Computer Science 186:
Economics and Computation
An Analysis of Automated Market Makers

Peter Bang, Joon Yang, Howard Zhang

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Professor David Parkes
Harvard College
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Abstract: Prediction markets have been proven to more accurately predict outcomes than opinion polling. Most prediction markets usually use an automatic market maker (AMM), which is able to provide liquidity to the market. Market makers in prediction markets can employ different scoring rules, and there is an inherent trade-off between the ability to converge to the true value and the ability to adapt quickly to market shocks. We studied the Logarithmic Market Scoring Rule (LMSR), Liquidity Sensitive LMSR (LS LMSR), and Bayesian Market Maker (BMM) and analyzed how they perform. As shown previously in literature, we were able to run simulations to confirm that an LMSR market maker adapts to market shocks quickly but fails to converge to the true value, while the LS LMSR market maker converges to the true but fails to adapt quickly to market shocks, resulting in large losses. Although we were not able to implement a BMM market maker, it has been previously noted in literature that BMM is able to perform reasonably well in both convergence and adaptation to market shocks by giving up the guarantee of bounded loss. Finally, we propose our own variant of an automatic market maker, motivated by the results from the adaptability of the BMM. We attempt to theoretically modify the LS LMSR to be more flexible with respect to shocks.

1. INTRODUCTION

Prediction markets have been of significant interest in recent years. Companies including Google and Microsoft have used internal prediction markets using virtual currency to forecast sales numbers and release dates. Real money prediction markets such as Iowa Electronic Markets, TradeSports, Hollywood Stock Exchange, and Economic Derivatives have consistently performed better than opinion polling in predicting outcomes in politics, sports, and entertainment. [Wolfers and Zitzewitz, 2004]. These exchanges use a variety of contract types as well as different market mechanisms and trading rules. We are interested in exploring recent developments in market mechanisms that are used to create the trading rules in prediction markets.

Two common mechanisms for markets include the *continuous double auction (CDAs)* and *automated market makers (AMMs)*. In a CDA market, the exchange takes on no risk, and all trades occur between participants. Buyers submit bids and sellers submit asks with desired quantities, and trades occur whenever the two sides reach an agreeable price. However, the liquidity in this type of market can be severely limited, and it has been observed in practice that prediction markets are only able to attract traders with information when there are sufficient uninformed traders in the market, which is difficult to achieve.

In a market with an AMM, all trades occur through a central market maker which is provided by the exchange. The automated market maker is always willing to both buy and sell the contract, providing liquidity to the market. In prediction markets, the market maker can often run at a loss, but the cost is justified as the price of information elicitation. In particular, the use of Logarithmic Market Scoring Rule (LMSR) allows this loss to be bounded, which has been recognized to be a desirable property (Hanson, 2003, 2007). For this reason, the LMSR has become somewhat of a standard for subsidized prediction markets.

Financial exchanges cannot run at a loss and are run through CDAs. To ensure sufficient liquidity in the market, many exchanges including the New York Stock Exchange (NYSE) and the NASDAQ Stock Market select one or more Designated Market Makers (DMMs) responsible for supplying liquidity for each stock. Recent developments in electronic trading systems have even led proprietary trading firms to behave as market makers without formally being designated to do so, simultaneously buying and selling stocks in an attempt to profit from the difference in the buying price (*bid*) and selling price (*ask*).

Hence we are intrigued by the adaption of ideas from financial economics that could be applied to automated market makers in prediction markets. Our goal in this paper is to understand and evaluate the motivations behind the crossover of the two fields and explain the reasons for the shift in paradigms for using Logarithmic Market Scoring Rules (LMSR) to the Liquidity Sensitive LMSR model (LS LMSR) (Othman *et al.*, 2010), and finally to the Bayesian Market Maker (BMM) (Brahma *et al.*, 2012) by assessing their relative strengths and weaknesses. We construct a stylistic model based on Glosten and Milgrom (1985) to evaluate how the LMSR and LS LMSR models perform. Specifically we look at two facets, their ability to adapt to market shocks as well as their ability to converge to the true values of the assets.

2. AUTOMATED MARKET MAKERS

2.1 Logarithmic Market Scoring Rule (LMSR)

LMSR is adapted from the strictly proper logarithmic scoring rule and has several appealing qualities. It has a deterministic loss bound that the market maker can suffer, it is difficult to manipulate as it is purely based on inventory, and given that traders are rational and learn from prices, the market maker will converge to the rational expectations equilibrium price. The LMSR model discussed in *Chapter 18: Prediction Markets* of Parkes and Seuken provides the basis for the LMSR market makers (MM) that are discussed by Hanson (2007).

For simplicity, consider an event with binary outcomes, which can be expressed as o_1 and o_2. An example of this could be the outcome of a specific NBA game, such as Golden State Warriors versus the Memphis Grizzlies. We can then define a contract for each outcome that will pay \$1 if the corresponding event occurs. Hence an *automated market maker* needs to be willing to buy or sell any quantity of each contract at any given time. We define $q_1(t)$ to be the quantity of the contract the market maker has sold for outcome o_1 at a given time index t and define $q_2(t)$ correspondingly. We restrict $q_1(t)$ and $q_2(t)$ to be non-negative. The *spot price*, or the cost of buying or selling an infinitesimally small amount of $q_1(t)$ is $p(q_1(t))$, is given by

$$p(q_1(t)) = \frac{e^{q_1(t)/\beta}}{e^{q_1(t)/\beta} + e^{q_2(t)/\beta}}$$

where $\beta > 0$. Correspondingly, we can define

$$p(q_2(t)) = \frac{e^{q_2(t)/\beta}}{e^{q_1(t)/\beta} + e^{q_2(t)/\beta}}$$

We can make two immediate observations. The first is that $p(q_1(t)) + p(q_2(t)) = 1$ and second, that $p(q_1(t))$ and $p(q_2(t))$ can be interpreted as the expected probability that outcome o_1 and o_2 will occur, respectively.

We also have corresponding cost functions. The cost function to trade quantity Q of the contract for outcome o_1 at time $t+1$ is given by

$$C(q_1(t) + Q, q_2(t)) - C(q_1(t), q_2(t)) =$$
$$\beta(ln(e^{(q_1(t)+Q)/\beta} + e^{q_2(t)/\beta}) - ln(e^{q_1(t)/\beta} + e^{q_2(t)/\beta}))$$

Hence, given a starting inventory $q_0 = 0$, the maximum loss is bounded by $\beta \ln 2$.

We have seen that the liquidity parameter β bounds the loss and determines the adaptability of the MM. Note that β is the only free parameter, but remains a constant in the classical LMSR model. Deciding what value β should be is more of an art than a science (Othman *et al.*, 2010), as the trade-offs are indisputable, and it became a matter of which to prioritize. A small β bounds the loss at a smaller value and allows the MM to be more adaptive to market shocks, but would also result in less liquidity because prices respond more quickly to trades. A large β, in contrast, bounds the loss at a higher value and is less adaptive to market shocks, but it provides greater liquidity to the market.

The LMSR model has been a standard for prediction markets since it satisfies several desirable properties. In addition to the property of bounded loss, there are several other desiderata for cost functions, recognized in both prediction markets and finance literature (Othman 2012).

(1) Monotonicity: $C(x) \leq C(y)\ \forall x, y$ such that $x_i \leq y_i$. This prevents simple arbitrage opportunities, such as a trader's purchase of a zero-cost contract that never results in a loss but sometimes results in gains. It also provides incentives for myopic traders to trade with the MM until they find no more valuable offers because the marginal price of a bet never decreases.
(2) Convexity: $C(\lambda x + (1-\lambda)y) \leq \lambda C(x) + (1-\lambda)C(y)\ \forall x, y$ and $\lambda \in [0, 1]$. This encourages the MM to diversify its risk.
(3) Positive Homogeneity: $C(\gamma x) = \gamma C(x)\ \forall x$ and scalar $\gamma > 0$. This ensures a scale-invariant, currency-independent price response; from a risk measurement perspective, it ensures that doubling a risk doubles its cost.

Despite the boons that LMSR provides, it also has some major shortcomings that called forth various improved models. One such major downside is LMSR is *liquidity-insensitive*, unable to adjust its price elasticity in response to the volume of market activity (Othman *et al.*, 2010). Another perspective on liquidity is that it may not make sense for the "1,000,001st dollar to move the price as much as the first" (Othman *et al.*, 2010).

2.2 Liquidity Sensitive LMSR

As a modification to the LMSR model, the literature has considered *liquidity-sensitive* market makers. Othman *et al.* (2010) propose to make β equal to

$$\beta(\mathbf{q}) = \alpha \sum_{i=1}^{k} q_i(t)$$

where k denotes the total number of contracts and $\mathbf{q}$ is a vector of the quantities of each contract that the market has sold at time t. We restrict $q_i(t)$ to be non-negative for all i and all t. Hence for our simple case of a binary outcome, we have

$$\beta(q_1(t), q_2(t)) = \alpha(q_1(t) + q_2(t))$$

This will mean that a small investment will have a large impact in an illiquid market and a small impact in a liquid market.

Taking this new β into account, the spot price for $q_i(t)$, $p(q_i(t))$, is given by

$$p(q_i(t)) = \alpha log \left(\sum_{j}^{k} e^{q_j/b(\mathbf{q})} \right) + \frac{\sum_j q_j e^{q_i/b(\mathbf{q})} - \sum_j e^{q_j/b(\mathbf{q})}}{\sum_j q_j \sum_j e^{q_j/b(\mathbf{q})}}$$

Then we have that in our binary outcome case, the spot price for $q_1(t)$ is given by

$$p(q_1(t)) = \alpha log \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right) + \frac{e^{q_1/b(\mathbf{q})}(q_1 + q_2) - q_1 e^{q_1/b(\mathbf{q})} - q_2 e^{q_2/b(\mathbf{q})}}{(q_1 + q_2) \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right)}$$

which simplifies to

$$p(q_1(t)) = \alpha log \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right) + \frac{q_2(e^{q_1/b(\mathbf{q})} - e^{q_2/b(\mathbf{q})})}{(q_1 + q_2) \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right)}$$

and the spot price for $q_2(t)$, $p(q_2(t))$, is given by

$$p(q_2(t)) = \alpha log \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right) + \frac{q_1(e^{q_2/b(\mathbf{q})} - e^{q_1/b(\mathbf{q})})}{(q_1 + q_2) \left(e^{q_1/b(\mathbf{q})} + e^{q_2/b(\mathbf{q})} \right)}$$

Othman *et al.* (2010) show that these cannot be interpreted directly as probabilities, since

$$1 \leq \sum_i p_i(\mathbf{q}) \leq 1 + \alpha log(n)$$

This seems like a significant issue since the law of one price is violated. However, Othman *et al.* (2010) argue that it is precisely dropping this condition that allows the market maker to expect a profit. More specifically, if a market maker sets prices that are not the actual expected final prices, then the market maker on average will suffer a loss. Thus, in many real prediction markets, the market maker charges the prices on the outcomes so that the sum of the prices is greater than unity, attempting to profit from trading on both sides of the market.

Othman *et al.* also show that no pricing rule has the properties of no-arbitrage, path independence and liquidity sensitive. The typical LMSR is path independent and no-arbitrage. Consequently, by having a liquidity sensitive market maker, they drop the law of one price assumption as well as the path independence property.

We also have corresponding cost functions. The cost function to trade quantity Q of the contract for outcome o_1 at time $t + 1$ is given by

$$C(q_1(t) + Q, q_2(t)) - C(q_1(t), q_2(t)) = \beta(\mathbf{q})(ln(e^{(q_1(t)+Q)\beta(\mathbf{q})} + e^{q_2(t)/\beta(\mathbf{q})}) - ln(e^{q_1(t)/\beta(\mathbf{q})} + e^{q_2(t)/\beta(\mathbf{q})})$$

The authors show theoretically that this market maker has several desirable qualities. The first is that there is a bounded loss of

$$C(\mathbf{q^0})$$

where $\mathbf{q^0}$ can be considered the "initial subsidy" to the market maker. Second, α an be thought of as the market maker's commission, where a higher α means a larger commission. In reality, they find that market makers take between 5 percent and 20 percent commission. Thus, we can simply set

$$\alpha = \frac{v}{nlog(n)}$$

where n is the number of states and v is the desired maximum commission. Hence for our simple binary case with 5 percent commission, we would set

$$\alpha = \frac{0.05}{2log(2)} = 0.0361$$

Nevertheless, what happens when there are shocks? Brahma *et al.* (2012) explore this and find that the LS LMSR is extremely slow to adapt to large shocks due to its "overconfidence." This will be discussed in more detail in the next section.

2.3 Bayesian Market Maker (BMM)

In order to understand BMMs, it is necessary to review some of the assumptions that motivate this model in Das

(2005) and Das and Magdon-Ismail (2008). Recall that the *No-Trade theorem* from Milgrom and Stokey (1982) tells us there needs to be information asymmetry for trading to occur. At any given time t, the only information the market maker has is some prior on the value of the stock $p_t(v)$ and a trader signal s with some variance, which represents the uncertainty in the signal. This information asymmetry can be quantified by the following metrics: (i) information disadvantage of the MM, (ii) ratio of variance in $p_t(v)$, and (iii) trader's uncertainty.

In the sequential bid model developed by Glosten and Milgrom (1985), the MM sets bid and ask prices at each trading period $t \geq 0$, and each trader can either buy ($x_t = 1$ for a unit) or sell ($x_t = -1$ for a unit) some quantity Q, or not participate in the trade ($x_t = 0$) upon observing the instantaneous spot price p_t. The classic Zero Profit (ZP) model for a competitive MM is a *single-period* decision making problem in a perfect competition where the MM does not expect expect any profit from ask or bid side, because if she did, a competing MM could place bid or ask prices obtaining less profit, and wiping out her advantage. This yields belief updates that solve the two non-linear equations for an ask price a_t and bid price b_t at time t:

$$a_t = E_{p_t(v)}[v|s > a_t] \quad b_t = E_{p_t(v)}[v|s < b_t]$$

In a sequential model, we update the prior belief after each period, which represents a new state for the MM, while the setting of bid and ask prices represents her actions, which trades off profit making and price discovery.

Recall that the state space for the MM is determined by her belief about the value V, which we describe using a density function p_t at time t. The action space, which is continuous, is determined by the prior in the form of bid and ask prices (b_t, a_t). These two combined, the MM's belief about V becomes a probability density function that is a complex product of error functions. Finding an exact solution to the dynamic programming problem is thus intractable, which warrants the use of the Bellman equation in a Gaussian state space evolution in the sequential game.

The MM is set as a Gaussian belief for market value $V : N(\mu_t, \sigma_t^2)$, and the trader signal is normally distributed around V such that $s \sim N(V, \sigma_t^2)$. With this in mind, our task is to find the value functional V in the following Bellman equation, which encapsulates the optimal strategy for the MM in the Bellman equation (Das and Magdon-Ismail, 2008):

$$V(p_t|\pi) = E[r_0|p_t, b_t^\pi(p_t), a_t^\pi(p_t)] + \gamma E[V(p_{t+1}|\pi)|p_t, b_t^\pi(p_t), a_t^\pi(p_t)]$$

where

$$p_t(v) = p_0(v) \prod_{\tau=1}^{t-1} \frac{q_\tau(v|b_\tau, a_\tau)}{\mathcal{A}_\tau}$$

such that

$$\mathcal{A}_t = \int_{-\infty}^{\infty} p_t(v)\ q_t(v|b_t, a_t)\ dv$$

π is the policy, γ is the discount factor where $0 < \gamma < 1$, and

$$q_t(V|b_t, a_t) = F_\epsilon(z_t^+ - V) - F_\epsilon(z_t^- - V)$$

where (z_t^+, z_t^-) is $(+\infty, a_t)$, (a_t, b_t), and $(b_t, -\infty)$ for $x_t = 1, 0, -1$, respectively, where F_ϵ is a symmetric distribution function in which $\{\epsilon_t\}$ are zero mean identically and independently distributed random variables for the noisy signal $s = V + \epsilon_t$.

The given Bellman equation is a function of both short-term expected return and discounted long-term expected return, both of which are affected by her bid ask prices. Because V is a value functional (translating a vector space onto a scalar field), this leads to several problems. One such troubles is in the informational difficulty, such as the difficulty in choosing the appropriate discount rate, γ. Another aspect of concern is in computational complexity, as the curse of dimensionality arises from the numerous state variables that need to be taken into account to choose the optimal strategy (or policy). The problem is essentially path dependent, which grows exponential to the number of trading periods, t. In order to make this problem tractable, we introduce a Gaussian approximation for the state space evolution.

By forcing the MM to maintain a Gaussian distribution of belief over the true value, empirical evidence from Das and Magdon-Ismail (2008) show that Gaussian state update very closely approximates true state updates. The state space is reduced to a function class with only two parameters, mean (μ_t) and variance (σ_t^2), where the value function is independent of μ_t. The new Bellman equation is given by:

$$V(\sigma_t) = \max_\delta\{r_t(\sigma_t, \delta) + \gamma\ E[V(\sigma_{t+1})|\delta]\}$$

such that the optimal actions ask and bid prices are $a_t = \mu_t + \delta_t$ and $b_t = \mu_t + \delta_t$, respectively. Using standard normal density and distribution functions N and Φ, we arrive at:

$$\mu_{t+1} = \mu_t + \sigma_t \cdot \frac{B}{A} \tag{1}$$

$$\sigma_{t+1}^2 = \sigma_t^2(1 - \frac{AC - B^2}{A^2}) \tag{2}$$

where A, B, and C are functions of $z^-, z^+, \mu_t, \rho_t, \sigma_\epsilon$ according to the following Gaussian integrals and normalization constants (Das and Magdon-Ismail, 2008):

$$
\begin{aligned}
I(\alpha,\beta) &= \int_{-\infty}^{\infty} dx\, N(x) \int_{-\infty}^{\alpha-\beta x} dy\, N(y) \\
&= \Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right), \\
J(\alpha,\beta) &= \int_{-\infty}^{\infty} dx\, x\cdot N(x) \int_{-\infty}^{\alpha-\beta x} dy\, N(y) \\
&= -\sqrt{\frac{\beta^2}{1+\beta^2}}\cdot N\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right), \\
K(\alpha,\beta) &= \int_{-\infty}^{\infty} dx\, x^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\alpha-\beta x} dy\, \frac{e^{-y^2/2}}{\sqrt{2\pi}} \\
&= I(\alpha,\beta) - \frac{\alpha\beta^2}{(1+\beta^2)^{3/2}}\cdot N\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right) \\
L(\alpha,\beta) &= I(\alpha,\beta) - K(\alpha,\beta) \\
A(z^+,z^-) &= I\left(\frac{z^+-\mu_t}{\sigma_\epsilon},\rho_t\right) - I\left(\frac{z^--\mu_t}{\sigma_\epsilon},\rho_t\right) \\
B(z^+,z^-) &= J\left(\frac{z^+-\mu_t}{\sigma_\epsilon},\rho_t\right) - J\left(\frac{z^--\mu_t}{\sigma_\epsilon},\rho_t\right) \\
C(z^+,z^-) &= L\left(\frac{z^+-\mu_t}{\sigma_\epsilon},\rho_t\right) - L\left(\frac{z^--\mu_t}{\sigma_\epsilon},\rho_t\right)
\end{aligned}
$$

One of the strengths of this model is that unlike ZP, BMM can quote a price for varying trade sizes. By implementing a heuristic of divvying up Q shares into fixed sizes of α, the market maker updates her mean and variance after each mini-trade i with the same μ_{i+1} and σ_{i+1}^2 values as those given by (1) and (2). The price quoted in each micro-trade is

$$p_i = \mu_t + \sigma_\epsilon Q(\rho_i)\sqrt{1+\rho_i^2}$$

where ρ signifies the information disadvantage of the market maker such that $\rho_t = \frac{\sigma_t}{\sigma_\epsilon}$, or the uncertainty ratio between the MM and the trader. In the end, for an ask price

$$a_t = p(Q) = \frac{1}{Q}\sum_{i=1}^{k} \alpha_i p_i$$

and updating the MM's belief based on whether the trade was accepted or not. The bid price is its mirror.

In addition to handling varying trade quantities with a fictitious division and sequential belief updates, BMM also addresses the fault of the classical ZP algorithm that constantly decreases the belief variance, which trades high liquidity with low adaptability in multiple market shocks due to overconfidence of the MM. The magnitude of each mean belief update is proportional to the variance of the MM's belief. Shortly after a jump, such as an IPO or a scandalous news coverage, trades will occur very one-sidedly. This generates the inconsistency with the MM's belief of the old valuation during a time of high confidence (low variance). BMM fixes this by allowing the MM to become less confident in these one sided trades that are not consistent with the previous history. The solution, in essence, is to increase the MM's belief variance during a period of low consistency by the consistency index

$$C(history) = L(\mu_t, 2\sigma_t) - L(\mu_t, \sigma_t)$$

where L, the probability of a sequence of trades over a window, is defined as

$$L(\mu,\sigma) = \int_{-\infty}^{\infty} N(v,\mu,\sigma)\cdot\prod_{i=1}^{s}(\Phi(z_i^+,v,\sigma_\epsilon) - \Phi(z_i^-,v,\sigma_\epsilon))dv$$

3. SIMULATION

Following Brahma *et al.* (2012) and Glosten and Milgrom (1985), we create a simulation of a stylistic market making model. We assume that the market maker is monopolistic (i.e. a pure dealer market). For simplicity, we consider a binary outcome, denoted by outcome 1 and outcome 2. Each contract then pays of \$100, and hence the instantaneous price will range between 0 and 100. The true value, or the probability of outcome 1 occurring, is initialized from $N(50, 12)$, or a Normal distribution with mean 50 and standard deviation 12. Following the literature, we consider two types of shocks.

- *Gaussian shock*: At each time period, a shock to the true value occurs with a probability of 0.01. This shock is distributed $N(0, 5)$.
- *Uniform shock*: A single shock is drawn uniformly from 0 to 40 at one time period (chosen to be in the middle of the simulation).

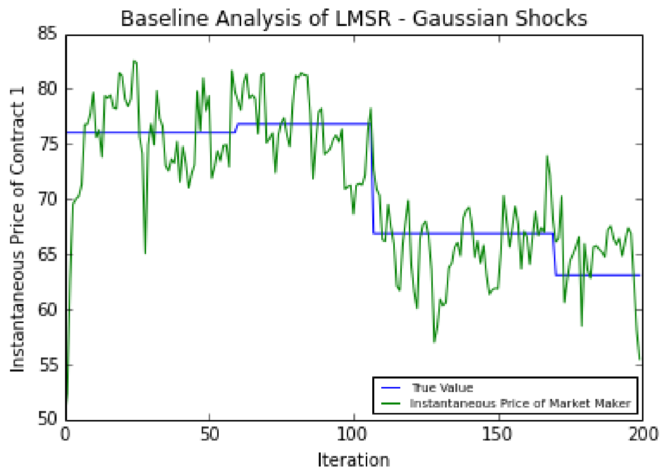
At each time period, a single trader arrives (this can be thought equivalently as selecting one trader out of a pool of traders). This is warranted in the LMSR model since the market maker's cost function and prices depend only on the current inventory. In other words, the market maker in a LMSR model does not absorb information when a trade is refused (unlike the case in the BMM). The trader has a noisy signal, $w_1(t)$ of the value of a contract on outcome 1 at time t. This is sampled from a Normal distribution with the mean equal to the true value of the contract, μ_t and with the standard deviation equal to 5.

The trader then chooses a quantity that they want to buy. As noted in lecture, traders will have myopic preferences and will be willing to trade until the instantaneous price of the contract is equal to $w_1(t)$. To generalize this, we also consider budget constraints. Hence each trader samples a quantity from an exponential model with a rate parameter of 0.05. Then we take the minimum of this random quantity and the quantity obtained from the myopic assumption. As a result, if the "budget condition" does not hold, then the trader is assumed to be myopic. However, if the "budget condition" does hold, then the trader trades until she reaches her maximum amount. The market maker calculates a price based on this quantity. If the trader accepts the price, then the order is executed.

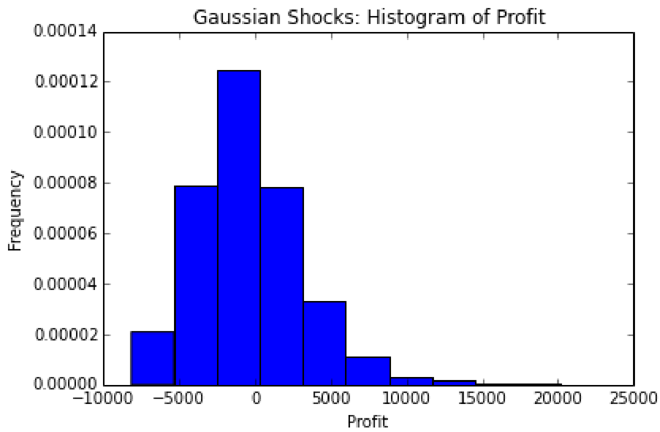
Unlike Brahma *et al.* (2012), we do not allow short selling due to the set-up of the LS LMSR model as the quantity of each contract cannot go below \$0. Brahma *et al.* only offer a single contract which requires the ability to short sell. Nevertheless, this should not make a difference in the results since we offer two contracts, which allows a trader to express her views by buying on either side. The equivalence is also due to the fact that the traders are able to sell any quantity of either contract that they had bought back to the market maker.

3.1 LMSR

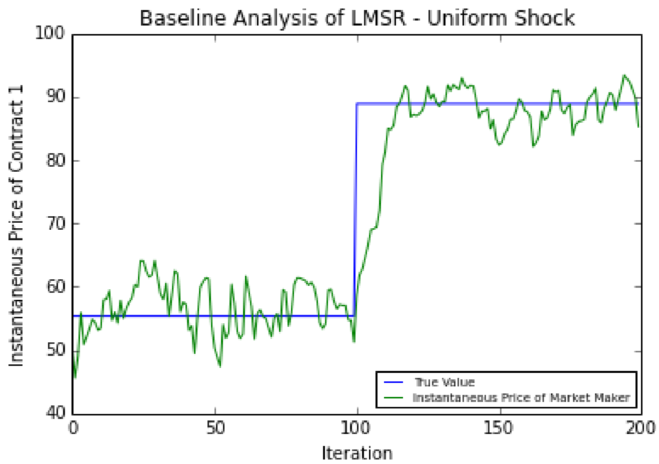
Under the base case scenario, we consider $\beta = 125$ under Gaussian shocks. We repeat the simulation 1000 times with 200 time periods in each simulation. The following figure is a sample simulation.



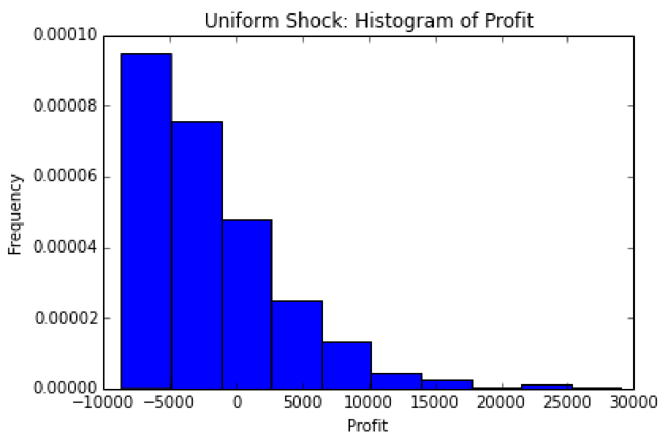
As shown above, the instantaneous price is quite volatile around the true value and does not seem to *converge* to the true value. However, it does seem to track it quite well, especially around the drop around time period 100. This is very similar to the results of Brahma *et al.* (2012), suggesting that the simulation was implemented correctly. We also examine the distribution of the market maker's total profit across the 1000 simulations. Profit for each simulation is calculated by simulating the final outcome from the probability of outcome 1 at the last time period. If the final outcome is 1, then we subtract the number of contracts for outcome 1, or q_1 (times 100 since our contract pays out \$100) from the total payments made to the market maker. If the final outcome is 0, then we subtract the number of contracts for outcome 2, or q_2 (times 100 since our contract pays out \$100) from the total payments made to the market maker.



As seen above, the median profit is negative. The maximum loss is bounded (we will see that the bound is correct later), whereas in some cases the market maker is able to gain large profits (intuitively, this occurs when the event that was unlikely to occur is realized). The following figure is a sample simulation using the *uniform shock* model:



Again, we see that the instantaneous price is volatile around the true value and we do not observe convergence. We also see that the price reacts to the market shock well. Now, we plot the distribution of profits for the market maker.



The market maker seems to lose more on average under this model.

In the following table, we display the the average profit, the standard deviation of the profits, the maximum loss in any of the 1000 simulations, and the average root mean square deviation (RMSD) of the infinitesimal price from the true value. We perform 1000 simulations with 200 time periods. The following table shows $\beta = 10, 125, 250, 500$ with Gaussian shocks.

	Gaussian Shocks			
	10	125	250	500
Average Profit	-211.9	-349.3	-877.2	-1969.0
SD of Profit	3019.4	3725.7	6735.2	13384.1
Maximum Loss	-6824.4	-8598.1	-15771.0	-32777.2
Average RMSD	3.85	3.69	3.25	3.12

As shown above, increasing β yields a larger maximum loss. Recall that the loss should be bounded at $\beta ln(2)$. As we are considering the contract paying out \$100, then we can multiply this by 100. Consider $\beta = 125$. The maximum loss should then be $125 * ln(2) * 100 = 8664$. Our worst case of 8282.4 is relatively close to this worst case bound. We also see that there is "more convergence" with increasing β., as reflected in the lower average RMSD Hence we have been able to theoretically show through this stylized model the trade-off for β, with a worse maximum loss but increased convergence to the true value. This can be seen as a "larger subsidy" required to produce a more accurate result.

In the following table, we consider $\beta = 10, 125, 250, 500$ with the uniform shock at the halfway point, or time period 100.

	Uniform Shock			
	10	125	250	500
Average Profit	-159.8	-1859.1	-3276.3	-5592.8
SD of Profit	409.3	5906.9	11387.9	24768.3
Maximum Loss	-693.0	-8663.1	-17326.2	-34588.6
Average RMSD	4.81	4.42	4.83	5.70

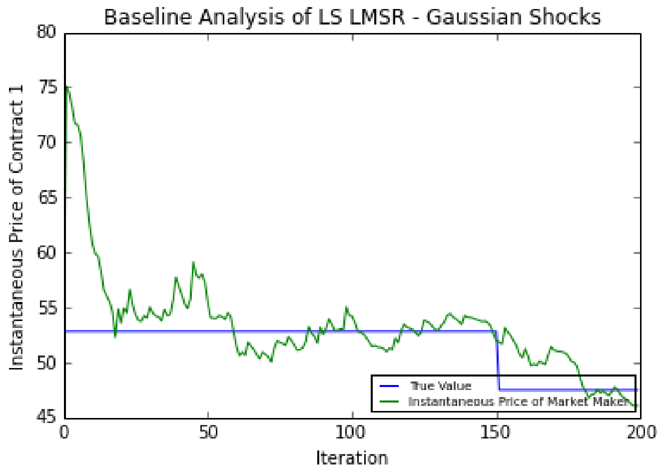
Again, we see a similar pattern as before. The average profit is negative but is correctly bounded.

3.2 LS LMSR

Under the base case, we consider $\alpha = 0.3$ under Gaussian shocks with an initial subsidy of 1000 for each contract. We repeat the simulation 1000 times with 200 time periods in each simulation. Note that we were not able to consider myopic traders. This is because of the complexity of obtaining the myopic quantity. Consider the case when the trader's value for the contract for outcome 1, $w_1(t)$ at time t, is greater than the market maker's instantaneous price for the contract for outcome 1. Hence we must solve for q_1 in the following equation:

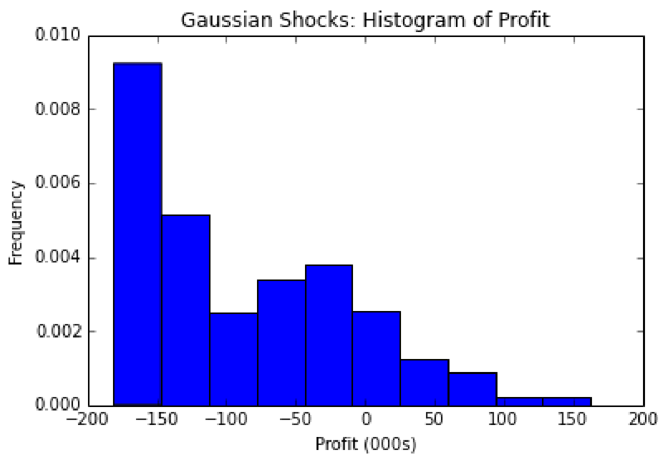
$$w_t(1) = \alpha log\left(exp(q_1/b(\mathbf{q})) + exp(q_2/b(\mathbf{q}))\right) + \frac{q_2(exp(q_1/b(\mathbf{q})) - exp(q_2/b(\mathbf{q})))}{(q_1+q_2)\left(exp(q_1/b(\mathbf{q})) + exp(q_2/b(\mathbf{q}))\right)}$$

This is analytically very difficult. Using a numerical estimator (e.g. Brent's Method or Newton's Method) produces inconsistent results and furthermore takes an extremely long time. Hence we do not consider myopic traders and simply draw the quantity from an exponential distribution with a rate parameter of 0.05. Because of this, the results produced here may not be precisely comparable to the more realistic results previously shown for the LMSR model. The following figure is a sample simulation.

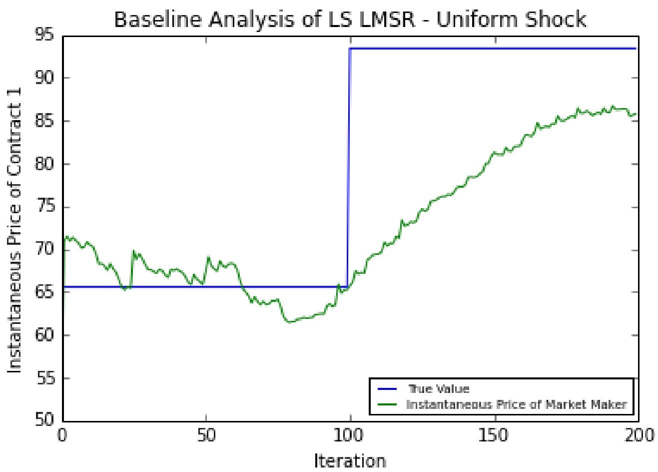


As seen above, the instantaneous price is much less volatile relative to the typical LMSR around the true value and seems to converge to the true value. Note also that it is slower to adapt to the decrease in the true value around time period 150, which is in line with expectations.

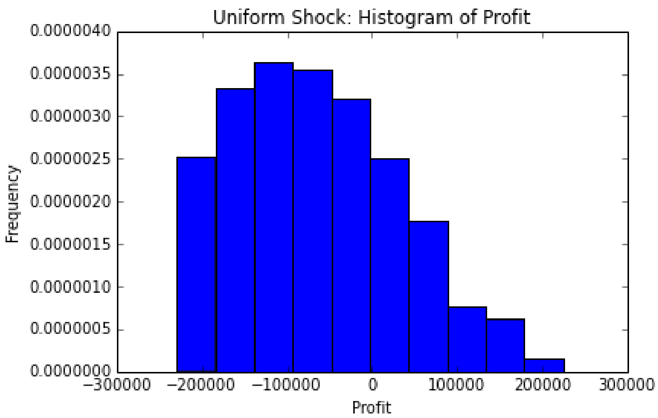
In the following figure, we plot the histogram across the 1000 simulations of the profit.



As seen, the most frequent outcome is a large significant loss between -\$150,000 and -\$200,000. The losses are much More significant than the LSMR model, due to the large subsidy of 1000 for each contract at the beginning of each simulation. However, the loss is still bounded, which is expected from the theory. The subsidy of 1000 is in line with the literature (see Othman *et al.*, 2010). The following figure is a sample simulation with uniform shock:



In the following figure, we plot the histogram across the 1000 simulations of the profit.



From the histogram, we can see a relatively different distribution of profits as the Gaussian shocks. Proportionally, the number of losses between -\$50,000 and -\$150,000. Moreover, the loss remains bounded.

In the following table, we display the average profit, the standard deviation of the profits, the maximum loss in any of the 1000 simulations, and the average root mean square deviation (RMSD) of the infinitesimal price from the true value. We perform 1000 simulations, considering $\alpha = 0.05, 0.1, 0.3, 0.05$ with Gaussian shocks. We set the initial subsidy per contract to be 1000.

Gaussian Shocks	0.05	0.1	0.3	0.5
Average Profit	-100225.1	-99573.7	-84268.6	-59501.5
SD of Profit	8231.3	18113.1	75712.2	151726.9
Maximum Loss	-115280.1	-130693.9	-182730.1	-232262.5
Average RMSD	5.01	3.64	6.32	15.94

From the previous table, we can see that increasing the α parameter seems to increase the average profit, from -\$100,000 at $\alpha = 0.05$ to -\$60,000 at $\alpha = 0.5$. Intuitively, this makes sense as α an be thought of as the market maker's commission, where a higher α means a larger commission. However, it is unknown why the standard deviation increases substantially, as well as the maximum loss. Two potential explanations for this pattern are the slower adaption to shocks or a very slow initial adaption to the price. This is also reflected in the RMSD increasing from 3.64 for $\alpha = 0.5$. to 15.94 for $\alpha = 0.5$. Note that with an initial subsidy of 1000 for each contract, our β is effectively around 600 at the start of the simulation for $\alpha = 0.3$.

In the following table, we consider $\alpha = 0.05, 0.1, 0.3, 0.05$ with a single uniform shock at the halfway point, or time period 100. We set the initial subsidy per contract to be 1000.

Uniform Shock	0.05	0.1	0.3	0.5
Average Profit	-102650.5	-106598.1	-101630.0	-66267.6
SD of Profit	11759.2	19064.8	46926.1	98986.7
Maximum Loss	-116682.6	-134286.4	-185007.4	-228783.1
Average RMSD	5.92	5.67	9.26	16.45

Comparing these results to the Gaussian shock, we see that there are similar patterns, with an increasing RMSD with increasing α as well as a small decrease in expected profit and an increase in the maximum loss.

In the following table, we consider different initial subsidies per each contract of $100, 500, 1000, 5000$ with Gaussian shocks. We set $\alpha = 0.5$.

Gaussian Shocks	100	500	1000	5000
Average Profit	31986.9	-4221.8	-52422.8	-471218.2
SD of Profit	113627.8	135198.4	154403.9	191071.3
Maximum Loss	-89949.5	-157675.4	-233038.6	-703563.6
Average RMSD	8.03	12.33	16.13	26.87

As seen from the previous table, we have that increasing the initial subsidy causes the maximum loss to grow as well as a decrease in the expected profit of the market maker. This is in line with expectations since Othman *et al.* (2008) derived that the maximum loss is $C(\mathbf{q^0})$, where $\mathbf{q^0}$ is the initial subsidy.

In the following table, we consider different initial subsidies per each contract of $100, 500, 1000, 5000$ with a single uniform shock at the halfway point, or time period 100. We set $\alpha = 0.5$.

From the previous table, we have similar trends for increasing initial subsidies. The consistency of these results support the general theory as well as demonstrate that the simulations were implemented correctly.

	Uniform Shock			
	100	500	1000	0.5
Average Profit	21359.8	-15944.4	-63724.5	-461608.7
SD of Profit	72865.7	88823.1	101949.0	143456.8
Maximum Loss	-77259.0	-154675.4	-228752.4	-699602.7
Average RMSD	8.93	13.38	16.57	23.70

4. PROPOSED LMSR*

Using what we learned about the BMM model, we were motivated to create a more flexible $\beta(\mathbf{q})$ in the liquidity sensitive LMSR model to address shocks. Hence we propose a modified LS LMSR model, denoted by LMSR*, that takes into account a window of size w of historical trades. Intuitively, we should be able to define $\beta(\mathbf{q})$ to be a function of the directional trades. We initially attempted to model this using a "switch" to reset β to some small value if there is evidence of a shock to the true value.

We considered several ways to define this "evidence." One proposed method is if the past 10 trade histories were all in one direction (unilaterally buys or sells) and at a large enough quantity. The former condition signifies that all the traders would want to trade in one direction, and the second condition will guarantee that it is not noise and that traders are trying to buy "as much as possible", taking into account the budget constraint. Note that this proposed method may be subject to collusion by traders. For instance, imagine if there are only 10 traders. Then they can mutually agree for one of the traders to not buy in the same direction. This will ensure that the market maker does not decrease β. By doing so, the market maker's instantaneous price will not adapt quickly to the shock and the traders will be able to guarantee a larger profit that they can then share among themselves. However, this may be unlikely in a large enough market with active participants and competition.

To make the description more concrete, consider time period T where the past w trades were all buy or all sell. Then we can reset

$$\beta(\mathbf{q}) = \beta(\mathbf{q_{new}})$$

where $\mathbf{q_{new}}$ denotes a new quantity of contracts that the market maker holds. Note that we cannot simply reset the quantities of the contracts to the initial values, since we would then lose all information about the prices. Given this issue, we attempted to find a possible $q_{new,1}$ given $q_{new,2} = q_2(0)$ such that the instantaneous price at time T would be equal to the instantaneous price at time $T-1$ for $q_{new,1}$ and $q_{new,2}$. Nevertheless, the attempted root finding numerical methods did not work, often because we were unable to find both an a and b such that $f(a)$ and $f(b)$ had different signs. However, if this is achievable, then we can let after time period T,

$$\beta(\mathbf{q}) = \alpha \sum_{i=1}^{k} (q_i(t) + q_{new,i}(0) - q_i(T))$$

for $t \geq T$.

After this, the next time that β can be reset must be at a minimum of w trades after T. After the next T' is chosen, the previous equations continue to apply, but replacing T with T'.

Another possible implementation may be some function of the quantity of contract 1 and contract 2 that were bought in a window of size w of historical trades. If we let x_1 denote the quantity of contract 1 that was bought and x_2 denote the quantity of contract 2 that was bought, then we can define

$$\beta(\mathbf{q}, x_1, x_2) = \alpha \left(\sum_i q_i \right) - (x_1 - x_2)^2$$

Thus, this may avoid the issue of the large, discrete jumps in the instantaneous price that was faced before. For future work, it may be interesting to see how these types of functions can be modified and whether they retain any of the theoretical properties of the LS LMSR, including loss boundedness and whether traders still have myopic incentives.

5. CONCLUSION

Providing liquidity in securities markets has motivated our survey over various AMMs. Looking into the LMSR and LS LMSR, both of which has the characteristic of bounded loss, we were able to confirm through simulations the trade offs between a quick convergence to the true value and a quick adaption to market shocks. BMMs, on the other hand, was proposed and confirmed in previous literature, both through theory and empirical research with human subjects, as solving both problems. However, finding the solutions to the Bellman equation even after forcing the MM to maintain a Gaussian belief distribution was highly taxing. As an alternative, we were able to borrow the unique property of a BMM that takes into account the history of transactions and applied it to the LS LMSR model as a tractable solution to the problem of adjusting to multiple shocks while maintaining LMSR's advantage of bounded loss.

6. CONTRIBUTION AND THANKS

We are grateful to the help and support of Professor Parkes as well as the Teaching Fellows. We greatly enjoyed learning more about recent developments to the LMSR model, as well as the intersection of financial economic theory and market making from an algorithmic standpoint. The

complexity of the subject is staggering but it seems like there is significant room for more research. We are excited to see how the theory will continue to develop and be adopted into real-life prediction markets.

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7. APPENDIX

Simulation.ipynb - The Notebook file that contains all of the simulations that were used to construct the stylized market making model. It contains algorithms for market makers based on LMSR and the LS LMSR, as well as showing analysis with respect to both Gaussian shocks as well as Uniform shocks. We have the outline of a model for the proposed LMSR* model. However, we were unable to achieve our complete vision due to the lack of success with the numerical root finding.