Mathematical Theory of Infectious Disease Epidemics

G. Hovhannisyan A. Abrahamyan
M. Khachatryan

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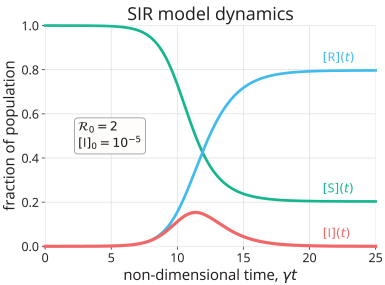
Figure: Example of SIR



Mathematical Formulation

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- Types of Variables
- Important Assumptions

Independent and Dependent Variables

The only dependent variable is t(time) measured in days.

The first set of dependent variables are functions of time, which count people in each group.

$S(t)$ is the number of *susceptible* individuals at given time,

$I(t)$ is the number of *infected* individuals at give time,

$R(t)$ is the number of *recovered* individuals at give time.

Independent and Dependent Variables

The second set of dependent variables represents the *fraction* of the total population in each of the three categories. So, if N is the total population we have:

$$s(t) = \frac{S(t)}{N} \text{ the } \textit{susceptible fraction} \text{ of the population,}$$

$$i(t) = \frac{I(t)}{N} \text{the } \textit{infected fraction} \text{ of the population, and}$$

$$r(t) = \frac{R(t)}{N} \text{the } \textit{recovered fraction} \text{ of the population.}$$

Important Assumptions

- Natural births, deaths, immigration and other similar factors are being ignored.
- The only way an individual leaves the susceptible group is by becoming infected.
- The rate of change of the number of susceptible individuals, $S(t)$, over time depends on the existing number of susceptibles, the number of individuals currently infected, and the level of interaction between susceptibles and infected individuals.
- Each infected individual initiates a fixed number β of contacts per day that can potentially transmit the disease.
- A uniformly mixed population, the proportion of these contacts involving susceptibles is denoted by $s(t)$. Therefore, on average, each infected individual gives rise to $\beta \cdot s(t)$ new daily infections.
- A fixed fraction γ of the infected group will be recovered during any given day.

Important Assumptions

These assumptions inform the derivatives of our dependent variables.

- $\frac{\partial s}{\partial t} = -\beta s(t)i(t)$
- $\frac{\partial r}{\partial t} = \gamma i(t)$

As the sum of *susceptable*, *infected* and *recovered* people gives the whole population, it means that

$$\frac{\partial s}{\partial t} + \frac{\partial i}{\partial t} + \frac{\partial r}{\partial t} = 0 \tag{1}$$

To get the differential equation for infecteds it is enough to plug equations (1) and (2) into equation (3). The result will be:

$$\frac{\partial i}{\partial t} = -\beta s(t)i(t) - \gamma i(t) \tag{2}$$

Important Assumptions

Combining all three equations will give the following system of differential equations:

$$\begin{cases} \frac{\partial s}{\partial t} = -\beta s(t) i(t) \\ \\ \frac{\partial i}{\partial t} = -\beta s(t) i(t) - \gamma i(t) \\ \\ \frac{\partial r}{\partial t} = \gamma i(t) \end{cases}$$

Analytical Solution

Analytical Solution

Analytical Solution

$$\begin{cases} x' = -\beta xy \\ y' = \beta xy - \gamma y \\ z' = \gamma z \\ x + y + z = N \\ x_0 = N_1, y_0 = N_2, z_0 = N_3 \end{cases}$$

Step 1) Differentiating x with respect to time t, and substituting y'.

$$x'' = -\beta \left(x' \left(\frac{-x'}{\beta x} \right) + xy' \right) \implies \frac{-x''}{\beta} = \frac{-(x')^2}{\beta x} + xy'$$

$$\implies xy' = \frac{-x''}{\beta} + \frac{-(x')^2}{\beta x}$$

$$\implies y' = \frac{-1}{\beta} \left(\frac{-x''}{x} - \left(\frac{x'}{x} \right)^2 \right)$$

Analytical Solution

Step 2) Next we insert $y = \frac{-x'}{\beta x}$ into y' and equate it to Step (1)

$$-x' + \frac{\gamma x'}{\beta x} = \frac{-1}{\beta}\left(\frac{-x''}{x} - \left(\frac{x'}{x}\right)^2\right)$$

$$\implies \beta x' - \frac{\gamma x'}{x} = \frac{x''}{x} - \left(\frac{x'}{x}\right)^2$$

$$\implies \frac{x''}{x} - \left(\frac{x'}{x}\right)^2 + \frac{\gamma x'}{x} - \beta x' = 0$$

Step 3) We also have that $y = \frac{-x'}{\beta x}$ and $y = \frac{z'}{\gamma}$. Equating them we get

$$\frac{-x'}{\beta x} = \frac{z'}{\gamma} \implies z' = -\frac{\gamma}{\beta}\left(\frac{x'}{x}\right)$$

Analytical Solution

Step 4) Next we integrate Step (3)

$$
\begin{aligned}
c_1 + z = -\frac{\gamma}{\beta}\int \frac{x'}{x}\,dt \implies & \; c_1 + z = -\frac{\gamma}{\beta}\left(\ln(x) + c_2\right) \\
\implies & \; \ln(x) = -\frac{\beta c_1}{\gamma} - \frac{\beta z}{\gamma} - c_2 \\
\implies & \; x = e^{-\frac{\beta z}{\gamma}} \cdot e^{-\frac{\beta c_1}{\gamma} - c_2} \\
\implies & \; x = x_0 e^{-\frac{\beta z}{\gamma}}
\end{aligned}
$$

Where x_0 is an integration constant.

Analytical Solution

Step 5) From Step(4) we get that

$$x' = -\frac{x_0\beta}{\gamma} z' e^{-\frac{\beta z}{\gamma}}$$

Step 6) Instead of integrating we differentiate Step (3) and get

$$z'' = -\frac{\gamma}{\beta}\left(\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right)$$

Step 7) Finally if we combine Step(6), Step(5) and Step(3) into Step(2) we get

$$z'' = x_0\beta z' e^{-\frac{\beta z}{\gamma}} - \gamma z'$$

Analytical Solution

All the other steps can be found in the paper in Appendix B, but eventually we get the parametric form

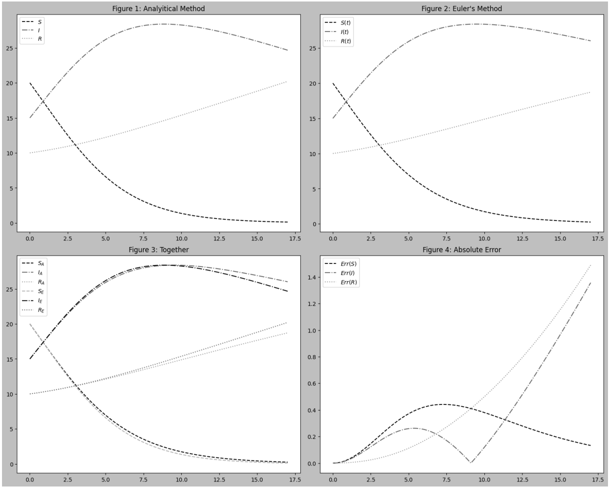
$$\begin{aligned} x &= x_0 u, \\ y &= \frac{\gamma}{\beta} \ln u - x_0 u - \frac{C_1}{\beta}, \\ z &= -\frac{\gamma}{\beta} \ln u \end{aligned}$$

Where $u_0 = e^{-\frac{\beta}{\gamma} N_3}$, $C_1 = -\beta N$.
And the relation between u and t being as follows

$$t = \int_{u_0}^{u} \frac{du}{u(C_1 - \gamma \ln u + x_0 \beta u)}$$

Analytical Solution

Figure: Graphical Examples of the Analytical Solution



Plotted Using Values

$N_1 = 20, \quad N_2 = 15, \quad N_3 = 10, \quad \beta = 0.01, \quad \gamma = 0.02$

Numerical Solutions

- Euler's Method
- Backward Euler's Method

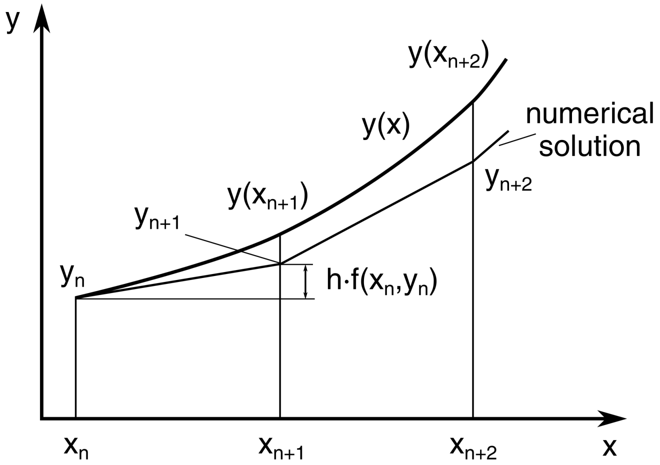
Euler's Method

By using the extension $y'(x) \approx \frac{y_{k+1} - y_k}{h}$ and having the y_0, we can iteratively predict the $y_{k+1} = y(x+h)$, by using the system

$$\begin{cases} y_{k+1} = y_k + h \cdot f(x_k, y_k) \\ y(x_0) = y_0 \end{cases}$$

Euler's Method

Figure: Euler Method Geometrical Interpretation



Euler's Method

In case of SIR, we are solving 3 ODE's, hence we are solving a system for s_{k+1}, i_{k+1}, and r_{k+1}, given the s_0, i_0, r_0

$$\begin{cases} s_{k+1} = s_k - \beta \cdot h \cdot s_k \cdot i_k \\ i_{k+1} = i_k + (\beta \cdot s_k \cdot i_k - \gamma \cdot i_k) \cdot h \\ r_{k+1} = r_k + \gamma \cdot h \cdot i_k \end{cases}$$

Backward Euler's Method

This time, the Method uses a different approach, namely

$$y'(x) \approx \frac{y_k - y_{k+1}}{h}$$

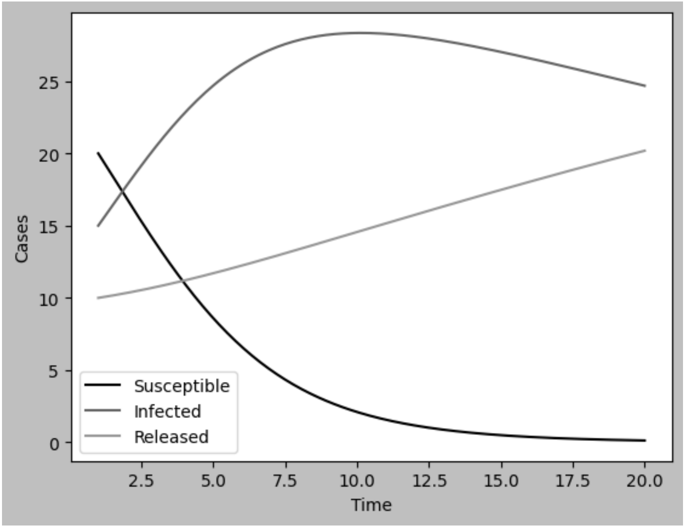
Plugging the SIR ODE's, we will get the system

$$\begin{cases} s_{t+1} = s_t - h \cdot \beta s_{t+1} i_{t+1} \\ i_{t+1} = i_t + h \cdot (\beta s_{t+1} i_{t+1} - \gamma i_{t+1}) \\ r_{t+1} = r_t + h \cdot \gamma i_{t+1} \end{cases}$$

By solving each one of these SLE's, we will get the next approximations for s_k, i_k, r_k

Backward Euler's Method

Figure: Backward Euler Method Example

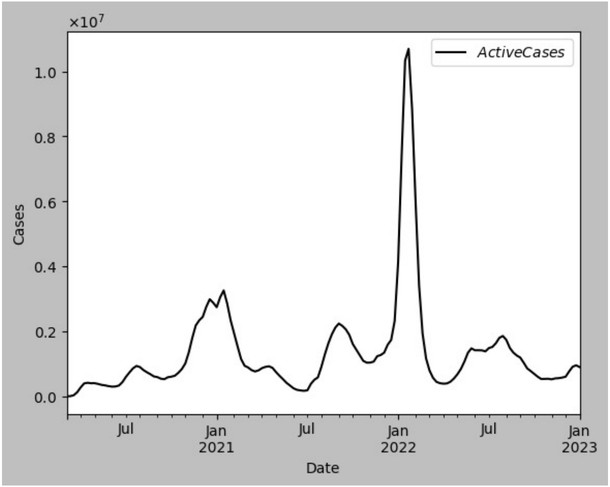


Simulations

Simulations on USA's COVID-19 Data

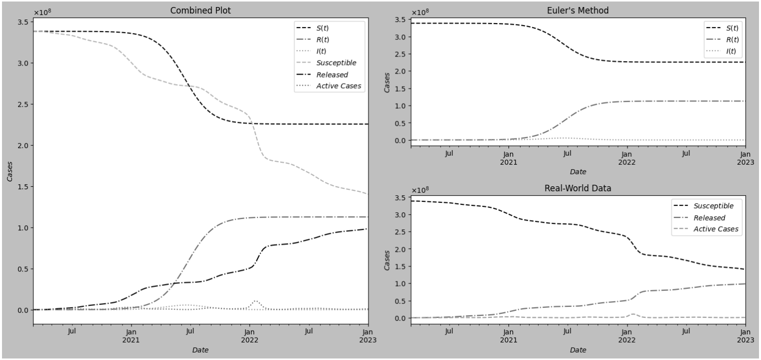
Active Cases

Figure: Active Cases According to USA's COVID-19 Data



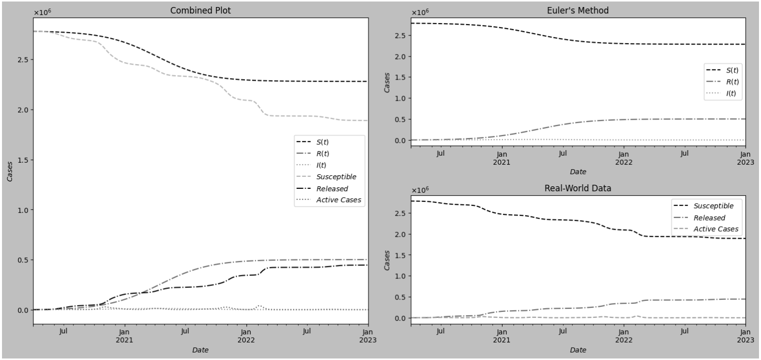
Simulations on USA's COVID-19 Data

Figure: Comparison Plot of Real data and Prediction by SIR



Simulations on Armenia's COVID-19 Data

Figure: Comparison Plot of Real data and Prediction by SIR



Thank You For Attention!
More Information is available at Our Github Repository