Distribution of the W statistic for Normal distribution

In the following hypothesis test, $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown.

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The W statistic is

$$W = \frac{\hat{\mu} - \mu}{\sqrt{\widehat{var}(\hat{\mu})}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{s_n^2}}$$

where

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{\mathrm{X}}_{\mathrm{n}})^2$$

Rewrite it as (divide both the numerator and the denominator by σ)

$$W = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{s_n^2}} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)/\sigma}{\sqrt{s_n^2}/\sigma} = \frac{\sqrt{n}(\bar{X}_n - \mu_0)/\sigma}{\sqrt{\frac{s_n^2}{\sigma^2}}}$$

The numerator can be written as (under the H_0)

$$\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = Z \sim N(0,1)$$

The denominator can be written as

$$\sqrt{\frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{\mathrm{X}}_{\mathrm{n}})^2}{\sigma^2}} = \sqrt{\frac{1}{n-1} \times \sum_{i=1}^{n}\frac{(X_i - \bar{\mathrm{X}}_{\mathrm{n}})^2}{\sigma^2}} = \sqrt{\frac{1}{n-1} \times \sum_{i=1}^{n}\left(\frac{X_i - \bar{\mathrm{X}}_{\mathrm{n}}}{\sigma}\right)^2}$$

According to Cochran's theorem,

$$\sum_{i=1}^{n}\left(\frac{X_i - \bar{\mathrm{X}}_{\mathrm{n}}}{\sigma}\right)^2 \sim \chi^2_{n-1}$$

The degree of freedom is $n-1$ because $\bar{\mathrm{X}}_{\mathrm{n}}$ is not a random given $X_1 \ldots X_n$

The complete proof of the Cochran's theorem is not provided here.

Below is an example of the Cochran's theorem for $n = 2$:

$$\sum_{i=1}^{n}\left(\frac{X_i - \bar{\mathrm{X}}_{\mathrm{n}}}{\sigma}\right)^2 = \left(\frac{X_1 - \frac{1}{2}X_1 - \frac{1}{2}X_2}{\sigma}\right)^2 + \left(\frac{X_2 - \frac{1}{2}X_1 - \frac{1}{2}X_2}{\sigma}\right)^2 = 2 \times \left[\frac{\frac{1}{2}(X_1 - X_2)}{\sigma}\right]^2$$

$$= \left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 = Y^2$$

Since $\sqrt{2}\sigma$ is a constant, X_1, X_2 follows Normal distribution, Y also follows Normal distribution.

$$E(Y) = E\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right) = 0$$

$$Var(Y) = Var\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right) = \left(\frac{1}{\sqrt{2}\sigma}\right)^2 \times 2 \times Var(X_1) = \frac{1}{2\sigma^2} \times 2 \times \sigma^2 = 1$$

Therefore $Y \sim N(0,1)$, and

$$\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 = Y^2 \sim \chi_1^2$$

So, the W statistic can be rewritten as

$$W_n = \frac{\sqrt{n}(\bar{X}_n - \mu_0)/\sigma}{\sqrt{\frac{s_n^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{V}{n-1}}} \sim t_{n-1}$$

where $Z \sim N(0,1),\ V \sim \chi_{n-1}^2$

So, the test statistic below follows the student's t distribution.

$$\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{s_n} \sim t_{n-1}$$