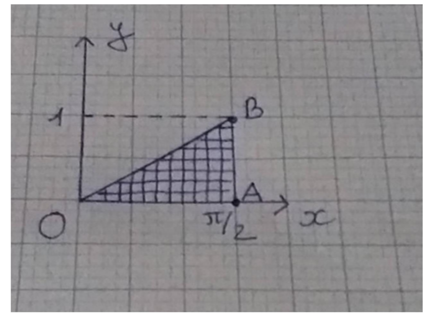
GIẢI ĐỀ THI 2018 – 2019

<u>Câu 1.</u>

a) Tính tích phân sau:



$$\oint_C (\boldsymbol{y} - \boldsymbol{sinx})\boldsymbol{dx} + \boldsymbol{cosxdy}$$

Trong đó (C) là tam giác OAB trong mặt phẳng $\boldsymbol{xOy}$ với tọa độ các đỉnh: $\boldsymbol{O(0,0), A\left(\frac{\pi}{2},0\right), B\left(\frac{\pi}{2},1\right)}$.

Giải

$$\oint_{\Delta OAB}[(y - sinx)dx + cosxdy] = \oint_{\Delta OAB} D = \oint_{OA} D + \oint_{AB} D + \oint_{OB} D$$

$$\oint_{OA} D = \int_0^{\frac{\pi}{2}} -\sin x dx = -1$$

$$\oint_{AB} D = \int_0^1 \cos x\, dy = \int_0^1 0dy = 0(\text{vì } x = \frac{\pi}{2} \to \cos x = 0)$$

Gọi $BO: y = ax + b: \begin{cases} 1 = a.\frac{\pi}{2} + b \\ 0 = a.0 + b \end{cases} \to \begin{cases} a = \frac{2}{\pi} \\ b = 0 \end{cases} \to y = \frac{2}{\pi}x \to dy = \frac{2}{\pi}dx$

$$\oint_{BO} D = \oint_{BO}[(y - sinx)dx + cosxdy] = \int_{\pi/2}^{0}\left(\frac{2}{\pi}x - sinx + \frac{2}{\pi}cosx\right)dx = \frac{x^2}{\pi} + cosx + \frac{2}{\pi}sinx\Big|_{\pi/2}^{0}$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$\to \oint_C [(y - sinx)dx + cosxdy] = -\frac{\pi}{4} - \frac{2}{\pi}$$

b) Nghiệm lại định lý Green trong mặt phẳng đối với tích phân trên:

Từ câu (a) ta có:

$$\oint_C [(y - sinx)dx + cosxdy] = -\frac{\pi}{4} - \frac{2}{\pi}$$

Đặt $\begin{cases} P(x,y) = y - sinx \\ Q(x,y) = cosx \end{cases} \rightarrow \begin{cases} P'_y = 1 \\ Q'_x = -sinx \end{cases}$

$$\oint_C [(y - sinx)dx + cosxdy] = \iint_D (Q'_x - P'_y)dA =$$

$$\iint_D (-sinx - 1)dxdy = \int_{y=1}^{0} \int_{x=\frac{\pi}{2}}^{\frac{y\pi}{2}} (-sinx - 1)dxdy = \int_{y=1}^{0} \left(cosx - x \Big|_{\frac{\pi}{2}}^{\frac{y\pi}{2}} \right) dy$$

$$= \int_{y=1}^{0} \left[\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2} + \frac{\pi}{2} \right] dy = \left[\frac{2}{\pi} sin\left(\frac{y\pi}{2}\right) - \frac{y^2\pi}{4} + \frac{y\pi}{2} \right] \Bigg|_1^0$$

$$= -\frac{2}{\pi} + \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} - \frac{\pi}{2} \rightarrow \text{định } lý\ Green \text{ được } kiểm\ chứng$$

Câu 1 (2019 – 2020): Cho trường vector

$$\vec{A} = (r\cos\theta)\vec{\imath_r} + (r\sin\theta)\vec{\imath_\theta} + (r\sin\theta\cos\varphi)\vec{\imath_\varphi}$$

a. **Tính div của $\vec{A}$.**

Ta có:

$$div\,\vec{A} = \nabla.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

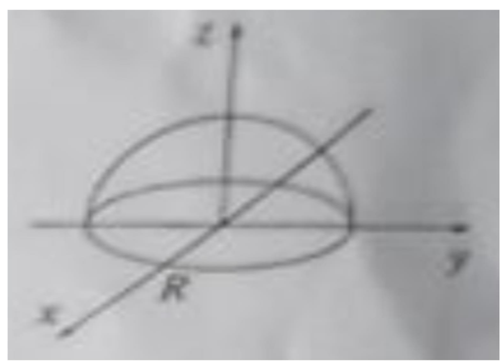
$$= \frac{1}{r^2}\frac{\partial(r^2r\cos\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,r\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi}$$

$$= \frac{1}{r^2}\frac{\partial(r^3\cos\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(r\sin^2\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi}$$

$$= 5\cos\theta - \sin\varphi$$

b. **Nghiệm lại định lý divergence trong thể tích V là ½ khối cầu bán kính R, có tâm ở gốc tọa độ (hình vẽ).**

$$\oint_S \vec{A}d\vec{S} = \oint_V div\vec{A}\,dV$$



$$\oint_V div\vec{A}\,dV = \oint_V (5\cos\theta - \sin\varphi)r^2\sin\theta drd\theta d\varphi$$

$$= \int_{\varphi=0}^{2\pi}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{R}(5\cos\theta - \sin\varphi)r^2\sin\theta drd\theta d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{R} r^2 \left(\frac{5}{2} sin2\theta - sin\theta sin\varphi\right) drd\theta d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{5}{2} sin2\theta - sin\theta sin\varphi\right) d\theta d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=0}^{2\pi} (-\frac{5}{4} cos2\theta + cos\theta sin\varphi)\Big|_{\theta=0}^{\frac{\pi}{2}} d\varphi = \frac{R^3}{3} \int_{\varphi=0}^{2\pi} \left(\frac{5}{2} - sin\varphi\right) d\varphi = \frac{5}{3}\pi R^3$$

$$\oint_S \vec{A} d\vec{S} = \oint_{S_1} \vec{A} d\overrightarrow{S_1} + \oint_{S_2} \vec{A} d\overrightarrow{S_2} \text{ với } S_1\text{: mặt bao ngoài}, S_2\text{: mặt đáy}$$

$$\rightarrow d\overrightarrow{S_1} = r^2 sin\theta d\theta d\varphi \overrightarrow{\iota_r}, d\overrightarrow{S_2} = rsin\theta drd\varphi \overrightarrow{\iota_\theta}$$

$$\oint_{S_1} \vec{A} d\overrightarrow{S_1} = \oint_{S_1} \left[(rcos\theta)\overrightarrow{\iota_r} + (rsin\,\theta)\overrightarrow{\iota_\theta} + (rsin\,\theta cos\varphi)\overrightarrow{\iota_\varphi}\right] r^2 sin\theta d\theta d\varphi \overrightarrow{\iota_r}$$

$$= \oint_{S_1} \frac{r^3 sin2\theta}{2} d\theta d\varphi = \frac{R^3}{2} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} sin2\theta d\theta d\varphi = \frac{R^3}{2} \int_{\varphi=0}^{2\pi} d\varphi = \pi R^3$$

$$\oint_{S_2} \vec{A} d\overrightarrow{S_2} = \oint_{S_2} \left[(rcos\theta)\overrightarrow{\iota_r} + (rsin\,\theta)\overrightarrow{\iota_\theta} + (rsin\,\theta cos\varphi)\overrightarrow{\iota_\varphi}\right] rsin\theta drd\varphi \overrightarrow{\iota_\theta}$$

$$= \oint_{S_2} r^2 \sin^2\theta\, drd\varphi = \int_{\varphi=0}^{2\pi} \int_{r=0}^{R} r^2 drd\varphi \left(\text{chọn } \theta = \frac{\pi}{2}\right) = \frac{R^3}{3} \int_{\varphi=0}^{2\pi} d\varphi = \frac{2\pi R^3}{3}$$

$$\rightarrow \oint_S \vec{A} d\vec{S} = \oint_{S_1} \vec{A} d\overrightarrow{S_1} + \oint_{S_2} \vec{A} d\overrightarrow{S_2} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5}{3}\pi R^3 = \oint_V div\vec{A}\, dV (\text{đpcm})$$