

To show that the fully disordered free energy,

$$F_{fd}(T) = \frac{1}{\Omega} \sum_{i=1}^{\Omega} e_i - k_B T \ln(\Omega);, \quad (1)$$

is greater than or equal to the Helmholtz free energy,

$$F(T) = -k_B T \ln(Z) = -k_B T \ln\left(\sum_{i=1}^{\Omega} e^{-\frac{e_i}{k_B T}}\right), \quad (2)$$

it is sufficient to show that,

$$f(N, T) - F(T) \geq 0. \quad (3)$$

Where  $f(N, T)$  is the subspace kernel function given by,

$$f(N, T) = \frac{1}{N} \sum_{i=1}^N e_i - k_B T \ln(N), \quad (4)$$

where  $N \leq \Omega$ . So the writing  $f(N, T) - F(T) \geq 0$  explicitly gives,

$$\frac{1}{N} \sum_{i=1}^N e_i + k_B T [\ln(Z) - \ln(N)] \geq 0;. \quad (5)$$

Subtracting the sum, dividing by  $k_B T$ , then exponentiating, and then multiplying by  $N$  gives,

$$Z = \sum_{i=1}^{\Omega} e^{-\frac{e_i}{k_B T}} \geq N e^{-\frac{1}{k_B T N} \sum_{i=1}^N e_i}. \quad (6)$$

With the set of energies is defined as begin ordered  $\{e_i\}_{\Omega} \equiv e_1 \leq e_i \leq e_{\Omega}$  and the fact that all terms on the left side of the inequality are positive it is sufficient to show that,

$$\sum_{i=1}^N e^{-\frac{e_i}{k_B T}} \geq N e^{-\frac{1}{k_B T N} \sum_{i=1}^N e_i}, \quad (7)$$

to prove that the subspace kernel function is greater than or equal to the exact Helmholtz free energy. Mathematically this is equivalent to proving,

$$\sum_{i=1}^N e^{-x_i} \geq N e^{-\frac{1}{N} \sum_{i=1}^N x_i}. \quad (8)$$

Where  $x_i$  comes from a set of real numbers  $\{x_i\}_N$ .