ĐỀ THI TOÁN CHUYÊN NGÀNH 2013 – 2014

Câu 1. (1,5 điểm) Tìm chuỗi Laurent của hàm số $f(z) = \frac{3z+1}{z^2-2z-15}$

a. Trong miền: $3 < |z| < 5$

$Ta\ có$: $f(z) = \dfrac{3z+1}{z^2-2z-15} = \dfrac{1}{z+3} + \dfrac{2}{z-5}$

$Với\ |z| < 5 \leftrightarrow \left|\dfrac{z}{5}\right| < 1. Ta\ có$:

$$\frac{1}{z-5} = -\frac{1}{5}.\frac{1}{1-\frac{z}{5}} = -\frac{1}{5}\sum_{n=0}^{+\infty}\left(\frac{z}{5}\right)^n = -\sum_{n=0}^{+\infty}\frac{z^n}{5^{n+1}}$$

$Với\ |z| > 3 \leftrightarrow \left|\dfrac{3}{z}\right| < 1. Ta\ có$:

$$\frac{1}{z+3} = \frac{1}{z}.\frac{1}{1+\frac{3}{z}} = \frac{1}{z}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{3}{z}\right)^n = \sum_{n=0}^{+\infty}\frac{(-3)^n}{z^{n+1}}$$

$$Vậy\ f(z) = \sum_{n=0}^{+\infty}\frac{(-3)^n}{z^{n+1}} - 2\sum_{n=0}^{+\infty}\frac{z^n}{5^{n+1}}$$

b. Trong miền: $2 < |z+1| < 6$

$Với\ |z+1| < 6 \leftrightarrow \left|\dfrac{z+1}{6}\right| < 1. Ta\ có$:

$$\frac{1}{z-5} = \frac{1}{z+1-6} = -\frac{1}{6}.\frac{1}{1-\frac{z+1}{6}} = -\frac{1}{6}\sum_{n=0}^{+\infty}\left(\frac{z+1}{6}\right)^n = -\sum_{n=0}^{+\infty}\frac{(z+1)^n}{6^{n+1}}$$

$Với\ |z+1| > 2 \leftrightarrow \left|\frac{2}{z+1}\right| < 1. Ta\ có$:

$$\frac{1}{z+3} = \frac{1}{z+1+2} = \frac{1}{z+1}.\frac{1}{1+\frac{2}{z+1}} = \frac{1}{z+1}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{2}{z+1}\right)^n = \sum_{n=0}^{+\infty}\frac{(-2)^n}{(z+1)^{n+1}}$$

$$Vậy\ f(z) = \sum_{n=0}^{+\infty}\frac{(-2)^n}{(z+1)^{n+1}} - 2\sum_{n=0}^{+\infty}\frac{(z+1)^n}{6^{n+1}}$$

c. Trong miền: $\sqrt{\mathbf{10}} < |\boldsymbol{z}+\boldsymbol{i}| < \sqrt{\mathbf{26}}$

$Với\ |z+i| < \sqrt{26} \leftrightarrow \left|\frac{z+i}{5+i}\right| < 1. Ta\ có$:

$$\frac{1}{z-5} = \frac{1}{z+i-(5+i)} = -\frac{1}{5+i}.\frac{1}{1-\frac{z+i}{5+i}} = \frac{1}{5+i}\sum_{n=0}^{+\infty}\left(\frac{z+i}{5+i}\right)^n = \sum_{n=0}^{+\infty}\frac{(z+i)^n}{(5+i)^{n+1}}$$

$Với\ |z+i| > \sqrt{10} \leftrightarrow \left|\frac{3-i}{z+i}\right| < 1. Ta\ có$:

$$\frac{1}{z+3} = \frac{1}{z+i+3-i} = \frac{1}{z+i}.\frac{1}{1+\frac{3-i}{z+i}} = \frac{1}{z+i}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{3-i}{z+i}\right)^n = \sum_{n=0}^{+\infty}\frac{(-3+i)^n}{(z+i)^{n+1}}$$

$$\boldsymbol{Vậy\ f(z)} = \sum_{n=0}^{+\infty}\frac{(-3+i)^n}{(z+i)^{n+1}} + 2\sum_{n=0}^{+\infty}\frac{(z+i)^n}{(5+i)^{n+1}}$$

Câu 2. (1,5 điểm) Sử dụng thặng dư, tính tích phân sau:

$$\boldsymbol{I = \int_{-\infty}^{+\infty} \frac{x\sin 4x}{x^2 - 8x + 17} dx}$$

$$Ta\ có: I = Im \int_{-\infty}^{+\infty} \frac{xe^{j4x}}{x^2 - 8x + 17}$$

$$Ta\ có: R(z) = \frac{z}{z^2 - 8z + 17}. Giải\ phương\ trình\ z^2 - 8z + 17 = 0 \leftrightarrow z = 4 \pm j$$

$$Cực\ điểm\ z = 4 + i\ nằm\ trong\ nữa\ mặt\ phẳng\ trên. Ta\ có:$$

$$\int_{-\infty}^{+\infty} \frac{x\sin 4x}{x^2 - 8x + 17} dx = 2\pi j. Res\left[\frac{ze^{j4z}}{z^2 - 8z + 17}, 4 + j\right] = 2\pi j \left.\frac{ze^{j4z}}{2z - 8}\right|_{4+j}$$

$$= 2\pi j \frac{(4 + j)e^{-4+16}}{2j} = \pi e^{-4}[-4\cos 16 + \sin 16] + j\pi[\cos 16 + 4\sin 16]$$

$$Từ\ đó\ suy\ ra\ I = \int_{-\infty}^{+\infty} \frac{x\sin 4x}{x^2 - 8x + 17} dx = \pi[\cos 16 + 4\sin 16]$$

Câu 3. (1,5 điểm): Cho hàm số $\boldsymbol{f(x) = x(\pi - x)(0 < x < \pi), T = 2l = \pi \to l = \frac{\pi}{2}}$

a. Tìm chuỗi Fourier của hàm $\boldsymbol{f(x)}$ theo các hàm sine

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos 2nx + b_n \sin 2nx)$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\pi x - x^2) dx = \frac{2}{\pi} \left(\frac{\pi x^2}{2} - \frac{x^3}{3} \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\pi} \left(-\frac{\pi^3}{12} \right) = -\frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\pi x - x^2) \cos 2nx\, dx$$