Statistics Session 03: Probabilistic Distributions

Continues Distributions

STATISTICS

Topics

- Probabilistic Distributions
- Continuous Distributions
- Normal Distribution
- Uniform Distribution
- Exponential Distribution

What Is a Probabilistic Distributions

In Data Analytics, we **rarely** know outcomes **with certainty**.

Instead of saying:

"Tomorrow's sales will be exactly 120 units"

We say:

"Sales will most likely be around 120, but could reasonably vary"

A **probabilistic distribution** is a formal way to describe this uncertainty by answering to the below three questions:

- What values can a variable take?
- How likely is each value (or range of values)?
- How is uncertainty spread across those values?

From Raw Data to Distribution

When we observe data repeatedly:

- Customer purchases
- Session durations
- Delivery times

Patterns emerge.

A distribution is a **model** that summarizes those patterns instead of listing every observation.

Random Variables

A **random variable** is a numerical description of an uncertain outcome.

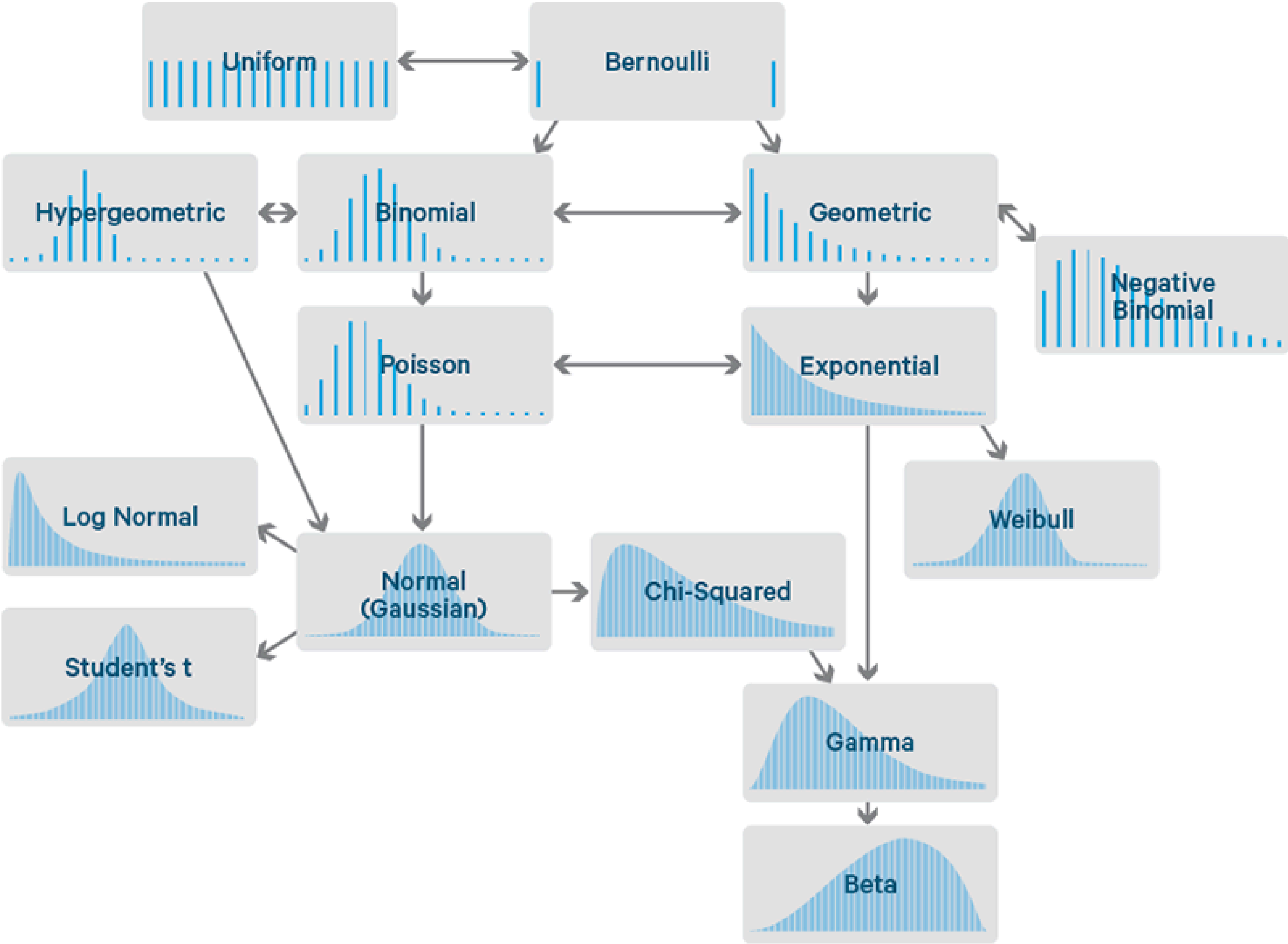
Examples:

- Number of purchases today
- Time (in minutes) until a customer churns
- Whether a user clicks an ad (1 or 0)

From the above example we could notice that the `Random Variable` could be

- **Discrete:** take counts , `yes` / `no` outcomes
  - Number of compaints per day
  - Number of items sold
  - Email opened or not
  - etc..
- **Continuous:** measured not counted
  - Revenue
  - Time
  - Weight
  - Distance
  - etc..

Well-Known Distributions



Continues Distribution

Many real-world business variables are measured: *Revenue, Time, Cost, Duration, Distances etc..*

A distribution is **continuous** if:

- The variable can take **any real value** in a range
- There are infinitely many possible values
- Exact values are not meaningful on their own

> **Remember**
>
> For a continuous random variable X:
>
> $$P(X = x) = 0$$
>
> **This is not a mistake.**
>
> Probability only makes sense over **intervals**:
>
> $$P(a \leq X \leq b)$$

To understand a Continuous probability distribution, we start with a simple experiment:

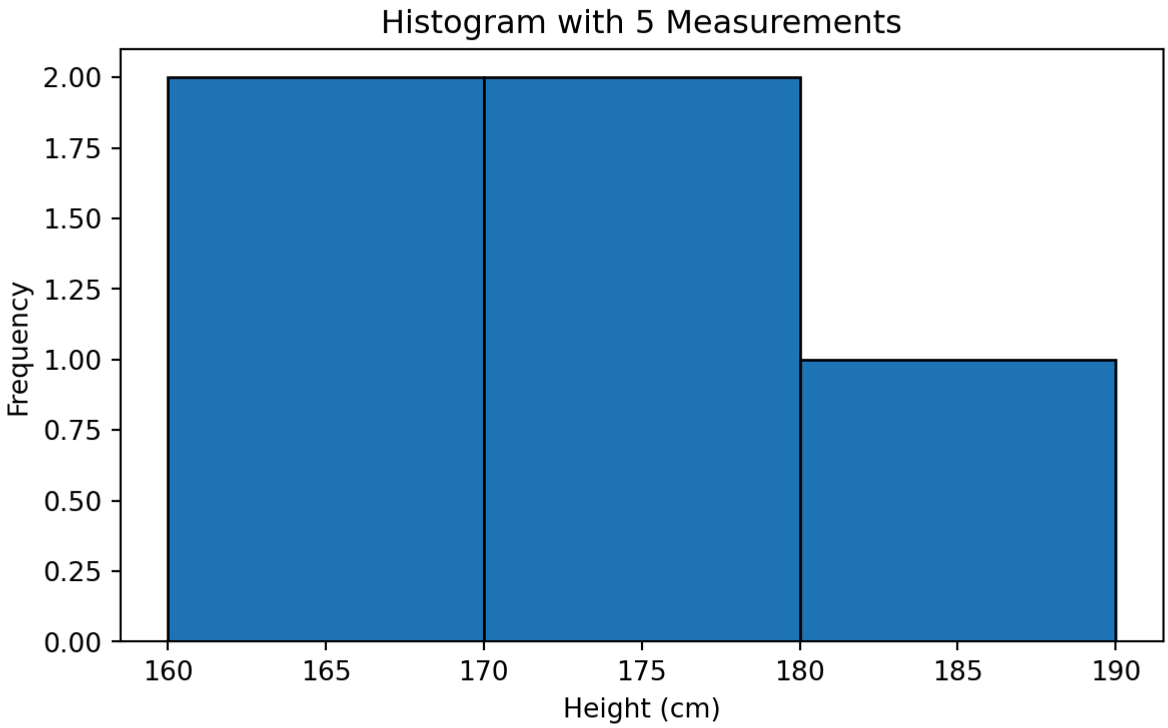
We go outside and measure people's heights, one person at a time. Assume the true average height in the population is around **170 cm**.

We begin with just a few measurements and gradually build the distribution.

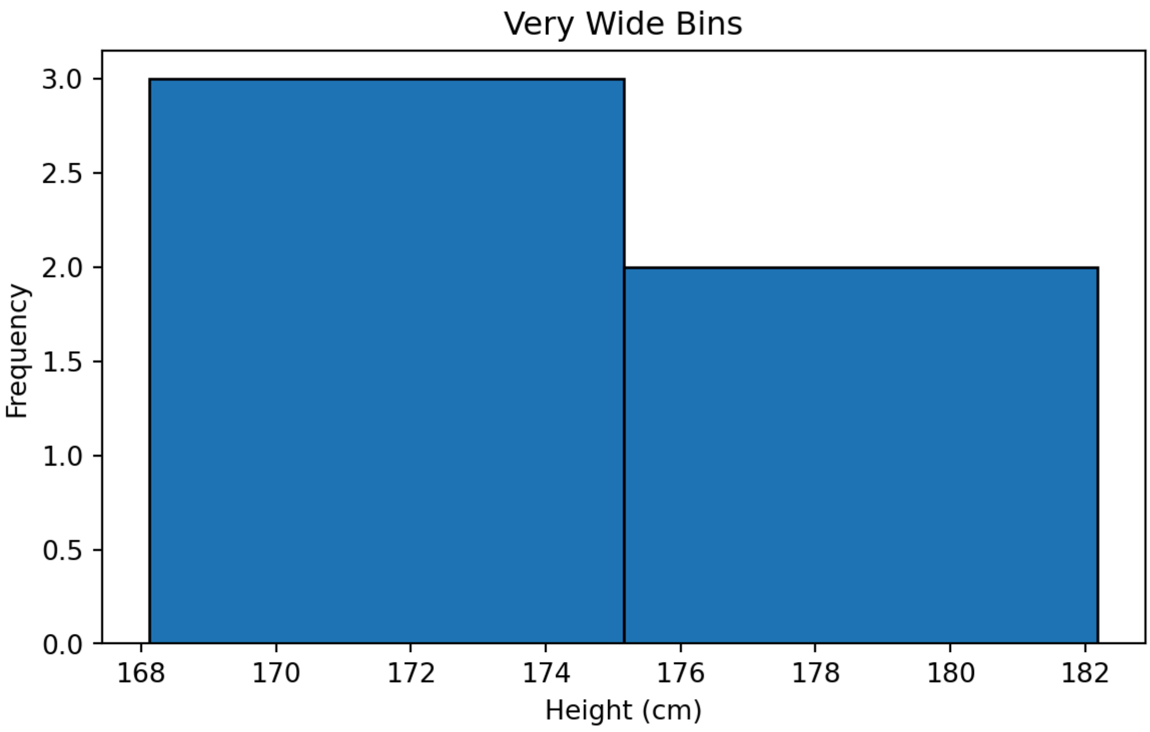
The heights of the first 5 People:

```
[174, 169, 175, 182, 168]
```

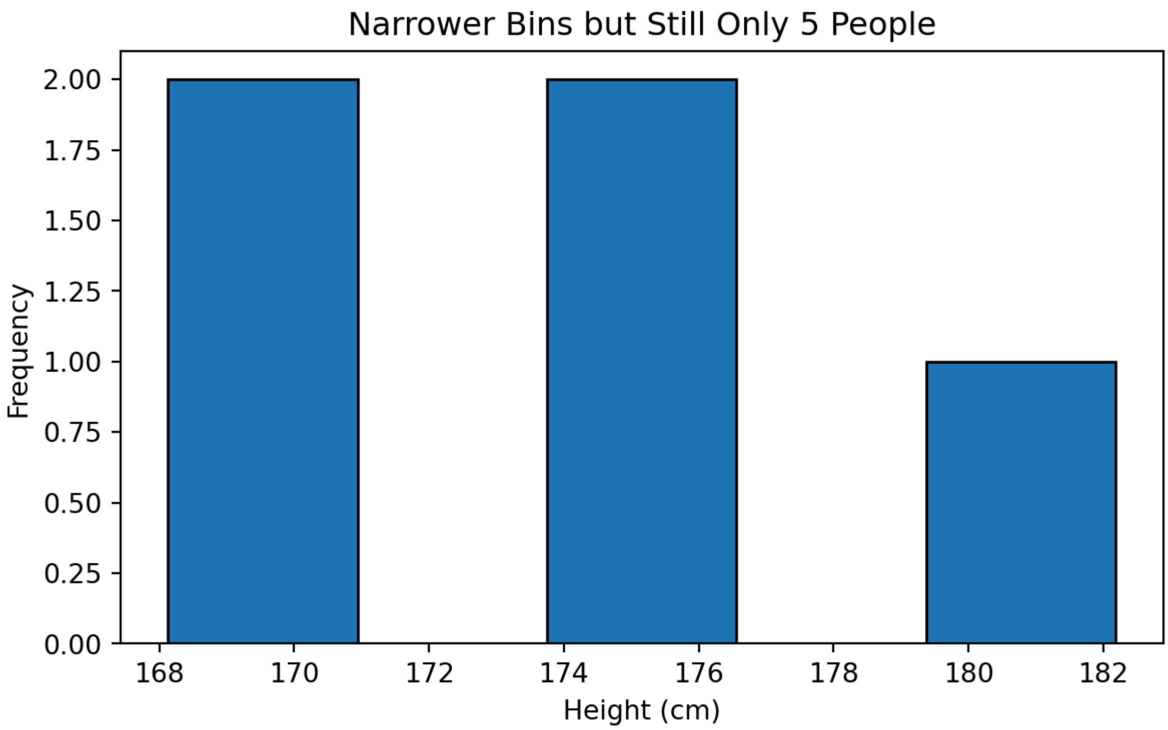
Histogram with 5 People



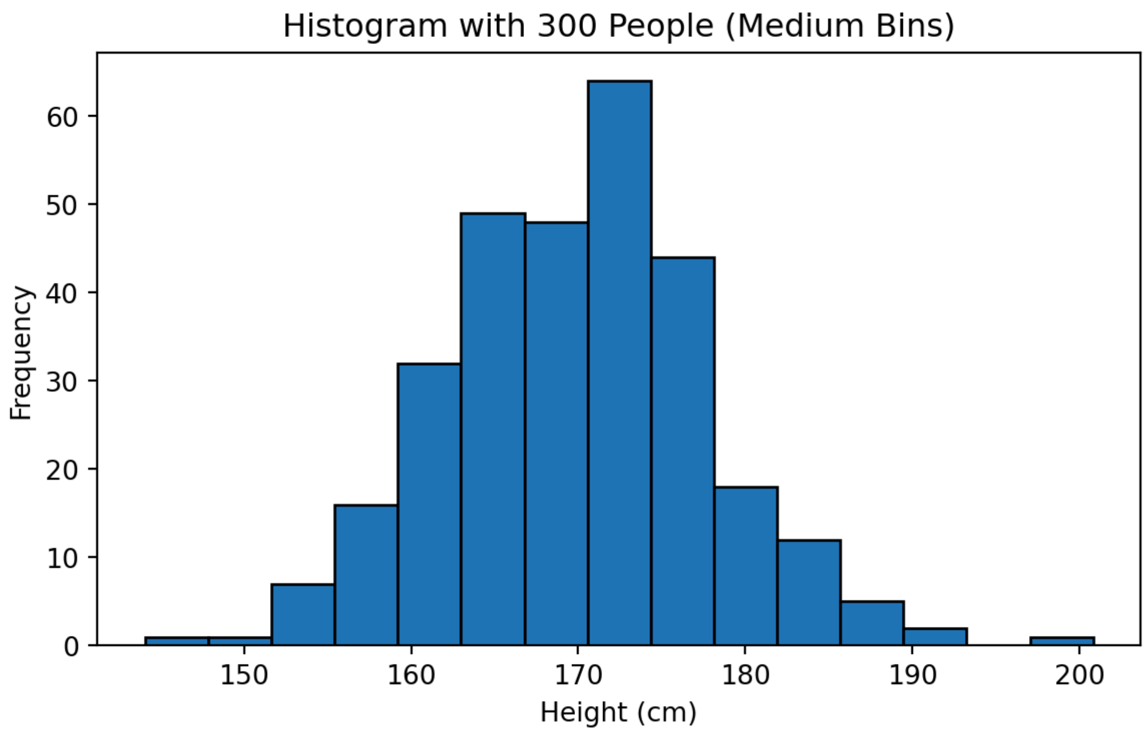
Wider Bins



Narrower Bins



Increasing Sample Size to 300



Probability Density Function (PDF)

Continuous distributions are described by a **Probability Density Function (PDF)**.

The PDF:

- Is always non-negative

- Integrates to 1 over its entire range
- Describes how “dense” probability is around a value

Mathematically:

Imagine as calculating the area under the curves!

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

> **Important**
>
> - **A higher PDF value means:** Values around that point are more likely
> - It does **not** mean: The probability at that exact point is higher

Expected Value (Mean)

The **expected value** represents the long-run average outcome.

For a continuous distribution:

$$E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

> **Business Interpretation**
>
> **Expected value answers:**
>
> “If we repeated this process many times, what average outcome should we expect?”

Variance

Variance measures **spread** or **uncertainty** around the mean.

For a continuous distribution:

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

where $\mu = E[X]$.

> **Business Interpretation**
>
> - Low variance → predictable outcomes
> - High variance → risky or unstable outcomes

Why Continuous Distributions Are Powerful in Analytics

They allow us to:

- Model natural variability
- Estimate probabilities over ranges
- Build confidence intervals
- Perform forecasting and optimization

Continuous Distributions in Practice

In this course, we focus on:

- Normal Distribution
- Uniform Distribution
- Exponential Distribution

Normal Distribution

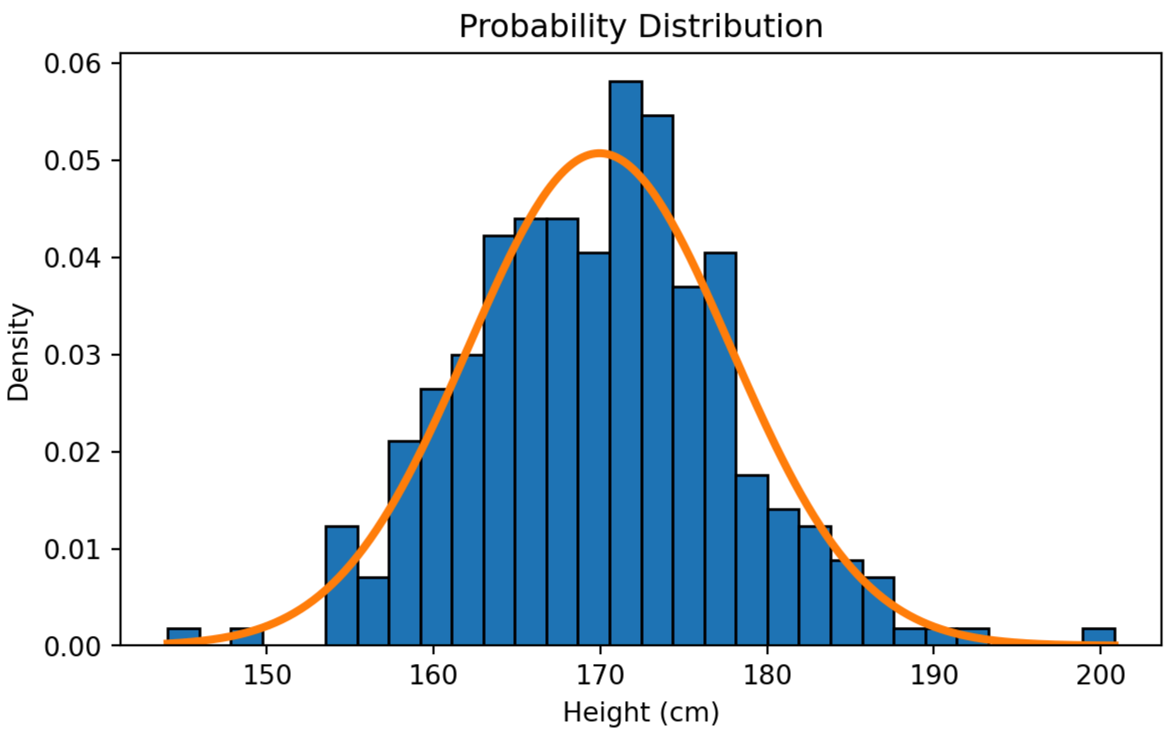
As the sample size increased and bins became finer, the histogram began to resemble a **smooth, symmetric, bell-shaped curve**. This shape corresponds to the **Normal Distribution**.

A **Normal Distribution** is a continuous distribution that is:

- Symmetric around its mean
- Bell-shaped
- Fully described by **two parameters**
  - Mean μ
  - Variance σ^2

It is denoted as:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- $f(x)$ gives the probability density at value x. It describes how concentrated the distribution is around that point.
- μ is the mean.
- σ is the standard deviation.
  - small σ makes the curve narrow and tall,
  - large σ makes it wide and flat.

The term $\frac{1}{\sigma\sqrt{2\pi}}$ ensures that the total area under the curve equals **1**, as required for any probability distribution.

Expecteed Value

For a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$E[X] = \mu$$

> **Business Interpretation**
>
> The expected value represents the **typical** or **average** outcome.
>
> **Examples:**
>
> - Average customer height
> - Average daily revenue
> - Average delivery time

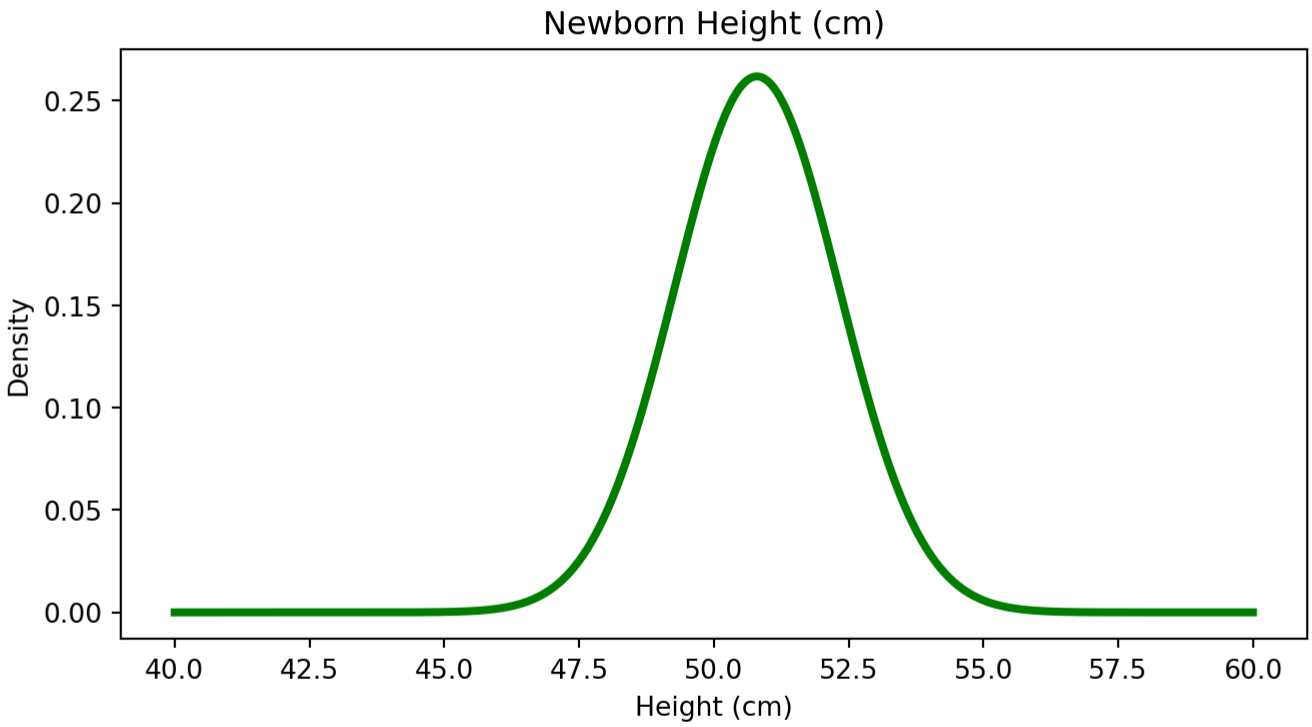
Variance of the Normal Distribution

For a normal random variable:

$$Var(X) = \sigma^2$$

> **Business Interpretation**
>
> Variance controls **spread**:
>
> - Small σ^2 → values tightly clustered around the mean
> - Large σ^2 → values widely spread and more uncertain

Two Normal Distributions



Spreadsheet Demostration

1. **Generation:** `=NORM.INV(RAND(),170,8)`
2. **Mean:** `=AVERAGE(range)`
3. **Standard Deviation:** `=STDEV.P(range)`
4. **Variance:** `=VAR.P(range)`
5. **PDF:** `=NORM.DIST(x, mean, std, FALSE)`
6. **Cumulative Probability CDF:** `=NORM.DIST(x, mean, std, TRUE)`

> **Tip**
>
> Checkout the Normal Distribtion on practice here

Standard Normal Distribution

The **Standard Normal Distribution** is a special case of the normal distribution where:

$$\mu = 0 \quad \text{and} \quad \sigma = 1$$

It is denoted as:

$$Z \sim \mathcal{N}(0, 1)$$

Any normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ can be transformed into a standard normal variable using **standardization**:

$$Z = \frac{X - \mu}{\sigma}$$

> **Recall**
>
> **Standardization allows us to:**

> - Compare values measured on different scales
> - Compute probabilities using a single reference distribution
> - Interpret how many **standard deviations** a value is from the mean

Normal vs Gaussian Distribution

The **Normal Distribution** and the **Gaussian Distribution** are **the same thing**.

They are two names for the **same mathematical distribution**.

- `Gaussian` is the name used in mathematics and physics, after Carl Friedrich Gauss
- `Normal` is the name commonly used in statistics and data analytics

Both refer to the distribution defined by the PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

They are **completely interchangeable terms**.

Uniform Distribution

The **Uniform Distribution** models situations where **all values within a range are equally likely**.

There is:

- No center
- No peak
- No value more likely than another

Imagine the following situations:

- A random number generator picks a number between 0 and 1
- A customer arrives at a store at a random time between 09:00 and 10:00
- A system assigns users randomly to time slots between 0 and 30 minutes

In all these cases:

> Every value in the interval is equally likely

Another example:

Imagine a telecom system that assigns a customer to one of several identical support bots **randomly**.
The system waits somewhere between **0 and 10 seconds** before routing the customer, and **every value in that interval is equally likely**.

This kind of process has **no preference**:

- not more likely to assign earlier,
- not more likely to assign later.

Definition

A **Uniform Distribution** on the interval $[a, b]$ is denoted as:

$$X \sim \text{Uniform}(a, b)$$

where:

- a is the minimum possible value
- b is the maximum possible value

Probability Density Function (PDF)

If $X \sim U(a, b)$, then:

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

- $f(x) = 0$ outside the interval
- $[a, b]$ is equally likely.

Suppose we observe a sample $x_1, x_2, \ldots, x_n$ from a uniform distribution $U(a, b)$.
The likelihood of the parameters (a, b) given the data is:

$$L(a, b \mid x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$$

Because the PDF is constant inside the interval:

- If **all** observations lie in $[a, b]$:

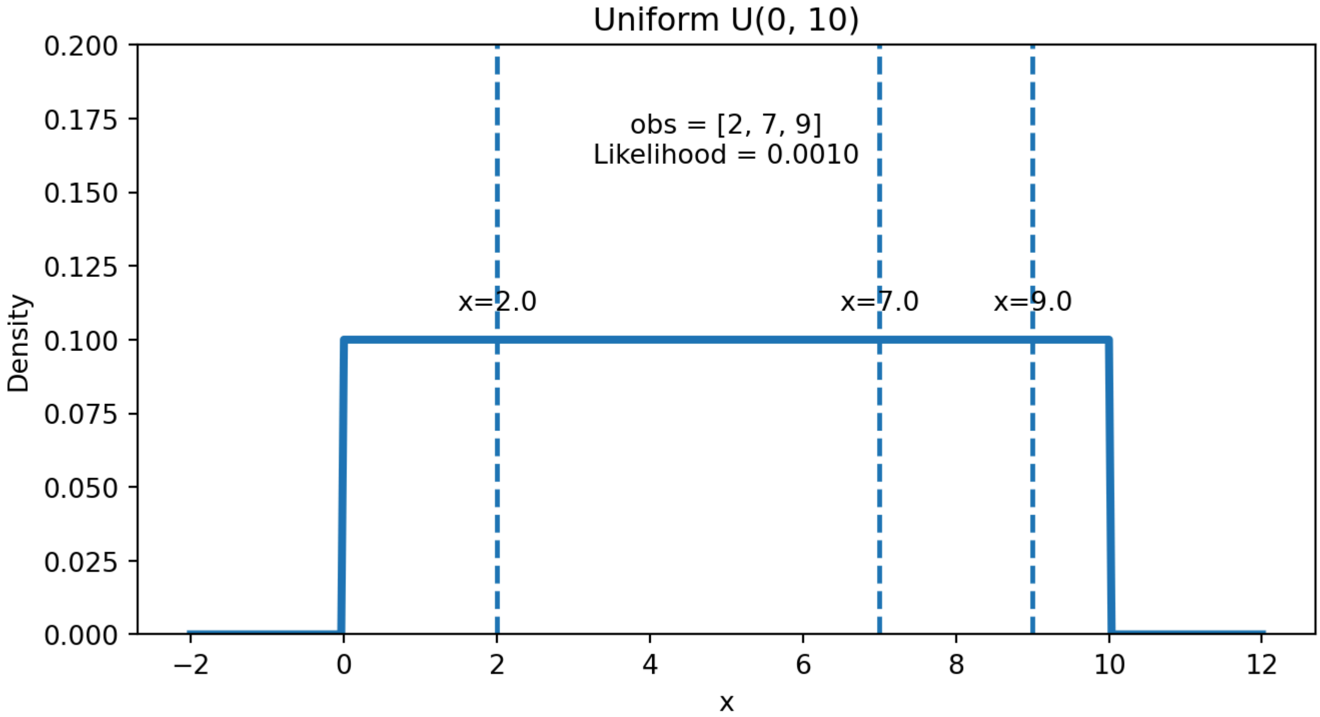
$$L(a, b \mid x_1, \ldots, x_n) = \left(\frac{1}{b-a}\right)^n$$

- If **any** observation lies outside $[a, b]$:

$$L(a, b \mid x_1, \ldots, x_n) = 0$$

So uniform likelihood is simple: **constant if all points are inside, zero otherwise.**

Visualizing Uniform Distribution



Important Interpretation

Because the PDF is flat:

- No value inside $[a, b]$ is more likely than another
- Probability depends **only on interval length**, not position

Expected Value (Mean)

For a uniform random variable $X \sim \text{Uniform}(a, b)$:

$$E[X] = \frac{a+b}{2}$$

Expected Value (Mean)

Business Interpretation

The expected value is simply the **midpoint** of the interval.

Example:

If arrival time is uniformly distributed between 0 and 60 minutes,
the average arrival time is **30 minutes**.

Variance

For a uniform distribution:

$$Var(X) = \frac{(b-a)^2}{12}$$

Business Interpretation

- Wider interval → higher uncertainty
- Narrow interval → more predictable outcomes

If $X \sim \text{Uniform}(0, 10)$:

$$P(2 \leq X \leq 5) = \frac{5-2}{10-0} = 0.3$$

Business Applications of the Uniform Distribution

Uniform distributions are used when:

- Random assignment is required
- No prior preference exists

Examples:

- A/B testing randomization
- Load balancing
- Simulation baselines
- Random sampling assumptions

Probability as Area

Because the density is constant:

$$P(c \leq X \leq d) = \frac{d-c}{b-a}$$

This is simply the **fraction of the interval covered**.

Spreadsheet Demonstration (Uniform Distribution)

1. **Generate Uniform Data:** `=RAND()*(b-a)+a`
    - Example for $[0, 10]$: `=RAND()*10`
2. **Expected Value:** `=(a+b)/2`
3. **Variance:** `=(b-a)^2/12`
4. **Probability Between Two Values c and d:** `=(d-c)/(b-a)`

Important

The uniform distribution assumes:

- No structure
- No memory
- No preferred values

Exponential Distribution

We now move to a continuous distribution that models **waiting time until an event occurs**.

This distribution is fundamentally different from Normal and Uniform distributions because:

- It is **not symmetric**
- It is **right-skewed**
- It explicitly models **time-to-event behavior**

Consider a supermarket or retail chain.
Customers arrive at the checkout lanes **randomly**, and the store wants to model:

How long until the next customer arrives at the counter?

If arrivals are independent and have no memory, then the waiting time** between customer arrivals follows an **Exponential distribution**.**

- if you've been waiting 4 minutes already, the next customer is **not** due
- every moment is a fresh start
- the past does NOT influence the future (Markov Chain)

This is very common in retail analytics:

- *time until next customer walks into the store,*
- *time until next person reaches a self-checkout station,*
- *time until next event in an online store: purchase, add-to-cart, click, etc.*

All of these waiting times are modeled by the *Exponential distribution.*

Definition

A random variable X follows an **Exponential Distribution** if:

$$X \sim \text{Exp}(\lambda)$$

where:

- $\lambda > 0$ is the **arrival rate** (events per unit time)
- $\frac{1}{\lambda}$ is the **average waiting time**

Probability Density Function (PDF)

If $X \sim \text{Exp}(\lambda)$:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Where:

- λ = customer arrival **rate** (customers per minute)
- $1/\lambda$ = **average waiting time**

Expected Value (Mean)

$$E[X] = \frac{1}{\lambda}$$

Business Interpretation

If customers arrive at a rate of: $\lambda = 2$ customers per minute then the expected waiting time is: $1/2 = 0.5$ minutes

---

Variance

$$Var(X) = \frac{1}{\lambda^2}$$

Business Interpretation

- High arrival rate → lower variability
- Low arrival rate → higher uncertainty in waiting times

Likelihood for Observed Retail Data

Suppose we measure actual waiting times between customer arrivals:

$$x_1, x_2, \ldots, x_n$$

The likelihood of λ given the data is:

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

This simplifies to:

$$L(\lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

Log-likelihood:

$$\ell(\lambda) = n \ln(\lambda) - \lambda \sum x_i$$

Maximum Likelihood Estimate (MLE):

$$\hat{\lambda} = \frac{n}{\sum x_i}$$

Interpretation:

- fast arrivals → large λ
- slow arrivals → small λ

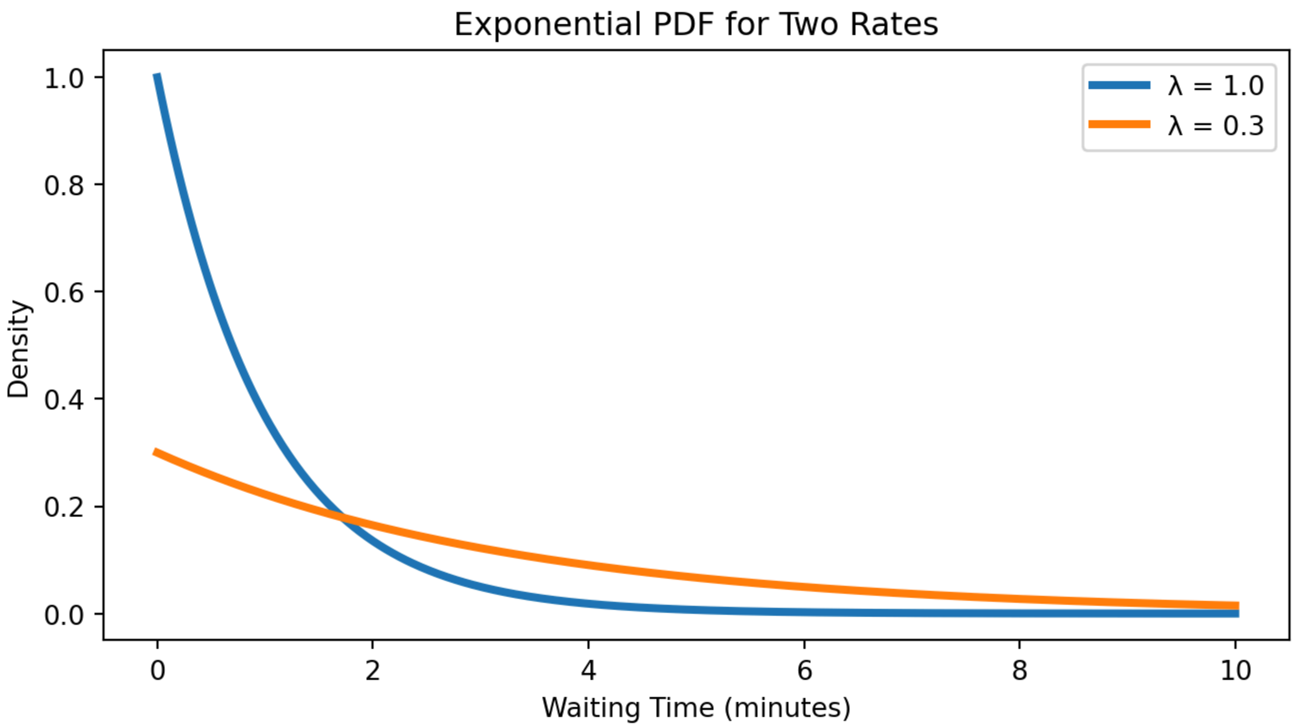
Just like checkout traffic in a retail store.

Visualization

We use example waiting times in minutes: `obs = [1.2, 0.5, 2.0, 0.8]`

These could be times between customers reaching a checkout lane.

- Left plot → PDF comparison for $\lambda = 1$ and $\lambda = 0.3$
- Right plot → Likelihood curve for the observed retail dat



Interpretation

- When customers arrive quickly and consistently, the waiting times shrink → the likelihood favors a **large** λ.
- When customers arrive sporadically or slowly, the waiting times grow → the likelihood favors a **small** λ.

In our observed data:

- Average waiting time = $4.5/4 = 1.125$ minutes
- MLE: $\hat{\lambda} = 4/4.5 = 0.889$ customers/minute

Meaning:

- the best-fitting model suggests approximately **0.89 customers per minute**,
- which corresponds to **one customer roughly every 1.1 minutes**.

This type of analysis is central in retail analytics for understanding staffing requirements, managing checkout lanes, predicting peak hours, and optimizing store operations.

Spreadsheet Demonstration (Exponential Distribution)

1. **Generate exponential data:** `=-LN(1-RAND())/lambda`
2. **Expected value:** `=1/lambda`
3. **Variance:** `=1/(lambda^2)`
4. **CDF (event occurs within x):** `=1-EXP(-lambda*x)`

Watching Materials

1. [Normal Distribution]
2. [Exponential Distribution]