LUBIN-TATE THEORY AND MORAVA STABILIZER GROUPS

Landweber exactness, Morava stabilizer group, and Morava E-theory

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Abstract. **Speaker**: Daishi Kiyohara (Harvard)

We have previously seen how a complex-oriented cohomology theory gives rise to a formal group law. In this talk, we begin by stating Landweber's exactness theorem, which provides a precise criterion ensuring that a formal group law can be realized by a generalized homology theory. Landweber's criterion is expressed geometrically in terms of the moduli stack of formal groups, a central object in chromatic homotopy theory. We then investigate this moduli stack more closely by examining its height stratification. In particular, we explicitly describe the open strata via the action of a certain profinite group called the Morava stabilizer group. Finally, as an important application of Landweber's exactness theorem, we construct Morava E-theory by leveraging Lubin-Tate deformation theory of formal groups of fixed height.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

We briefly review chromatic homotopy theory for the admitted students in attendance.

1. Introduction

We start with complex-oriented cohomology theories. Instead of defining these objects, we give a property:

$$E^*(\mathbb{C}P^\infty) \simeq E^*(*)[\![t]\!]$$

The choice of t is a choice of complex orientation. $\mathbb{C}P^\infty$ is a classifying space of line bundles. For a line bundle on a topological space X, then we write $\mathcal{L} = f^*\mathcal{O}(1)$ to be the pullback of $f : X \to \mathbb{C}P^\infty$. We define a version of the Chern character:

$$c_1^E(\mathcal{L}) = f^*t \in E^*(X)$$

Example 1.1. For ordinary cohomology, $E^*(X) = H^*(X;\mathbb{Z})$, $c_1^E(\mathcal{L})$ recovers the usual Chern class. The tensor product on line bundles becomes additive on Chern classes:

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$$

$\triangleleft$

Question 1.2. Can we find a formula for $c_1^E(\mathcal{L} \otimes \mathcal{L}')$ for any complex-oriented cohomology theory E? $\triangleleft$

We can look at the universal example:

$$\begin{tikzcd} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}P^\infty & & \mathbb{C}P^\infty \end{tikzcd}$$

We can compare $\pi_1^*\mathcal{O}(1)$, $\pi_2^*\mathcal{O}(1)$, and $\mathcal{O} = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$. In particular,

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq (\pi_* E)[\![u, v]\!]$$

with $u = c_1^E\left(\pi_1^*\mathcal{O}(1)\right), v = c_1^E\left(\pi_1^*\mathcal{O}(1)\right)$. We then have:

$$c_1^E(\mathcal{O}) = f(u, v)$$

which is a power series with coefficients in $\oplus_{n \geq 0} \pi_{2n} E$.

Corollary 1.3. $c_1^E(\mathcal{L} \otimes \mathcal{L}') = f\left(c_1^E(\mathcal{L}), c_2^E(\mathcal{L})\right)$. *Moreover, $f(x, y)$ defines a formal group law over $\oplus_{n \geq 0} \pi_{2n} E$.*

There is a universal example of complex-oriented cohomology theories called complex bordism. This gives a formal group law over $\pi_*\mathrm{MU}$. A formal group law corresponds to a ring homomorphism from the Lazard ring which carries the universal formal group law, so we get a map:

$$L \to \pi_*\mathrm{MU}$$

Theorem 1.4 (Quillen). *This canonical map $L \to \pi_*\mathrm{MU}$ is an isomorphism.*

We can ask about the opposite direction:

Question 1.5. Given a graded ring map $L \to R$ which gives a certain formal group law over R, when can we find a cohomology theory E with $\pi_*E \simeq R$. ◁

Idea 1.6. For a space X, we can set:

$$E_*(X) = \mathrm{MU}_*(X) \underset{\pi_*\mathrm{MU}}{\otimes} R$$

◁

The problem is we need to satisfy an exactness condition to be a cohomology theory. We will find a weaker condition than flatness which guarantees this.

Solution 1.7. Landweber exactness criterion, which involves $\mathcal{M}_{\mathrm{fg}}$ (the moduli stack of formal groups). ◁

Once we have this, we can build a lot of cohomology theories through maps out of the Lazard ring.

2. Landweber's exactness

To a spectrum X we can associate an L-module $\mathrm{MU}_*^{\mathrm{ev}}(X)$. This descends to $\mathcal{F}_X$, a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{fg}}$. $\mathcal{M}_{\mathrm{fg}}(R)$ corresponds to formal groups over R, which locally look like formal group laws (Zariski).

Definition 2.1. A quasi-coherent sheaf on $\mathcal{M}_{\mathrm{fg}}$:

$$\begin{aligned} \mathcal{M}_{\mathrm{fg}}(R) &\to \mathsf{Mod}_R \\ \eta &\mapsto \mathcal{F}(\eta) \end{aligned}$$

that is compatible with base changes $R \to R'$. That is, if $\eta \mapsto \eta' \in \mathcal{M}_{\mathrm{fg}}(R')$, then $\mathcal{F}(\eta') = \mathcal{F} \underset{R}{\otimes} R'$. ◁

Definition 2.2. A quasicoherent sheaf is **flat** or **faithfully flat** if each module is flat or faithfully flat.
We can test this on formal group laws, and in fact we can just test it on $\eta_0 \in \mathcal{M}_{\mathrm{fg}}(L)$, the universal formal group law on the Lazard ring. ◁

Pick a point $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$, as in $\eta \in \mathcal{M}_{\mathrm{fg}}(R)$. Consider the pullback

$$\begin{aligned} q^* : \mathrm{QCoh}\big(\mathcal{M}_{\mathrm{fg}}\big) &\to \mathrm{QCoh}(\operatorname{Spec} R) = \mathsf{Mod}_R \\ \mathcal{F} &\mapsto \mathcal{F}(\eta) \end{aligned}$$

Proposition 2.3. *The pullback q^* admits a right adjoint, q_*:*

$$q_* : \mathsf{Mod}_R \to \mathrm{QCoh}\big(\mathcal{M}_{\mathrm{fg}}\big)$$

Proof sketch. Let $N \in \mathsf{Mod}_R$, we describe q_*N. Take $p : \operatorname{Spec} R' \to \mathcal{M}_{\mathrm{fg}}$.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} R' \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M}_{\mathrm{fg}} \end{tikzcd}$$

B is a $R \otimes R'$ -algebra which classifies isomorphisms between η and η'. We set:

$$\begin{aligned} q_*N &= N \underset{R}{\otimes} B \in \mathsf{Mod}_{R'} \\ &= (q')_*(p')^*N \end{aligned}$$

by viewing B as a module over R'. □

Definition 2.4. Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$ and $N \in \mathsf{Mod}_R$. We say N is **flat** or **faithfully flat** if q_*N is flat or faithfully flat over $\mathcal{M}_{\mathrm{fg}}$. ◁

Proposition 2.5. *Let N be a flat R-module over $\mathcal{M}_{\mathrm{fg}}$ where $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$. Then the functor:*

$$\begin{aligned} \mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg}}\right) &\to \mathsf{Mod}_R \\ \mathcal{F} &\mapsto q^*\mathcal{F} \underset{R}{\otimes} N \end{aligned}$$

is exact.

Proof. We may assume q corresponds to a formal group law, and form the following pullback diagram.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} L \arrow[d, "p"] \\ \operatorname{Spec} R \arrow[r, "q"'] & \mathcal{M}_{\mathrm{fg}} \end{tikzcd}$$

Lemma 2.6. *You can show that $p : \operatorname{Spec} L \to \mathcal{M}_{\mathrm{fg}}$ is faithfully flat. In fact, $B \simeq R[b_0^{\pm 1}, b_1, b_2, \ldots]$*

It suffices to show exactness after pulling back:

$$\mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg}}\right) \to \mathsf{Mod}_R \xrightarrow{-\underset{R}{\otimes} B} \mathsf{Mod}_B$$

The composite sends $\mathcal{F} \in \mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg}}\right)$ to:

$$\begin{aligned} (p')^*(q^*F \otimes N) &= (q')^* p^* F \underset{B}{\otimes} (p')^* N \\ &= \left(p^*\mathcal{F} \underset{L}{\otimes} B\right) \underset{B}{\otimes} (p')^* N \\ &= p^*\mathcal{F} \underset{L}{\otimes} (q')_*(p')^* N \\ &= p^*\mathcal{F} \underset{L}{\otimes} q_* N(\eta_0) \end{aligned}$$

with the last term being a flat L-module. Therefore, the composite is exact. □

Corollary 2.7. *Let M be a graded module over L. If M is flat over $\mathcal{M}_{\mathrm{fg}}$, then:*

$$\begin{aligned} \mathsf{Sp} &\to \mathsf{Mod}_L \\ X &\mapsto \mathrm{MU}_*(X) \underset{L}{\otimes} M \end{aligned}$$

is a homology theory and is therefore representable by some spectrum.

Proof of exactness.

$$\begin{aligned} \mathsf{Sp} \to \mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg}}\right) &\to \mathsf{Mod}_L \\ X \mapsto \mathcal{F}_X = \mathrm{MU}_*(X) &\mapsto q^*\mathcal{F}_X \underset{L}{\otimes} M \end{aligned}$$

which is exact by the previous proposition. □

Recall 2.8. $L \simeq \mathbb{Z}[c_{ij}]/\{\text{relations}\}$ with the power series $f(x,y) = \sum c_{ij} x^i y^j$. The relations are such that f is a formal group law.

We write $v_n \in L$ for the coefficient of t^{p^n} in $[p]_f(t) = t +_f t + \ldots +_f t$. We say a formal group law f over R, corresponding to a map $L \xrightarrow{\phi} R$, has **height** n if $\phi(v_i)$ vanishes for all $i < n$ ("height $\geq n$"), and $\phi(v_n)$ is invertible in R. ◁

Remark 2.9. By this definition, not all formal group laws have a height. This is a different condition from having infinite height. ◁

Theorem 2.10 (Landweber's exactness criterion)**.** *A graded L-module M is flat over $\mathcal{M}_{\mathrm{fg}}$ if and only if for every prime p, $(p, v_1, v_2, \ldots)$ defines a regular sequence for M.*

Example 2.11. Let $R = \mathbb{Z}[\beta^{\pm 1}]$ with $\deg\beta = 2$. We consider the formal group law $f(x,y) = x + y + \beta xy$ over R. Then R is Landweber exact because p is a non-zero divisor and $v_1 = \beta^{p-1} \mod p$ is invertible on R/pR. This corresponds to complex K-theory KU. Namely, $\tau_{\geq 0}\mathrm{KU}$ is not Landweber exact. However, it is still complex-orientable nonetheless. $\mathrm{H}\mathbb{Z}$ is also not Landweber exact. ◃

3. Morava stabilizer group

Goal 3.1. We can understand $\mathcal{M}_{\mathrm{fg}}^{n} = \mathcal{M}_{\mathrm{fg}}^{\geq n} \setminus \mathcal{M}_{\mathrm{fg}}^{\geq n+1} \subset \mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}$, open strata of height n, through some profinite group. ◃

Recall 3.2. There is a unique isomorphism class of formal group laws of height n over $\overline{\mathbb{F}_p}$.

Fact 3.3. $\operatorname{Spec}\overline{\mathbb{F}_p} \to \mathcal{M}_{\mathrm{fg}}^{n}$ *is faithfully flat.*

We can use $\operatorname{Spec}\overline{\mathbb{F}_p}$ as an atlas for $\mathcal{M}_{\mathrm{fg}}$.

$$\begin{tikzcd}
\Gamma_n \arrow[r, equal] & \operatorname{Spec}B \arrow[r] \arrow[d] & \operatorname{Spec}\overline{\mathbb{F}_p} \arrow[d] \\
& \operatorname{Spec}\overline{\mathbb{F}_p} \arrow[r] & \mathcal{M}_{\mathrm{fg}}^{n}
\end{tikzcd}$$

A point of Γ_n is identified with a pair (σ,α), or $\operatorname{Aut}\left(f,\overline{\mathbb{F}_p}\right)$.

$$\begin{cases} \sigma : \overline{\mathbb{F}_p} \simeq \overline{\mathbb{F}_p} \\ \alpha : \text{isomorphism between } f \text{ and } \sigma(f) \end{cases}$$

We have a short exact sequence:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Aut}(f) & \to & \operatorname{Aut}\left(f,\overline{\mathbb{F}_p}\right) & \to & \operatorname{Gal}\left(\overline{\mathbb{F}_p}/\mathbb{F}_p\right) & \to 0 \\ & \alpha & \to & (\mathrm{id},\alpha) & & & \\ & & & (\sigma,\alpha) & \to & \sigma & \end{array}$$

◃

Definition 3.4. We call Γ_n the Morava stabilizer group. ◃

Corollary 3.5. $\mathcal{M}_{\mathrm{fg}}^{n} \simeq \operatorname{Spec}\overline{\mathbb{F}_p}/\Gamma_n$.

Construction 3.6. A formal group law f over $\overline{\mathbb{F}_p}$ can be mapped to $f^p = F(f)$ by the Frobenius, which is a height n formal group law over $\overline{\mathbb{F}_p}$. Then, $\exists v : f \to f^p$ that is an isomorphism of formal group laws.

$$v(f(x,y)^p) = vf^p(x^p,y^p) = f(v(x^p), v(y^p))$$

so that $\pi(t) = v(t^p)$ is an automorphism of f. ◃

Proposition 3.7. $\operatorname{End}(f)$ *is a local ring with a maximal ideal given by the left ideal generated by v. Moreover, the residue field is $\mathbb{F}_{p^n}$.*

Proposition 3.8. *Let $D = \operatorname{End}(f)\left[\frac{1}{p}\right]$. It is a central division algebra over $\mathbb{Q}_p$ with Hasse invariant $1/n$.*

$$\begin{aligned} v : D \setminus \{0\} &\to \mathbb{Z}_{\geq 0} \\ \pi^n(x_0) &\mapsto n \end{aligned}$$

for $x_0 \in \operatorname{End}(f)^{\times}$

$$\eta(D) = \frac{v(\pi)}{v(p)} = \frac{1}{n}$$

4. Lubin-Tate theory

Goal 4.1. We want to understand what $\mathcal{M}_{\mathrm{fg}}$ looks like on a neighborhood of a point in $\mathcal{M}_{\mathrm{fg}}^{n}$. ◁

Let k be a perfect field of characteristic p.

Definition 4.2. An **infinitesimal thickening** of k is a local Artinian ring A with residue field k.

A **deformation** of a formal group law f over k is a formal group law $\tilde{f}$ over A which lifts f, and we say two deformations are **isomorphic** if $\exists g(t)$, an isomorphism between the formal group laws such that $g(t) = t \mod \mathfrak{m}_A$ for $\mathfrak{m}_A = \ker(A \to k)$. ◁

Let $R = W(k)[\![u_1, \dots, u_{n-1}]\!]$, then giving a formal group law over R is equivalent to givinga map $L_{(p)} \to R$. Note that $L_{(p)} = \mathbb{Z}_{(p)}[t_1, \dots]$ with $v_i = t_{p^i - 1}$. This allows us to lift $L_{(p)} \to k$ given a formal group law to $L_{(p)} \to R$ such that $v_i \mapsto u_i$. Let us write $\tilde{f}$ for such a formal group law. This is the universal example.

Theorem 4.3 (Lubin-Tate)**.** *For any infinitesimal thickening A,*

$$\mathrm{Hom}_{W(k)}(R, A) \leftrightarrow \mathrm{Def}(A)$$

are in bijection.

By constructions, $L_{(p)} \to R$ exhibits R as Landweber exact (v_n acts invertibly on $R/(p, u_1, \dots, u_{n-1}) \simeq k$).

Definition 4.4. The corresponding spectrum is called the Morava E-theory spectrum, E_n. ◁

Remark 4.5. E_n depends on your choice of k and the choice of formal group law of height n. For this reason, some people will write $E(\Gamma)$. We will usually take $k = \overline{\mathbb{F}_p}$. ◁