THE SLICE, REDUCTION, AND GAP THEOREMS

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Abstract

We wish to show that after localizing at a certain element D in its $\mathrm{RO}(C_8)$-graded homotopy groups, $\mathrm{MU}^{((C_8))}$ satisfies the gap property - its homotopy groups vanish at non-equivariant degrees -1, -2, and -3. This fact - the Gap Theorem - was one of the three key developments Hill-Hopkins-Ravenel used to show maps of spectral sequences involving MU, $\mathbb{S}$, and $D^{-1}\mathrm{MU}^{((C_8))}$ put enough algebraic constraints in order to settle the Kervaire Invariant One problem. We have previously constructed a refinement of homotopy A that approximates $\mathrm{MU}^{((C_8))}$. In fact, after taking a relative smash product with the sphere spectrum $\mathbb{S}$ over A, we get back the familiar Mackey-functor-valued Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ - this is the Reduction Theorem. We use this to characterize the homotopy fibers of the slice tower of $\mathrm{MU}^{((C_8))}$ through the Slice Theorem. The Gap Theorem will quickly follow from these and previous results.

References.

- Sections 6-8 of HHR
- Sections 7-8 of Haynes Miller's Seminaire Bourbaki report on HHR
- Doug Ravenel's notes from a talk on the problem at Tokyo City University

1. Introduction

A framed k-dim manifold in Ω^{fr} corresponds to a stable homotopy class $\pi_k(\mathbb{S})$. The Kervaire invariants are defined at dimensions $4k+2$. Maps between spectral sequences will give us restrictions on $\pi_k(\mathbb{S})$. HHR create a spectrum that has enough nice properties that we can say a lot more about what elements in $\pi_*(\mathbb{S})$ have Kervaire invariant one. This spectrum is *almost* $\mathrm{MU}^{((C_8))}$ - we just need to flip some homotopy class. One of the desired properties is the gap property:

Theorem 1.1 (Gap Theorem). *Let $G = C_{2^n}$, ρ_G be its regular representation, and ℓ arbitrary. For any choice of:*

$$D \in \pi_{\ell\rho_G}\mathrm{MU}^{((G))}$$

We have, for $-4 < i < 0$:

$$\pi_i(D^{-1}\mathrm{MU}^{((G))}) = 0$$

Remark 1.2. Really, all we need is the $i = -2$ case.

We start by the following intermediate result:

Theorem 1.3 (Slice Theorem). $\mathrm{MU}^{((G))}$ *is isotropic and pure*

There's another big intermediate result we will prove that deserves to be called a theorem. Earlier, we developed a refinement of homotopy A of $\mathrm{MU}^{((G))}$, given by a wedge of slice cells. Relative to the refinement, $\mathrm{MU}^{((G))}$ is "close to" the standard equivariant Eilenberg-MacLane spectrum:

Theorem 1.4 (Reduction Theorem). *By smashing over the refinement, we get:*

$$\mathrm{MU}^{((G))} \wedge_A \mathbb{S} \simeq H\underline{\mathbb{Z}}$$

The Slice Theorem will follow from the Reduction Theorem, and the Gap Theorem will quickly follow from the Slice Theorem and results we've already developed. The Reduction Theorem is a very technical result, and will require a "converse" to the Slice Theorem. I will conveniently push it to the end to spend as little time talking about it as I can.

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Avoiding problems.

Note 1.5. We will implicitly use fibrant or cofibrant replacement of spectra when needed.

Note 1.6. We will fix some finite, abelian group G, usually C_{2^n}.

Note 1.7. Spectra will be assumed to be G-spectra.

Note 1.8. All representations will be real and orthogonal

Note 1.9. We will pretend $0 \in \mathbb{N}$. In this paper, I have redefined my usual `\N` command to be $\mathbb{N}_0$, so this doesn't matter if you're reading these notes. I will certainly forget the subscript in the presentation.

Reminders. Some notation:

- $\mathrm{Res}^G_H(-)$ has a left adjoint $\mathrm{Ind}^G_H(-)$. HHR call this $G_+ \underset{H}{\wedge} X$.

Definition 1.10. For ρ_K the regular representation of $K \subset G$, we write, for any m

$$\widehat{S}(m, K) = \mathrm{Ind}^G_K S^{m\rho_K}$$

A slice cell is a G-spectrum (weakly) of the form $\widehat{S}(m, K)$ (regular) or $\Sigma^{-1}\widehat{S}(m, K)$ (irregular).

Definition 1.11. The dimension of $\widehat{S}(m, K)$ is $m|K|$ and of $\Sigma^{-1}\widehat{S}(m, K)$ is $m|K| - 1$.

Definition 1.12. A slice cell is called induced if it is also $\mathrm{Ind}^G_H \widehat{S}$ if $\widehat{S}$ is a slice cell for H. An induced cell is free if $H = 1$ and isotropic otherwise.

Recall the slice tower, with n-slices being the homotopy fibers on the right:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \cdots \arrow[l] \\
 & P^3X \arrow[d] & P^3_3X \arrow[l] \\
 & P^2X \arrow[d] & P^2_2X \arrow[l] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & P^1X & P^1_1X \arrow[l]
\end{tikzcd}$$

Definition 1.13. We say a n-slice is cellular if it is of the form $H\underline{\mathbb{Z}} \wedge \widehat{W}$ where $\widehat{W}$ is a wedge of slice cells of dimension n.

Definition 1.14. A cellular n-slice is isotropic if the slice cells in $\widehat{W}$ are isotropic. It is pure if the slice cells in $\widehat{W}$ can be made to be regular.

Definition 1.15. A spectrum is cellular if all its n-slices are cellular.

Definition 1.16. A cellular spectrum is isotropic (resp., pure) if all its n-slices are isotropic (resp., pure)

Reminder, for $G = C_{2^n}$:

$$\mathrm{MU}^{((G))} = N^{C_{2^n}}_{C_2} \mathrm{MU}_{\mathbb{R}}$$

Using twisted monoid ring nonsense, we had a refinement of homotopy

$$A = \mathbb{S}[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \ldots] \to \mathrm{MU}^{((G))}$$

This is some wedge of slice cells at some dimensions. The "sub-wedge" of dimension k cells, A_k, has isomorphisms in dimension k homotopy of the underlying spectra:

$$\pi^u_k(A_k) \xrightarrow{\sim} \pi^u_k(\mathrm{MU}^{((G))})$$

Remark 1.17. We will implicitly replace $\mathrm{MU}^{((G))}$ with a cofibrant A-module.

Let $R(\infty)$ denote the relative smash product of $\mathrm{MU}^{((G))}$ and $\mathbb{S}$ over A:

$$R(\infty) = \mathrm{MU}^{((G))} \wedge_A \mathbb{S}$$

Remark 1.18 (Might be lies)**.** A is an (equivariant) $\mathbb{E}_\infty$ spectrum/commutative algebra object in $G-$Sp. Norm functors descend onto the CAlg subcategories, and $\mathrm{MU}_\mathbb{R}$ is $\mathbb{E}_\infty$ (much the same argument as the non-equivariant case - you use the Thom spectrum construction). $\mathbb{S}$ is also $\mathbb{E}_\infty$ like the non-equivariant case. Since A is $\mathbb{E}_\infty$, we are allowed to smash relative to it. I believe the relative smash product should literally be a pushout

$$\begin{tikzcd} A \arrow[r, "\mathrm{triv}"] \arrow[d, "\mathrm{refine}"'] & \mathbb{S} \arrow[d, dashed] \\ \mathrm{MU}^{((G))} \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & \mathrm{MU}^{((G))} \wedge_A \mathbb{S} \end{tikzcd}$$

Recall the structure of A. We let J be the (left) G-set:

$$J = \coprod_{i=1}^{\infty} G/C_2$$

with a bunch of copies.

$$A = \bigvee_{f: J \to \mathbb{N}_0} \mathbb{S}^{\rho_f}$$

where $G \curvearrowright J$ induces the G-action on the indices. ρ_f is a multiple of the regular representation of the stabilizer in G of f. We will force it to have dimension

$$\dim f = 2 \sum_{j \in J} j f(j)$$

(it really should be $\dim(\rho_f)$ but I digress)

2. The Slice Theorem

Our goal is to prove the following:

Theorem 2.1 (Slice Theorem)**.** $\mathrm{MU}^{((G))}$ *is isotropic and pure*

Proof. Let $M_d \subset A$ be the monomial ideal that has $\mathbb{S}^{\rho_f}$ when $\dim(f) \geq d$. Obviously this thing grows as d gets larger. We force the dimensions to be even, so $M_{2k} = M_{2k-1}$. Clearly,

$$M_{2d}/M_{2d-2} = \bigvee_{\dim(f)=2d} \mathbb{S}^{\rho_f} = \widehat{W}_{2d}$$

This is a wedge of regular isotropic cells. Now, we define:

$$K_{2d} = \mathrm{MU}^{((G))} \wedge_A M_{2d}$$

We have a sequence:

$$\cdots \hookrightarrow K_{2d} \hookrightarrow K_{2d-2} \hookrightarrow \cdots$$

Basically, the ρ_fs fit into bigger representations.

Fact 2.2.

$$\underbrace{K_{2d+2} \hookrightarrow K_{2d} \to \overbrace{K_{2d}/K_{2d+2}}}\to \mathrm{MU}^{((G))}/K_{2d+2} \to \mathrm{MU}^{((G))}/K_{2d}$$

are weakly equivalent to cofibration sequences.

Notice:

$$R(\infty) \wedge \widehat{W}_{2d} \simeq \mathrm{MU}^{((G))} \wedge_A \mathbb{S} \wedge \widehat{W}_{2d} \simeq \mathrm{MU}^{((G))} \wedge_A (M_{2d}/M_{2d+2}) \simeq K_{2d}/K_{2d+2}$$

We are entitled to the last bit via the cofibration sequence.

We have a fibration sequence:

$$K_{2d+2} \to \mathrm{MU}^{((G))} \to \mathrm{MU}^{((G))}/K_{2d+2}$$

Fact 2.3. *K_{2d+2} is slice 2d-positive*

Fact 2.4. *Reduction* $\implies \mathrm{MU}^{((G))}/K_{2d+2} \leq 2d$

Then, we get that in fact, we have a weak equivalence to the fiber sequence of P^n in the slice tower:

$$\begin{tikzcd}
\mathrm{hofib}(\xrightarrow{\text{bous}}) \arrow[d, equal] & & \\
2d \leq P_{2d}\mathrm{MU}^{((G))} \arrow[r] & \mathrm{MU}^{((G))} \arrow[r, "\text{bous}"] \arrow[d, equal] & P^{2d}\mathrm{MU}^{((G))} \leq 2d \\
2d \leq K_{2d+2} \arrow[r] \arrow[u, dashed, "\wr"] & \mathrm{MU}^{((G))} \arrow[r] & \mathrm{MU}^{((G))}/K_{2d+2} \leq 2d \arrow[u, dashed, "\wr"']
\end{tikzcd}$$

Therefore,

$$P^{2d+1}\mathrm{MU}^{((G))} \simeq P^{2d}\mathrm{MU}^{((G))} \simeq \mathrm{MU}^{((G))}/K_{2k+2}$$

Upshot 2.5. The odd slices of $\mathrm{MU}^{((G))}$ are contractible. The earlier cofibration sequence meant the even fibers are equivalent to $R(\infty) \wedge \widehat{W}_{2d}$. Reduction tells us this is just $H\underline{\mathbb{Z}}$ smashes with a bunch of slice cells. We already knew $\widehat{W}_{2d}$ just had regular isotropic cells. Therefore, $\mathrm{MU}^{((G))}$ is pure and isotropic by definition.

$\square$

3. The Gap Theorem

4. The Reduction Theorem