$\mathbf{Ber(p)}$

By Central Limit Theorem,

$$\sqrt{n}(\bar{X}_n - p) \sim N(0, p(1-p))$$

The MLE of p is

$$\hat{p} = \bar{X}_n$$

So, the confidence intervals of p are in the form of

$$\text{Confidence Intevral } = \left(\bar{X}_n \pm q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}}\right)$$

1. Conservative Bound

$$\sqrt{p(1-p)} \leq \sqrt{0.5(1-0.5)} = 0.5$$

$$\Downarrow$$

$$CI_{cons} = \left(\bar{X}_n \pm q_{1-\alpha/2} \times \frac{0.5}{\sqrt{n}}\right)$$

2. Solving

$$\bar{X}_n - q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}} \leq p \leq \bar{X}_n + q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

$$\Downarrow$$

$$-q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}} \leq p - \bar{X}_n \leq q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

$$\Downarrow$$

$$(p - \bar{X}_n)^2 \leq \left(q_{1-\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}}\right)^2$$

$$\Downarrow$$

$$Ap^2 + Bp + C \leq 0$$

where

$$A = 1 + \frac{\left(q_{1-\alpha/2}\right)^2}{n}, \qquad B = -2\bar{X}_n - \frac{\left(q_{1-\alpha/2}\right)^2}{n}, \qquad C = (\bar{X}_n)^2$$

$$\Downarrow$$

$$CI_{solve} = \left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

3. Plug-In

$$CI_{plug-in} = \left(\bar{X}_n \pm q_{1-\alpha/2}\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\right) = \left(\bar{X}_n \pm q_{1-\alpha/2}\frac{\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}\right)$$

$\mathbf{Exp(\lambda)}$

By Central Limit Theorem,

$$\sqrt{n}\left(\bar{X}_n - \frac{1}{\lambda}\right) \sim N\left(0, \frac{1}{\lambda^2}\right)$$

The MLE of λ is

$$\hat{\lambda} = \frac{1}{\bar{X}_n}$$

By Delta Method,

$$g(x) = 1/x,\ \ g'(x) = -x^{-2},\ \ g'(\mu)^2 = x^{-4}, g'(\tfrac{1}{\lambda})^2 = \lambda^4$$

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \lambda\right) \sim N\left(0, \frac{1}{\lambda^2} \times \lambda^4\right) = N(0, \lambda^2)$$

So, the confidence intervals of λ are in the form of

$$\text{Confidence Intevral } = \left(\frac{1}{\bar{X}_n} \pm q_{1-\alpha/2}\frac{\sqrt{\lambda^2}}{\sqrt{n}}\right) = \left(\frac{1}{\bar{X}_n} \pm q_{1-\alpha/2}\frac{\lambda}{\sqrt{n}}\right)$$

1. Conservative Bound
   $\lambda > 0$ is not bounded
   $CI_{cons} = (-\infty, +\infty)$

2. Solving
   $$\hat{\lambda} - q_{1-\alpha/2}\frac{\lambda}{\sqrt{n}} \leq \lambda \leq \hat{\lambda} + q_{1-\alpha/2}\frac{\lambda}{\sqrt{n}}$$
   $\Downarrow$
   $$\lambda \geq \hat{\lambda}\left(1 + \frac{q_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right)^{-1} \text{ and } \lambda \leq \hat{\lambda}(1 - \frac{q_{1-\frac{\alpha}{2}}}{\sqrt{n}})^{-1}$$
   $\Downarrow$
   $$CI_{solve} = \left(\hat{\lambda}\left(1 + \frac{q_{1-\alpha/2}}{\sqrt{n}}\right)^{-1}, \hat{\lambda}\left(1 - \frac{q_{1-\alpha/2}}{\sqrt{n}}\right)^{-1}\right)$$
   $$= \left(\frac{1}{\bar{X}_n}\left(1 + \frac{q_{1-\alpha/2}}{\sqrt{n}}\right)^{-1}, \frac{1}{\bar{X}_n}\left(1 - \frac{q_{1-\alpha/2}}{\sqrt{n}}\right)^{-1}\right)$$

3. Plug-in
   $$CI_{plug-in} = \left(\hat{\lambda} \pm q_{1-\alpha/2}\frac{\hat{\lambda}}{\sqrt{n}}\right) = \left(\frac{1}{\bar{X}_n}\left(1 - \frac{q_{1-\alpha/2}}{\sqrt{n}}\right), \frac{1}{\bar{X}_n}\left(1 + \frac{q_{1-\alpha/2}}{\sqrt{n}}\right)\right)$$

$\mathbf{Gamma}(\boldsymbol{\alpha}, \mathbf{1}/\boldsymbol{\alpha})$

For $X \sim \text{Gamma}(\alpha, 1/\alpha)$,

$$\mu = E(X) = \frac{\alpha}{1/\alpha} = \alpha^2$$

$$\sigma^2 = Var(X) = \frac{\alpha}{(1/\alpha)^2} = \alpha^3$$

By Central Limit Theorem,

$$\sqrt{n}(\bar{X}_n - \alpha^2) \xrightarrow{d} N(0, \alpha^3)$$

The MLE of α is

$$\hat{\alpha} = \sqrt{\bar{X}_n}$$

By Delta Method,

$g(x) = \sqrt{x},\ g'(x) = \frac{1}{2}x^{-\frac{1}{2}},\ g'(x)^2 = \frac{1}{4}x^{-1}, g'(\alpha^2)^2 = \frac{1}{4\alpha^2}$

$$\sqrt{n}\left(\sqrt{\bar{X}_n} - \alpha\right) \xrightarrow{d} N\left(0, \alpha^3 \times \frac{1}{4\alpha^2}\right) = N\left(0, \frac{\alpha}{4}\right)$$

So, the confidence intervals of λ are in the form of

$$\text{Confidence Intevral } = \left(\sqrt{\bar{X}_n} \pm q\frac{\sqrt{\alpha/4}}{\sqrt{n}}\right) = \left(\sqrt{\bar{X}_n} \pm \frac{q\sqrt{\alpha}}{2\sqrt{n}}\right)$$

1. Conservative Bound
   $\alpha > 0$ is not bounded
   $CI_{cons} = (-\infty, +\infty)$

2. Solving
   $\hat{\alpha} - \frac{q\sqrt{\alpha}}{2\sqrt{n}} \le \alpha \le \hat{\alpha} + \frac{q\sqrt{\alpha}}{2\sqrt{n}}$
   $\Downarrow$
   $\alpha - \hat{\alpha} \le \frac{q\sqrt{\alpha}}{2\sqrt{n}}$
   $\Downarrow$
   $(\alpha - \hat{\alpha})^2 \le \left(\frac{q\sqrt{\alpha}}{2\sqrt{n}}\right)^2 = \frac{q^2\alpha}{4n}$
   $\Downarrow$

$$A\alpha^2 + B\alpha + c \leq 0$$

where

$$A = 1, \qquad B = -2\hat{\alpha} - \frac{q^2}{4n} = -2\sqrt{\bar{X}_n} - \frac{q^2}{4n}, \qquad C = \hat{\alpha}^2 = \bar{X}_n$$

$\Downarrow$

$$CI_{solve} = \left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

3. Plug-in

$$CI_{plug-in} = \left(\sqrt{\bar{X}_n} \pm \frac{q\sqrt{\hat{\alpha}}}{2\sqrt{n}}\right) = \left(\sqrt{\bar{X}_n} \pm \frac{q\sqrt{\sqrt{\bar{X}_n}}}{2\sqrt{n}}\right)$$