# Delta Functions Generated by Complex Exponentials

Here we consider four important relationships showing the fact that the Kronecker and Dirac delta functions can be generated respectively as the sum and integral of certain complex exponential functions. These formulas are widely used in our future discussions of different forms of the Fourier transform.

- Dirac delta as an integral of a complex exponential:

$$\int_{-\infty}^{\infty} e^{\pm j2\pi ft} dt = \delta(f) \tag{1}$$

**Proof:**

Consider the *sinc function* which can be obtained by the following integral:

$$\int_{-a/2}^{a/2} e^{\pm j2\pi ft} dt = \frac{1}{\pm j2\pi f} e^{\pm j2\pi ft} |_{-a/2}^{a/2} = \frac{\sin(\pi fa)}{\pi f} = a\ sinc(af)$$

where $sinc(x)$ is the sinc function commonly defined as:

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

In particular when $x = 0$, we have $\lim_{x\rightarrow 0} sinc(x) = 1$. When $a$ is increased, the function $a\ sinc(af)$ becomes narrower but taller, until when $a \rightarrow \infty$, it becomes infinity at $f = 0$ but zero everywhere else. Also, as the integral of this sinc function is unity:

$$\int_{-\infty}^{\infty} \frac{\sin(\pi fa)}{\pi f} df = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\pi fa)}{af} d(af) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = 1$$

Now we see that Eq.1 represents a Dirac delta:

$$\int_{-\infty}^{\infty} e^{\pm j2\pi ft} dt = \lim_{a\rightarrow\infty} a\ sinc(af) = \delta(f)$$

This result can be interpreted intuitively. The integral of any sinusoid over all time is always zero, except if $f = 0$ then $\sin(0) = 0$ but

$\cos(0) = 1$, and the integral becomes infinity. Alternatively, if we integrate the complex exponential with respect to $f$, we get:

$$\int_{-\infty}^{\infty} e^{j2\pi ft} df = \delta(t) \tag{2}$$

which can be interpreted intuitively as a superposition of infinitely many sinusoidal functions with progressively higher frequency $f$. These sinusoids cancel each other at any time $t \neq 0$ except when $t = 0$, where all cosine functions equal to 1 and their superposition becomes infinity.

- Kronecker delta as an integral of a complex exponential:

$$\frac{1}{T}\int_T e^{\pm j2\pi kt/T} dt = \delta[k] \tag{3}$$

**Proof:**

$$\frac{1}{T}\int_T e^{j2\pi kt/T} dt = \frac{1}{T}[\int_T \cos(\frac{2\pi}{T/k}t)dt \pm j\int_T \sin(\frac{2\pi}{T/k}t)dt]$$

The sinusoids have period $T/k$ and their integral over $T$ is zero, except if $k = 0$ then $\cos 0 = 1$ and $\int_T dt/T = 1$, i.e., it is a Kronecker delta.

- A train of Dirac deltas with period $F$ as a summation of a complex exponential:

$$\frac{1}{F}\sum_{k=-\infty}^{\infty} e^{\pm j2k\pi f/F} = \sum_{n=-\infty}^{\infty} \delta(f - nF) \tag{4}$$

**Proof:** First we note that if $f = nF$ is any multiple of $F$, then $e^{\pm j2k\pi f/F} = e^{\pm j2\pi nk} = 1$, and the summation on the left-hand side is infinity. Next we consider the following summation with the assumptions that $|a| < 1$ and $f \neq nF$:

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (ae^{jx})^k &= \sum_{k=0}^{\infty}(ae^{jx})^k + \sum_{k=0}^{\infty}(ae^{-jx})^k - (e^{jx})^0 \\
&= \frac{1}{1-ae^{jx}} + \frac{1}{1-ae^{-jx}} - 1 = \frac{2 - a(e^{jx}+e^{-jx})}{1 - a(e^{jx}+e^{-jx}) + a^2} - 1
\end{aligned}$$

The first equal sign is due to the fact that the following power series converges when $|a| < 1$:

$$\sum_{k=0}^{\infty}(ae^x)^k = \frac{1}{1-ae^x}$$

However, when $a \to 1$, this summation becomes $1-1=0$. Now we see that the summation on the left-hand side of Eq.4 is zero except when $f = nF$, in which case is is infinity. In other words, the summation is actually an impulse train with a gap $F$. Moreover, we can further show that the integral of each impulse with respect to $f$ over one period $F$ is 1:

$$\begin{aligned} & \int_F \frac{1}{F} \sum_{k=-\infty}^{\infty} e^{\pm j2k\pi f/F} df = \sum_{k=-\infty}^{\infty} \frac{1}{F} \int_F e^{\pm j2k\pi f/F} df = \sum_{k=-\infty}^{\infty} \delta[k] \\ = & \cdots \delta[-1] + \delta[0] + \delta[1] \cdots = \cdots + 0 + 1 + 0 + \cdots = 1 \end{aligned}$$

Here we have used the result of Eq.3. Now we see that Eq.4 holds to represent a train of unit impulses or Dirac deltas with period $F$.

- A train of Kronecker deltas with period $N$ as a summation of complex exponential:

$$\begin{aligned} & \frac{1}{N}\sum_{n=0}^{N-1} e^{\pm j2\pi nm/N} \\ = & \frac{1}{N}\sum_{n=0}^{N-1}\cos(2\pi nm/N) \pm \frac{j}{N}\sum_{n=0}^{N-1}\sin(2\pi nm/N) \\ = & \sum_{k=-\infty}^{\infty}\delta[m-kN] \end{aligned} \tag{5}$$

and

$$\sum_{n=0}^{N-1}\sin(2\pi nm/N) = 0$$

**Proof:**

The summation is obviously equal to 1 if $m = kNi$ for all integer $k$, otherwise the above becomes

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{\pm j2\pi nk/N} = \frac{1}{N}\sum_{n=0}^{N-1}(e^{\pm j2\pi k/N})^n = \frac{1}{N}\frac{1-e^{\pm j2\pi kN/N}}{(1-e^{\pm j2\pi k/N})} = 0$$

The second equal sign is due to the geometric series formula:

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x} \tag{6}$$

In other words, the summation in Eq.5 represents a periodic and discrete Kronecker delta with period $N$.

The proof of these four important identities is left as homework problems for the reader.

The integral in Eq.1 can also be written as:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\pm j2\pi ft} dt &= \int_{-\infty}^{\infty} [\cos(2\pi ft) \pm j\sin(2\pi ft)] dt \\ = \int_{-\infty}^{\infty} \cos(2\pi ft) dt &= 2\int_{0}^{\infty} \cos(2\pi ft) dt = \delta(f) \end{aligned} \tag{7}$$

The second equal sign is due to the fact that $\sin(2\pi ft) = -\sin(-2\pi ft)$ is odd and its integral over all time $-\infty < t < \infty$ is zero, and the third equal sign is due to the fact that $\cos(2\pi ft) = \cos(-2\pi ft)$ is even and its integral over all time is equal to twice of the integral over half time $0 < t < \infty$.

This result can be interpreted intuitively. The integral of any sinusoid over all time $-\infty < t < \infty$ is always zero, except if $f = 0$ then $\sin(0) = 0$ but $\cos(0) = 1$, and the integral over all time becomes infinity. Alternatively, if we integrate the complex exponential with respect to $f$, we get:

$$\int_{-\infty}^{\infty} e^{j2\pi ft} df = 2\int_{0}^{\infty} \cos(2\pi ft) df = \delta(t) \tag{8}$$

which can also be interpreted intuitively as a superposition of infinitely many cosine functions with progressively higher frequency $f$. These sinusoids cancel each other at any time $t \neq 0$ except when $t = 0$, where all cosine functions equal to 1 and their superposition becomes infinity. Similar arguments can also be made for the other three cases.