# 1 Linear Algebra

## 1)

Because $M$ is real symmetric matrix, we can use eigendecomposition to decompose $M$ into: $M = UDU^T$, in which $U$ is an orthogonal matrix, and $D$ is a diagonal matrix containing all eigenvalues of $M : D = diag(\lambda_1, \lambda_2, ..., \lambda_k)$. Hence,

$$\begin{aligned} x^T M x &= x^T (UDU^T) x \\ &= (x^T U) D (x^T U)^T \end{aligned}$$

Set $y = x^T U$, we have:

$$\begin{aligned} x^T M x &= y D y^T \\ &= \sum_i^k \lambda_i \cdot y_i^2 \qquad \text{(Because } D \text{ is diagonal matrix)} \end{aligned}$$

So,

$$\begin{aligned} &\iff \lambda_{min} \sum_i^k y_i^2 \leq x^T M x \leq \lambda_{max} \sum_i^k y_i^2 \\ &\iff \lambda_{min} \cdot y^T y \leq x^T M x \leq \lambda_{max} \cdot y^T y \end{aligned}$$

On the other hand, $y^T y = (x^T U)(x^T U)^T = x^T U U^T x = x^T x$ (Because $U$ is orthogonal). Then we have:

$$\begin{aligned} &\lambda_{min} \cdot x^T x \leq x^T M x \leq \lambda_{max} \cdot x^T x \\ &\iff \lambda_{min} \cdot \|x\|_2^2 \leq x^T M x \leq \lambda_{max} \cdot \|x\|_2^2 \\ &\iff \lambda_{min} \leq \frac{x^T M x}{\|x\|_2^2} \leq \lambda_{max} \end{aligned}$$

## 2)

Because $M$ is real symmetric matrix, we can use eigendecomposition to decompose $M$ into: $M = UDU^T$, in which $U$ is an orthogonal matrix, and $D$ is a diagonal matrix containing all eigenvalues of $M$. Hence,

$$\begin{aligned} M^{100} &= (UDU^T)^{100} \\ &= UD^{100}U^T \qquad (because\ UU^T = I, since\ U\ is\ orthogonal) \end{aligned}$$

To calculate $D^{100}$, assume $D = diag(\lambda_1, \lambda_2, .., \lambda_k)$, then $D^{100} = diag(\lambda_1^{100}, \lambda_2^{100}, .., \lambda_k^{100})$.

## 3)

(i) We can see that this matrix is symmetric and contains only real number, so it is a Hermitian matrix, hence, it is positive definite iff its principal minors are positive, more formally:

- $\begin{vmatrix} 2 \end{vmatrix} > 0$ (Obvious)
- $\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} > 0$ (Obvious)
- $\begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & x \end{vmatrix} > 0 \iff x > \frac{7}{5}$

$\Rightarrow$ We conclude that $x > \frac{7}{5}$

(ii) We transform matrix into row echelon form:

$$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & x \end{vmatrix}$$
$$\equiv \begin{vmatrix} 1 & -0.5 & -0.5 \\ 0 & 2.5 & -1.5 \\ 0 & -1.5 & \frac{2x-1}{2} \end{vmatrix}$$
$$\equiv \begin{vmatrix} 1 & -0.5 & -0.5 \\ 0 & 1 & -0.6 \\ 0 & 0 & \frac{2x-1}{2} - 0.9 \end{vmatrix}$$

$\Rightarrow$ We conclude that to have rank-2 matrix:

$$\frac{2x-1}{2} - 0.9 = 0$$
$$x = \frac{7}{5}$$

## 4)

(i) Vector (1,1,1) is in null space of matrix, hence,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -5 & 3 & y \end{bmatrix} . \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$
$$y = -2$$

(ii) Sum of eigenvalues is equal to the trace of matrix $\Rightarrow$ to have sum of eigenvalues is equal to 0, we need $2 + 2 + y = 0 \iff y = -4$.

## 5)

Suppose vector $x$ has number of dimensions $k$. We have:

$$\|x\|_1 = \sum_i^k |x_i|$$
$$\text{and } \|x\|_\infty = \max_i^k |x_i|$$

Hence, we can conclude:

$$\|x\|_\infty \leq \|x\|_1 \leq k\|x\|_\infty$$

# 2 Probability Theory

## 1)

A = {2, 4, 6}; B = {1, 2, 3, 4} and C = {1, 3, 5}

$P(A) = \frac{3}{6} = \frac{1}{2}$

$P(B) = \frac{4}{6} = \frac{2}{3}$

$P(C) = \frac{3}{6} = \frac{1}{2}$

$P(A) \cdot P(B) = \frac{1}{3}$

$P(A) \cdot P(C) = \frac{1}{4}$

$P(A, B) = P(\{2, 4\}) = \frac{2}{6} = \frac{1}{3}$

$P(A, C) = P(\{\}) = 0$

Because $P(A) \cdot P(B) = P(A, B)$, A and B are independent. Because $P(A) \cdot P(C) \neq P(A, C)$, A and C are **not** independent.

## 2)

We have facts:
$P(Macintosh) = 0.3; P(Windows) = 0.5; P(Linux) = 0.2$
$P(virus|Macintosh) = 0.65; P(virus|Windows) = 0.82; P(virus|Linux) = 0.5$

Then, we need to calculate $P(Windows|virus)$. We have:

$$\begin{aligned}
P(Windows|virus) &= \frac{P(Windows, virus)}{Pvirus} \\
&= \frac{P(virus|Windows).P(Windows)}{P(virus, Macnitosh) + P(virus, Windows) + P(virus, Linux)} \\
&= \frac{P(virus|Windows).P(Windows)}{P(virus|Macnitosh).P(Macnitosh) + P(virus|Windows).P(Windows) + P(virus|Linux).P(Linux)} \\
&= \frac{0.82 \times 0.5}{0.65 \times 0.3 + 0.82 \times 0.5 + 0.5 \times 0.2} \\
&\approx 0.58 = 58\%
\end{aligned}$$

## 3)

(i) With:

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{1+x}, & \text{otherwise} \end{cases}$$

We need to verify whether $\int_{-\infty}^{\infty} f(x)dx = 1$.
We have:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{0}^{\infty} \frac{1}{1+x}dx && \text{Because } f(x) = 0 \text{ with } x < 0 \\
&= ln(x+1)\Big|_0^{\infty} \\
&= ln(\infty + 1) - ln(0 + 1) \\
&= ln(\infty) \\
&= \infty > 1
\end{aligned}$$

Hence, $f(x)$ is not a valid PDF.

(ii) With:

$$g(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{(1+x)^2}, & \text{otherwise} \end{cases}$$

We need to verify whether $\int_{-\infty}^{\infty} g(x)dx = 1$.
We have:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &= \int_{0}^{\infty} \frac{1}{(1+x)^2}dx && \text{Because } g(x) = 0 \text{ with } x < 0 \\
&= \frac{-1}{1+x}\Big|_0^\infty \\
&= \frac{-1}{1+\infty} - \frac{-1}{1+0} \\
&= 1 - \frac{1}{\infty} = 1
\end{aligned}$$

Hence, $g(x)$ is a valid PDF. Mean of $g(x)$:

$$\begin{aligned}
E(X) &= \int_{0}^{\infty} x\frac{1}{(1+x)^2}dx && \text{Because } g(x) = 0 \text{ with } x < 0 \\
&= \int_{0}^{\infty} \frac{1+x}{(1+x)^2}dx - \int_{0}^{\infty} \frac{1}{(1+x)^2}dx \\
&= \int_{0}^{\infty} \frac{1}{(1+x)}dx - \int_{0}^{\infty} \frac{1}{(1+x)^2}dx \\
&= ln(x+1)\Big|_0^\infty - \frac{-1}{1+x}\Big|_0^\infty \\
&= \infty - 1 = \infty
\end{aligned}$$

## 4)

We have:
$P(Y = 1) = b$ and $P(Z = 1) = (1 - a)$ (Because x is uniformly distributed between (0,1)).
Also, $P(Y = 1, Z = 1) = P((0 < x < b)\ \&\ (a < x < 1)) = b - a$

Hence:
$P(Y = 1).P(Z = 1) = b(1 - a) = b - ab \neq b - a = P(Y = 1, Z = 1)$
So, Y and Z are **not** independent.

## 5)

Because X and Y are independent, we have:

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Hence,

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty} x \cdot y \cdot f_{XY}(x, y)dxdy \\
&= \int_{-\infty}^{\infty} x \cdot y \cdot f_X(x) \cdot f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} x \cdot f_X(x)dx \cdot \int_{-\infty}^{\infty} y \cdot f_Y(y)dy \\
&= E(X) \cdot E(Y)
\end{aligned}$$

Covariance:

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X) \cdot E(Y) \\ &= 0 && \text{Because X and Y are independent, so E(XY) = E(X) $\cdot$ E(Y)} \end{aligned}$$

# 3 Multivariable Calculus

Step 1 - Find critical points, i.e. both partial derivatives must be zero at those points:

$$J_f(x,y) = [f_x, f_y] = [3x^2 - 3y^2, -6xy] = [0,0]$$

Which means

$$3x^2 - 3y^2 = 0 \tag{1}$$
$$-6xy = 0 \tag{2}$$

Equation (2) is satisfied if $x = 0$ or $y = 0$. We consider these two solutions as two separate cases. For each case we will find solutions for equation (1).

Case 1: Let $x = 0$. We plug that into equation (1): $3 \cdot 0^3 - 3y^2 = 0 \Leftrightarrow y = 0$. So if $x = 0$, then $y = 0$ in order to fulfill both equations. Therefore, $(0,0)$ is a critical point.

Case 2: Let $y = 0$. We plug that into equation (1): $3x^3 - 3 \cdot 0^2 = 0 \Leftrightarrow x = 0$. So we again get $(0,0)$ as our critical point. Therefore, $(0,0)$ is our only critical point.

Step 2 - Classify critical point:
In order to classify the critical point, we can try to do the second partial derivative test for two variables. For that we first need to compute the Hessian matrix of $f$:

$$H_f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}$$

Then we need to determine the defiteness of $H_f$ at the critical point $(0,0)$:

$$\det(H_f(0,0)) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Since $h_{11} = 0$ and $\det(H_f) = 0$, the second partial derivative test is unfortunately inconclusive.

We can try to find the type of the point by a more geometric approach by looking what values $f$ takes arbitrarily near $(0,0)$:

For $(x,y) = (a,0), a > 0 : f(x,y) = a^3 > 0$.
For $(x,y) = (a,0), a < 0 : f(x,y) = a^3 < 0$.
For $(x,y) = (0,b), b > 0 : f(x,y) = 0$.
For $(x,y) = (0,b), b < 0 : f(x,y) = 0$.

Since $f$ stays 0 for $f(0,y)$ but takes values of $f(x,0) < 0$ for $x < 0$ and $f(x,0) > 0$ for $x > 0$, we can confirm that $(0,0)$ is a saddle point.

The maximum value $f$ can take is $\infty$, since $\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} x^3 = \infty$.
Analog, the minimum value $f$ can take is $-\infty$, since $\lim_{x\to-\infty} f(x,0) = \lim_{x\to-\infty} x^3 = -\infty$.