Seat No.______ Major/Class______ School______ Student ID □□□□□□□□□□□□ Name______

(DONNOT WRITE YOUR ANSWER IN THIS AREA)

Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

Mathematical Analysis II Exam Paper B（2019-2020-2）

Notice:
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	I	II	III	IV	Sum
Score					

Score

I. Please fill the correct answers in the following blanks. (**24 points:** $4' \times 6 = 24'$)

1. (Convergent or divergent) The series $\sum_{n=1}^{+\infty}\left(\frac{n}{n+1}\right)^{n^2}$ is Convergent.

and the series $\sum_{n=1}^{+\infty}\frac{2n^2+3n}{\sqrt{5+n^5}}$ is divergent.

2. Let $S(x)$ be the sum of the Fourier series of the periodic function

$$f(x)=\begin{cases}-1, -\pi < x \leqslant 0,\\ 1+x^2, 0 < x \leqslant \pi.\end{cases}$$ Then $S(\pi)=\frac{\pi^2}{2}$, $S(4\pi)=0$.

3. The equation of the tangent plane at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$

is $3x-6y+2z+18=0$ or $-(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$.
or $-x+2y-\frac{2}{3}z=6$

4. Let $I_1=\iint\limits_{x^2+y^2\leqslant 1}|xy|dxdy$, $I_2=\iint\limits_{|x|+|y|\leqslant 1}|xy|dxdy$, $I_3=\iint\limits_{|x|\leqslant 1,|y|\leqslant 1}|xy|dxdy$, order them by

using "<" $I_2<I_1<I_3$.

5. Let Γ be the intersection of the sphere $x^2+y^2+z^2=a^2$ and plane $x+y+z=0$ then the

line integral $\int_\Gamma (x^2+y^2)ds=$ $\frac{4\pi a^3}{3}$.

6. The general solution of $y''-3y'+2y=e^{2x}$ is $y(x)=C_1e^x+C_2e^{2x}+xe^{2x}$.

给出 $C_1e^x+C_2e^{2x}$ 给2分.

II. **Finish the following calculations. (42 points.** 7-11: $6' \times 5 = 30'$, 12:$12' \times 1 = 12'$)

Score

7. Find the sum of the power series $\sum_{n=1}^{+\infty} \frac{x^n}{n}$.

Solution: The Convergence radius is

$$R = \lim_{n\to+\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to+\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1.$$

When $x=1$, $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent.

When $x=-1$, The series $\sum \frac{(-1)^n}{n}$ is convergent, according to the alternating series Test.

Then the convergence set is $[-1, 1)$. —— 2'

Let $S(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}$, then $S(x)$ is continuous on $[-1, 1)$.

$$S'(x) = \left(\sum_{n=1}^{+\infty} \frac{x^n}{n}\right)' = \sum_{n=1}^{+\infty} \left(\frac{x^n}{n}\right)' = \sum_{n=1}^{+\infty} x^{n-1} = \frac{1}{1-x},\ x \in (-1, 1).$$

$$S(x) = S(x) - S(0) = \int_0^x S'(t)\,dt$$ —— 4'

$$= \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x) \quad x \in (-1, 1).$$

Because $S(x)$ is continuous at $x=-1$.

Then $S(-1) = \lim_{x\to(-1)} S(x) = \lim_{x\to(-1)} \left(-\ln(1-x)\right) = -\ln 2.$

$$\sum_{n=1}^{+\infty} \frac{x^n}{n} = -\ln(1-x), \quad x \in [-1, 1)$$ —— 6'

8. Let $z = f\left(xy, \dfrac{x}{y}\right)$, the function f has continuous second derivatives,

Find $\dfrac{\partial z}{\partial y}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

Let $u = xy$ $v = \dfrac{x}{y}$.

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = f_u x + f_v\left(-\frac{x}{y^2}\right) \text{ --- } 3'$$

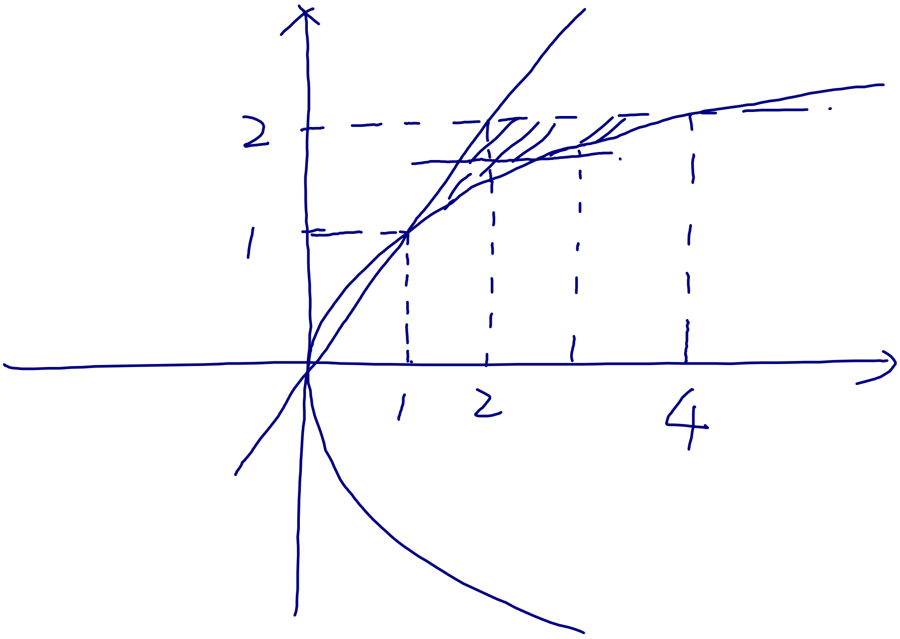
$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial y}\right)\cdot x$$

$$+\left(\frac{\partial^2 f}{\partial v \partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)\left(-\frac{x}{y^2}\right) + f_v\left(2y^{-3}x\right)$$

$$= f_{uu}x^2 + 2f_{uv}\left(-\frac{x^2}{y^2}\right) + \frac{\partial^2 f}{\partial v^2}\cdot\frac{x^2}{y^4} + \frac{2x}{y^3}f_v$$

$$\text{——————— } 6'$$

9. Evaluate $\int_1^2 dx\int_{\sqrt{x}}^{x} e^{\frac{x}{y}}dy+\int_2^4 dx\int_{\sqrt{x}}^{2} e^{\frac{x}{y}}dy$



— 2'

$$=\int_1^2\int_y^{y^2} e^{\frac{x}{y}}\,dx\,dy.$$ — 4'

$$=\int_1^2 ye^{\frac{x}{y}}\Big|_y^{y^2}dy$$

$$=\int_1^2 (ye^y-ye)\,dy$$

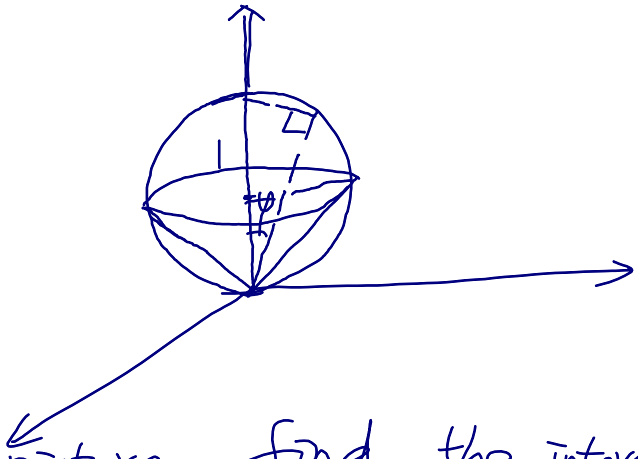
$$=\int_1^2 y\,de^y-\frac{y^2}{2}\Big|_1^2 e$$

$$=ye^y\Big|_1^2-\int_1^2 e^y dy-\frac{3}{2}e$$

$$=2e^2-e-e^2+e-\frac{3}{2}e$$ — 6'

$$=e^2-\frac{3}{2}e$$

10. Find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



Solution: Draw a picture, find the intersection of the two surfaces.

$$\begin{cases} x^2+y^2+z^2=z \\ z=\sqrt{x^2+y^2} \end{cases} \Rightarrow z=\frac{1}{2}. \quad \text{———— } 2'$$

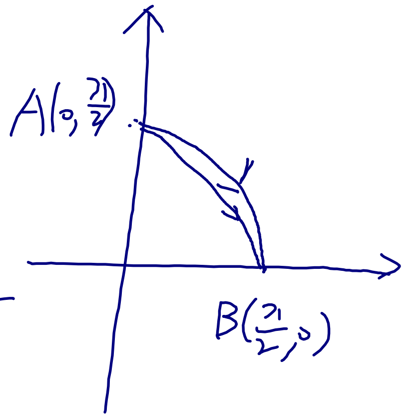
$$V=\iiint_{\Omega} 1\, dV$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta. \quad \text{———— } 4'$$

$$=\frac{\pi}{8}. \quad \text{———— } 6'$$

11. Find the work done by the force $\vec{F}=\dfrac{y\vec{i}-x\vec{j}}{\sqrt{x^2+y^2}}$ in moving a particle from $A\left(0,\dfrac{\pi}{2}\right)$ along the curve $y=\dfrac{\pi}{2}\cos x$ to $B\left(\dfrac{\pi}{2},0\right)$.

改为 $\vec{F}=\dfrac{y\vec{i}-x\vec{j}}{x^2+y^2}$.



Solution: $\vec{F}=\dfrac{y\vec{i}-x\vec{j}}{x^2+y^2}\triangleq(P,Q)$.

$$\frac{\partial P}{\partial y}=\frac{1}{x^2+y^2}-\frac{y}{(x^2+y^2)^2}\cdot 2y=\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x}=\frac{-1}{x^2+y^2}+(-x)\cdot\left(-\frac{1}{(x^2+y^2)^2}\cdot 2x\right)$$

$$=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}\Rightarrow\vec{F}$ is conservative. ——— 2'

Then we can chang the path to another path from $A(0,\frac{\pi}{2})$ to $B(\frac{\pi}{2},0)$ along $x^2+y^2=(\frac{\pi}{2})^2$

$$\int_{\overset{\frown}{AB}}\vec{F}\cdot d\vec{r}=\int_{\overset{\frown}{AB}}\vec{F}\cdot d\vec{r}$$

$$=\int_{\overset{\frown}{AB}}\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy \quad \text{——— } 4'$$

$$\underset{\substack{y=\frac{\pi}{2}\sin t\\(0\le t\le\frac{\pi}{2})}}{\overset{x=\frac{\pi}{2}\cos t}{=\!=\!=}}-\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}\sin t\left(-\frac{\pi}{2}\sin t\right)-\frac{\pi}{2}\cos t\,\frac{\pi}{2}\cos t\,dt}{\left(\frac{\pi}{2}\cos t\right)^2+\left(\frac{\pi}{2}\sin t\right)^2}$$

$$=\int_0^{\frac{\pi}{2}}1\,dt=\frac{\pi}{2}. \quad \text{——— } 6'$$

12. Evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = x\vec{i} + y\vec{j} + (z^2 - 2z)\vec{k}$ and S is the surface of the cone $z = \sqrt{x^2+y^2}$ with downward normal vector by using

$0 \le z \le 1$

1) direct calculation of surface integral of vector field.
2) the divergence theorem

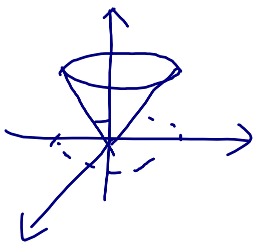
Solution: 1) $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$.

$z = \sqrt{x^2+y^2}$.

$$\vec{n} = \frac{\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} = \frac{\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} \quad \text{— } 4'$$

$$= \iint_D \left\{ x\cdot\frac{x}{\sqrt{x^2+y^2}} + y\cdot\frac{y}{\sqrt{x^2+y^2}} - \left(\left(\sqrt{x^2+y^2}\right)^2 - 2\sqrt{x^2+y^2}\right)\right\} dx\,dy$$

$$= \iint_D \left\{3\sqrt{x^2+y^2} - (x^2+y^2)\right\} dx\,dy.$$



$$= \int_0^{2\pi}\int_0^1 (3r - r^2)\cdot r\, dr\, d\theta.$$

$$= 2\pi\cdot\left(r^3 - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\cdot\left(1-\frac{1}{4}\right) = \frac{3}{2}\pi. \quad \text{— } 6'$$

2) Add one plane $z=1$. such that we can get a closed surface.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S+S_1}\vec{F}\cdot d\vec{S} - \iint_{S_1}\vec{F}\cdot d\vec{S} \quad \text{— } 4'$$

$$= \iiint_\Omega \operatorname{div}\vec{F}\, dV - \iint_{S_1}\vec{F}\cdot d\vec{S} \qquad \vec{n}_1 = (0,0,1)$$

$$= \iiint_\Omega (2z)\, dV - \iint_{S_1}(z^2-2z)\, dS$$

$$= \iint_D \left(\int_{\sqrt{x^2+y^2}}^1 2z\, dz\right) dx\,dy + \iint_D 1\, dx\,dy$$

$$= \iint_D \left[1-(x^2+y^2)\right] dx\,dy + \iint_D 1\, dx\,dy = \frac{3}{2}\pi. \quad 6'$$

III. Prove the following conclusions. (24 points. 13-15: $8' \times 3 = 24'$ **)**

Score

13. Let $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$.

Show that both $f_x(0,0)$ and $f_y(0,0)$ exist, but $f(x,y)$ is not differentiable at $(0,0)$.

Proof.

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0.$$ —— $4'$

$$f_y(0,0)=\lim_{\Delta x\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=0.$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(0+\Delta x,0+\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}.$$ —— $8'$

This limit does not exist.

then $f(x,y)$ is not differentiable at $(0,0)$.

14. State the definition of uniform convergence for a series of functions and prove that the sum of the series $\sum_{n=1}^{+\infty} \frac{\cos nx}{2^n}$ is continuous.

Proof. The definition of uniform convergence.

Denote $\sum_{n=1}^{+\infty} u_n(x) = S(x)$.

$$\sum_{k=1}^{n} u_k(x) = S_n(x).$$

$\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, when $n > N(\varepsilon)$, we have

$$|S(x) - S_n(x)| < \varepsilon.$$

i.e., $\left| \sum_{n=1}^{+\infty} u_n(x) - \sum_{k=1}^{n} u_k(x) \right| < \varepsilon$. ———— 4'.

$\sum_{n=1}^{+\infty} \frac{\cos nx}{2^n}$.

Since (i) $\left| \frac{\cos nx}{2^n} \right| \leq \frac{1}{2^n}$. ———— 6'

$\sum_{n=1}^{+\infty} \frac{1}{2^n}$ is convergent, Then by M-test, we have $\sum_{n=1}^{+\infty} \frac{\cos nx}{2^n}$ is uniformly convergent.

(ii) $\frac{\cos nx}{2^n}$ is continuous.

$\Rightarrow \sum_{n=1}^{+\infty} \frac{\cos nx}{2^n}$ is continuous ———— 8'

15. Show that the vector field $\vec{F} = \sin y\vec{i} + (x\cos y + \cos z)\vec{j} - y\sin z\vec{k}$ is conservative and moreover find a function f such that $\vec{F} = \nabla f$

proof. $\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & (x\cos y + \cos z) & -y\sin z \end{vmatrix} = 0$ — 4'

$$\frac{\partial}{\partial x}(x\cos y + \cos z) - \frac{\partial}{\partial y}(\sin y) = 0.$$

$$\frac{\partial}{\partial y}(-y\sin z) - \frac{\partial}{\partial z}(x\cos y + \cos z) = 0$$

$$\frac{\partial}{\partial z}(\sin y) - \frac{\partial}{\partial x}(-y\sin z) = 0.$$

Then $\vec{F}$ is conservative.

$\sin y = f_x \Rightarrow f(x,y,z) = x\sin y + \varphi(y,z)$

$\frac{\partial f}{\partial y} = (x\cos y + \cos z) = x\cos y + \frac{\partial \varphi}{\partial y}.$

$\Rightarrow \frac{\partial \varphi}{\partial y} = \cos z \Rightarrow \varphi(y,z) = y\cos z + \psi(z).$

$\because \frac{\partial f}{\partial z} = -y\sin z + \psi'(z) = -y\sin z$

$\Rightarrow \psi'(z) = 0 \Rightarrow \psi(z) = C.$

$\Rightarrow f(x,y,z) = x\sin y + y\cos z + C$ — 8'

IV. Finish the following questions. (10 points. 16:10' **)**

Score

16. Find the points on the curve of the intersection of the two surfaces $z = x^2 + y^2$ and $xy = 1$ that are closest to the origin.

Solution: $d^2(x, y, z) = x^2 + y^2 + z^2$.

$L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - z) - \mu(xy - 1)$

$L_x = 2x - 2\lambda x - \mu y = 0$

$L_y = 2y - 2\lambda y - \mu x = 0$

$L_z = 2z + \lambda = 0$

$L_\lambda = x^2 + y^2 - z = 0$ ———— 9'

$L_\mu = xy - 1 = 0$

$x^2 = y^2 = \frac{\mu}{2(1-\lambda)}$ $\Rightarrow$ $x = y$

$\Rightarrow x = \pm 1$, $y = \pm 1$ $\Rightarrow z = 2$

$d_{min} = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$ ———— 10'