (n + a)\lambda = (n + a)x$, so $\beta = n + a$

So the posterior distribution is

$$\text{Gamma}(\alpha, \beta) = \text{Gamma}\left(\sum_{i=1}^{n} x_i + 1, \text{n} + \text{a}\right)$$