An Example with given a joint density function

Assume the joint density function of x and y is

$$h(x,y) = x + y, (x,y) \in [0,1]^2$$

First, it can be shown that

$$\int_0^1 \int_0^1 (x+y)\, dxdy = \int_0^1 \left(\int_0^1 (x+y)\, dy \right) dx = \int_0^1 \left(\int_0^1 (x+y)\, dy \right) dx = \int_0^1 \left(x + \frac{1}{2} \right) dx = \left(\frac{1}{2}x^2 + \frac{1}{2}x \right)\Big|_0^1$$
$$= 1$$

1) Now, find the **marginal density function of x**:

$$h(x) = \int_0^1 h(x,y)dy = \int_0^1 (x+y)dy = x + \frac{1}{2}$$

It can be shown

$$\int_0^1 h(x)dx = \int_0^1 \left(x + \frac{1}{2} \right) dx = \left(\frac{1}{2}x^2 + \frac{1}{2}x \right)\Big|_0^1 = 1$$

2) Then the **conditional density function of y given x** is

$$h(y|x) = \frac{x+y}{x + \frac{1}{2}}$$

It can be shown

$$\int_0^1 h(y|x)dy = \int_0^1 \left(\frac{x+y}{x+\frac{1}{2}} \right) dy = \left(\frac{1}{x+\frac{1}{2}} \right) \int_0^1 (x+y)dy = \left(\frac{1}{x+\frac{1}{2}} \right) \times \left[\left(xy + \frac{1}{2}y^2 \right)\Big|_0^1 \right] = \frac{x+\frac{1}{2}}{x+\frac{1}{2}} = 1$$

3) Then the expectation of y given x is

$$E(y|x) = \int_0^1 [yh(y|x)]dy = \int_0^1 \left(\frac{xy+y^2}{x+\frac{1}{2}} \right) dy = \left(\frac{1}{x+\frac{1}{2}} \right) \times \left[\int_0^1 (xy+y^2)dy \right] = \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}} = \frac{3x+2}{6x+3}$$

4) Based on symmetry,

$$E(x|y) = \frac{3y+2}{6y+3}$$

5) Consider the random variable X

$$E(X) = \int_0^1 [xh(x)]dx = \int_0^1 \left[x\left(x + \frac{1}{2} \right) \right] dx = \int_0^1 \left(x^2 + \frac{1}{2}x \right) dx = \left(\frac{1}{3}x^3 + \frac{1}{4}x^2 \right)\Big|_0^1 = \frac{7}{12}$$

6) Since the density function of X is

$$h(x) = x + \frac{1}{2}$$

7) The **cumulative density function of X** is

$$F_X(x) = \int_{-\infty}^x h(t)dt = \int_0^x \left(t + \frac{1}{2} \right) dt = \left(\frac{1}{2}t^2 + \frac{1}{2}t \right)\Big|_0^x = \frac{1}{2}x^2 + \frac{1}{2}x$$

where $x \in [0,1]$

8) The **inverse cumulative density function (quantile function) of X** is calculated as below

$$F_x(x) = y$$

where $x \in [0,1]$, $y \in [0,1]$

$$\frac{1}{2}x^2 + \frac{1}{2}x = y$$

Replacing x with y and replacing y with x

$$\frac{1}{2}y^2 + \frac{1}{2}y = x$$

The roots of this one-variable (y) quadratic equation are:

$$y_1 = \frac{-1 + \sqrt{1 + 8x}}{2}, \qquad y_2 = \frac{-1 - \sqrt{1 + 8x}}{2}$$

Only y_1 satisfies $x \in [0,1]$ and $y \in [0,1]$.

So the inverse CDF is

$$y = \frac{-1 + \sqrt{1 + 8x}}{2}$$

Since here $x \in [0,1]$ can be interpreted as a probability, we can rewrite the inverse CDF as

$$F_x{}^{-1}(p) = Q_x(p) = \frac{-1 + \sqrt{1 + 8p}}{2}$$

Checking:

- when $p = 0$, $F_x{}^{-1}(p) = \frac{-1+\sqrt{1+0}}{2} = 0$
- when $p = 1$, $F_x{}^{-1}(p) = \frac{-1+\sqrt{1+8}}{2} = 1$

These are consistent with the support of X.

9) Now consider the conditional quantile function of Y, i.e., given x, the quantile $q(p)$ satisfies

$$P(Y \leq q(p)|X = x) = p$$

I.e.,

$$p = \int_{-\infty}^{q(p)} h(y|x)dy$$

Denote $q(p) = q$

$$\int_0^q \left(\frac{x + y}{x + \frac{1}{2}}\right) dy = \left(\frac{1}{x + \frac{1}{2}}\right) \times \int_0^q (x + y)dy = \left(\frac{1}{x + \frac{1}{2}}\right) \times \left(xy + \frac{1}{2}y^2\right)\Big|_0^q = \left(\frac{1}{x + \frac{1}{2}}\right) \times \left(xq + \frac{1}{2}q^2\right) = p$$

$$\frac{1}{2}q^2 + xq - \left(x + \frac{1}{2}\right)p = 0$$

The root of this quadratic equation is

$$q_1 = -x + \sqrt{x^2 + 2p(x + 1/2)}$$
$$q_2 = -x - \sqrt{x^2 + 2p(x + 1/2)}$$

q_2 is dropped because it is not in [0,1].

So

$$F_{y|x}{}^{-1}(p) = Q_{y|x}(p) = -x + \sqrt{x^2 + 2p(x + 1/2)}$$

Checking: when $p = 0$, ${F_{y|x}}^{-1}(p) = 0$; when $p = 1$, ${F_x}^{-1}(p) = 1$. T

- when $p = 0$, ${F_{y|x}}^{-1}(p) = -x + \sqrt{x^2} = 0$
- when $p = 1$, ${F_{y|x}}^{-1}(p) = -x + \sqrt{x^2 + 2x + 1} = 1$

These are consistent with the support of Y.