To show that the fully disordered free energy,

$$F_{fd}(T) = \frac{1}{\Omega}\sum_{i=1}^{\Omega} e_i - k_B T \ln(\Omega); , \tag{1}$$

is greater than or equal to the Helmholtz free energy,

$$F(T) = -k_B T \ln(Z) = -k_B T \ln(Z) = -k_B T \ln(\sum_{i=1}^{\Omega} e^{-\frac{e_i}{k_B T}}) , \tag{2}$$

it is sufficient to show that,

$$f(N,T) - F(T) \geq 0. \tag{3}$$

Where $f(N,T)$ is the subspace kernel function given by,

$$f(N,T) = \frac{1}{N}\sum_{i=1}^{N} e_i - k_B T \ln(N) , \tag{4}$$

where $N \leq \Omega$. So the writing $f(N,T) - F(T) \geq 0$ explicitly gives,

$$\frac{1}{N}\sum_{i=1}^{N} e_i + k_B T[\ln(Z) - \ln(N)] \geq 0; . \tag{5}$$

Subtracting the sum, dividing by $k_B T$, then exponentiating, and then multiplying by N gives,

$$Z = \sum_{i=1}^{\Omega} e^{-\frac{e_i}{k_B T}} \geq N e^{-\frac{1}{k_B T N}\sum_{i=1}^{N} e_i} . \tag{6}$$

With the set of energies is defined as begin ordered $\{e_i\}_\Omega \equiv e_1 \leq e_i \leq e_\Omega$ and the fact that all terms on the left side of the inequality are positive it is sufficient to show that,

$$\sum_{i=1}^{N} e^{-\frac{e_i}{k_B T}} \geq N e^{-\frac{1}{k_B T N}\sum_{i=1}^{N} e_i} , \tag{7}$$

to prove that the subspace kernel function is greater than or equal to the exact Helmholtz free energy. Mathmaticaly this is equivalent to proving,

$$\sum_{i=1}^{N} e^{-x_i} \geq N e^{-\frac{1}{N}\sum_{i=1}^{N} x_i} . \tag{8}$$

Where x_i comes from a set of real numbers $\{x_i\}_N$.