《线性代数与解析几何》2019—2020年试卷(A)答案

1. (14 points) Compute the following determinants.

(1) $\begin{vmatrix} 3 & 5 & 7 & 9 \\ 11 & 13 & 15 & 17 \\ 19 & 21 & 23 & 25 \\ 27 & 29 & 31 & 33 \end{vmatrix}$; (2) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 \end{vmatrix}$.

Solution: (1) $\begin{vmatrix} 3 & 5 & 7 & 9 \\ 11 & 13 & 15 & 17 \\ 19 & 21 & 23 & 25 \\ 27 & 29 & 31 & 33 \end{vmatrix}$ (add (-1) ×row 1 to row 2)
(add (-1) ×row 3 to row 4)

$= \begin{vmatrix} 3 & 5 & 7 & 9 \\ 8 & 8 & 8 & 8 \\ 19 & 21 & 23 & 25 \\ 8 & 8 & 8 & 8 \end{vmatrix}$

$= 0.$

(2) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 \end{vmatrix}$ (add (-2) ×row 1 to row 2)
(add (-3) ×row 1 to row 3)
(add (-1) ×row 1 to row 4)

$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 6 & 24 & 78 \\ 0 & 3 & 15 & 63 \end{vmatrix}$ (Use the cofactor expansion down the first column)

$= \begin{vmatrix} 2 & 6 & 14 \\ 6 & 24 & 78 \\ 3 & 15 & 63 \end{vmatrix}$

$= 2 \times 6 \times 3 \begin{vmatrix} 1 & 3 & 7 \\ 1 & 4 & 13 \\ 1 & 5 & 21 \end{vmatrix}$ (add (-1) ×row 2 to row 3)
(add (-1) ×row 1 to row 2)

$= 36 \begin{vmatrix} 1 & 3 & 7 \\ 0 & 1 & 6 \\ 0 & 1 & 8 \end{vmatrix} = 36 \times (8 - 6) = 72.$

2. (8 points) Calculate the area of the parallelogram determined by the points $(-3,-5),(-1,0),(3,-4)$ and $(5,1)$.

Solution: Translate the parallelogram to one having the origin as a vertex.

For example subtract the vertex$(-3,-5)$ from each of four vertices. The new parallelogram has the same area and its vertices are $(0,0),(2,5),(6,1)$ and $(8,6)$.

The area of the parallelogram is $\left|\det\begin{bmatrix}2 & 6\\5 & 1\end{bmatrix}\right| = 28.$

The area can also be obtained by computing

$$\left|\det\begin{bmatrix}2 & 8\\5 & 6\end{bmatrix}\right| = 28.$$

$$\left|\det\begin{bmatrix}6 & 8\\1 & 6\end{bmatrix}\right| = 28.$$

3. (15 points) For two matrices $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & y \end{bmatrix}$,

(1) (6 points) if matrix equation $A\vec{x} = B\vec{x}$ has nonzero solutions, what is the value of y?

Solution: If matrix equation $A\vec{x} = B\vec{x}$ has nonzero solutions, then

$$\det(A - B) = \begin{vmatrix} 4 & 2 \\ -4 & 1 - y \end{vmatrix} = 4 \times (1 - y) + 8 = 0.$$

It implies that $y = 3$.

(2) (3 points) if A and B are similar to each other, what is the value of y?

Solution: A is similar to a diagonal matrix, which implies that both 3 and y are eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 7 - \lambda & 2 \\ -4 & 1 - \lambda \end{vmatrix} = (7 - \lambda)(1 - \lambda) + 8$$

$$= \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).$$

So $y = 5$.

(3) (6 points) Find a formula for A^k.

Solution:

Basis for the eigenspace of $\lambda = 3$: $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$;

Basis for the eigenspace of $\lambda = 5$: $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$A^k = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3^k + 2 \cdot 5^k & -3^k + 5^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$

4. (16 points) For the following quadratic form

$$\boldsymbol{x}^T A \boldsymbol{x} = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

(1) (4 points) Give the matrix A of the quadratic form, and indicate which type this quadratic form is?

Solution: The matrix A of the quadratic form is $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5-\lambda & 1 & 1 \\ 5-\lambda & 3-\lambda & 1 \\ 5-\lambda & 1 & 3-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)^2$$

The eigenvalues of A are 5 and 2. Both are positive, so A is positive definite.

(2) (12 points) find an orthogonal matrix P such that the change of variable $\boldsymbol{x} = P\boldsymbol{y}$ transforms $\boldsymbol{x}^T A\boldsymbol{x}$ into a new quadratic form with no cross-product term. Give the new quadratic form.

Solution:

Basis for the eigenspace of $\lambda = 2$: $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Basis for the eigenspace $\lambda = 5$: $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is not an orthogonal set.

Let $\boldsymbol{z} = \boldsymbol{v}_2 - Proj_{v_1}\boldsymbol{v}_2 = \boldsymbol{v}_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1}\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1/2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

then $\{\boldsymbol{v}_1, \boldsymbol{z}\}$ is an orthogonal basis for the eigenspace of $\lambda = 1$.

We then normalize $\boldsymbol{v}_1, \boldsymbol{z}, \boldsymbol{v}_3$ to obtain three orthonormal eigenvectors of A.

$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$; $\boldsymbol{u}_2 = \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$; $\boldsymbol{u}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

Let $P = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3]$

$$= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

Then the change of variable $\boldsymbol{x} = P\boldsymbol{y}$ transforms $\boldsymbol{x}^T A\boldsymbol{x}$ into a new quadratic form with no cross-product term.

The new quadratic form is $\boldsymbol{y}^T D\boldsymbol{y}$ = $2\boldsymbol{y}_1^2 + 2\boldsymbol{y}_2^2 + 5\boldsymbol{y}_3^2$.

5. (20 points) In vector space $\mathbb{p}_2$, the vector sets $B = \{-1, 2-t, -3-t-t^2\}$ and $C = \{1+t^2, t+t^2, -t\}$ are two bases.

(1) (6 points) Find the coordinate vector $[p]_B$ of $p(t) = 1 - 2t + t^2$ relative to B.

Solution: Suppose $[\boldsymbol{p}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then by definition of $[\boldsymbol{p}]_B$,

$$p(t) = -x_1 + x_2(2-t) + x_3(-3-t-t^2)$$

$$= (-x_1 + 2x_2 - 3x_3) + (-x_2 - x_3)t + (-x_3)t^2$$

That is $\begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

And thus $[\boldsymbol{p}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$

(2) (6 points) Find the coordinate vector $[p]_C$ of $p(t) = 1 - 2t + t^2$ relative to C.

Solution: Suppose $[\boldsymbol{p}]_C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then by definition of $[\boldsymbol{p}]_C$,

$$\begin{aligned} p(t) &= x_1(1+t^2) + x_2(t+t^2) + x_3(-t) \\ &= x_1 + (x_2 - x_3)t + (x_1 + x_2)t^2 \end{aligned}$$

That is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And thus $[\boldsymbol{p}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(3) (8 points) Find a matrix M, such that for any $p \in \mathbb{P}_2$, there is $[\boldsymbol{p}]_B = M[\boldsymbol{p}]_C$.

Solution: Clearly, $\mathrm{M} = P_{B\leftarrow C} = [[1+t^2]_B \quad [t+t^2]_B \quad [-t]_B]$.

To determine the three columns in M, we need to solve three linear systems. And we can solve them together

$$\begin{bmatrix} -1 & 2 & -3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & 0 & 1 & -1 \\ 0 & 0 & -1 & & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & \vdots & -1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & -1 & 1 \\ 0 & 0 & 1 & & -1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & & 2 & 3 & 0 \\ 0 & 1 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 0 & 1 & & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & & 4 & 1 & 2 \\ 0 & 1 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 0 & 1 & & -1 & -1 & 0 \end{bmatrix}$$

And thus $M = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

6. (11 points) For the vector space

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4b + 4d \\ -2c - d \\ -a + 8b - 6c + 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

(1) (3 points) what is the dimension for H?

Solution: It is easy to see that H is the set of all linear combinations of the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 8 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ -6 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

$$[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4] = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 4 & 0 & 4 \\ 0 & 0 & -2 & -1 \\ -1 & 8 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4]$ has three pivot columns, and thus the dimension for H is 3.

(2) (8 points) Find a set of basis for the orthogonal compliment $H^{\perp}$ of H.

Solution: A vector $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in $H^{\perp}$ if and only if

$\boldsymbol{u} \cdot \boldsymbol{v}_1 = 0$, $\boldsymbol{u} \cdot \boldsymbol{v}_2 = 0$ and $\boldsymbol{u} \cdot \boldsymbol{v}_3 = 0$. That is

$$\begin{bmatrix} 1 & 5 & 0 & -1 \\ -3 & 4 & 0 & 8 \\ 6 & 0 & -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 5 & 0 & -1 \\ -3 & 4 & 0 & 8 \\ 6 & 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -44/19 \\ 0 & 1 & 0 & 5/19 \\ 0 & 0 & 1 & -75/19 \end{bmatrix}$$, which implies that

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 44/19 \\ -5/19 \\ 75/19 \\ 1 \end{bmatrix}, x_4 \in \mathbb{R}.$ And thus a set of basis for $H^{\perp}$ is $\begin{bmatrix} 44 \\ -5 \\ 75 \\ 19 \end{bmatrix}$.

7. (8 points) $M = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$ is a 6×6 matrix. Find M^{-1}.

Solution: M can be written as

$$M = \begin{bmatrix} M_{11} & \mathbf{0}_{3\times3} \\ I_{3\times3} & M_{22} \end{bmatrix},$$

where $M_{11} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

It can be shown that both M_{11} and M_{22} are invertible and

$$M_{11}^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_{22}^{-1} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let a matrix X = $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, such that $AX = I$.

$$AX = \begin{bmatrix} M_{11} & \mathbf{0}_{3\times3} \\ I_{3\times3} & M_{22} \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11}X_{11} & M_{11}X_{12} \\ X_{11} + M_{22}X_{21} & X_{12} + M_{22}X_{22} \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & I_{3\times3} \end{bmatrix}.$$ That means

$$M_{11}X_{11} = I_{3\times3} \quad \Rightarrow \quad X_{11} = M_{11}^{-1}$$

$$M_{11}X_{12} = \mathbf{0}_{3\times3} \quad \Rightarrow \quad X_{12} = \mathbf{0}_{3\times3}$$

$$X_{11} + M_{22}X_{21} = \mathbf{0}_{3\times3} \quad \Rightarrow \quad X_{21} = -M_{22}^{-1}X_{11} = -M_{22}^{-1}M_{11}^{-1}$$

$$X_{12} + M_{22}X_{22} = I_{3\times3} \quad \Rightarrow \quad X_{22} = M_{22}^{-1}$$

$$-M_{22}^{-1}M_{11}^{-1} = \begin{bmatrix} 1 & 4 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$ So $M^{-1} = X = \begin{bmatrix} 1 & 2 & -3 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & -8 & -1 & -2 & 3 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$

8. (8 points) For a matrix A, $\det(A) = s$ and $s \neq 0$. If A has an eigenvalue λ, prove that the adjoin matrix A^* has an eigenvalue $\frac{s}{\lambda}$.

Proof. If A has an eigenvalue λ, then there is a nonzero vector $\boldsymbol{x}$ such that

$A\boldsymbol{x} = \lambda\boldsymbol{x}$.

We left multiply both sides of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ by $\det(A)A^{-1}$ to obtain

$det(A)\boldsymbol{x} = \lambda\det(A)A^{-1}\boldsymbol{x}$.

Observe that $A^* = \det(A)A^{-1}$. We find that

$det(A)\boldsymbol{x} = \lambda A^*\boldsymbol{x}$.

Since $\det(A) = s \neq 0$, $\lambda \neq 0$.

And thus $A^*\boldsymbol{x} = \frac{s}{\lambda}\boldsymbol{x}$. That means the adjoin matrix A^* has an eigenvalue $\frac{s}{\lambda}$.