1)

```
def my_pow(x, m):
  if(m == 0):
    return 1
  if(m == 1):
    return x
  pow_half = my_pow(x*x, m//2)
  elif(m % 2 == 0):
    return pow_half
  else:
    return x * pow_half
```

We prove using induction.
The base case:
m=0, the function return 1. $x^0 = 1$. where x is not 0. This is trivial.
m=1. the functions return x. $x^1 = x$. This is trivial.

We assume the algorithms works and is correct for all $m = 0, 1, 2, 3, ..., k$.

We now see that if $k + 1$ is even, the algorithm calls $my\_pow(x * x, m//2)$, where $m//2$ is floor division. By the induction hypothesis, the algorithm is correct for $m//2$. Therefore, the algorithm is correct for $k + 1$ when m is even.

We see that if $k + 1$ is odd, the algorithm calls $x * my\_pow(x * x, m//2)$. By the induction hypothesis, the algorithm is correct for $m//2$. Therefore, the algorithm is correct for $k + 1$ when m is odd.

Since we covered both cases, the algorithm is correct for all m.

2)
For copying we see: $\sum_{j=1}^{k} 3^{j-1} = (1/2)(3^k - 1)$
$T_{copy}(M) = (1/2)(3^{(log_3(M)+1)} - 1) = (1/2)(3M - 1)$ where M is power of 3.
$T_{push}(M) = M + M + (1/2)(3M - 1) = O(M)$
$T_{pop} = M = O(M)$
So with 3 as factor, M stack operations is still $O(M)$.

3)

```
def find_start(li):
  lo = 0
  hi = len(li)-1
  mid = 0
  while(hi>=lo):
    mid = (hi+lo) //2
    if(li[mid] > li[hi]):
      lo = mid+1
    elif(li[mid] < li[hi]):
      hi = mid
    else:
      return mid
```

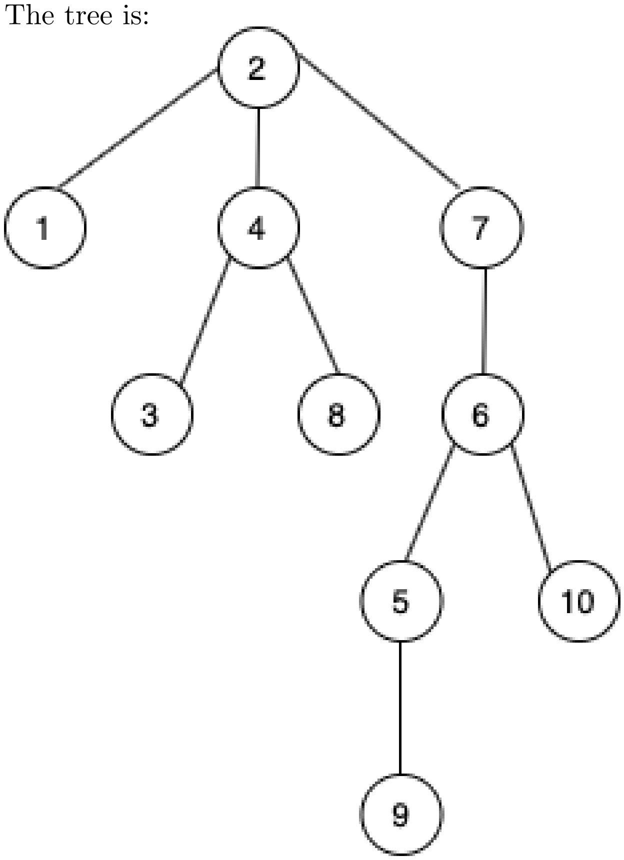
We prove using induction.
Base case: If list is 1 element, lo, hi, and mid are 0, so find_start(li) returns the mid element, which is the lowest starting number. If list is 2 elements, the algorithm picks index 0 (since floor division) as mid and compares that with index 1, so lo = 0, mid=0, hi=1. Comparing li[mid] to li[hi] in the if statements,

we either update lo = 0+1 or hi = 0 based on the comparison, and from there, returns mid by first base case.

We assume the algorithm works and is correct the for all lists of size $n = 0, 1, 2, 3, ..., k$.
We need to show it works for k+1. With k+1, find_start(li) starts with lo=0, hi=k, mid = (hi+lo)//2 in first iteration of loop and compares li[mid] to li[hi]. If li[mid] > li[hi], lo=mid+1 and If li[mid] < li[hi], hi = mid. So we throw away half of the list and the we only look at elements lo=0 to hi=(hi+lo)//2 or mid = (hi+lo)//2 + 1 to hi = k. By the inductive hypothesis, the algorithm is correct for either of these 2 cases. Therefore, the algorithm is correct for all lists.

4)
For copying we see:
$\sum_{j=1}^{k} 11^{j-1} = (1/10)(11^k - 1)$
$T_{copy}(M) = (1/10)(11^{(log_{11}(M)+1)} - 1) = (1/10)(11M - 1)$ where M is power of 11.
$T_{push}(M) = M + M + (1/10)(11M - 1) = O(M)$
$T_{pop} = M = O(M)$
So with 11 as factor, M stack operations is still $O(M)$.

5)
The tree is:




6)
The find array cannot be result of running weighted quick union. It's height is 5 from nodes 2,7,6,5,9. With weighted quick union the height is at most log_2 (N), so log_2 (10) is 3. In this problem, the height 5 is greater than 3.

7) This problem doesn't specify the type of running algorithm. The complexity of a sequence of M add_new_set, union, and find operations will depend on the algorithm and if is M >> N or N << M.
There is a pretty detailed paper I read about this. With empty data structure, pay it forward, naive linking of M sequence and M $>= N$, the complexity is $O(MlogN)$. The proof is complex and nontrivial.
`https://e-maxx.ru/bookz/files/dsu/Worst-Case%20Analysis%20of%20Set%20Union%20Algorithms.%20Tarjan,%20Leeuwen.pdf`

The naive set union algorithm runs in $O(N + MN)$ for a sequence M of the 3 operations, since add_new_set and link is $O(1)$, Union = $O(N)$ and find=$O(1)$.