18.906 - Characteristic Classses

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These are notes for one of my presentations from Prof. Haynes Miller's 18.906 (Algebraic Topology II) reading group at MIT in Fall 2024, and follows chapters 71-73 from his book [Mil22]. As they are presentation notes, they are not meant to be complete, as some material is cut for time. 18.906 was not taught this academic year, and the reading group sought out to address this gap.

Note: a lot of the proofs here are cut for time. I offer my condolences.

1 Chern Classes, Stiefel-Whitney Classes, and the Leray-Hirsch Theorem

Remember: $\mathrm{Bun}_G(X)$ is the set(!) of isomorphism classes of principal G-bundles over X. $\mathrm{Bun}_G(-)$ is a homotopy invariant and contravariant functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$. However, it can be extremely horrible to compute.

Goal: define some easier invariant. Here's another nice homotopy invariant and contravariant functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$: cohomology. We will focus on n-plane bundles ($\mathrm{GL}_n(\mathbb{F})$-bundles, for example). All our bundles will be numerable.

Definition 1.0.1. A characteristic class is a natural transformation of morphisms $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$:

$$c : \mathrm{Bun}_G(-) \to H^n(-, A)$$

i.e. an assignment, to each G-bundle $\xi : E \to X$, of some cohomology class that satisfies naturality with respect to pullbacks:

$$c(f^*\xi) = f^*c(\xi)$$

Definition 1.0.2 (Reminder)**.** Euler class $e(\xi)$ for a R-oriented n-plane bundle (for some $R \in \mathrm{CRing}$). For $\mathbb{Z}$, this is an oriented bundle, and for $\mathbb{Z}/2$, this is just any bundle. For a bundle $\xi : E \to X$, the cohomological Serre spectral sequence gives:

$$E_2^{s,t} = H^s(X; H^t(p^{-1}(-))) \underset{s}{\implies} H^{s+t}(E)$$

The orientation gives us a class $\sigma \in H^0(B; H^{n-1}(F))$ and we define:

$$e(\xi) = d_n(\sigma) \in H^n(B; R)$$

Remark 1.0.3. $\mathrm{Bun}_G(-)$ is representable through classifying spaces BG: $\mathrm{Bun}_G(X) \simeq [X, BG]$. Therefore, we may pull a characteristic class to a universal class in $H^n(BG)$. For example, all characteristic classes of integral complex line bundles are inside of $H^\bullet(BU(1)) \simeq H^\bullet(\mathbb{C}\mathbb{P}^\infty) \simeq \mathbb{Z}[e]$ where e is the Euler class of $EU(1) \to BU(1)$.

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A (numerable) complex vector bundle has a Hermitian metric, which gives a retraction, as $U(n)$ retracts to $\mathrm{GL}_n(\mathbb{C})$, so we get $BU(n) \simeq B\,\mathrm{GL}_n(\mathbb{C})$. Therefore,

Remark 1.0.4. Complex vector bundles have a preferred real orientation. We pick an orientation based on an ordering $(e_1, ie_1, \ldots, e_n, ie_n)$ in $\mathbb{R}^{2n}$. Therefore, complex line bundles have Euler classes.

1.1 Chern Classes

Theorem 1.1.1. *For n-plane complex vector bundles and $k \in \mathbb{N}$, we have a characteristic class $c_k^{(n)}(\xi) \in H^{2k}(X;\mathbb{Z})$ called the Chern classes of ξ. They are uniquely characterized by:*

$$c_0^{(n)}(\xi) = 1$$

$$c_0^{(1)}(\xi) = -e(\xi)$$

and the Whitney sum formula:

$$c_k^{(p+q)}(\xi \oplus \eta) = \sum_{i+j=k} c_i^{(p)}(\xi) \smile c_k^{(q)}(\eta)$$

For the universal n-plane bundle ξ_n, we have:

$$H^\bullet(BU(n)) \simeq \mathbb{Z}[c_1^{(n)}(\xi_n), \ldots, c_n^{(n)}(\xi_n)]$$

As in, there are no relations between the Chern classes, and all characteristic classes are polynomials in these.

Chern classes are stable. For a trivial bundle $\epsilon : X \times \mathbb{C}^q \to X$, we have a pullback:

$$\begin{tikzcd} X \times \mathbb{C}^q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C}^q \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

Naturality then tells us that $c_j^{(n)}(q\epsilon) = 0$, for $j \geq 1$. The Whitney sum tells us that:

$$c_j^{(n+q)}(\xi \oplus q\epsilon) = c_j^{(n)}(\xi)$$

We also have maps going down in the superscript, as the maps $BU(n) \to BU(n+1)$ classify $\xi \oplus \epsilon$ and send the Chern classes down in degree. Therefore, the Chern classes depend only on the stable equivalence class of ξ, and we will drop the superscripts.

1.2 Grothendieck's Construction

For an n-plane bundle ξ (as in, an element of $\mathrm{Vect}_n(B)$), we may principalize it as:

$$P(\xi)_b = \{\text{ordered bases for } E(\xi)_b\}$$

We may then form the projectivization:

Definition 1.2.1. For an n-plane bundle ξ, we form its projectivization $\mathbb{P}(\xi)$ using the balanced pruduct:

$$\mathbb{P}(\xi) = P(\xi) \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}\,\mathbb{P}^{n-1}$$

We may then ask for the cohomology of projectivizations:

Theorem 1.2.2 (Leray-Hirsch)**.** *Let $R \in \mathrm{CRing}$, $\pi : E \to B$ be a fibration, with path connected B, so that we have a fiber F defined up to homotopy. Assume:*

- *$H^t(F)$ is a free R-module*
- *The restriction $H^\bullet(E) \to H^\bullet(F)$ is surjective*

We get a splitting, using the free structure,

$$s : H^\bullet(F) \to H^\bullet(E)$$

$H^\bullet(E)$ is a $H^\bullet(B)$-module by the projection. We extend it $H^\bullet(B)$-linearly:

$$\bar{s} : H^\bullet(B) \otimes_R H^\bullet(F) \to H^\bullet(E)$$

This map is a $H^\bullet(B)$-module isomorphism.

Proof sketch. Remember: in the cohomological Serre spectral sequence, we had:

$$E_2^{s,t} = H^s(B; H^t(F)) \underset{s}{\Longrightarrow} H^{s+t}(E)$$

if $\pi_1(B)$ acts trivially on $F = \pi^{-1}(*)$. That is, the local coefficient system was constant. But, the restriction $H^\bullet(E) \to H^\bullet(F)$ has image in the $\pi_1(B)$-invariant subgroup. But the map is surjective by assumption, so that we get to use the Serre spectral sequence. One can show (I won't) that:

$$E^2_{s,t} = H^s(B) \otimes_R H^t(F)$$

The base generators survive since differentials hit 0. The fiber generators survive by assumption (for example, because of the surjection). Therefore, the spectral sequence collapses at the E_2 page.

We give a new filtration:

$$F_t H^n(E) = F^{n-t} H^n(E)$$

We have:

$$\operatorname{gr}_t H^n(E) = F^{n-t} H^n(E) / F^{n-t+1} H^n(E) = E_\infty^{n-t,t}$$

For the sake of time, I won't compute the proof, but just wanted to show you the grading you use. □

Upshot: when the projectivizations are nice (we are just missing surjection of the restriction map), we get:

$$H^\bullet(\mathbb{P}(\xi)) = H^\bullet(X) \left\langle 1, e, \ldots, e^{n-1} \right\rangle$$

is freely generated as a $H^\bullet(X)$-module, indexed by the Euler classes $e(\lambda) \in H^2(\mathbb{P}(\xi))$. What is e^n? In fact, Euler classes satisfy the Chern polynomial:

$$c_\xi(t) = \sum_{k=0}^{n} t^{n-k} c_k(\xi)$$

and $c_\xi(e) = 0$

Remark 1.2.3. We can do this mod 2 and not require orientability. These are called Stiefel-Whitney classes. That's all I want to say here.

2 $H^\bullet(BU(n))$ and the Splitting Principle

Theorem 2.0.1 (Alternative characterization of Chern classes)**.** *For complex line bundles ξ, ζ, if $\xi \simeq \zeta \oplus (n-i)\varepsilon$, then:*

$$c_i(\xi) = (-1)^i e(\zeta)$$

These generate all characteristic classes and have no algebraic relations between them.

Proof. We calculate $H^\bullet(BU(n))$, inductively. We embed $U(n-1) \hookrightarrow U(n)$ into the top-left. Then, the orbit of e_n under $U(n) \curvearrowright \mathbb{C}^n$

$$S^{2n-1} \simeq U(n)/U(n-1)$$

Clearly:

$$BU(n-1) = EU(n)/U(n-1) = EU(n) \times_{U(n)} S^{2n-1}$$

Then, $p : BU(n-1) \to BU(n)$ is the unit sphere bundle.

$$BU(n-1) \to BU(n)$$

classifies $\xi_{n-1} \oplus \epsilon$. ☐

Fact 2.0.2. *I won't prove this for the sake of time.*

$$H^\bullet(BU(n)) \simeq \mathbb{Z}[c_1, \dots, c_n]$$

We let Fl_n be orthogonal flags: decompositions of $\mathbb{C}^n$ into n 1D subspaces. Alternatively,

$$\mathrm{Fl}_n = U(n)/T^n$$

Theorem 2.0.3 (Splitting Principle). *Let $\xi : E \downarrow X$ be a complex n-plane bundle. There is a map $f : \mathrm{Fl}(\xi) \to X$ s.t.*

1. $f^\bullet(\xi) \simeq \lambda_1 \oplus \dots \oplus \lambda_n$
2. $f^\bullet : H^\bullet(X) \to H^\bullet(\mathrm{Fl}(\xi))$ *is monic.*

Proof. We already saw the spectral sequence collapses at E_2, so that the projection map induces a monomorphism on cohomology. For the tautological bundle λ on $\mathbb{P}(\xi)$, we can canonically embed $\lambda \hookrightarrow \pi^*\xi$, [won't give details but this gives (1)] ☐

So that

$$\mathrm{Fl}(\xi_n) = EU(n) \times_{U(n)} (U(n)/T^n) = EU(n)/T^n = BT^n = (\mathbb{C}\mathbb{P}^\infty)^n$$

We have Σ_n actions from the symmetric group which lives inside of $U(n)$, via conjugation. Therefore, we get:

$$H^\bullet(BU(n)) \hookrightarrow H^\bullet(BT^n)^{\Sigma_n}$$

By an algebraic argument, also cut for time, we have that this is in fact an isomorphism!

Definition 2.0.4. We define the elementary symmetric polynomials σ_i as the coefficients:

$$\prod_{i=1}^n (t - t_i) = \sum_{j=0}^n \sigma_j t^{n-j}$$

This means that Chern classes get identified with elementary symmetric polynomials!

3 Thom Class and Whitney Sum Formula

3.1 Thom Space and Thom Class

Definition 3.1.1. For a real n-bundle $\xi : E \xrightarrow{p} B$, we may form the Thom space $\mathrm{Th}(E)$ by taking the one-point compactification at each fiber, $E(\mathrm{Th}(\xi)) = E_x \sqcup \infty$. For a compact Hausdorff space, this is the same as $E \sqcup \infty$.

If we pick a metric, we get:

Definition 3.1.2 (Alternative definition of Thom construction)**.**

$$\mathrm{Th}(\xi) = \mathrm{D}(\xi)/\,\mathrm{S}(\xi)$$

Remark 3.1.3. We can think of $(\mathrm{D}(\xi), \mathrm{S}(\xi)) \simeq (E(\xi), E(\xi) - Z)$ for the image of the zero section, Z.

Fun things:

$$\mathrm{Th}(\xi \times \eta) = \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta)$$
$$\mathrm{Th}(\xi \oplus n\epsilon) = \Sigma^n \,\mathrm{Th}(\xi)$$

We can think of Thom spaces as twisted suspensions.

We get maps between Thom spaces using pullbacks:

$$\bar{f} : \mathrm{Th}(f^*\xi) \to \mathrm{Th}(\xi)$$

The bundle $0 \times \xi$ over $B \times B$ is pulled back from $\mathrm{pr}_2 : B \times B \to B$. The diagonal map $\Delta : B \to B \times B$ then gives us

$$\mathrm{Th}(\xi) \to \mathrm{Th}(0) \wedge \mathrm{Th}(\xi) = B_+ \wedge \mathrm{Th}(\xi)$$

We get a relative cup product:

$$\smile : H^\bullet(B) \otimes \bar{H}^\bullet(\mathrm{Th}(\xi)) \to \bar{H}^\bullet(\mathrm{Th}(0) \wedge \mathrm{Th}(\xi)) \to \bar{H}^*(\mathrm{Th}(\xi))$$

The first map comes from the fact that $\bar{H}^\bullet\,\mathrm{Th}(\xi)$ is a $\bar{H}^\bullet(B_+) = H^\bullet(B)$-module and a $H^\bullet(\mathrm{Th}(\xi))$-module.

Proposition 3.1.4 (Thom Isomorphism Theorem)**.** *Let $R \in \mathrm{CRing}$, ξ is a R-oriented n-plane bundle over B. We have a unique class, the Thom class, $U \in \bar{H}^n(\mathrm{Th}(\xi); R)$ that restricts, on each fiber, to the dual of the orientation class. Also,*

$$- \smile U : H^\bullet(B) \to \bar{H}^\bullet(\mathrm{Th}(\xi))$$

is an isomorphism

Proof. Cut for time, uses a relative spectral sequence □

3.2 Thom and Euler

We have a map $\pi : B \to \mathrm{D}(\xi) \to \mathrm{Th}(\xi)$. The first map is the zero section, and the second is the quotient.

Lemma 3.2.1. *It coincides with the Euler class, up to sign.*

$$\pi^* U = \pm e$$

Proof. Cut for time, also much of it is in the exercises or the next chapter. □

We may benefit from calling *this* the Euler class. We get a Gysin sequence, a long exact sequence with the Thom isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{s-1}(B) \arrow[r, "p^*"] & H^{s-1}(E) \arrow[r, "\delta"] \arrow[dr, "p_*"'] & \bar{H}^s(\mathrm{Th}(\xi)) \arrow[r, "\pi^*"] & H^s(B) \arrow[r] & H^s(E) \arrow[r] & \cdots \\
& & & H^{s-n}(B) \arrow[u, "\smile U"'] \arrow[ur, "\smile e"'] & & &
\end{tikzcd}$$

3.3 Cut for time

$$e(\xi \oplus \eta) = e(\xi) \smile e(\eta)$$

also, proof of Whitney sum.

Notation 3.3.1. Sometimes we will denote $\mathrm{Th}(\xi_n)$ as $MSO(n)$

References

[Mil22] Haynes R. Miller. *Lectures on algebraic topology.* World Scientific Publishing Co, New Jersey, 2022.