Significant level for multiple hypothesis testing in a regression

Now consider a null hypothesis which is a combination of two (or more) different hypotheses:

$$H_0 : 0 \leq \beta_1 \leq \beta_2$$

Note that this null is equivalent to two different hypotheses combined by "AND"

$$H_0^{(1)} : \beta_1 \geq 0 \textbf{ AND } H_0^{(2)} : \beta_1 \leq \beta_2$$

And the alternative hypothesis is

$$H_0 : \beta_1 < 0 \textbf{ OR } \beta_1 > \beta_2$$

Let the significant level of H_0, $H_0^{(1)}$ and $H_0^{(2)}$ be α, α_1 and α_2.

We have

$$\alpha \leq \alpha_1 + \alpha_2$$

Proof

$$\begin{aligned}
\alpha &= \text{Prob}(\text{reject } H_0 | H_0 \text{ is true}) \\
&= \text{Prob}\left(\text{reject } H_0^{(1)} \text{ OR reject } H_0^{(2)} \middle| H_0^{(1)} \text{ is true AND } H_0^{(2)} \text{ is true}\right) \\
&= \text{Prob}\left(\text{reject } H_0^{(1)} \cup \text{ reject } H_0^{(2)} \middle| H_0^{(1)} \text{ is true } \cap \ H_0^{(2)} \text{ is true}\right) \\
&\leq \text{Prob}\left(\text{reject } H_0^{(1)} \middle| H_0^{(1)} \text{ is true } \cap \ H_0^{(2)} \text{ is true}\right) \\
&+ \text{Prob}\left(\text{reject } H_0^{(2)} \middle| H_0^{(1)} \text{ is true } \cap \ H_0^{(2)} \text{ is true}\right)
\end{aligned}$$

The inequality is because

$$\text{Prob}(\text{A} \cup \text{B}) = \text{Prob}(\text{A}) + \text{Prob}(\text{B}) - P(A \cap \text{B}) \leq \text{Prob}(\text{A}) + \text{Prob}(\text{B})$$

Note that

$$\text{Prob}\left(\text{reject } H_0^{(1)} \middle| H_0^{(1)} \text{ is true } \cap \ H_0^{(2)} \text{ is true}\right) \leq \text{Prob}\left(\text{reject } H_0^{(1)} \middle| H_0^{(1)} \text{ is true}\right) = \alpha_1$$

where α_1 is the significant level of $H_0^{(1)}$.

Similarly, $\text{Prob}\left(\text{reject } H_0^{(1)} \middle| H_0^{(1)} \text{ is true } \cap \ H_0^{(2)} \text{ is true}\right) \leq \alpha_2$

where α_2 is the significant level of $H_0^{(2)}$.

Plugging this back yields

$$\alpha \leq \alpha_1 + \alpha_2$$

So if we bound $\alpha_1 + \alpha_2$ at 5%, we can guarantee that

$$\alpha \leq \alpha_1 + \alpha_2 \leq 5\%$$