ĐỀ THI TOÁN CHUYÊN NGÀNH 2019 – 2020

Câu 1. (2đ) Cho hàm số $f(z) = \frac{2z+5}{z^2-4z+3}$, tìm chuỗi Laurent của $f(z)$ trong các miền sau:

a. Trong miền $1 < |z| < 3$

$$f(z) = \frac{2z+5}{z^2-4z+3} = \frac{2z+5}{(z-1)(z-3)} = -\frac{7}{2}.\frac{1}{z-1} + \frac{11}{2}.\frac{1}{z-3}$$

Với $|z| > 1 \Leftrightarrow \left|\frac{1}{z}\right| < 1$ ta có:

$$\frac{1}{z-1} = \frac{1}{z}.\frac{1}{1-\frac{1}{z}} = \frac{1}{z}.\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{+\infty}\frac{1}{z^{n+1}}$$

Với $|z| < 3 \Leftrightarrow \left|\frac{z}{3}\right| < 1$ ta có:

$$\frac{1}{z-3} = -\frac{1}{3}.\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}.\sum_{n=0}^{+\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{+\infty}\frac{(-z)^n}{3^{n+1}}$$

$$\text{Vậy } f(z) = \frac{2z+5}{z^2-4z+3} = -\frac{7}{2}\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} + \frac{11}{2}\sum_{n=0}^{+\infty}\frac{(-z)^n}{3^{n+1}}$$

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Câu 1. (2đ) Cho hàm số $f(z) = \frac{2z+5}{z^2-4z+3}$, tìm chuỗi Laurent của $f(z)$ trong các miền sau:

b. Trong miền: $1 < |z-4| < 3$

$$f(z) = \frac{2z+5}{z^2-4z+3} = \frac{2z+5}{(z-1)(z-3)} = -\frac{7}{2}.\frac{1}{z-1} + \frac{11}{2}.\frac{1}{z-3}$$

Với $|z-4| > 1 \Leftrightarrow \left|\frac{1}{z-4}\right| < 1$ ta có:

$$\frac{1}{z-3} = \frac{1}{z-4+1} = \frac{1}{z-4}.\frac{1}{1+\frac{1}{z-4}} = \frac{1}{z-4}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{1}{z-4}\right)^n = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(z-4)^{n+1}}$$

Với $|z-4| < 3 \Leftrightarrow \left|\frac{z-4}{3}\right| < 1$ ta có:

$$\frac{1}{z-1} = \frac{1}{3+z-4} = \frac{1}{3}.\frac{1}{1+\frac{z-4}{3}} = \frac{1}{3}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{z-4}{3}\right)^n = \sum_{n=0}^{+\infty}\frac{[-(z-4)]^n}{3^{n+1}}$$

$$f(z) = \frac{2z+5}{z^2-4z+3} = -\frac{7}{2}\sum_{n=0}^{+\infty}\frac{[-(z-4)]^n}{3^{n+1}} + \frac{11}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(z-4)^{n+1}}$$

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Câu 1. (2đ) Cho hàm số $f(z) = \frac{2z+5}{z^2-4z+3}$, tìm chuỗi Laurent của $f(z)$ trong các miền sau:

c. Trong miền $\sqrt{2} < |z+i| < \sqrt{10}$

$$f(z) = \frac{2z+5}{z^2-4z+3} = \frac{2z+5}{(z-1)(z-3)} = -\frac{7}{2}.\frac{1}{z-1} + \frac{11}{2}.\frac{1}{z-3}$$

Với $|z+i| > \sqrt{2} \Leftrightarrow \left|\frac{1+i}{z+i}\right| < 1$ ta có:

$$\frac{1}{z-1} = \frac{1}{z+i-(1+i)} = \frac{1}{z+i}.\frac{1}{1-\frac{1+i}{z+i}} = \frac{1}{z+i}.\sum_{n=0}^{+\infty}\left(\frac{1+i}{z+i}\right)^n = \sum_{n=0}^{+\infty}\frac{(1+i)^n}{(z+i)^{n+1}}$$

Với $|z+i| < \sqrt{10} \Leftrightarrow \left|\frac{z+i}{3+i}\right| < 1$ ta có:

$$\frac{1}{z-3} = \frac{1}{z+i-(3+i)} = -\frac{1}{3+i}.\frac{1}{1-\frac{z+i}{3+i}} = -\frac{1}{3+i}.\sum_{n=0}^{+\infty}\left(\frac{z+i}{3+i}\right)^n = -\sum_{n=0}^{+\infty}\frac{(z+i)^n}{(3+i)^{n+1}}$$

$$Vậy\ f(z) = \frac{2z+5}{z^2-4z+3} = -\frac{7}{2}\sum_{n=0}^{+\infty}\frac{(1+i)^n}{(z+i)^{n+1}} - \frac{11}{2}\sum_{n=0}^{+\infty}\frac{(z+i)^n}{(3+i)^{n+1}}$$

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Câu 2. (2đ) Sử dụng thặng dư, tính các tích phân sau:

$$a)\ A=\int_0^{2\pi}\frac{cos2\emptyset}{5-4cos\emptyset}d\emptyset=\int_0^{2\pi}\frac{1-2sin^2\emptyset}{5-4cos\emptyset}d\emptyset$$

Đặt $z=e^{j\emptyset}\rightarrow dz=je^{j\emptyset}d\emptyset=jzd\emptyset\rightarrow d\emptyset=\frac{dz}{jz},\cos\emptyset=\frac{1}{2}\left(z+\frac{1}{z}\right),\sin\emptyset=-\frac{j}{2}\left(z-\frac{1}{z}\right)$

$$A=\int_0^{2\pi}\frac{1-2\sin^2\emptyset}{5-4cos\emptyset}d\emptyset=\oint_c\left(\frac{1-2.\left[\frac{-j}{2}\left(z-\frac{1}{z}\right)\right]^2}{5-4.\frac{1}{2}\left(z+\frac{1}{z}\right)}\right)\frac{dz}{jz}=\oint_c\left[\frac{1+\left(\frac{z^2}{2}-1+\frac{1}{2z^2}\right)}{5-2\left(z+\frac{1}{z}\right)}\right]\frac{dz}{jz}$$

$$=-\oint_c\left|\frac{\frac{z^2}{2}+\frac{1}{2z^2}}{5-2z-\frac{2}{z}}\right|\frac{jdz}{z}=-\oint_c\frac{z^4+1}{10z^2-4z^3-4z}\frac{dz}{jz}=-\frac{j}{2}\oint_c\frac{z^4+1}{z^2(-2z^2+5z-2)}dz$$

$$\oint_c\frac{z^4+1}{z^2(-2z^2+5z-2)}dz=2\pi j\left(Res\left[\frac{z^4+1}{z^2(-2z^2+5z-2)},0\right]+Res\left[\frac{z^4+1}{z^2(-2z^2+5z-2)},\frac{1}{2}\right]\right)$$

$$Res\left[\frac{z^4+1}{z^2(-2z^2+5z-2)},0\right]=\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}\left(z^2.\frac{z^4+1}{z^2(-2z^2+5z-2)}\right)=\lim_{z\to 0}\left(\frac{z^4+1}{-2z^2+5z-2}\right)'$$

$$=\lim_{z\to 0}\frac{4z^3(-2z^2+5z-2)-(5-4z)(z^4+1)}{(-2z^2+5z-2)^2}=-\frac{5}{4}$$

$$Res\left[\frac{z^4+1}{z^2(-2z^2+5z-2)},\frac{1}{2}\right]=\frac{z^4+1}{z^2(-4z+5)}\bigg|_{z=\frac{1}{2}}=\frac{17}{12}$$

$$\rightarrow A=-2\pi j.\frac{j}{2}.\left(-\frac{5}{4}+\frac{17}{12}\right)=\frac{\pi}{6}$$

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Câu 2. (2đ) Sử dụng thặng dư, tính các tích phân sau:

$$\boldsymbol{b)\ B = \int_{-\infty}^{+\infty} \frac{x\cos 2x}{x^2 - 4x + 8} dx}$$

$$Ta\ có: B = Re \int_{-\infty}^{+\infty} \frac{xe^{j2x}}{x^2 - 4x + 8} dx, R(z) = \frac{z}{z^2 - 4z + 8}$$

Giải phương trình $z^2 - 4z + 8 = 0 \leftrightarrow \left[\begin{smallmatrix} z=2+2j \\ z=2-2j \end{smallmatrix}\right.$ $là\ 2\ cực\ đơn\ của\ R(z)$

Cực điểm $z = 2 + 2j$ nằm trong nữa mặt phẳng trên. Ta có:

$$\int_{-\infty}^{+\infty} \frac{x\cos 2x}{x^2 - 4x + 8} dx = 2\pi j. Res\left[\frac{ze^{j2z}}{z^2 - 4z + 8}, 2 + 2j\right] = 2\pi j. \frac{ze^{j2z}}{2z - 4}\bigg|_{2+2j} = 2\pi j \frac{(2 + 2j)e^{-4+4j}}{4j}$$

$$= \pi(1 + j)e^{-4+4j} = \pi e^{-4}(\cos 4 - \sin 4) + j\pi e^{-4}(\cos 4 + \sin 4)$$

$$\text{Vậy } B = \pi e^{-4}(\cos 4 - \sin 4)$$