電腦視覺與應用
Computer Vision and Applications

Lecture07-Camera calibration

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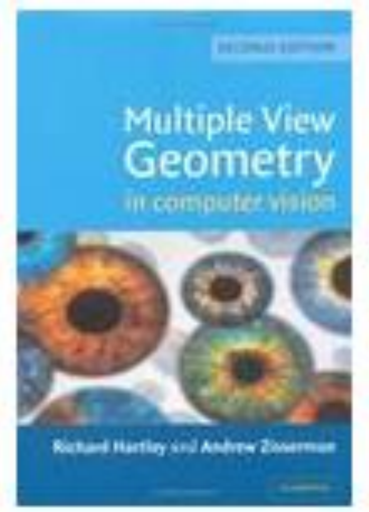
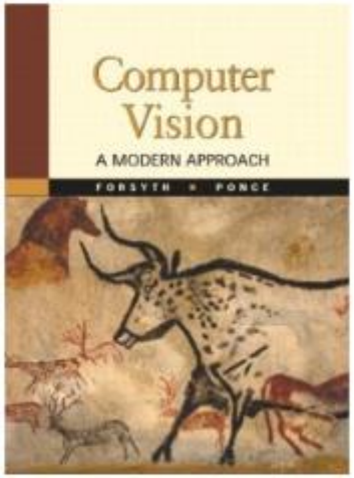
Camera calibration

- projection matrix
    - perspective projection
    - orthographic projection

Camera calibration

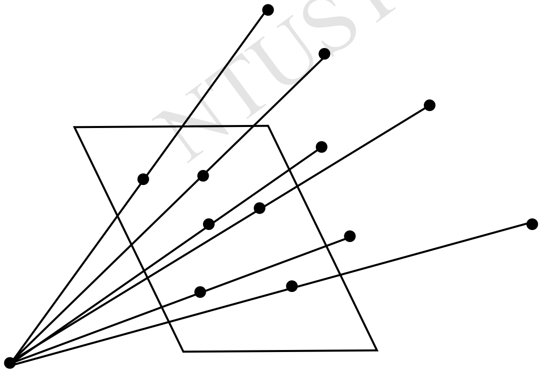
- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, (Chapter7)
  - Computer Vision A Modern Approach, Chapter 3 (Geometric Camera Calibration).

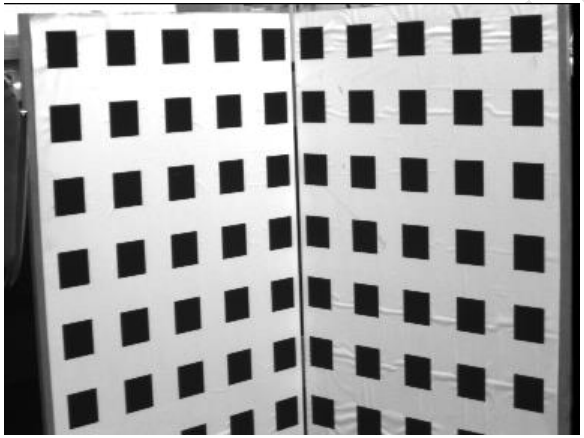




Camera calibration

- Problem description:
  - Camera calibration is to use precise geometric structure, then to adjust the camera parameter under measurements and constraints.
  - In short, given $\mathbf{X}_i \leftrightarrow \mathbf{x}_i$
  - then determine the mapping transformation





Camera calibration: A linear approach

- The camera mathematical model:

$$\mathbf{x}_{\text{img}} = \underline{\mathbf{K}[\mathbf{R}\,|\,\mathbf{t}]}\mathbf{X}_{\text{world}}$$

- To determine the mapping matrix (3x4), rewrite as

$$\mathbf{x} = \mathbf{P} \cdot \mathbf{X}$$

(請注意，此處符號與兩本課本定義稍微不同)

Camera calibration: A linear approach—cont.

- To determine **P** in $\mathbf{x} = \mathbf{P} \cdot \mathbf{X}$

3x4 mapping matrix

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{p}_1^{\mathrm{T}} \\ \mathbf{p}_2^{\mathrm{T}} \\ \mathbf{p}_3^{\mathrm{T}} \end{bmatrix}_{3\times 4} \cdot \mathbf{X}_i$$

1x4 vector

1x4 vector

1x4 vector

i-th 3D point in space

$$\mathbf{X}_i = [X_{i1} \quad X_{i2} \quad X_{i3} \quad X_{i4}]^{\mathrm{T}}$$

$$\mathbf{x}_i = [u_i \quad v_i \quad 1]^{\mathrm{T}}$$

i-th 2D point on image

inner product (dot) of two vectors

4x1 vector

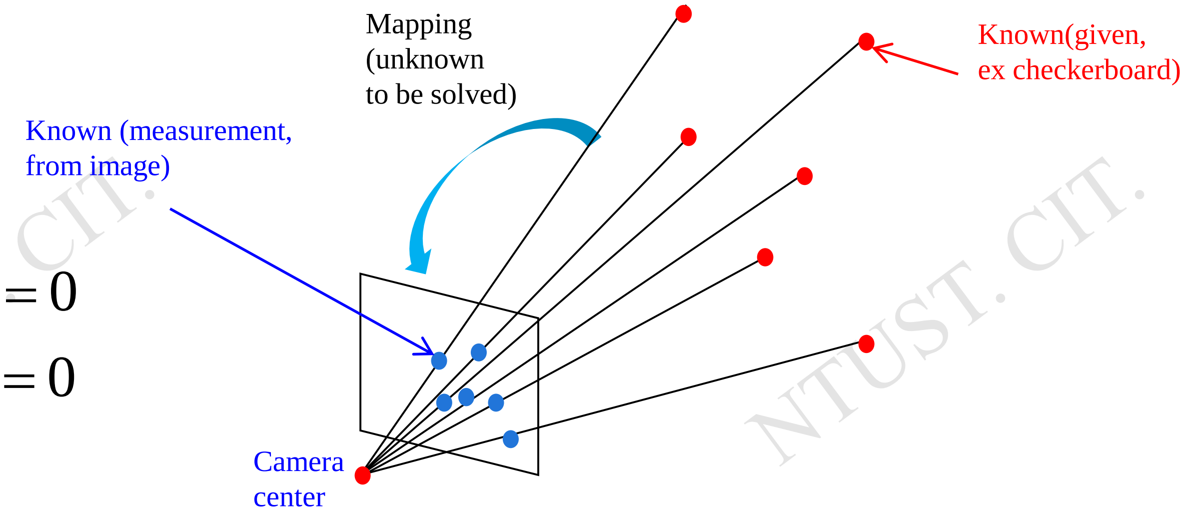
$$\begin{cases} u_i = \dfrac{\mathbf{p}_1 \cdot \mathbf{X}_i}{\mathbf{p}_3 \cdot \mathbf{X}_i} \\ v_i = \dfrac{\mathbf{p}_2 \cdot \mathbf{X}_i}{\mathbf{p}_3 \cdot \mathbf{X}_i} \end{cases} \rightarrow \begin{cases} u_i \mathbf{p}_3 \cdot \mathbf{X}_i = \mathbf{p}_1 \cdot \mathbf{X}_i \\ v_i \mathbf{p}_3 \cdot \mathbf{X}_i = \mathbf{p}_2 \cdot \mathbf{X}_i \end{cases} \rightarrow \begin{cases} (\mathbf{p}_1 - u_i \mathbf{p}_3) \cdot \mathbf{X}_i = 0 \\ (\mathbf{p}_2 - v_i \mathbf{p}_3) \cdot \mathbf{X}_i = 0 \end{cases}$$

Camera calibration: A linear approach—cont.

- Again, one feature has two constraints.
- Note : one feature means that you have already known a 3D point (in 3D space) and a projected 2D point (on image)

$$\begin{cases}(\mathbf{p}_1 - u_i\mathbf{p}_3)\cdot\mathbf{X}_i = 0 \\ (\mathbf{p}_2 - v_i\mathbf{p}_3)\cdot\mathbf{X}_i = 0\end{cases}$$

$$\rightarrow \begin{cases}[\mathbf{p}_1 + 0\mathbf{p}_2 + (-u_i)\mathbf{p}_3]\cdot\mathbf{X}_i = 0 \\ [0\mathbf{p}_1 + \mathbf{p}_2 + (-v_i)\mathbf{p}_3]\cdot\mathbf{X}_i = 0\end{cases}$$



Camera calibration: A linear approach—cont.

$$\begin{cases} [\mathbf{p}_1 + 0\mathbf{p}_2 + (-u_i)\mathbf{p}_3]\cdot \mathbf{X}_i = 0 \\ [0\mathbf{p}_1 + \mathbf{p}_2 + (-v_i)\mathbf{p}_3]\cdot \mathbf{X}_i = 0 \end{cases}$$

- So, u_i, v_i and $\mathbf{X}_i$ are known, and $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ (12 elements) are unknown and to be solved.
- Re-arrange the equations:

$$\begin{pmatrix} \mathbf{X}_i{}^{\mathrm{T}} & 0^{\mathrm{T}} & -u_i\mathbf{X}_i{}^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_i{}^{\mathrm{T}} & -v_i\mathbf{X}_i{}^{\mathrm{T}} \end{pmatrix}_{2\times 12} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}_{12\times 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{2\times 1}$$

Camera calibration: A linear approach—cont.

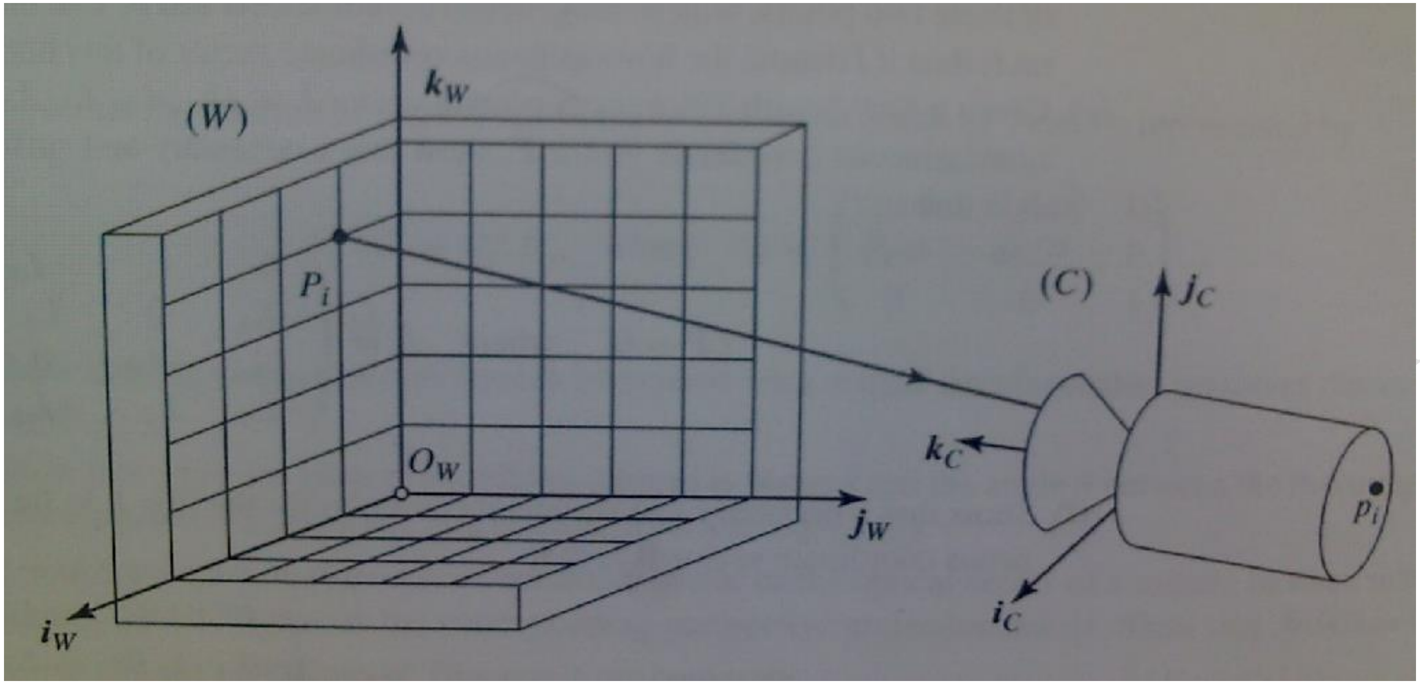
- In practice, given *n* features:

$$\begin{bmatrix} \begin{pmatrix} \mathbf{X}_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1\mathbf{X}_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_1^{\mathrm{T}} & -v_1\mathbf{X}_1^{\mathrm{T}} \end{pmatrix} \\ \begin{pmatrix} \mathbf{X}_2^{\mathrm{T}} & 0^{\mathrm{T}} & -u_2\mathbf{X}_2^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_2^{\mathrm{T}} & -v_2\mathbf{X}_2^{\mathrm{T}} \end{pmatrix} \\ \cdots\cdots \\ \begin{pmatrix} \mathbf{X}_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n\mathbf{X}_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_n^{\mathrm{T}} & -v_n\mathbf{X}_n^{\mathrm{T}} \end{pmatrix} \end{bmatrix}_{2n\times12} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}_{12\times1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}_{2n\times1}$$

- Solution? At least 6 features are needed for solving 12-1 unknowns.
- Least square or SVD…

Camera calibration: A linear approach—cont.

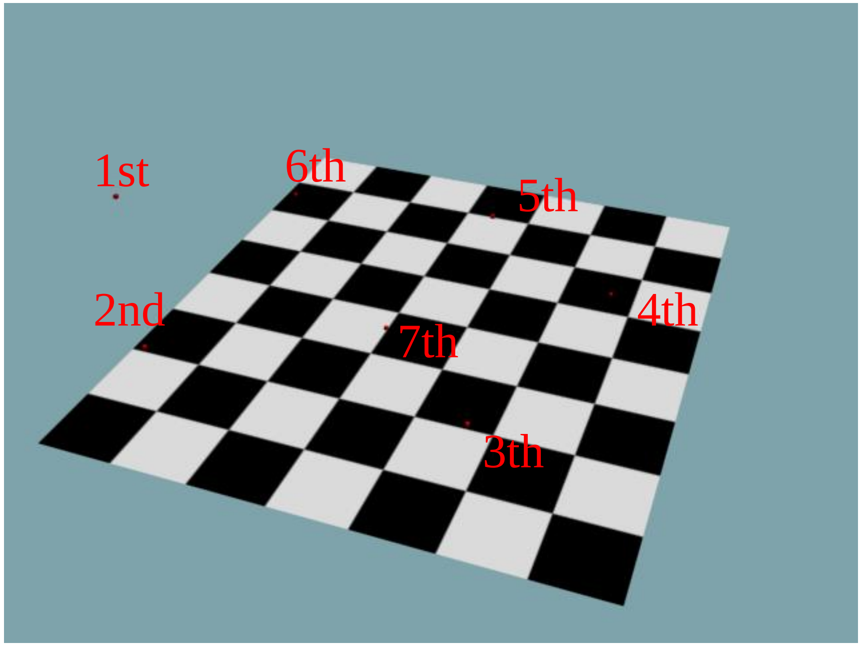
- A simple framework:
  - 3D Checkerboard → known 3D points
  - 2D feature → projected on image



Picture from [Foysyth 2003]

Camera calibration: A linear approach

- Example



3D points	2D points (on image)
X1=[0,0,75,1]^T	uv1=[83,146,1]^T
X2=[0,0,25,1]^T	uv2=[103,259,1]^T
X3=[100,0,25,1]^T	uv3=[346,315,1]^T
X4=[120,90,15,1]^T	uv4=[454,218,1]^T
X5=[90,50,60,1]^T	uv5=[365,161,1]^T
X6=[0,100,25,1]^T	uv6=[218,144,1]^T
X7=[60,40,20,1]^T	uv7=[286,244,1]^T

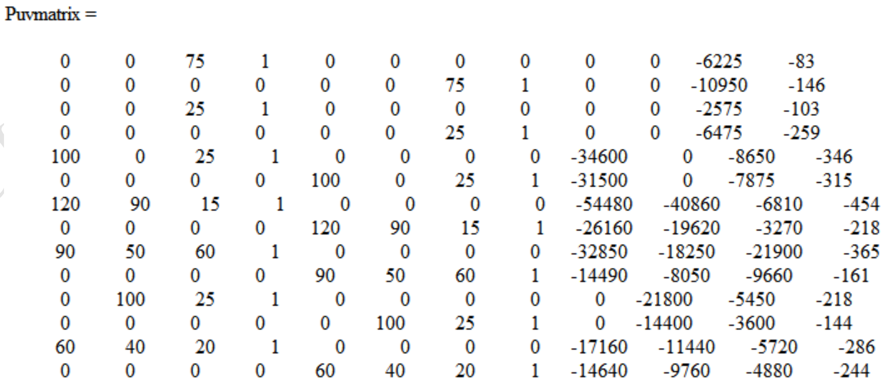
To determine a transformation **P**

Camera calibration: A linear approach

- Example—cont.
- Calculation using Matlab

```
Puvmatrix=
[  X1'    zero' -uv1(1).*X1';
   zero'  X1'   -uv1(2).*X1';
   X2'    zero' -uv2(1).*X2';
   zero'  X2'   -uv2(2).*X2';
   X3'    zero' -uv3(1).*X3';
   zero'  X3'   -uv3(2).*X3';
   X4'    zero' -uv4(1).*X4';
   zero'  X4'   -uv4(2).*X4';
   X5'    zero' -uv5(1).*X5';
   zero'  X5'   -uv5(2).*X5';
   X6'    zero' -uv6(1).*X6';
   zero'  X6'   -uv6(2).*X6';
   X7'    zero' -uv7(1).*X7';
   zero'  X7'   -uv7(2).*X7']
```

→



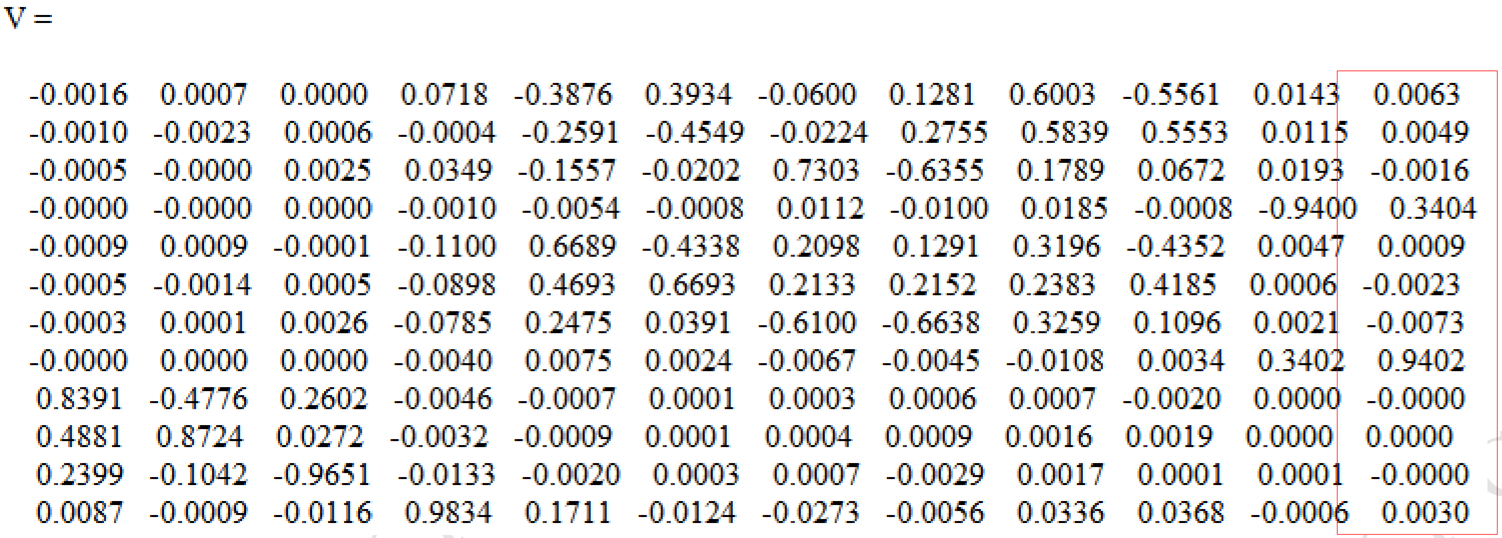
```
Puvmatrix =

     0      0     75      1      0      0      0      0       0       0    -6225     -83
     0      0      0      0      0      0     75      1       0       0   -10950    -146
     0      0     25      1      0      0      0      0       0       0    -2575    -103
     0      0      0      0      0      0     25      1       0       0    -6475    -259
   100      0     25      1      0      0      0      0  -34600       0    -8650    -346
     0      0      0      0    100      0     25      1  -31500       0    -7875    -315
   120     90     15      1      0      0      0      0  -54480  -40860    -6810    -454
     0      0      0      0    120     90     15      1  -26160  -19620    -3270    -218
    90     50     60      1      0      0      0      0  -32850  -18250   -21900    -365
     0      0      0      0     90     50     60      1  -14490   -8050    -9660    -161
     0    100     25      1      0      0      0      0       0  -21800    -5450    -218
     0      0      0      0      0    100     25      1       0  -14400    -3600    -144
    60     40     20      1      0      0      0      0  -17160  -11440    -5720    -286
     0      0      0      0     60     40     20      1  -14640   -9760    -4880    -244
```

$$* \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}_{12\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}_{14\times 1}$$

Camera calibration: A linear approach

- Example—cont.

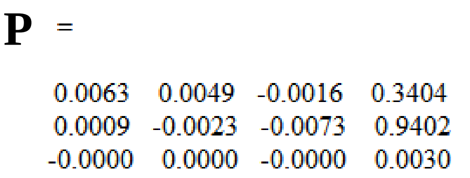
Solve by SVD:



```
V =

-0.0016   0.0007   0.0000   0.0718  -0.3876   0.3934  -0.0600   0.1281   0.6003  -0.5561   0.0143   0.0063
-0.0010  -0.0023   0.0006  -0.0004  -0.2591  -0.4549  -0.0224   0.2755   0.5839   0.5553   0.0115   0.0049
-0.0005  -0.0000   0.0025   0.0349  -0.1557  -0.0202   0.7303  -0.6355   0.1789   0.0672   0.0193  -0.0016
-0.0000  -0.0000   0.0000  -0.0010  -0.0054  -0.0008   0.0112  -0.0100   0.0185  -0.0008  -0.9400   0.3404
-0.0009   0.0009  -0.0001  -0.1100   0.6689  -0.4338   0.2098   0.1291   0.3196  -0.4352   0.0047   0.0009
-0.0005  -0.0014   0.0005  -0.0898   0.4693   0.6693   0.2133   0.2152   0.2383   0.4185   0.0006  -0.0023
-0.0003   0.0001   0.0026  -0.0785   0.2475   0.0391  -0.6100  -0.6638   0.3259   0.1096   0.0021  -0.0073
-0.0000   0.0000   0.0000  -0.0040   0.0075   0.0024  -0.0067  -0.0045  -0.0108   0.0034   0.3402   0.9402
 0.8391  -0.4776   0.2602  -0.0046  -0.0007   0.0001   0.0003   0.0006   0.0007  -0.0020   0.0000  -0.0000
 0.4881   0.8724   0.0272  -0.0032  -0.0009   0.0001   0.0004   0.0009   0.0016   0.0019   0.0000   0.0000
 0.2399  -0.1042  -0.9651  -0.0133  -0.0020   0.0003   0.0007  -0.0029   0.0017   0.0001   0.0001  -0.0000
 0.0087  -0.0009  -0.0116   0.9834   0.1711  -0.0124  -0.0273  -0.0056   0.0336   0.0368  -0.0006   0.0030
```

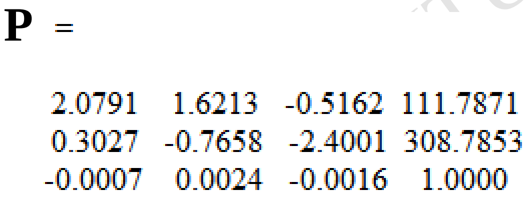
Get **P**:

P =[V(1:4,12)';V(5:8,12)';V(9:12,12)']



```
P =

 0.0063   0.0049  -0.0016   0.3404
 0.0009  -0.0023  -0.0073   0.9402
-0.0000   0.0000  -0.0000   0.0030
```

→ (normalized)

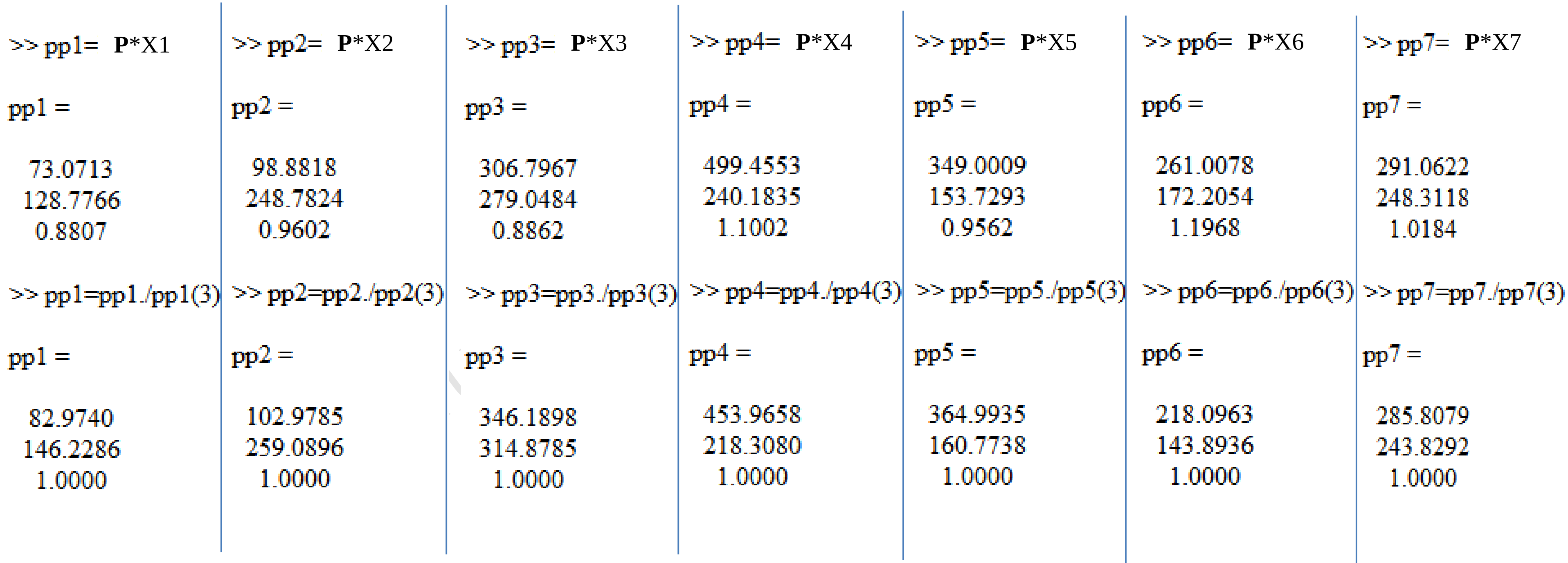


```
P =

 2.0791   1.6213  -0.5162  111.7871
 0.3027  -0.7658  -2.4001  308.7853
-0.0007   0.0024  -0.0016    1.0000
```

Camera calibration: A linear approach

- Example—cont.
  - Verify solution (in Matlab):

3D points	2D points (on image)
$X1=[0,0,75,1]^T$	$uv1=[83,146,1]^T$
$X2=[0,0,25,1]^T$	$uv2=[103,259,1]^T$
$X3=[100,0,25,1]^T$	$uv3=[346,315,1]^T$
$X4=[120,90,15,1]^T$	$uv4=[454,218,1]^T$
$X5=[90,50,60,1]^T$	$uv5=[365,161,1]^T$
$X6=[0,100,25,1]^T$	$uv6=[218,144,1]^T$
$X7=[60,40,20,1]^T$	$uv7=[286,244,1]^T$



```
>> pp1= P*X1

pp1 =

   73.0713
  128.7766
    0.8807

>> pp1=pp1./pp1(3)

pp1 =

   82.9740
  146.2286
    1.0000
```

```
>> pp2= P*X2

pp2 =

   98.8818
  248.7824
    0.9602

>> pp2=pp2./pp2(3)

pp2 =

  102.9785
  259.0896
    1.0000
```

```
>> pp3= P*X3

pp3 =

  306.7967
  279.0484
    0.8862

>> pp3=pp3./pp3(3)

pp3 =

  346.1898
  314.8785
    1.0000
```

```
>> pp4= P*X4

pp4 =

  499.4553
  240.1835
    1.1002

>> pp4=pp4./pp4(3)

pp4 =

  453.9658
  218.3080
    1.0000
```

```
>> pp5= P*X5

pp5 =

  349.0009
  153.7293
    0.9562

>> pp5=pp5./pp5(3)

pp5 =

  364.9935
  160.7738
    1.0000
```

```
>> pp6= P*X6

pp6 =

  261.0078
  172.2054
    1.1968

>> pp6=pp6./pp6(3)

pp6 =

  218.0963
  143.8936
    1.0000
```

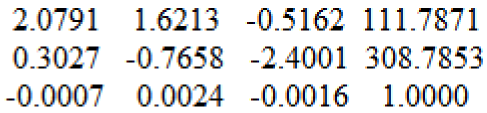
```
>> pp7= P*X7

pp7 =

  291.0622
  248.3118
    1.0184

>> pp7=pp7./pp7(3)

pp7 =

  285.8079
  243.8292
    1.0000
```

$uv1=[83,146,1]^T$ $uv2=[103,259,1]^T$ $uv3=[346,315,1]^T$ $uv4=[454,218,1]^T$ $uv5=[365,161,1]^T$ $uv6=[218,144,1]^T$ $uv7=[286,244,1]^T$

Camera calibration: A linear approach

- Example—cont.
  - Compared with "openCV calibration result"

P =



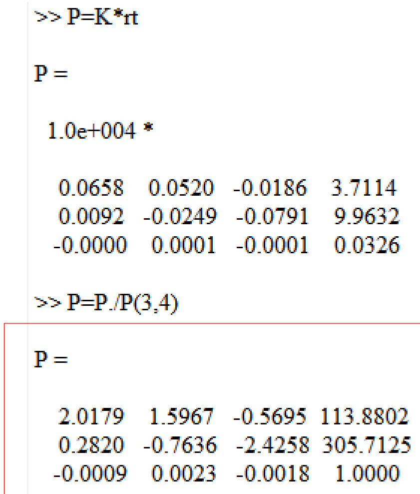
```
 2.0791    1.6213   -0.5162  111.7871
 0.3027   -0.7658   -2.4001  308.7853
-0.0007    0.0024   -0.0016    1.0000
```

Intrinsic parameter 3x3 Matrix (for all views)
797.467667 0.000000 318.980339
0.000000 797.569342 243.459839
0.000000 0.000000 1.000000

Distortion factor K
-0.002244 0.030546 -0.000019 -0.000223 0.0

view1.bmp: Extrinsic parameter 3x4 matrix
0.937760 0.347283 -0.000317 -83.818298
0.201549 -0.544979 -0.813865 25.437572
-0.282814 0.763146 -0.581054 325.901184

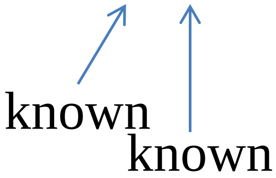
Compare data



```
>> P=K*rt

P =

  1.0e+004 *

    0.0658    0.0520   -0.0186    3.7114
    0.0092   -0.0249   -0.0791    9.9632
   -0.0000    0.0001   -0.0001    0.0326

>> P=P./P(3,4)

P =

    2.0179    1.5967   -0.5695  113.8802
    0.2820   -0.7636   -2.4258  305.7125
   -0.0009    0.0023   -0.0018    1.0000
```

Camera calibration: A linear approach

- Decomposition for $\mathbf{K}[\mathbf{R}|\mathbf{t}]$
- In case of: $\mathbf{P}=\mathbf{K}[\mathbf{R}|\mathbf{t}]$



- then, $[\mathbf{R}|\mathbf{t}]=\mathbf{K}^{-1}\mathbf{P}$ →up to scale

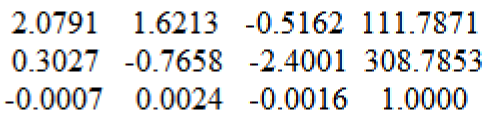
Note! Constraints for **R** are orthogonal and unit column/row vectors. In simple, you need to scale it.

- $[\mathbf{R}|\mathbf{t}]= s\mathbf{K}^{-1}\mathbf{P}$

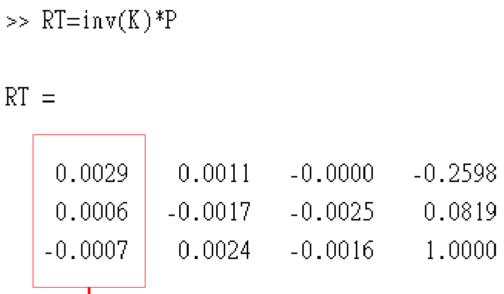
Camera calibration: A linear approach

- Example—cont.

P =

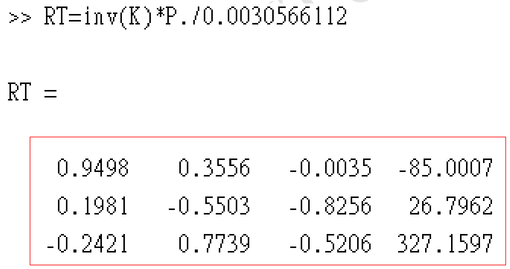


```
  2.0791    1.6213   -0.5162  111.7871
  0.3027   -0.7658   -2.4001  308.7853
 -0.0007    0.0024   -0.0016    1.0000
```



```
>> RT=inv(K)*P

RT =

    0.0029    0.0011   -0.0000   -0.2598
    0.0006   -0.0017   -0.0025    0.0819
   -0.0007    0.0024   -0.0016    1.0000
```

Length=0.0030566112
Not a unit vector from inv(K)*P. Need a scale.

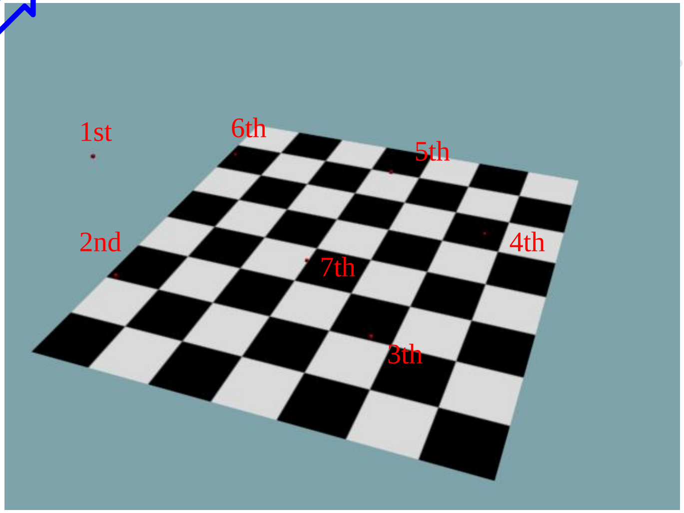


```
>> RT=inv(K)*P./0.0030566112

RT =

    0.9498    0.3556   -0.0035  -85.0007
    0.1981   -0.5503   -0.8256   26.7962
   -0.2421    0.7739   -0.5206  327.1597
```

Intrinsic parameter 3x3 Matrix (for all views)
797.467667 0.000000 318.980339
0.000000 797.569342 243.459839
0.000000 0.000000 1.000000

Distortion factor K
-0.002244 0.030546 -0.000019 -0.000223 0.0

view1.bmp: Extrinsic parameter 3x4 matrix
0.937760 0.347283 -0.000317 -83.818298
0.201549 -0.544979 -0.813865 25.437572
-0.282814 0.763146 -0.581054 325.901184



Gold Standard algorithm

- **Objective**
- Given n≥6 3D to 2D point correspondences $\{\mathbf{X}_i \leftrightarrow \mathbf{x}_i'\}$, determine the Maximum Likelyhood Estimation of mapping transformation **P**
- **Algorithm**
  - (i) Linear solution:
    - (a) Normalization:
    - (b) DLT (using SVD or least-square …) → get $\tilde{\mathbf{P}}$
  - (ii) Minimization of geometric error: using the linear estimate as a starting point minimize the geometric error. (RANSIC)
  - (iii) Denormalization.

$$\mathbf{P} = \left([\mathbf{s}][\mathbf{t}]\right]_{3\times3}\Big)^{-1}\tilde{\mathbf{P}} \cdot \left([\mathbf{S}][\mathbf{T}]\right]_{4\times4}\Big)$$

Algorithm 7.1 [Hartley04]

Gold Standard algorithm—cont.

$$\mathbf{x} = \mathbf{P} \cdot \mathbf{X}$$

normalization

$$\tilde{\mathbf{x}} = \tilde{\mathbf{P}} \cdot \tilde{\mathbf{X}}$$

Step-1a $\tilde{\mathbf{X}} = \left[[\mathbf{S}][\mathbf{T}]\right]_{4\times4} \mathbf{X}$ →normalization (3D points)

Translate center to the origin

Scale all points to the unit cube (or average the vector distance to be 1.414 or $\sqrt{2}$ →textbook suggestion value)

$\tilde{\mathbf{x}} = \left[[\mathbf{s}][\mathbf{t}]\right]_{3\times3} \mathbf{x}$ →normalization (2D points)

Gold Standard algorithm—cont.

Step-1b

$$\tilde{\mathbf{x}} = \tilde{\mathbf{P}} \cdot \tilde{\mathbf{X}}$$

→solve $\tilde{\mathbf{P}}$ by SVD or least-square method

Hint:

$$\begin{bmatrix} \begin{pmatrix} \mathbf{X}_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1\mathbf{X}_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_1^{\mathrm{T}} & -v_1\mathbf{X}_1^{\mathrm{T}} \end{pmatrix} \\ \begin{pmatrix} \mathbf{X}_2^{\mathrm{T}} & 0^{\mathrm{T}} & -u_2\mathbf{X}_2^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_2^{\mathrm{T}} & -v_2\mathbf{X}_2^{\mathrm{T}} \end{pmatrix} \\ \cdots\cdots \\ \begin{pmatrix} \mathbf{X}_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n\mathbf{X}_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}_n^{\mathrm{T}} & -v_n\mathbf{X}_n^{\mathrm{T}} \end{pmatrix} \end{bmatrix}_{2n\times12} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}_{12\times1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}_{2n\times1}$$

Step-2 →minimize the re-projection error, if necessary.

Gold Standard algorithm—cont.

Step-3 De-normalization (determine **P**)

$$\mathbf{x} = \mathbf{P} \cdot \mathbf{X}$$

$$\mathbf{P} = \left(\left[[\mathbf{s}][\mathbf{t}]\right]_{3\times3}\right)^{-1} \tilde{\mathbf{P}} \cdot \left(\left[[\mathbf{S}][\mathbf{T}]\right]_{4\times4}\right)$$

$$\tilde{\mathbf{x}} = \tilde{\mathbf{P}} \cdot \tilde{\mathbf{X}}$$

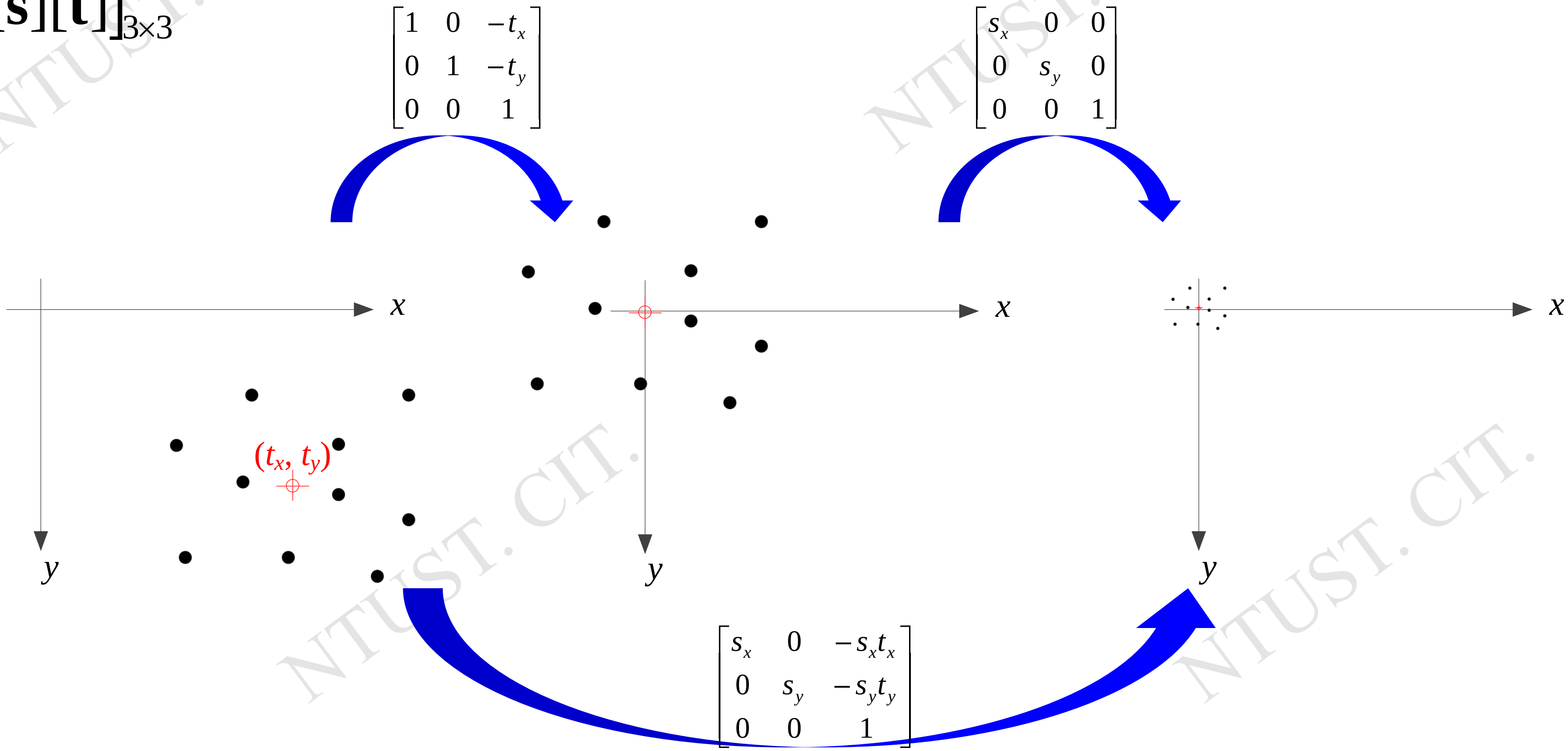
$$\left[[\mathbf{s}][\mathbf{t}]\right]_{3\times3} \mathbf{x} = \tilde{\mathbf{P}}\left(\left[[\mathbf{S}][\mathbf{T}]\right]_{4\times4} \mathbf{X}\right)$$

$$\mathbf{x} = \left(\left[[\mathbf{s}][\mathbf{t}]\right]_{3\times3}\right)^{-1} \tilde{\mathbf{P}}\left(\left[[\mathbf{S}][\mathbf{T}]\right]_{4\times4}\right)\mathbf{X}$$

$$\therefore \mathbf{P} = \left(\left[[\mathbf{s}][\mathbf{t}]\right]_{3\times3}\right)^{-1} \tilde{\mathbf{P}} \cdot \left(\left[[\mathbf{S}][\mathbf{T}]\right]_{4\times4}\right)$$

Gold Standard algorithm—remark

$$\left[[\mathbf{s}][\mathbf{t}]\right]_{3\times 3}$$



$$\begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 & -s_x t_x \\ 0 & s_y & -s_y t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Camera calibration: A linear approach

- Special case:
  - Affine camera model for mapping transform

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{p}_1{}^{\mathrm{T}} \\ \mathbf{p}_2{}^{\mathrm{T}} \\ \mathbf{p}_3{}^{\mathrm{T}} \end{bmatrix}_{3\times 4} \cdot \mathbf{X}_i$$

$$\mathbf{x}_i = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \cdot \mathbf{X}_i$$

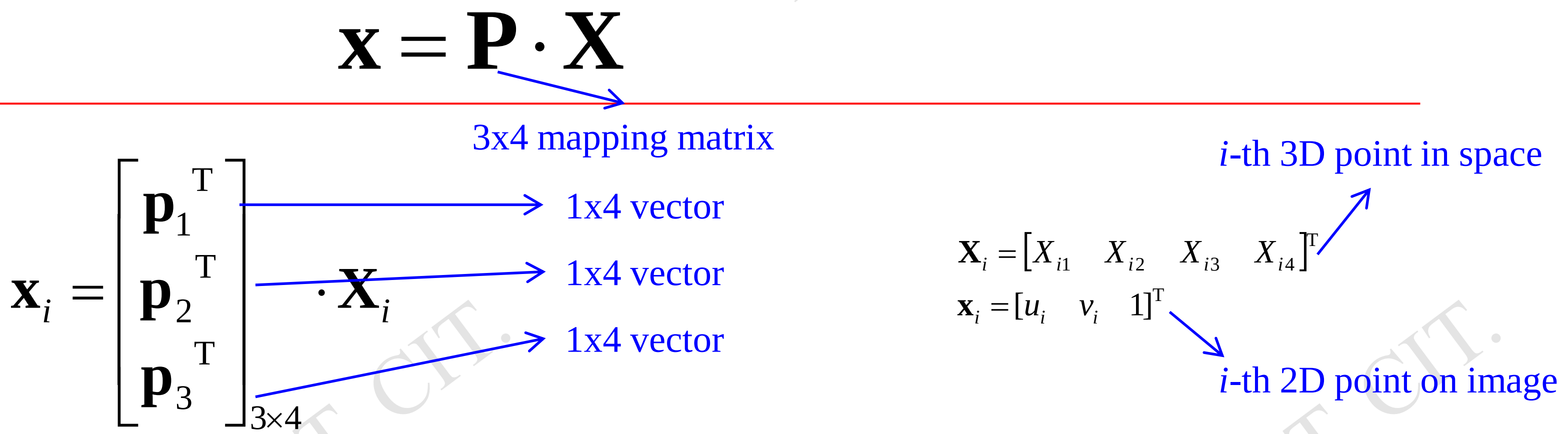
Projective camera

$$\mathbf{x}_i = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{X}_i$$

Affine camera (orthography)

Camera calibration: A linear approach

- Special case:
  - Affine camera model for mapping transform



$$\mathbf{x}=\mathbf{P}\cdot\mathbf{X}$$

3x4 mapping matrix

$$\mathbf{x}_i=\begin{bmatrix}\mathbf{p}_1{}^{T}\\ \mathbf{p}_2{}^{T}\\ \mathbf{p}_3{}^{T}\end{bmatrix}_{3\times4}\cdot\mathbf{X}_i$$

1x4 vector

1x4 vector

1x4 vector

$$\mathbf{X}_i=\left[X_{i1}\quad X_{i2}\quad X_{i3}\quad X_{i4}\right]^{T}$$

i-th 3D point in space

$$\mathbf{x}_i=[u_i\quad v_i\quad 1]^{T}$$

i-th 2D point on image

If in special case (affine camera): $\mathbf{p}_3{}^{T}=\left[0\quad 0\quad 0\quad 1\right]$

$$\begin{bmatrix}\begin{pmatrix}\mathbf{X}_1{}^{T} & 0^{T} & -u_1\mathbf{X}_1{}^{T}\\ 0^{T} & \mathbf{X}_1{}^{T} & -v_1\mathbf{X}_1{}^{T}\end{pmatrix}\\ \begin{pmatrix}\mathbf{X}_2{}^{T} & 0^{T} & -u_2\mathbf{X}_2{}^{T}\\ 0^{T} & \mathbf{X}_2{}^{T} & -v_2\mathbf{X}_2{}^{T}\end{pmatrix}\\ \cdots\cdots\\ \begin{pmatrix}\mathbf{X}_n{}^{T} & 0^{T} & -u_n\mathbf{X}_n{}^{T}\\ 0^{T} & \mathbf{X}_n{}^{T} & -v_n\mathbf{X}_n{}^{T}\end{pmatrix}\end{bmatrix}_{2n\times12}\begin{pmatrix}\mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{pmatrix}_{12\times1}=\begin{pmatrix}0\\0\\0\\0\\ \cdots\\0\\0\end{pmatrix}_{2n\times1}$$

$\rightarrow$ **Reduce to 8 unknowns**

$$\begin{bmatrix}\begin{pmatrix}\mathbf{X}_1{}^{T} & 0^{T}\\ 0^{T} & \mathbf{X}_1{}^{T}\end{pmatrix}\\ \begin{pmatrix}\mathbf{X}_2{}^{T} & 0^{T}\\ 0^{T} & \mathbf{X}_2{}^{T}\end{pmatrix}\\ \cdots\cdots\\ \begin{pmatrix}\mathbf{X}_n{}^{T} & 0^{T}\\ 0^{T} & \mathbf{X}_n{}^{T}\end{pmatrix}\end{bmatrix}_{2n\times8}\begin{pmatrix}\mathbf{p}_1\\ \mathbf{p}_2\end{pmatrix}_{8\times1}=\begin{pmatrix}u_1\\v_1\\u_2\\v_2\\ \cdots\\u_n\\v_n\end{pmatrix}_{2n\times1}$$

Gold Standard algorithm

- **Objective**
- Given $n \geq 4$ 3D to 2D point correspondences $\{\mathbf{X}_i \leftrightarrow \mathbf{x}_i'\}$, determine the Maximum Likelyhood Estimation of **P**, here $\mathbf{p}_3^T=[0,0,0,1]$.
- **Algorithm**

(i) Normalization:

(ii) Correspondence (governing equation)

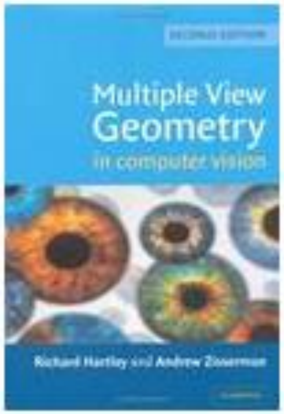
(iii) Determine **P** by SVD or DLT.

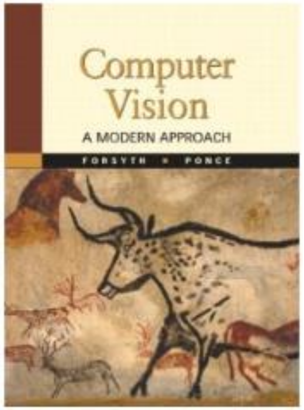
(iv) Denormalization:

Algorithm 7.2 [Hartley04]

Camera calibration

- Intrinsic parameter & extrinsic parameter
- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, (Chapter8.4, 8.5)
  - Computer Vision A Modern Approach, NA
  - Select paper: Zhang, Z. 1999. Flexible camera calibration by viewing a plane from unknown orientations. *IEEE International Conference on Computer Vision*. 1, (1999), 666-673.





Camera calibration

- Intrinsic parameter calibration
  1. From absolute conic
  2. From homography

Camera calibration

- From absolute conic
  - Points on the plane at infinity and absolute conic

$$\mathbf{x}=\mathbf{P}\cdot\mathbf{X}_\infty$$

$$\mathbf{x}=\mathbf{K}[\mathbf{R}\,|\,\mathbf{t}]\cdot\mathbf{X}_\infty$$

$\mathbf{X}_\infty$ is one 3D point at infinity. Let $\mathbf{X}_\infty=\begin{bmatrix}\mathbf{d}_{3\times1}\\ 0\end{bmatrix}$

$$\mathbf{x}=\mathbf{K}[\mathbf{R}\,|\,\mathbf{t}]\cdot\mathbf{X}_\infty=\mathbf{K}[\mathbf{R}\,|\,\mathbf{t}]\cdot\begin{bmatrix}\mathbf{d}_{3\times1}\\ 0\end{bmatrix}$$

$$\mathbf{x}=\underline{\mathbf{KR}}\mathbf{d}\longrightarrow$$ 2D point to 2D point mapping → homography

$$\downarrow$$

$$\mathbf{x}=\mathbf{H}\mathbf{d}$$

Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic
  - The mapping between $\boldsymbol{\pi}_{\infty}$ and an image is given by the planar homography $\mathbf{x} = \mathbf{H}\mathbf{d}$ with

$$\mathbf{H} = \mathbf{K}\mathbf{R}$$

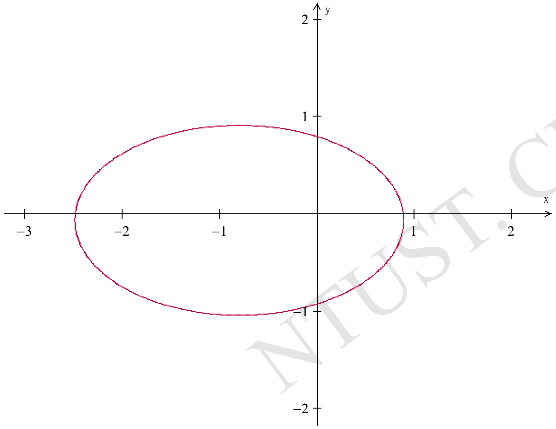
Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic
- Recall the property of the conic: if one point is on the conic C, it forms

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0$$

- For example, one ellipse:

$$\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0$$



Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic

- Absolute conic $\mathbf{\Omega}_\infty$: the a (point) conic on $\boldsymbol{\pi}_\infty = (0,0,0,1)^\mathrm{T}$

$$\mathbf{X}_\infty = (X_1, X_2, X_3, X_4)$$

$$\begin{cases} \boxed{X_1^{\ 2} + X_2^{\ 2} + X_3^{\ 2} = 0} \\ X_4 = 0 \end{cases}$$

$$\rightarrow (X_1, X_2, X_3)\mathbf{I}(X_1, X_2, X_3)^\mathsf{T} = 0$$

$$\downarrow$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic
- **The *image of the absolute* conic (called IAC) is the conic :**

$$\boxed{\boldsymbol{\omega} = (\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}}$$

- Proof: $\mathbf{x}' = \mathbf{H}\mathbf{x}$ →points **x** are mapped to **x**' by homography **H**

  $\mathbf{C}' = \mathbf{H}^{-\mathrm{T}}\mathbf{C}\mathbf{H}^{-1}$ →conic **C** will be mapped to **C**' (according this eqs)
- Since we want to determine the mapping result (says **ω** as a new notation) of the absolute conic (**C**=**I**), the **H**=**KR**

$$\boldsymbol{\omega} = \mathbf{H}^{-\mathrm{T}}\mathbf{I}\mathbf{H}^{-1} = (\mathbf{KR})^{-\mathrm{T}}\mathbf{I}(\mathbf{KR})^{-1} = \mathbf{K}^{-\mathrm{T}}\mathbf{R}^{-\mathrm{T}}\mathbf{I}\mathbf{R}^{-1}\mathbf{K}^{-1}$$

$$= \mathbf{K}^{-\mathrm{T}}\mathbf{R}\mathbf{I}\mathbf{R}^{-1}\mathbf{K}^{-1} = \mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1} = (\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}$$

to determine **K** from a known **ω**, Cholesky factorization is used.

Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic
- **Absolute conic:** points $[1 \quad \pm i \quad 0]$ are on absolute conic (imaginary point).

$$[1 \quad \pm i \quad 0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\boxed{\begin{bmatrix}1\\\pm i\\0\end{bmatrix}}=0$$

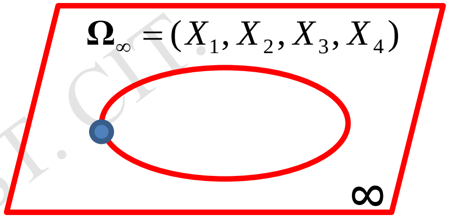
Points at infinity plane
(and on absolute conic)

- Suppose these points can be mapped by a known **H**, the mapped points will be $\mathbf{h}_1 \pm i\mathbf{h}_2$

$$\boxed{\mathbf{H}\begin{bmatrix}1\\\pm i\\0\end{bmatrix}}=[\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3]\begin{bmatrix}1\\\pm i\\0\end{bmatrix}=\mathbf{h}_1 \pm i\mathbf{h}_2$$

Points mapping back (via homography) to image plane

3D Points on the plane at infinity and absolute conic



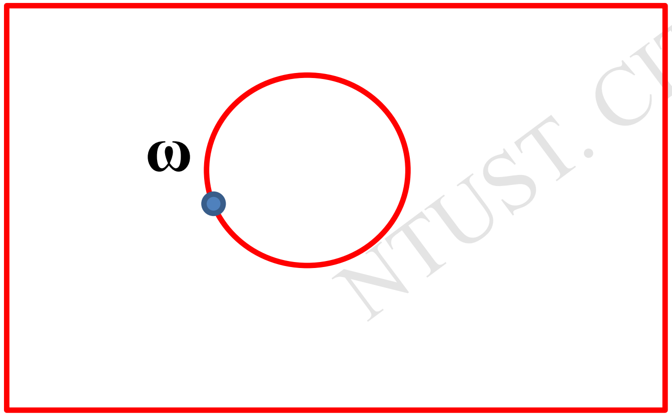
$$\mathbf{\Omega}_\infty = (X_1, X_2, X_3, X_4)$$

$$\mathbf{X}_\infty = (X_1, X_2, X_3, X_4) \quad \boldsymbol{\pi}_\infty = (0,0,0,1)^\mathrm{T}$$

$$\mathbf{H} = \mathbf{KR}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Image plane (taken pic)



$$\mathbf{x}' = \mathbf{Hx}$$

$$\mathbf{C}' = \mathbf{H}^{-\mathrm{T}}\mathbf{C}\mathbf{H}^{-1}$$

$$\boldsymbol{\omega} = \mathbf{H}^{-\mathrm{T}}\mathbf{\Omega}_\infty\mathbf{H}^{-1}$$

$$\boldsymbol{\omega} = (\mathbf{KR})^{-\mathrm{T}}\mathbf{\Omega}_\infty(\mathbf{KR})^{-1}$$

Camera calibration

- From absolute conic—cont.
  - Points on the plane at infinity and absolute conic

- According the "points on conic" equation $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x}=0$

$$(\mathbf{h}_1 \pm i\mathbf{h}_2)^{\mathrm{T}}\boldsymbol{\omega}\boxed{(\mathbf{h}_1 \pm i\mathbf{h}_2)} = 0$$

The mapping back points will be on the conic

- Deal with real part & imaginary part individually:

$$\underline{\mathbf{h}_1{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_1 + i^2\mathbf{h}_2{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2} + 2i(\underline{\mathbf{h}_1{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2}) = 0$$

$$\mathbf{h}_1{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2 = 0$$

$$\mathbf{h}_1{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_1 = \mathbf{h}_2{}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2$$

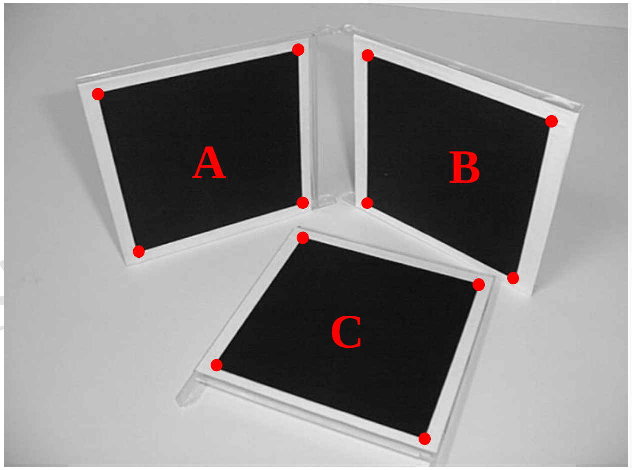
Camera calibration

- Summary: A simple calibration from the absolute conic
  - Step-1: compute **H** for each square →for example 3 squares induce 3 **H** (unit corners (0,0),(1,0),(0,1),(1,1))
  - Step-2: compute the imaged circular points **H**$(1,\pm i,0)^T$
  - Step-3: fit a conic to 6 circular points **ω** → SVD
  - Step-4: compute **K** from **ω** through Cholesky factorization

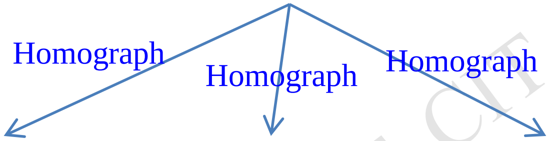
[algorithm from Hartley textbook example8.18]

Camera calibration

- Example: from absolute conic

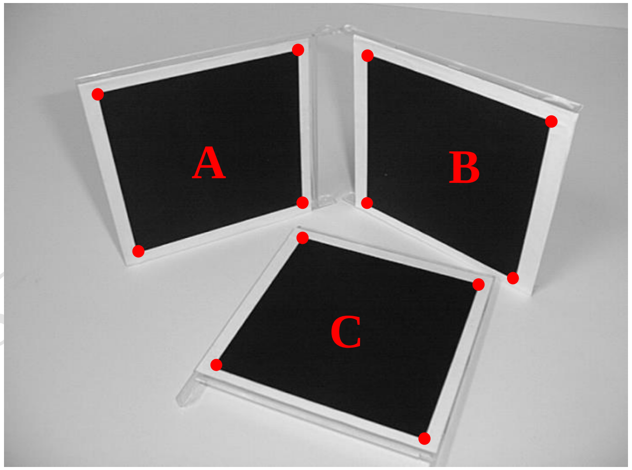


unit square
0,0
0,1
1,1
1,0



A-square	**B-square**	**C-square**
152,149	596,84	490,387
218,413	596,334	343,602
490,332	838,458	689,722
482,77	898,195	780,465

Camera calibration



- Example: from absolute conic—cont.

Homography (unit square to **A-square**)
379.199677 111.830818 152.000000
-64.140297 350.826263 149.000000
0.102074 0.210233 1.000000

Homography (unit square to **B-square**)
168.271439 125.763161 596.000000
81.960945 320.478027 84.000000
-0.148918 0.211012 1.000000

Homography (unit square to **C-square**)
228.969971 -209.572891 490.000000
41.616714 105.178200 387.000000
-0.078244 -0.182428 1.000000

Camera calibration

- Example: from absolute conic—cont.
  - To determine **ω**, remember this is a "conic", a symmetric 3x3 matrix form

$$\mathbf{h}_1^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2=0$$

$$\mathbf{h}_1^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_1=\mathbf{h}_2^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2$$

$$\mathbf{h}_1=\begin{bmatrix}h_{11}\\h_{12}\\h_{13}\end{bmatrix},\mathbf{h}_2=\begin{bmatrix}h_{21}\\h_{22}\\h_{23}\end{bmatrix}$$

$$\mathbf{h}_1^{\mathrm{T}}\begin{bmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{bmatrix}\mathbf{h}_2=0$$

$$\mathbf{h}_1^{\mathrm{T}}\begin{bmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{bmatrix}\mathbf{h}_1=\mathbf{h}_2^{\mathrm{T}}\begin{bmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{bmatrix}\mathbf{h}_2$$

$$h_{11}h_{21}\omega_{11}+h_{11}h_{22}\omega_{12}+h_{11}h_{23}\omega_{13}+h_{12}h_{21}\omega_{21}+h_{12}h_{22}\omega_{22}+h_{12}h_{23}\omega_{23}+h_{13}h_{21}\omega_{31}+h_{13}h_{22}\omega_{32}+h_{13}h_{23}\omega_{33}=0$$

$$h_{11}h_{11}\omega_{11}+h_{11}h_{12}\omega_{12}+h_{11}h_{13}\omega_{13}+h_{12}h_{11}\omega_{21}+h_{12}h_{12}\omega_{22}+h_{12}h_{13}\omega_{23}+h_{13}h_{11}\omega_{31}+h_{13}h_{12}\omega_{32}+h_{13}h_{13}\omega_{33}=$$
$$h_{21}h_{21}\omega_{11}+h_{21}h_{22}\omega_{12}+h_{21}h_{23}\omega_{13}+h_{22}h_{21}\omega_{21}+h_{22}h_{22}\omega_{22}+h_{22}h_{23}\omega_{23}+h_{23}h_{21}\omega_{31}+h_{23}h_{22}\omega_{32}+h_{23}h_{23}\omega_{33}$$

Note! **ω** is symmetric

Camera calibration

- Example: from absolute conic—cont.

$$h_{11}h_{21}\omega_{11}+(h_{11}h_{22}+h_{12}h_{21})\omega_{12}+(h_{11}h_{23}+h_{13}h_{21})\omega_{13}+h_{12}h_{22}\omega_{22}+(h_{12}h_{23}+h_{13}h_{22})\omega_{23}+h_{13}h_{23}\omega_{33}=0$$

$$(h_{11}h_{11}-h_{21}h_{21})\omega_{11}+(h_{11}h_{12}+h_{12}h_{11}-h_{21}h_{22}-h_{22}h_{21})\omega_{12}+(h_{11}h_{13}+h_{13}h_{11}-h_{21}h_{23}-h_{23}h_{21})\omega_{13}+(h_{12}h_{12}-h_{22}h_{22})\omega_{22}$$
$$+(h_{12}h_{13}+h_{13}h_{12}-h_{22}h_{23}-h_{23}h_{22})\omega_{23}+(h_{13}h_{13}-h_{23}h_{23})\omega_{33}=0$$

$$\Rightarrow \begin{bmatrix} h_{11}h_{21} & (h_{11}h_{22}+h_{12}h_{21}) & (h_{11}h_{23}+h_{13}h_{21}) & h_{12}h_{22} & (h_{12}h_{23}+h_{13}h_{22}) & h_{13}h_{23} \\ (h_{11}{}^2-h_{21}{}^2) & 2(h_{11}h_{12}-h_{21}h_{22}) & 2(h_{11}h_{13}-h_{21}h_{23}) & (h_{12}{}^2-h_{22}{}^2) & 2(h_{12}h_{13}-h_{22}h_{23}) & (h_{13}{}^2-h_{23}{}^2) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{13} \\ \omega_{22} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = 0$$

↘ Two constraints from one homography

6 unknowns (actually 5 unknowns upto scale), so it needs at least **3** homography

```
A=[
h(1,1)*h(2,1) h(1,1)*h(2,2)+h(1,2)*h(2,1) h(1,1)*h(2,3)+h(1,3)*h(2,1) h(1,2)*h(2,2) h(1,2)*h(2,3)+h(1,3)*h(2,2) h(1,3)*h(2,3);
h(1,1)^2-h(2,1)^2 2*(h(1,1)*h(1,2)-h(2,1)*h(2,2)) 2*(h(1,1)*h(1,3)-h(2,1)*h(2,3)) h(1,2)^2-h(2,2)^2
2*(h(1,2)*h(1,3)-h(2,2)*h(2,3)) h(1,3)^2-h(2,3)^2 ]
```

BE CAREFUL! This notation is different from Matlab.

Camera calibration

- Example: from absolute conic—cont.
  - To determine K in $\boldsymbol{\omega} = (\mathbf{KK}^{\mathrm{T}})^{-1}$

Assume $\mathbf{K} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{\omega}^{-1} = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix} = s^2 \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$

$$= s^2 \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & 1 \end{bmatrix} = s^2 \begin{bmatrix} a^2+b^2+c^2 & bd+ce & c \\ bd+ce & d^2+e^2 & e \\ c & e & 1 \end{bmatrix}$$

$$\boldsymbol{\omega}^{-1} = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix} = s^2 \begin{bmatrix} a^2+b^2+c^2 & bd+ce & c \\ bd+ce & d^2+e^2 & e \\ c & e & 1 \end{bmatrix}$$

$$s = \pm\sqrt{\varpi_{33}}$$

Camera calibration

- Example: from absolute conic—cont.

$$\begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix} = s^2 \begin{bmatrix} a^2+b^2+c^2 & bd+ce & c \\ bd+ce & d^2+e^2 & e \\ c & e & 1 \end{bmatrix}$$

let $\varpi_{33} = 1$

then, $c = \varpi_{13}$

$e = \varpi_{23}$

Solve *d*: (by 2nd row, 2nd column) $d = \pm\sqrt{\varpi_{22} - e^2}$ (use positive value)

Solve *e*:(by 1st row, 2nd column) $b = (\varpi_{12} - ce)/d$

Solve *a*:(by 1st row, 1st column) $a = \pm\sqrt{\varpi_{11} - b^2 - c^2}$ (use positive value)

Finally, $\mathbf{K} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix}$

Another solution for $\boldsymbol{\omega} = (\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}$ is Cholesky factorization (matlab2011)

Camera calibration

- Example: from absolute conic—cont.

Of course you can assume $\mathbf{K}=\begin{bmatrix} a & 0 & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix}$, since most digital CCDs have NO (or tiny) skew effect.

$$\boldsymbol{\omega}^{-1}=\begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix}=s^2\begin{bmatrix} a^2+c^2 & ce & c \\ ce & d^2+e^2 & e \\ c & e & 1 \end{bmatrix}$$

let $\varpi_{33}=1$

then, $c=\varpi_{13}$

$e=\varpi_{23}$

Solve d: (by 2nd row, 2nd column) $d=\pm\sqrt{\varpi_{22}-e^2}$ (use positive value)

Solve a:(by 1st row, 1st column) $a=\pm\sqrt{\varpi_{11}-c^2}$ (use positive value)

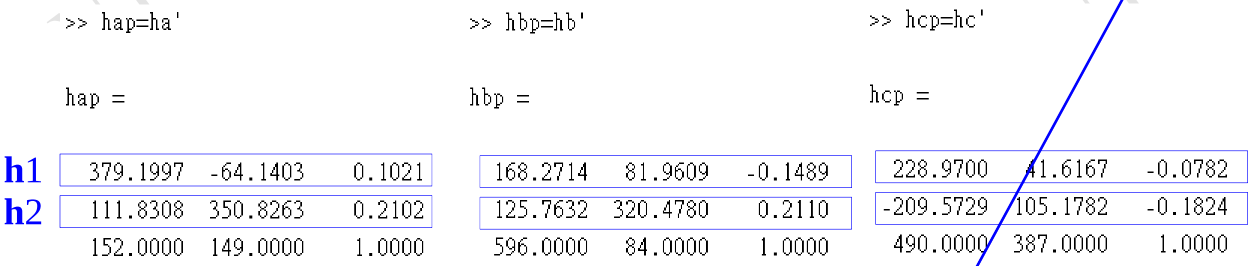
Check $|\varpi_{12}-ce|$ value, if it is not neglected, this assumption is poor.

Solution in Matlab (for example)

$$\begin{bmatrix} h_{11}h_{21} & (h_{11}h_{22}+h_{12}h_{21}) & (h_{11}h_{23}+h_{13}h_{21}) & h_{12}h_{22} & (h_{12}h_{23}+h_{13}h_{22}) & h_{13}h_{23} \\ (h_{11}{}^2-h_{21}{}^2) & 2(h_{11}h_{12}-h_{21}h_{22}) & 2(h_{11}h_{13}-h_{21}h_{23}) & (h_{12}{}^2-h_{22}{}^2) & 2(h_{12}h_{13}-h_{22}h_{23}) & (h_{13}{}^2-h_{23}{}^2) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{13} \\ \omega_{22} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = 0$$

Step-1 determine every homography
(in this example 3 square→ 3 homography)

Step-2 determine every **h**1 & **h**2 from 3 homography. One homography gives two equations (constraints)
(NOTE! The transpose operation is ONLY for notation purpose in Matlab)



h1
h2

```
>> hap=ha'

hap =

  379.1997  -64.1403    0.1021
  111.8308  350.8263    0.2102
  152.0000  149.0000    1.0000
```

```
>> hbp=hb'

hbp =

  168.2714   81.9609   -0.1489
  125.7632  320.4780    0.2110
  596.0000   84.0000    1.0000
```

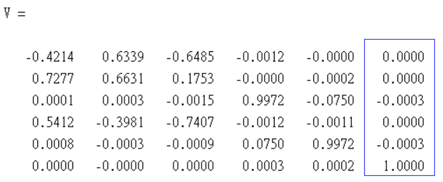
```
>> hcp=hc'

hcp =

  228.9700   41.6167   -0.0782
 -209.5729  105.1782   -0.1824
  490.0000  387.0000    1.0000
```

Step-3 construct the matrix form, then solve it by SVD for determine **ω**.

```
A=[hap(1,1)*hap(2,1) hap(1,1)*hap(2,2)+hap(1,2)*hap(2,1) hap(1,1)*hap(2,3)+hap(1,3)*hap(2,1) hap(1,2)*hap(2,2) hap(1,2)*hap(2,3)+hap(1,3)*hap(2,2) hap(1,3)*hap(2,3);
hap(1,1)^2-hap(2,1)^2 2*(hap(1,1)*hap(1,2)-hap(2,1)*hap(2,2)) 2*(hap(1,1)*hap(1,3)-hap(2,1)*hap(2,3)) hap(1,2)^2-hap(2,2)^2 2*(hap(1,2)*hap(1,3)-hap(2,2)*hap(2,3)) hap(1,3)^2-hap(2,3)^2;
hbp(1,1)*hbp(2,1) hbp(1,1)*hbp(2,2)+hbp(1,2)*hbp(2,1) hbp(1,1)*hbp(2,3)+hbp(1,3)*hbp(2,1) hbp(1,2)*hbp(2,2) hbp(1,2)*hbp(2,3)+hbp(1,3)*hbp(2,2) hbp(1,3)*hbp(2,3);
hbp(1,1)^2-hbp(2,1)^2 2*(hbp(1,1)*hbp(1,2)-hbp(2,1)*hbp(2,2)) 2*(hbp(1,1)*hbp(1,3)-hbp(2,1)*hbp(2,3)) hbp(1,2)^2-hbp(2,2)^2 2*(hbp(1,2)*hbp(1,3)-hbp(2,2)*hbp(2,3)) hbp(1,3)^2-hbp(2,3)^2;
hcp(1,1)*hcp(2,1) hcp(1,1)*hcp(2,2)+hcp(1,2)*hcp(2,1) hcp(1,1)*hcp(2,3)+hcp(1,3)*hcp(2,1) hcp(1,2)*hcp(2,2) hcp(1,2)*hcp(2,3)+hcp(1,3)*hcp(2,2) hcp(1,3)*hcp(2,3);
hcp(1,1)^2-hcp(2,1)^2 2*(hcp(1,1)*hcp(1,2)-hcp(2,1)*hcp(2,2)) 2*(hcp(1,1)*hcp(1,3)-hcp(2,1)*hcp(2,3)) hcp(1,2)^2-hcp(2,2)^2 2*(hcp(1,2)*hcp(1,3)-hcp(2,2)*hcp(2,3)) hcp(1,3)^2-hcp(2,3)^2]
```

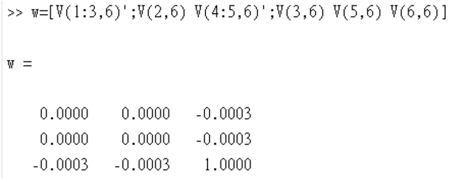
```
[U,S,V]=svd(A)
```

→



```
V =

  -0.4214    0.6339   -0.6485   -0.0012   -0.0000    0.0000
   0.7277    0.6631    0.1753   -0.0000   -0.0002    0.0000
   0.0001    0.0003   -0.0015    0.9972   -0.0750   -0.0003
   0.5412   -0.3981   -0.7407   -0.0012   -0.0011    0.0000
   0.0008   -0.0003   -0.0009    0.0750    0.9972   -0.0003
   0.0000   -0.0000    0.0000    0.0003    0.0002    1.0000
```

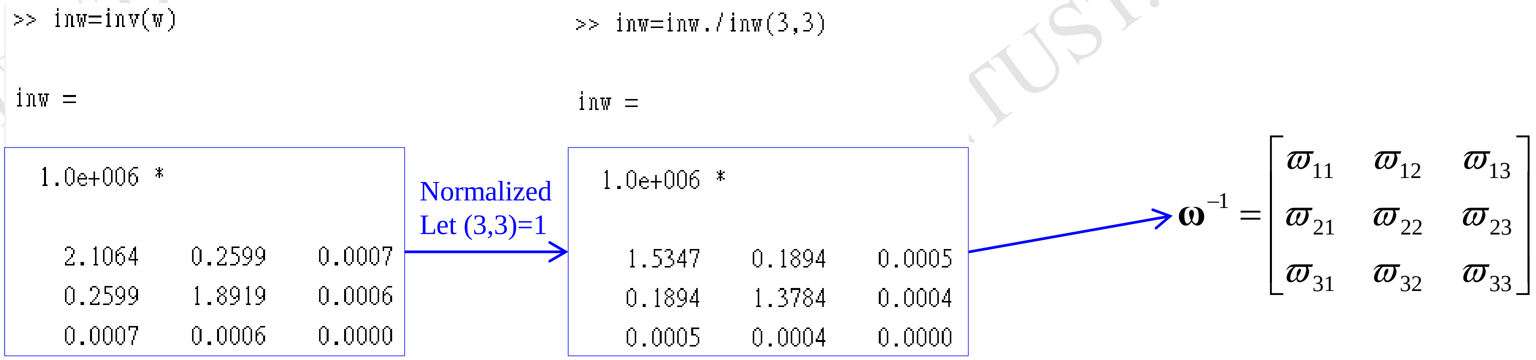
→



```
>> w=[V(1:3,6)';V(2,6) V(4:5,6)';V(3,6) V(5,6) V(6,6)]

w =

    0.0000    0.0000   -0.0003
    0.0000    0.0000   -0.0003
   -0.0003   -0.0003    1.0000
```

Solution in Matlab (for example)—cont.

Step-4 solve **K**, take inverse of **ω**, then use close-form solution

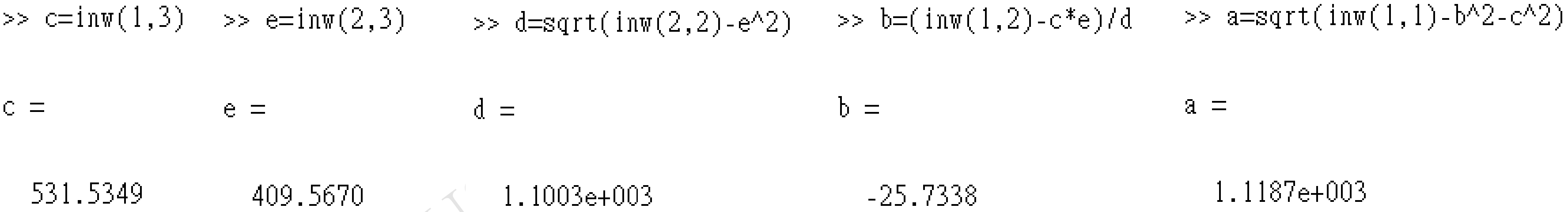


```
>> inw=inv(w)

inw =

  1.0e+006 *

    2.1064    0.2599    0.0007
    0.2599    1.8919    0.0006
    0.0007    0.0006    0.0000
```

Normalized
Let (3,3)=1

```
>> inw=inw./inw(3,3)

inw =

  1.0e+006 *

    1.5347    0.1894    0.0005
    0.1894    1.3784    0.0004
    0.0005    0.0004    0.0000
```

$$\boldsymbol{\omega}^{-1} = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix}$$



```
>> c=inw(1,3)

c =

  531.5349
```

```
>> e=inw(2,3)

e =

  409.5670
```

```
>> d=sqrt(inw(2,2)-e^2)

d =

  1.1003e+003
```

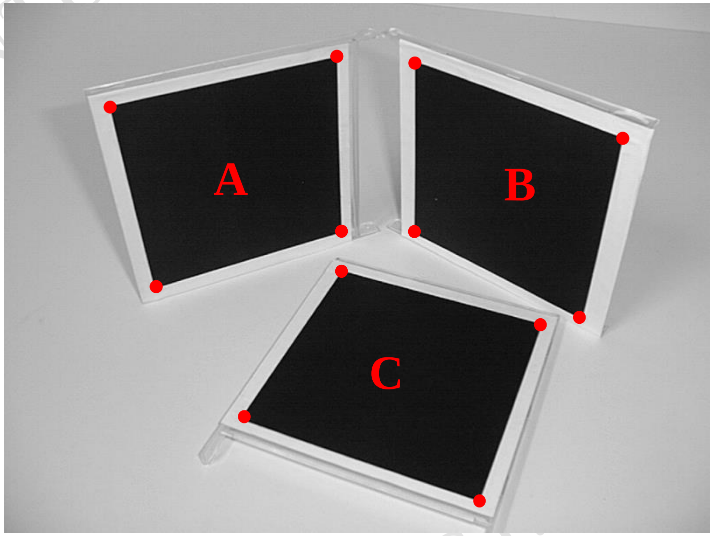
```
>> b=(inw(1,2)-c*e)/d

b =

  -25.7338
```

```
>> a=sqrt(inw(1,1)-b^2-c^2)

a =

  1.1187e+003
```

$$\mathbf{K} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1118.7 & -25.7 & 531.5 \\ 0 & 1100.3 & 409.6 \\ 0 & 0 & 1 \end{bmatrix}$$

Camera calibration

- Finally,



$$\mathbf{K} = \begin{bmatrix} 1108.3 & -9.8 & 525.8 \\ 0 & 1097.8 & 395.9 \\ 0 & 0 & 1 \end{bmatrix}$$

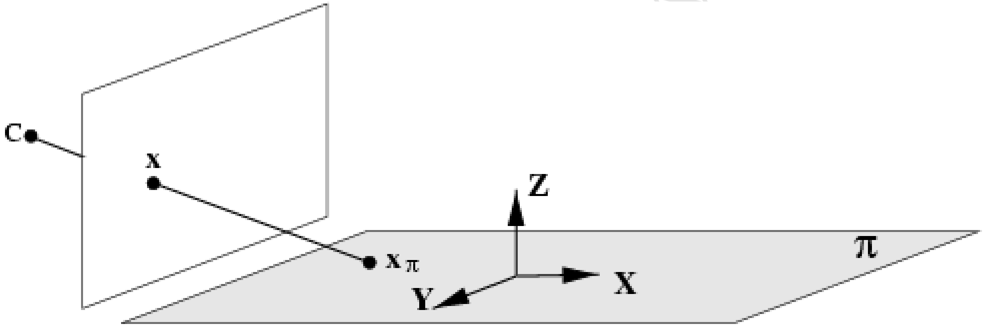
(from textbook)

What's NEXT?

Determine distortion & extrinsic parameter
Refer to the following paper:
Zhang, Z. 1999. Flexible camera calibration by viewing a plane from unknown orientations. *IEEE International Conference on Computer Vision*. 1, 666-673.

Camera calibration

- from homography—Zhang's method



$$\mathbf{x} = \mathbf{P}\mathbf{X} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

2D points on image plane

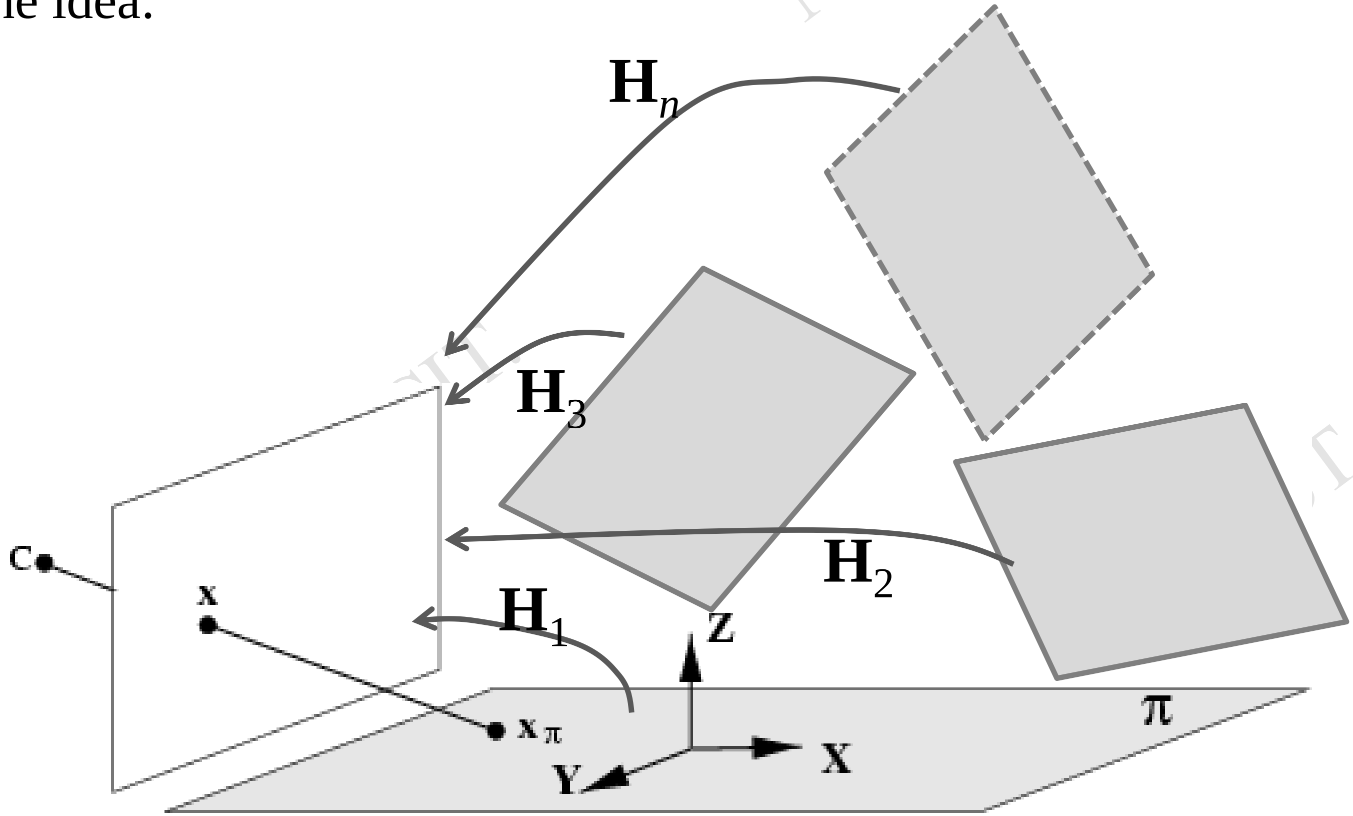
Homography

3D points on a plane
(considered as 2D points)

Zhang, Z. 1999. Flexible camera calibration by viewing a plane from unknown orientations. *IEEE International Conference on Computer Vision*. 1, 666-673.

Camera calibration

- from homography—Zhang's method—cont.
  - The idea:



Camera calibration

- from homography—Zhang’s method—cont.

Points on a 3D plane:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4]\begin{bmatrix}X\\Y\\0\\1\end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_4]\begin{bmatrix}X\\Y\\1\end{bmatrix}$$

Recall the pin-hole projection formula:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X}$$

K: intrinsic parameter.
[**R**|**t**]: extrinsic parameter

can be reduced to:

$$\mathbf{x} = s\begin{bmatrix}u\\v\\1\end{bmatrix} = \mathbf{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}]\begin{bmatrix}X\\Y\\1\end{bmatrix} = \mathbf{H}\begin{bmatrix}X\\Y\\1\end{bmatrix}$$

The mapping (indeed homography) can be defined as

$$\mathbf{H} = \mathbf{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}]$$

Remark: **H** consists of intrinsic and part of extrinsic parameters

1x3 column vector

Note the notation in paper.

Camera calibration

- from homography—Zhang’s method—cont.

Since homography is up to scale, we define 3 column vectors for **H** and rewrite the equation:

$$\mathbf{H} = \mathbf{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] \rightarrow \quad [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3] = \lambda \mathbf{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}]$$

here, some information is important:

$$\begin{cases} \mathbf{h}_1 = \lambda \mathbf{K}\mathbf{r}_1 \\ \mathbf{h}_2 = \lambda \mathbf{K}\mathbf{r}_2 \end{cases} \rightarrow \begin{cases} \mathbf{r}_1 = \frac{1}{\lambda}\mathbf{K}^{-1}\mathbf{h}_1 \\ \mathbf{r}_2 = \frac{1}{\lambda}\mathbf{K}^{-1}\mathbf{h}_2 \end{cases}$$

Recall the behavior of “**Rotation Matrix**”: orthogonal & unit length

Orthogonal: inner product=0 → $\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = 0$

unit length: the same length→ $\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_1 = \mathbf{r}_2^{\mathrm{T}}\mathbf{r}_2$

Camera calibration

- from homography—Zhang’s method—cont.

$$\begin{cases} \mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = 0 \\ \mathbf{r}_1^{\mathrm{T}}\mathbf{r}_1 = \mathbf{r}_2^{\mathrm{T}}\mathbf{r}_2 \end{cases} \qquad \begin{cases} (\mathbf{K}^{-1}\mathbf{h}_1)^{\mathrm{T}}(\mathbf{K}^{-1}\mathbf{h}_2) = 0 \\ (\mathbf{K}^{-1}\mathbf{h}_1)^{\mathrm{T}}(\mathbf{K}^{-1}\mathbf{h}_1) = (\mathbf{K}^{-1}\mathbf{h}_2)^{\mathrm{T}}(\mathbf{K}^{-1}\mathbf{h}_2) \end{cases}$$

$$\rightarrow \begin{cases} \mathbf{h}_1^{\mathrm{T}}(\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1})\mathbf{h}_2 = 0 \\ \mathbf{h}_1^{\mathrm{T}}(\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1})\mathbf{h}_1 = \mathbf{h}_2^{\mathrm{T}}(\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1})\mathbf{h}_2 \end{cases}$$

→ The same result with absolute conic method

$$\mathbf{h}_1^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2 = 0$$

$$\mathbf{h}_1^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_1 = \mathbf{h}_2^{\mathrm{T}}\boldsymbol{\omega}\mathbf{h}_2$$

$$\boldsymbol{\omega} = (\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}$$

Summary:

$$\mathbf{v}_{ij} = [h_{i1}h_{j1} \quad h_{i1}h_{j2} + h_{i2}h_{j1} \quad h_{i2}h_{j2} \quad h_{i3}h_{j1} + h_{i1}h_{j3} \quad h_{i3}h_{j2} + h_{i2}h_{j3} \quad h_{i3}h_{j3}]^{\mathrm{T}}$$

$$\begin{bmatrix} \mathbf{v}_{12}^{\mathrm{T}} \\ (\mathbf{v}_{11} - \mathbf{v}_{22})^{\mathrm{T}} \end{bmatrix} \boldsymbol{\omega} = 0 \quad \text{and} \quad \boldsymbol{\omega} = (\mathbf{K}\mathbf{K}^{\mathrm{T}})^{-1}$$

$$\mathbf{v}_{ij}^{\mathrm{T}}\mathbf{b} = 0$$

Note, the difference between **ω** and **b**

$$\mathbf{b} = [B_{11} \quad B_{12} \quad \underline{B_{22}} \quad \underline{B_{13}} \quad B_{23} \quad B_{33}]^{\mathrm{T}}$$

Camera calibration

- from homography—Zhang’s method—cont.

$$\mathbf{r}_1 = \lambda \mathbf{K}^{-1}\mathbf{h}_1$$

$$\mathbf{r}_2 = \lambda \mathbf{K}^{-1}\mathbf{h}_2$$

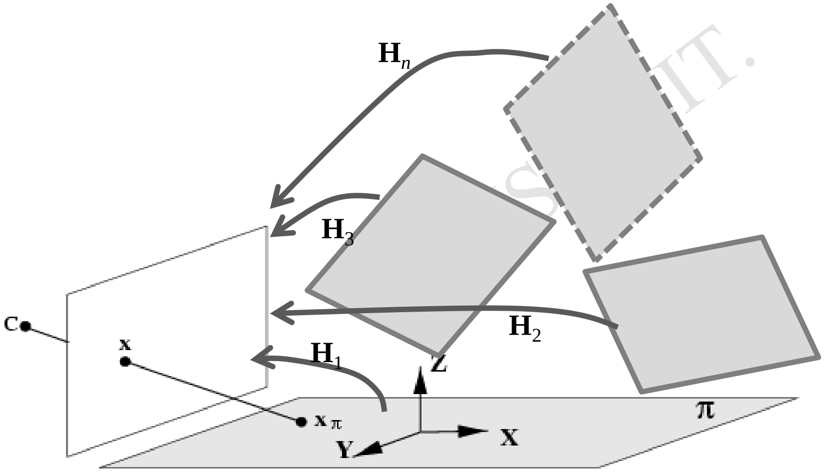
$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{t} = \lambda \mathbf{K}^{-1}\mathbf{h}_3$$

$$\lambda = \frac{1}{|\mathbf{K}^{-1}\mathbf{h}_1|} = \frac{1}{|\mathbf{K}^{-1}\mathbf{h}_2|}$$

$$\mathbf{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3]$$

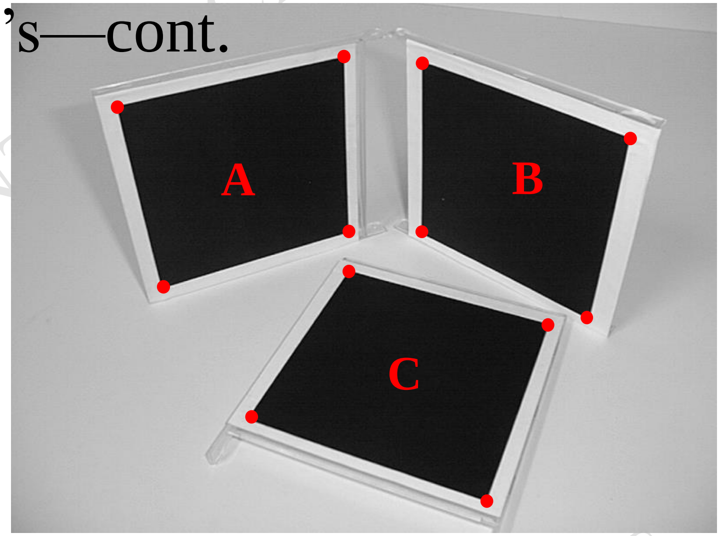
$$\mathbf{x} = \mathbf{P} \cdot \mathbf{X} = \mathbf{K}[\mathbf{R} \,|\, \mathbf{t}]\mathbf{X}$$



One square has one **H** (of course, one **R**|**t**)

Camera calibration

- Example: from homography—Zhang’s—cont.
  - follow the previous example



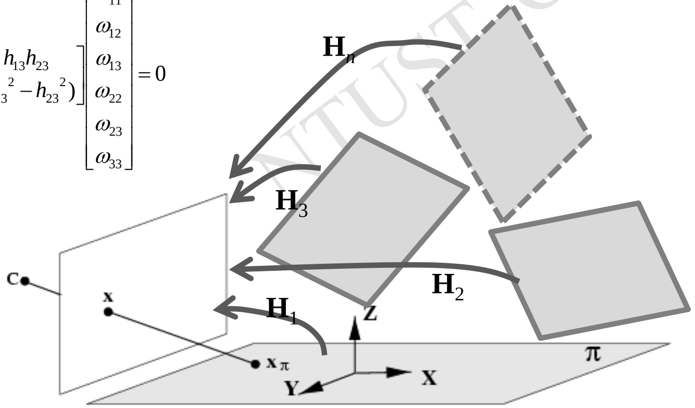
Homography (unit square to **A-square**)
379.199677 111.830818 152.000000
-64.140297 350.826263 149.000000
0.102074 0.210233 1.000000

Homography (unit square to **B-square**)
168.271439 125.763161 596.000000
81.960945 320.478027 84.000000
-0.148918 0.211012 1.000000

Homography (unit square to **C-square**)
228.969971 -209.572891 490.000000
41.616714 105.178200 387.000000
-0.078244 -0.182428 1.000000

Solve equation:
$$\begin{bmatrix} h_{11}h_{21} & (h_{11}h_{22}+h_{12}h_{21}) & (h_{11}h_{23}+h_{13}h_{21}) & h_{12}h_{22} & (h_{12}h_{23}+h_{13}h_{22}) & h_{13}h_{23} \\ (h_{11}{}^{2}-h_{21}{}^{2}) & 2(h_{11}h_{12}-h_{21}h_{22}) & 2(h_{11}h_{13}-h_{21}h_{23}) & (h_{12}{}^{2}-h_{22}{}^{2}) & 2(h_{12}h_{13}-h_{22}h_{23}) & (h_{13}{}^{2}-h_{23}{}^{2}) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{13} \\ \omega_{22} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = 0$$

Finally get:
$$\mathbf{K} = \begin{bmatrix} 1118.7 & -25.7 & 531.5 \\ 0 & 1100.3 & 409.6 \\ 0 & 0 & 1 \end{bmatrix}$$



What are **R**|**t** of square A, B and C?

Camera calibration

(for square A)

ha =

$$\begin{array}{ccc} 379.1997 & 111.8308 & 152.0000 \\ -64.1403 & 350.8263 & 149.0000 \\ 0.1021 & 0.2102 & 1.0000 \\ \downarrow & \downarrow & \\ \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{array}$$

(in Matlab)

ra=inv(K)*ha

$$\begin{array}{ccc} 0.2883 & 0.0056 & -0.3447 \\ -0.0963 & 0.2406 & -0.2368 \\ 0.1021 & 0.2102 & 1.0000 \end{array}$$

(length=0.3206) (length=0.3195)

(needs adjustment)

$\mathbf{r}_1$ $\mathbf{r}_2$

r1=		r2=		r3=cross(r1,r2)		r3=
0.8991		0.0175		-0.4359		-0.4373
-0.3004		0.7504		-0.5840		-0.5860
0.3184	→	0.6558	→	0.6800	→	0.6822
		(not unit length)		(not unit length)		(normalized)

r2=cross(r3,r1)

r2=cross(r3,r1)		t =		
0.0183		-1.0751		
0.7526	→	-0.7388	→	
0.6582		3.1192		
(unit length)		(inv(K)*h3./0.3206)		

$$[\mathbf{R}\,|\,\mathbf{t}] = \boxed{\begin{array}{cccc} 0.8991 & 0.0183 & -0.4373 & -1.0751 \\ -0.3004 & 0.7526 & -0.5860 & -0.7388 \\ 0.3184 & 0.6582 & 0.6822 & 3.1192 \end{array}}$$

Using “Gram-Schmidt” ortho-normalization

(for square B)

hb =

$$\begin{array}{ccc} 168.2714 & 125.7632 & 596.0000 \\ 81.9609 & 320.4780 & 84.0000 \\ -0.1489 & 0.2110 & 1.0000 \end{array}$$

(in Matlab)

rb=inv(K)*hb

$$\begin{array}{ccc} 0.2242 & 0.0171 & 0.0509 \\ 0.1299 & 0.2127 & -0.2959 \\ -0.1489 & 0.2110 & 1.0000 \end{array}$$

$$[\mathbf{R}\,|\,\mathbf{t}] = \boxed{\begin{array}{cccc} 0.7501 & 0.0565 & 0.6589 & 0.1702 \\ 0.4348 & 0.7086 & -0.5557 & -0.9902 \\ -0.4983 & 0.7033 & 0.5070 & 3.3463 \end{array}}$$

(for square C)

hc =

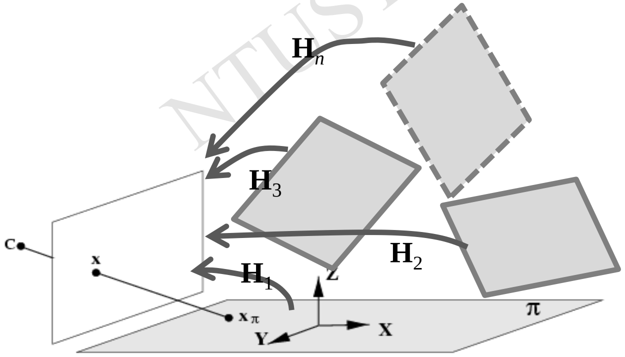
$$\begin{array}{ccc} 228.9700 & -209.5729 & 490.0000 \\ 41.6167 & 105.1782 & 387.0000 \\ -0.0782 & -0.1824 & 1.0000 \end{array}$$

(in Matlab)

rc=inv(K)*hc

$$\begin{array}{ccc} 0.2434 & -0.0969 & -0.0376 \\ 0.0670 & 0.1635 & -0.0205 \\ -0.0782 & -0.1824 & 1.0000 \end{array}$$

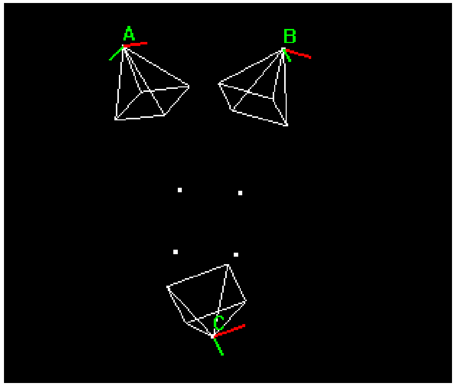
$$[\mathbf{R}\,|\,\mathbf{t}] = \boxed{\begin{array}{cccc} 0.9210 & -0.3896 & 0.0083 & -0.1422 \\ 0.2533 & 0.6148 & 0.7468 & -0.0777 \\ -0.2961 & -0.6857 & 0.6649 & 3.7839 \end{array}}$$

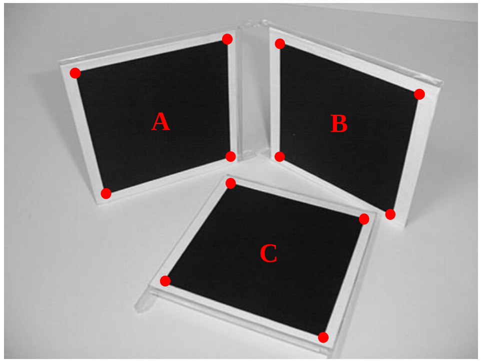


Camera calibration

- Draw camera positions relative to "Single Square"

$$\mathbf{X}_{\text{cam}} = \begin{bmatrix} u_x' & v_x' & w_x' & t_x \\ u_y' & v_y' & w_y' & t_y \\ u_z' & v_z' & w_z' & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \mathbf{X}_{\text{world}}$$

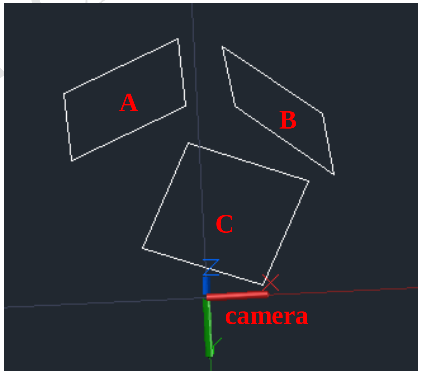


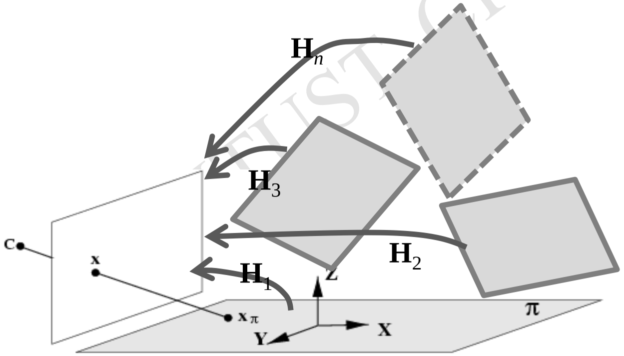


- Draw squares respective to Single camera

draw

$$[\mathbf{R}\,|\,\mathbf{t}]\begin{bmatrix}0\\0\\0\\1\end{bmatrix},[\mathbf{R}\,|\,\mathbf{t}]\begin{bmatrix}1\\0\\0\\1\end{bmatrix},[\mathbf{R}\,|\,\mathbf{t}]\begin{bmatrix}1\\1\\0\\1\end{bmatrix},[\mathbf{R}\,|\,\mathbf{t}]\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

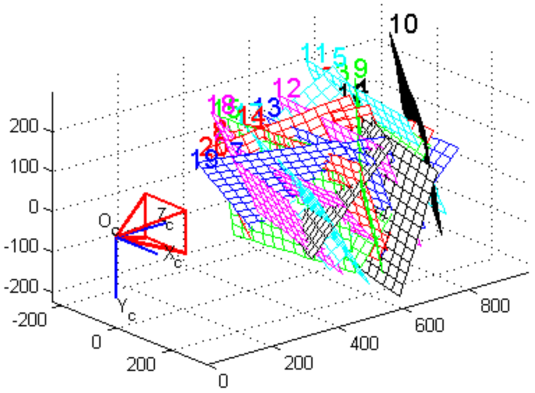


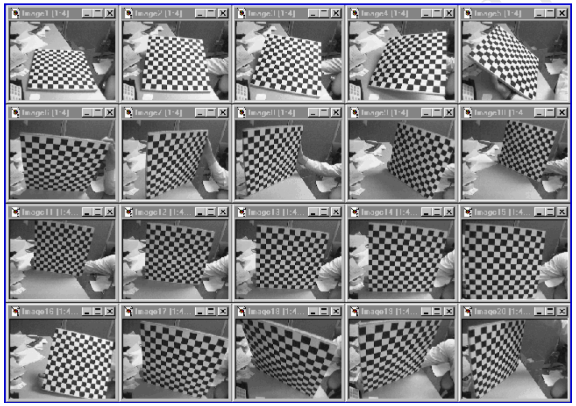


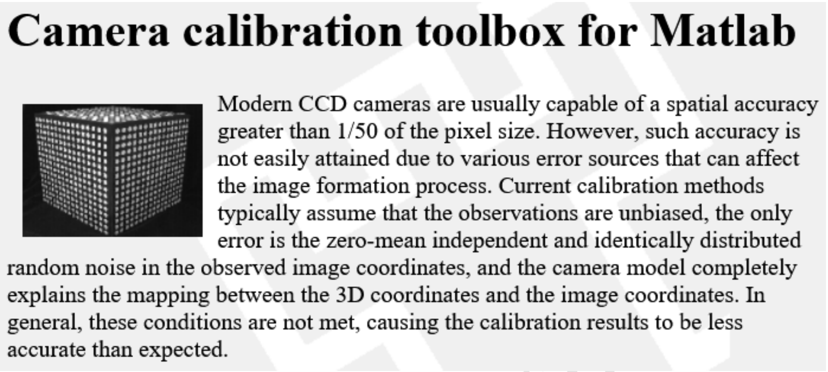
Note! the result is up to scale, unless the actual size is known

Camera calibration

- Other resources:







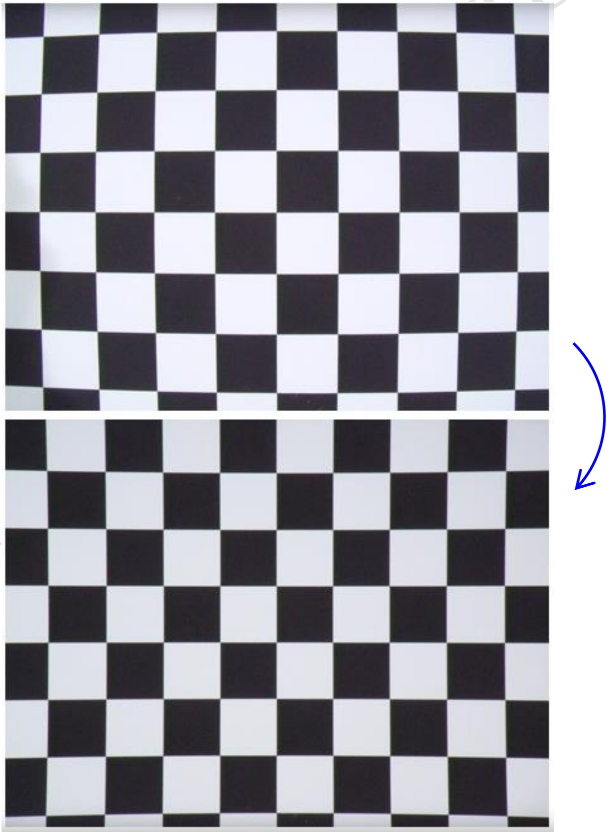
Camera calibration toolbox for Matlab

Modern CCD cameras are usually capable of a spatial accuracy greater than 1/50 of the pixel size. However, such accuracy is not easily attained due to various error sources that can affect the image formation process. Current calibration methods typically assume that the observations are unbiased, the only error is the zero-mean independent and identically distributed random noise in the observed image coordinates, and the camera model completely explains the mapping between the 3D coordinates and the image coordinates. In general, these conditions are not met, causing the calibration results to be less accurate than expected.

http://www.vision.caltech.edu/bouguetj/calib_doc/
http://www.ee.oulu.fi/~jth/calibr/

Camera calibration

- Deal with the lens distortion:



Traditional Method: Refer to this book
C. C. Slama, editor. Manual of Photogrammetry. American Society of Photogrammetry, 4th ed., 1980.

Camera calibration

- Lens distortion from Zhang’s method

$[u \quad v \quad 1]^T$ →ideal points on image (distortion-free)

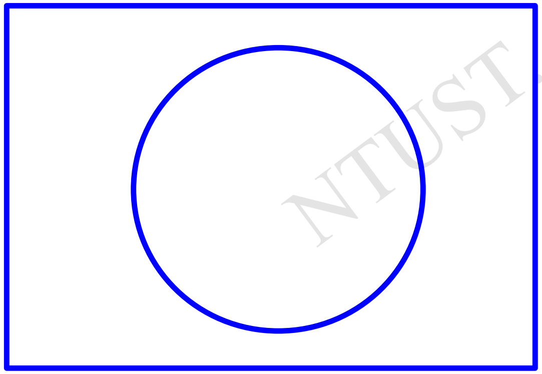
$[\breve{u} \quad \breve{v} \quad 1]^T$ →measurement points on image (distortion)

$[x \quad y \quad 1]^T$ →ideal points on real 3D space (normalized, distortion-free)

$[\breve{x} \quad \breve{y} \quad 1]^T$ →measurement on real 3D space (normalized, distortion)

Radial distortion model:

$$\begin{cases} \breve{x} = x + x[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2] \\ \breve{y} = y + y[k_1(x^2 + y^2) + k_2(x^2 + y^2)^2] \end{cases}$$



Camera calibration

- Lens distortion from Zhang's method—cont.

$$\mathbf{x} = \mathbf{P}\cdot\mathbf{X} = \underbrace{\mathbf{K}[\mathbf{R}\,|\,\mathbf{t}]\mathbf{X}}_{[x \quad y \quad 1]^{\mathrm{T}}}$$

$$\mathbf{K} = \begin{bmatrix} \alpha & c & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} \breve{x} = x + x[k_1(x^2+y^2) + k_2(x^2+y^2)^2] \\ \breve{y} = y + y[k_1(x^2+y^2) + k_2(x^2+y^2)^2] \end{cases}$$

$$\begin{cases} \breve{u} = u_0 + \alpha\breve{x} + c\breve{y} \\ \quad \breve{v} = v_0 + \beta\breve{y} \end{cases}$$

Initial guess

$$\begin{cases} \breve{u} = u + (u-u_0)[k_1(x^2+y^2) + k_2(x^2+y^2)^2] \\ \breve{v} = v + (v-v_0)[k_1(x^2+y^2) + k_2(x^2+y^2)^2] \end{cases}$$

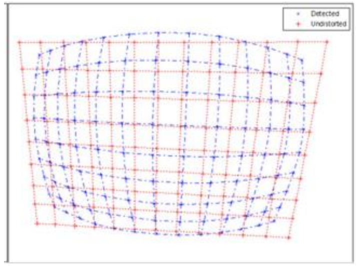
Alternatively solve k_1, k_2.

$$\begin{bmatrix} (u-u_0)(x^2+y^2) & (u-u_0)(x^2+y^2)^2 \\ (v-v_0)(x^2+y^2) & (v-v_0)(x^2+y^2)^2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \breve{u}-u \\ \breve{v}-v \end{bmatrix}$$

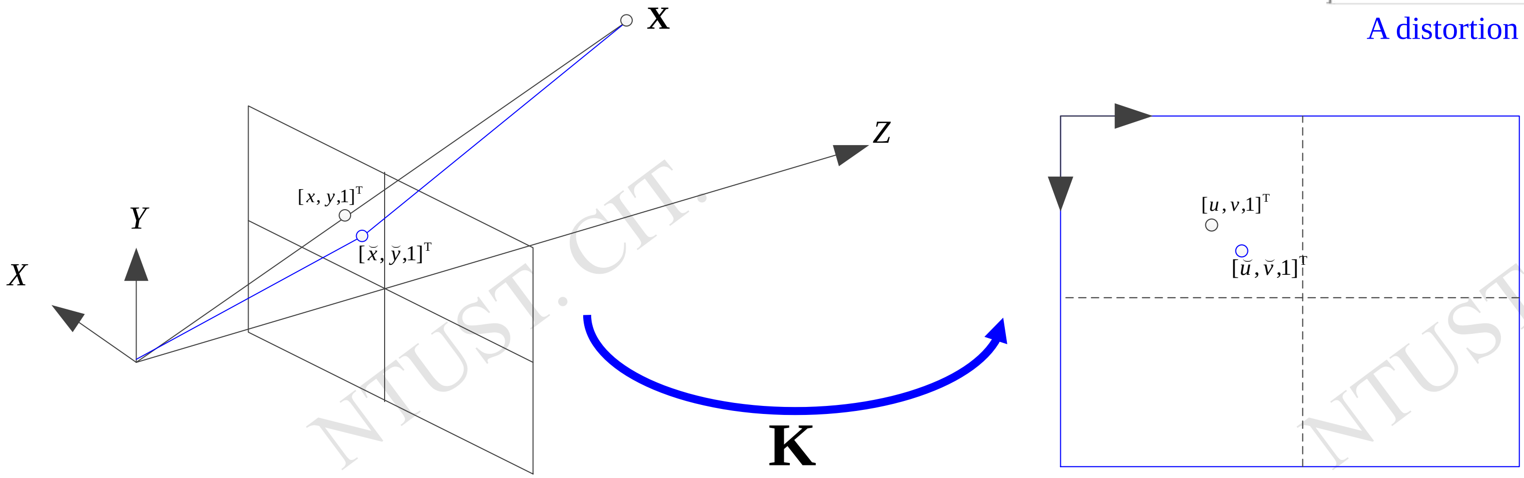
→solve by SVD, then iteratively minimize the error

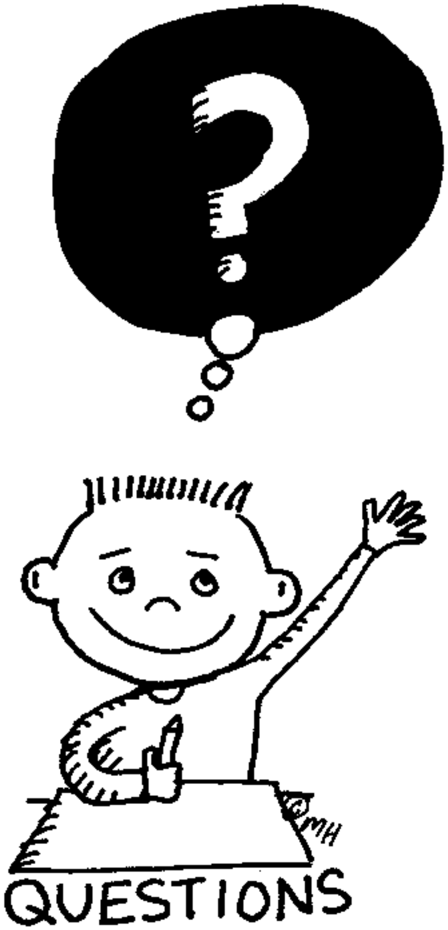
Camera calibration

- Lens distortion from Zhang’s method—cont.



A distortion example





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