Trong hệ toa độ Descartes:

$$\vec{A} = A_x\overrightarrow{\iota_x} + A_y\overrightarrow{\iota_y} + A_z\overrightarrow{\iota_z}$$

$$\nabla = \frac{\partial}{\partial x}\overrightarrow{\iota_x} + \frac{\partial}{\partial y}\overrightarrow{\iota_y} + \frac{\partial}{\partial z}\overrightarrow{\iota_z}$$

$$rot\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \overrightarrow{\iota_x} & \overrightarrow{\iota_y} & \overrightarrow{\iota_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}, div\,\vec{A} = \nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Trong hệ tọa độ Trụ:

$$\vec{A} = A_r\overrightarrow{\iota_r} + A_\varphi\overrightarrow{\iota_\varphi} + A_z\overrightarrow{\iota_z}$$

$$\nabla = \frac{\partial}{\partial r}\overrightarrow{\iota_r} + \frac{1}{r}\frac{\partial}{\partial \varphi}\overrightarrow{\iota_\varphi} + \frac{\partial}{\partial z}\overrightarrow{\iota_z}$$

$$rot\,\vec{A} = \nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \overrightarrow{\iota_r} & \overrightarrow{\iota_\varphi} & \overrightarrow{\iota_z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}, div\,\vec{A} = \nabla.\vec{A}$$

$$= \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Trong hệ tọa độ cầu:

$$\vec{A} = A_r\overrightarrow{\iota_r} + A_\theta\overrightarrow{\iota_\theta} + A_\varphi\overrightarrow{\iota_\varphi}$$

$$\nabla = \frac{\partial}{\partial r}\overrightarrow{\iota_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{\iota_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\overrightarrow{\iota_\varphi}$$

$$rot\,\vec{A} = \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{\iota_r} & r.\vec{\iota_\theta} & r\sin\theta.\vec{\iota_\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & A_\varphi \end{vmatrix},$$

$$div\,\vec{A} = \nabla.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

<u>Câu 1</u> (2019 – 2020): Cho trường vector $\vec{A} = (r\cos\theta)\vec{\iota_r} + (r\sin\theta)\vec{\iota_\theta} + (r\sin\theta\cos\varphi)\vec{\iota_\varphi}$

a. Tính div của $\vec{A}$.

Ta có:

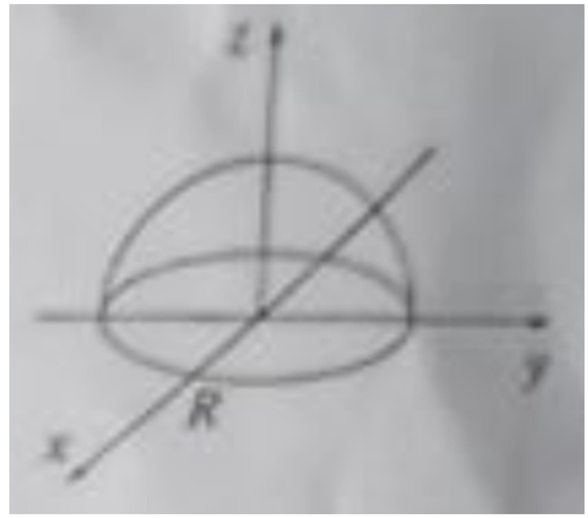
$$div\,\vec{A} = \nabla.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$= \frac{1}{r^2}\frac{\partial(r^2 r\cos\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, r\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi}$$

$$= \frac{1}{r^2}\frac{\partial(r^3\cos\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(r\sin^2\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(r\sin\theta\cos\varphi)}{\partial\varphi}$$

$$= 5\cos\theta - \sin\varphi$$

b. Nghiệm lại định lý divergence trong thể tích V là ½ khối cầu bán kính R, có tâm ở gốc tọa độ(hình vẽ).



$$\oint_S \vec{A} d\vec{S} = \oint_V div\vec{A}\, dV$$

$$\varphi: 0 \to 2\pi$$

$$\theta: 0 \to \frac{\pi}{2}$$

$$r: 0 \to R$$

$$\oint_V div\vec{A}\, dV = \oint_V (5\cos\theta - \sin\varphi) r^2 sin\theta dr d\theta d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{R} (5\cos\theta - \sin\varphi) r^2 sin\theta dr d\theta d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{R} r^2 \left(\frac{5}{2} sin2\theta - sin\theta sin\varphi\right) dr d\theta d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{5}{2} sin2\theta - sin\theta sin\varphi\right) d\theta d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=0}^{2\pi} \left(-\frac{5}{4} cos2\theta + cos\theta sin\varphi\right)\Big|_{\theta=0}^{\frac{\pi}{2}} d\varphi = \frac{R^3}{3} \int_{\varphi=0}^{2\pi} \left(\frac{5}{2} - sin\varphi\right) d\varphi$$

$$= \frac{5}{3}\pi R^3$$

$$\oint_S \vec{A}d\vec{S} = \oint_{S_1} \vec{A}d\overrightarrow{S_1} + \oint_{S_2} \vec{A}d\overrightarrow{S_2} \text{ với } S_1\text{: mặt bao ngoài}, S_2\text{: mặt đáy}$$

$$\rightarrow d\overrightarrow{S_1} = r^2 sin\theta d\theta d\varphi \overrightarrow{\iota_r}, d\overrightarrow{S_2} = rsin\theta drd\varphi\overrightarrow{\iota_\theta}$$

$$\oint_{S_1} \vec{A}d\overrightarrow{S_1} = \oint_{S_1} \left[(rcos\theta)\overrightarrow{\iota_r} + (rsin\ \theta)\overrightarrow{\iota_\theta}\right.$$

$$\left. + (rsin\ \theta cos\varphi)\overrightarrow{\iota_\varphi}\right] r^2 sin\theta d\theta d\varphi\overrightarrow{\iota_r} = \oint_{S_1} \frac{r^3 sin2\theta}{2} d\theta d\varphi$$

$$= \frac{R^3}{2} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} sin2\theta d\theta d\varphi = \frac{R^3}{2} \int_{\varphi=0}^{2\pi} d\varphi = \pi R^3$$

$$\oint_{S_2} \vec{A}d\overrightarrow{S_2} = \oint_{S_2} \left[(rcos\theta)\overrightarrow{\iota_r} + (rsin\ \theta)\overrightarrow{\iota_\theta} + (rsin\ \theta cos\varphi)\overrightarrow{\iota_\varphi}\right] rsin\theta drd\varphi\overrightarrow{\iota_\theta}$$

$$= \oint_{S_2} r^2 \sin^2\theta\, drd\varphi = \int_{\varphi=0}^{2\pi} \int_{r=0}^{R} r^2 drd\varphi \left(chọn\ \theta = \frac{\pi}{2}\right)$$

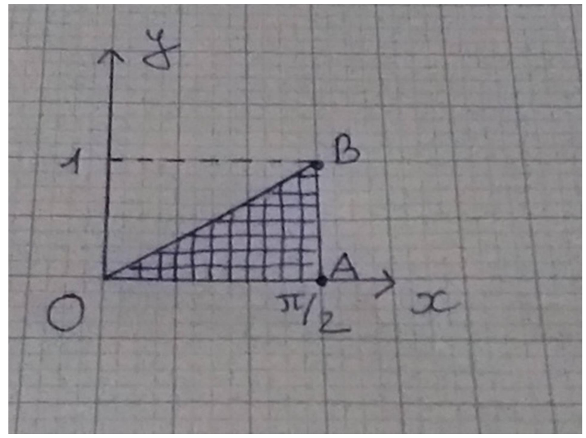
$$= \frac{R^3}{3} \int_{\varphi=0}^{2\pi} d\varphi = \frac{2\pi R^3}{3}$$

$$\rightarrow \oint_S \vec{A}d\vec{S} = \oint_{S_1} \vec{A}d\overrightarrow{S_1} + \oint_{S_2} \vec{A}d\overrightarrow{S_2} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5}{3}\pi R^3 = \oint_V div\vec{A}\, dV$$

$\boldsymbol{C}$**âu** $\boldsymbol{2}$. $\boldsymbol{T}$**ính** $\oint_C [(\boldsymbol{y} - \boldsymbol{sinx})\boldsymbol{dx}$

$+ \boldsymbol{cosxdy}]$, ***trong đó*** $(\boldsymbol{C})$ ***là tam giác OAB nằm trong mặt phẳng***

xOy biết $\boldsymbol{A\left(\frac{\pi}{2},0\right),B\left(\frac{\pi}{2},1\right)}$.



Giải

$$\oint_C [(y - sinx)dx + cosxdy] = \oint_C D$$

$$\oint_C D = \oint_{OA} D + \oint_{AB} D + \oint_{OB} D$$

$$\oint_{OA} D = \int_0^{\frac{\pi}{2}} -sin\, xdx = -1,\ \oint_{AB} D = \int_0^1 \cos x\, dy = \int_0^1 0dy = 0(v\text{ì}\ x = \frac{\pi}{2}$$
$$\to \cos x = 0)$$

$$\oint_{BO} D = \oint_{BO} [(y - sinx)dx + cosxdy] = \int_{\frac{\pi}{2}}^0 ydx - \int_{\frac{\pi}{2}}^0 \sin x\, dx + \int_1^0 \cos x\, dy$$
$$= -\frac{y\pi}{2} + 1 - \cos x$$

$$\to \oint_C [(y - sinx)dx + cosxdy] = -\frac{y\pi}{2} - \cos x$$

Câu 3. **Mật độ electron bên trong khối cầu bán kính 2m cho bởi quy luật** $n_e = \frac{1000}{r} cos\frac{\emptyset}{4}$ **(electron/m^3). Tìm điện tích của toàn bộ khối cầu biết điện tích của electron là** $-1,6.10^{-19}C$

Giải

Gọi $N = số\ electron\ chứa\ trong\ khối\ cầu, ta\ có$:

$$N = \oint_V n_e\, dV = \oint_V \frac{1000}{r} \cos\frac{\emptyset}{4} r^2 sin\theta dr d\theta d\emptyset$$

$$= \int_0^2 \frac{1000}{r} r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} \cos\frac{\emptyset}{4} d\emptyset = 16000(electron)$$

$\rightarrow$ $Điện\ tích\ khối\ cầu: Q = N.e = 16000.(-1,6).10^{-19}$
$= -2,56.10^{-1}\ \ (C)$

Câu 4. **Cho hàm vô hướng** $\emptyset(x, y, z) = 3x^2y - y^3z^2$**. Tìm** $grad(\emptyset)$ **tại điểm** $P(1, -2, -1)$**.**

- **Theo công thức:**

$$grad\phi = (\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z})(3x^2y - y^3z^2)$$

$$= 6xy.\vec{a}_x + (3x^2 - 3y^2z^2)\vec{a}_y - 2y^3z.\vec{a}_z$$

- **Tại P(1,-2,-1):** $grad\phi = -12.\vec{a}_x - 9.\vec{a}_y - 16.\vec{a}_z$

Câu 5. **Cho vector** $\vec{A} = x^2z.\vec{\iota_x} - 2y^3z^2.\vec{\iota_y} + xy^2z.\vec{\iota_z}$**. Tìm** $div\vec{A}$ **tại điểm** $P(1, -1, 1)$

- **Theo công thức:**

$$div\vec{A} = (\frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z))$$

$$= 2xz - 6y^2 z^2 + xy^2$$

- **Tại P(1,-1,1):** $div\vec{A} = -3$

<u>Câu 6.</u> Cho vector $\vec{A} = 3x\vec{\imath_x} + (y-3)\vec{\imath_y} + (2-z)\vec{\imath}_z$ và S là mặt hộp giới hạn bởi $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$. Nghiệm lại định lý divergence.

Giải

- Trên mặt $x = 0$, ta có:

$$\begin{cases} \vec{A} = (y-3)\vec{\imath}_y + (2-z)\vec{\imath}_z \\ \quad d\vec{S} = -dydz.\vec{\imath}_x \end{cases} \rightarrow \vec{A}d\vec{S} = 0 \rightarrow \oint_{x=0} \vec{A}d\vec{S} = 0$$

- Trên mặt $x = 1$, ta có:

$$\begin{cases} \vec{A} = 3\vec{\imath}_x(y-3)\vec{\imath}_y + (2-z)\vec{\imath}_z \\ \quad d\vec{S} = dydz.\vec{\imath}_x \end{cases} \rightarrow \vec{A}d\vec{S} = 3dydz \rightarrow \oint_{x=1} \vec{A}d\vec{S}$$

$$= \int_{y=0}^{2} \int_{z=0}^{3} 3dydz = 18$$

- Trên mặt $y = 0$, ta có:

$$\begin{cases}\vec{A} = 3x\overrightarrow{i_x} - 3\overrightarrow{i_y} + (2-z)\vec{i}_z \\ d\vec{S} = -dxdz.\overrightarrow{i_y}\end{cases} \rightarrow \vec{A}d\vec{S} = 3dxdz \rightarrow \oint_{y=0} \vec{A}d\vec{S}$$

$$= \int_{x=0}^{1}\int_{z=0}^{3} 3dxdz = 9$$

- Trên mặt $y = 2$, ta có:

$$\begin{cases}\vec{A} = 3x\overrightarrow{i_x} - \overrightarrow{i_y} + (2-z)\vec{i}_z \\ d\vec{S} = dxdz.\overrightarrow{i_y}\end{cases} \rightarrow \vec{A}d\vec{S} = -dxdz \rightarrow \oint_{y=2} \vec{A}d\vec{S}$$

$$= \int_{x=0}^{1}\int_{z=0}^{3} -dxdz = -3$$

- Trên mặt $z = 0$, ta có:

$$\begin{cases}\vec{A} = 3x\overrightarrow{i_x} + (y-3)\overrightarrow{i_y} + 2\vec{i}_z \\ d\vec{S} = -dxdy.\overrightarrow{i_z}\end{cases} \rightarrow \vec{A}d\vec{S} = -2dxdy \rightarrow \oint_{z=0} \vec{A}d\vec{S}$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} -2dxdy = -4$$

- Trên mặt $z = 3$, ta có:

$$\begin{cases}\vec{A} = 3x\overrightarrow{i_x} + (y-3)\overrightarrow{i_y} - \vec{i}_z \\ d\vec{S} = dxdy.\overrightarrow{i_z}\end{cases} \rightarrow \vec{A}d\vec{S} = -dxdy \rightarrow \oint_{z=3} \vec{A}d\vec{S}$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} -dxdy = -2$$

$$Vậy \oint_S \vec{A}d\vec{S} = 0 + 18 + 9 - 3 - 4 - 2 = 18$$

Ta có:

$$div\vec{A} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(y-3) + \frac{\partial}{\partial z}(2-z) = 3$$

$$dV = dxdydx$$

$$\rightarrow \oint_V div\vec{A}\,dV = \oint_V 3dxdydz = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} 3dxdydz = 18 = \oint_S \vec{A}d\vec{S}$$

$\rightarrow$ *Định lý Divergence được kiểm chứng.*

Câu 7. Trong hệ tọa độ cầu cho vector $\vec{A} = Cr.\vec{i}_r$ với $C = const, r \leq a$. Nghiệm lại định lý divergence.

$$\oint_S \vec{A}d\vec{S} = \oint C.b.\vec{i}_r.(b^2\sin\theta d\theta d\phi\vec{i}_r) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} C.b^3\sin\theta d\theta d\phi = 4\pi Cb^3$$

$$div\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Cr) = 3C$$

$$\int_V div\vec{A}dV = \int_V 3Cr^2\sin\theta drd\theta d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{b} 3Cr^2\sin\theta drd\theta d\phi = 4\pi Cb$$

$$\rightarrow \oint_S \vec{A}d\vec{S} = \int_V div\vec{A}.dV$$

Câu 8. Nghiệm lại định lý Stokes với $\vec{A} = x^2\vec{i}_y$ và S là hình chữ nhật có các đỉnh $M(g,0,0), N(g+\Delta,0,0), P(g+\Delta,h,0), Q(g,h,0)$.

Giải

Ta có:

$$\begin{cases} \vec{A} = x^2\vec{i}_y \\ d\vec{l} = dx\vec{i}_x + dy\vec{i}_y + dz\vec{i}_z \end{cases} \rightarrow \oint_C \vec{A}d\vec{l} = \oint_C x^2dy = \int_N^P x^2\,dy + \int_Q^M x^2\,dy$$

$$= x^2y|_0^h + x^2y|_h^0 = (g+\Delta)^2h - g^2h = (\Delta^2 + 2g\Delta)h$$

Ta có:

$$\begin{cases} rot\,\vec{A} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix} = 2x\vec{i}_z \\ d\vec{S} = dxdy\vec{i}_z \end{cases} \rightarrow \oint_S rot\,\vec{A}\,d\vec{S} = \oint_S 2xdxdy$$

$$= \int_g^{g+\Delta} 2xdx \int_0^h dy = x^2|_g^{g+\Delta}.\,y|_0^h = (\Delta^2 + 2g\Delta)h$$

$$= \oint_C \vec{A}d\vec{l}\,(\text{đ}pcm)$$