Solutions to

An Introduction to Analysis (Wade 4/e)

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Contents

Preface

These solutions were done as an undergraduate student when studying the book. And the materials now only contain some selected solutions. If there is something vague or incredible, it is possible that it doesn't make sense since it is wrong. If you have any suggestions or corrections to solutions, please direct to email "howevereeeee@gmail.com". Your intelligence will be highly appreciated.

Hope everybody good luck and have fun in solving problems!

Chapter 6

Infinite Series of Real Numbers

6.1 Introduction

Exercise 6.1.4

Solution. Compute

$$\sum_{k=1}^{\infty}(a_{k+1}-2a_k+a_{k-1}) = \lim_{n\to\infty}(a_0-a_1+a_{n+1}-a_n) = a_0-a_1$$

. $\square$

Exercise 6.1.5

Solution.

$$\begin{aligned}\sum_{k=1}^{\infty}\left(x^k-x^{k-1}\right)\left(x^k+x^{k-1}\right) &= \sum_{k=1}^{\infty}\left(x^{2k}-x^{2k-2}\right)\\ &= \lim_{n\to\infty}\left(x^{2n}-1\right)\end{aligned}$$

converges if and only if $|x|\le 1$. As a result,

$$\sum_{k=1}^{\infty}\left(x^k-x^{k-1}\right)\left(x^k+x^{k-1}\right) = \begin{cases}0 & \text{for } x=\pm 1\\ -1 & \text{for } x\in(-1,1)\end{cases}.$$

$\square$

Exercise 6.1.7

a) *Proof.* Since $f'(x)$ exists for all $x\in\mathbb{R}$, we conclude $F'(x)$ exists.
By the Mean Value Theorem, there is a number $c\in I$ so that

$$|F(x)-F(y)| = F'(c)(x-y),\ \forall x,y\in I$$

where

$$F'(c) = 1-\frac{f'(c)}{f'(a)}.$$

Since $c\in I$, we know $\frac{f'(c)}{f'(a)}\in[1-r,1]$; therefore

$$0\le F'(c) = 1-\frac{f'(c)}{f'(a)}\le r.$$

As a result, we conclude that

$$|F(x)-F(y)| = |F'(c)||x-y|\le r|x-y|,\ \forall x,y\in I.$$

$\square$

b) *Proof.* For $n = 1$, by definition of x_n, we get

$$|x_2 - x_1| = |F(x_1) - F(x_0)| \le r|x_1 - x_0|.$$

Assume for $n = k$,

$$|x_{k+1} - x_k| \le r^k|x_1 - x_0|$$

holds, then for $n = k + 1$,

$$|x_{k+2} - x_{k+1}| = |F(x_{k+1}) - F(x_k)| \le r|x_{k+1} - x_k| \le r^{k+1}|x_1 - x_0|$$

also holds.

By induction, we conclude

$$|x_{n+1} - x_n| \le r^n|x_1 - x_0|, \ \forall n \in \mathbb{N}.$$

□

c) *Proof.* Since $f(I) \subseteq I$ and $x_0 \in I$, we hold $x_n \in I$, $\forall n \in \mathbb{N}$. In addition, I is a closed interval, by Bolzano-Weierstrass Theorem, there is a subsequence $\{x_{n_k}\}$ which converges to a fixed number $b \in I$.

Moreover, by part b), we get $\{x_n\}$ is Cauchy. Hence

$$\lim_{n\to\infty} x_n = b.$$

Besides, F is differentiable on I, so F is continuous on I.

Consider the equation $x_n = F(x_{n-1})$ and notice F is continuous on I. Taking the limit on both sides as $n \to \infty$, we have $b = F(b)$ which implies that

$$b = b - \frac{f(b)}{f'(a)}.$$

Then we conclude $f(b) = 0$ as promised. □

Exercise 6.1.9

a) *Proof.* For all $n > N$,

$$\begin{aligned}
\left| nb - \sum_{k=1}^{n} b_k \right| &= \left| \sum_{k=1}^{n} (b - b_k) \right| \\
&\le \sum_{k=1}^{n} |b - b_k| \\
&\le \sum_{k=1}^{N} |b - b_k| + \sum_{k=N+1}^{n} M \\
&= \sum_{k=1}^{N} |b_k - b| + M(n - N).
\end{aligned}$$

□

b) *Proof.* Since $\lim_{n\to\infty} b_n = b$, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |b_n - b| < \epsilon$. Then when $n > N$, by part a), we have

$$\left|\frac{b_1 + b_2 + ... + b_n}{n} - b\right| = \left|\frac{b_1 + b_2 + ... + b_n - nb}{n}\right| \leq \left|\frac{\sum_{k=1}^{n} |b_k - b|}{n}\right| + \epsilon\left(1 - \frac{N}{n}\right).$$

Taking the limit of both sides as $n \to \infty$, we have

$$\limsup_{n\to\infty} \left|\frac{b_1 + b_2 + ... + b_n}{n} - b\right| \leq \epsilon.$$

This hold for any $\epsilon > 0$, so

$$\lim_{n\to\infty} \left|\frac{b_1 + b_2 + ... + b_n}{n} - b\right| = 0$$

which implies

$$\frac{b_1 + b_2 + ... + b_n}{n} \to b \text{ as } n \to \infty.$$

□

c) *Solution.* The counter-example is $b_k = (-1)^{k-1}$. □

Exercise 6.1.12

Proof. Let

$$b_n := \sum_{k=1}^{n} k a_k = \frac{n+1}{n+2}.$$

So

$$b_{n+1} - b_n = (n+1)a_{n+1} = \frac{1}{(n+2)(n+3)}.$$

We have

$$a_1 = \frac{2}{3} \text{ and } a_n = \frac{1}{n(n+1)(n+2)} \text{ for } n > 1.$$

Calculate the summation and we get

$$\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} a_k = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}.$$

□

6.2 Series with Nonnegative Terms

Exercise 6.2.1

a) *Proof.* Let
$$a_n := \frac{2n+5}{3n^3+2n-1} \text{ and } b_n := \frac{1}{n^2}.$$
Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-Series Test; besides,
$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3} < \infty.$$
By the Limit Comparison Test, we conclude
$$\sum_{k=1}^{\infty} \frac{2k+5}{3k^3+2k-1}$$
converges. □

d) *Proof.* Since $\log k < k$, $\forall k \in \mathbb{N}$, we have
$$\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k} \le \sum_{k=1}^{\infty} \frac{k^5}{e^k}.$$
Let
$$a_n := \frac{n^5}{e^n} \text{ and } b_n := \frac{1}{n^2}.$$
Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-Series Test; besides,
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^7}{e^n} = 0.$$
By the Limit Comparison Test and the Comparison Test, we conclude
$$\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k}$$
converges. □

Exercise 6.2.2

a) *Proof.* Let
$$a_n := \frac{3n^3+n-4}{5n^4-n^2+1} \text{ and } b_n := \frac{1}{n}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p-Series Test; besides,

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{3}{5} < \infty.$$

By the Limit Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{3k^3 + k - 4}{5k^4 - k^2 + 1}$$

diverges. □

d) *Proof.* Consider

$$\int_2^{\infty} \frac{1}{x \log^p x} = \left| (\log x)^{1-p} \right|_2^{\infty} + p \int_2^{\infty} \frac{1}{x \log^p x}.$$

Since $p \leq 1$, then $1 - p \geq 0$, we have

$$\int_2^{\infty} \frac{1}{x \log^p x} = \frac{\left| (\log x)^{1-p} \right|_2^{\infty}}{1-p} = \infty.$$

By the Integral Test, we conclude

$$\int_2^{\infty} \frac{1}{x \log^p x} \quad \text{for } p \leq 1$$

diverges. □

Exercise 6.2.3

Proof. Assume $a_k \leq M$, $\forall k \in \mathbb{N}$ where $M > 0$. Therefore

$$\frac{a_k}{(k+1)^p} < \frac{M}{k^p}.$$

Since $p > 1$, by the p-Series Test and the Limit Comparison Test, we know

$$\sum_{k=1}^{\infty} \frac{M}{k^p}$$

converges.

By the Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$$

converges. □

Exercise 6.2.5

Proof. Since for $p \geq 0$, we have

$$\frac{|a_k|}{k^p} \leq |a_k|.$$

By the Comparison Test, we know

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$$

converges.

When $p < 0$, the series might converge or diverge. e.g. set $p = -1$ and $a_k = \frac{1}{k}$, then

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p} = \sum_{k=1}^{\infty} 1$$

diverges.

For the same p, set $a_k = \frac{1}{k^3}$, then the series

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges by the p-Series Test. □

Exercise 6.2.7

Proof. Since the series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \leq M$ for all k. It follows that $0 \leq a_k b_k \leq M b_k$ and the series $\sum_{k=1}^{\infty} b_k$ converges. By the Comparison Test, we conclude $\sum_{k=1}^{\infty} a_k b_k$ converges. □

Exercise 6.2.9

Proof. ($\Longrightarrow$) Let $s_n = \sum_{k=1}^{\infty} a_k$. Since

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = S \in \mathbb{R}$$

converges, so does its partial sum. Then

$$\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) = \lim_{n\to\infty} (s_{2n+1} - a_1) = S - s_1 \in \mathbb{R}$$

converges.

($\Longleftarrow$) Since

$$\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) = \lim_{n\to\infty} (s_{2n+1} - a_1)$$

converges so does s_{2n+1}.

Consider

$$s_{2n+2} = s_{2n+1} + a_{2n+2}.$$

Taking the limit of both sides as $n \to \infty$, we have

$$\lim_{n\to\infty} s_{2n+2} = \lim_{n\to\infty} s_{2n+1}$$

which implies s_{2n+2} also converges.

Finally, no matter the number of terms is either odd or even, the series always converges; hence, we conclude

$$\sum_{k=1}^{\infty} a_k$$

converges. $\square$

6.3 Absolute Convergence

Exercise 6.3.3

a) *Solution.* Consider

$$\int_2^\infty \left|\frac{1}{x\log^p x}\right| = \int_2^\infty \frac{1}{x\log^p x} = \frac{\left|(\log x)^{1-p}\right|_2^\infty}{1-p}$$

converges by the Integral Test if and only if $1-p<0$. So

$$p \in (1,\infty).$$

$\square$

c) *Solution.* Let

$$a_k = \frac{k^p}{p^k}.$$

Consider

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\left(\frac{k+1}{k}\right)^p\left(\frac{1}{p}\right)\right| = \frac{1}{|p|} < 1$$

which implies $|p|>1$. So

$$p \in (-\infty,-1)\cup(1,\infty).$$

$\square$

e) *Solution.* Let

$$a_k := \sqrt{k^{2p}+1} - k^p = \frac{1}{\sqrt{k^{2p}+1}+k^p} \text{ and } b_k := \frac{1}{k^p}.$$

Since $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if $p>1$ by the p-Series Test; besides,

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \frac{1}{\sqrt{1+\frac{1}{k^{2p}}}+1} = \begin{cases} 0 & \text{if } p<0 \\ \frac{1}{2} & \text{if } p>0 \\ \frac{1}{\sqrt{2}+1} & \text{if } p=0 \end{cases}.$$

By the Limit Comparison Test, we know the series converges when $p > 1$. For $p < 0$, consider

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^{2p}+1}+k^p} \geq \sum_{k=1}^{\infty} \frac{1}{2\sqrt{k^{2p}+1}}.$$

Since

$$\lim_{k\to\infty} \frac{1}{2\sqrt{k^{2p}+1}} = \frac{1}{2},$$

by the Divergence Test and the Comparison Test, we know the series diverges when $p < 0$. Finally, we conclude

$$p \in (1, \infty).$$

□

Exercise 6.3.5

Proof. Since $0 < \frac{1}{k} \leq 1$ and $\sin\frac{1}{k} > 0$ for all $k \in \mathbb{N}$, we know

$$1 + k\sin\frac{1}{k} > 0.$$

Hence

$$\lim_{k\to\infty} \left|\frac{a_k}{a_{k-1}}\right| = \lim_{k\to\infty} \left(\frac{1}{1+k\sin\frac{1}{k}}\right) = \frac{1}{2} < 1$$

where

$$\lim_{k\to\infty} \left(k\sin\frac{1}{k}\right) = \lim_{k\to\infty} \left(\frac{\sin\frac{1}{k}}{\frac{1}{k}}\right) = \lim_{k\to\infty} \left(\cos\frac{1}{k}\right) = 1.$$

By the Ratio Test, we conclude

$$\sum_{k=1}^{\infty} a_k$$

converges absolutely. □

Exercise 6.3.6

a) *Proof.* For $N \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{N} a_{kj}\right) = \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} a_{kj}\right).$$

By the Limit Theorem. Then fix N, for any $K \in \mathbb{N}$,

$$\sum_{k=1}^{K} \left(\sum_{j=1}^{N} a_{kj}\right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj}\right).$$

If right hand side is finite, by the Monotone Convergence Theorem, we have

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{N} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right).$$

On the other hand if it is infinite, then it holds trivially; hence for any $N \in \mathbb{N}$,

$$\sum_{j=1}^{N}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right).$$

Using the Monotone Convergence Theorem again, we conclude

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right).$$

□

b) *Proof.* Set $B_j = \sum_{k=1}^{\infty} a_{kj}$. From part a),

$$\sum_{j=1}^{\infty} B_j = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty} A_k$$

converges so does $\sum_{j=1}^{\infty} B_j$ which implies both

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) \text{ and } \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) \leq \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right)$$

hold. So we conclude

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right).$$

□

c) *Proof.* Consider

$$a_{kj} = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } j = k+1 \\ 0 & \text{otherwise} \end{cases}.$$

Then the equation from part b) implies

$$1 = 0$$

which is obviously a contradiction. □

Exercise 6.3.8

a) *Proof.* Let $s := \liminf_{k\to\infty} x_k$ and $s_n := \inf_{k>n} x_k$. We observe that $\lim_{n\to\infty} s_n = s$.
If $s > x$ for some $x \in \mathbb{R}$,

$$\exists N \in \mathbb{N} \text{ such that } s_n > x.$$

i.e., $x_k > x,\ \forall k > N$ as promised. □

b) *Proof.* If x_k converges to x, given $\epsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } k \geq N \implies |x_k - x| < \epsilon.$$

i.e., for any $n \geq N$,

$$x_k < x - \epsilon,\ \forall k > n.$$

Taking the infimum of this last inequality over $k > n$, we see that

$$s_n \leq x + \epsilon \text{ for any } n \geq N.$$

Hence the limit of the s_n's satisfies $s \leq x + \epsilon$; thus $s \leq x$.

A similar argument proves that $s \geq x$, then we conclude

$$s = \liminf_{k\to\infty} x_k = x.$$

□

c) *Proof.* Let

$$b_n = \inf_{k>n} \frac{a_{k+1}}{a_k} \text{ and } b = \lim_{n\to\infty} b_n.$$

For any $N \in \mathbb{N}$ where $N < n$. By definition of b_n, we have

$$b_N \leq \frac{a_{k+1}}{a_k} \text{ for } k = N+1, N+2, ..., n-1.$$

Notice that $b_n > 0,\ \forall n \in \mathbb{N}$. Multiplying these $n - N + 1$ inequalities together, we have

$$b_N^{n-N-1} \leq \frac{a_{N+1}}{a_n}.$$

Then taking n-th root of both sides, we see that

$$b_N^{1-\frac{N+1}{n}} \times \sqrt[n]{a_{N+1}} \leq \sqrt[n]{a_n}.$$

Taking the limit infimum on both sides and since the limit exists on the left hand side,

$$b_N \leq \liminf_{n\to\infty} \sqrt[n]{a_n}.$$

Finally, taking the limit on both sides as $N \to \infty$, we get

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n}.$$

A similar argument proves that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

On the other hand, by the Limit Comparison Theorem,

$$\liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n}$$

holds trivially.

Combining all above discussions and replacing the notation n with k, we conclude

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \leq \liminf_{k\to\infty} \sqrt[k]{a_k} \leq \limsup_{k\to\infty} \sqrt[k]{a_k} \leq \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

□

d) *Proof.* Notice $|b_n| > 0$, $\forall n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = r \in \mathbb{R},$$

we have

$$\liminf_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \limsup_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = r.$$

From part c), it squeezes that

$$\liminf_{n\to\infty} \sqrt[n]{|b_n|} = \limsup_{n\to\infty} \sqrt[n]{|b_n|} = r.$$

which implies

$$\lim_{n\to\infty} |b_n|^{\frac{1}{n}} = r.$$

□

6.4 Alternating Series

Exercise 6.4.3

a) *Solution.* Let

$$a_k := \frac{(-1)^k k^3}{(k+1)!}.$$

Since

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{-1}{k+2}\left(\frac{k+1}{k}\right)^3\right| = 0 < 1,$$

by the Ratio Test, we conclude the series **converges absolutely**. □

b) *Solution.* Let

$$a_k := \frac{(-1)\times(-3)\times ... \times(1-2k)}{1\times 4\times ... \times(3k-2)}.$$

Since

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{-2k-1}{3k+1}\right| = \frac{2}{3} < 1,$$

by the Ratio Test, we conclude the series **converges absolutely**. □

c) *Solution.* Let

$$a_k := \frac{(k+1)^k}{p^k k!}.$$

Since

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{k+2}{p\,(k+1)}\left(1+\frac{1}{k+1}\right)^k\right| = \frac{e}{p}$$

where

$$\lim_{k\to\infty}\left(1+\frac{1}{k+1}\right)^k = e,$$

by the Ratio Test, we conclude the series **converges absolutely**. □

d) *Solution.* Let

$$a_k := \frac{\sqrt{k}}{k+1}.$$

Calculate

$$a_k' = \frac{-3k-1}{2\sqrt{k}\,(k+1)^2} < 0$$

since $k \in \mathbb{N}$. Moreover,

$$\lim_{k\to\infty} a_k = 0,$$

so we know

$$a_k \downarrow 0 \text{ as } k \to \infty.$$

By the Alternating Series Test, we have

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

which converges.

Is it absolutely convergent? Using the divergent p-Series

$$b_k := \frac{1}{k},$$

we have

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{\sqrt{k}}{1+\frac{1}{k}}\right) = \infty.$$

By the Limit Comparison Test, we know

$$\sum_{k=1}^{\infty} a_k$$

diverges. Then we conclude the series **converges conditionally**. □

e) *Solution.* Let
$$a_k := \frac{(-1)^k\sqrt{k+1}}{\sqrt{k}\,k^k}.$$
Since
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\sqrt{\frac{k+2}{k+1}}\sqrt{\frac{k}{k+1}}\left(\frac{1}{k+1}\right)\left(1-\frac{1}{k+1}\right)^k\right) = 0 < 1$$
where
$$\lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)^k = \frac{1}{e}.$$
By the Ratio Test, we conclude the series **converges absolutely**. □

Exercise 6.4.4

Proof. Consider
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty}\left(a_k(b_k-b) - a_k b\right) = \sum_{k=1}^{\infty}\left[a_k(b_k-b)\right] - b\sum_{k=1}^{\infty} a_k.$$
By the Dirichlet's Test, we know
$$\sum_{k=1}^{\infty}\left[a_k(b_k-b)\right]$$
converges and $\sum_{k=1}^{\infty} a_k$ converges by the assumption so does
$$b\sum_{k=1}^{\infty} a_k,$$
then we conclude
$$\sum_{k=1}^{\infty} a_k b_k$$
converges. □

Exercise 6.4.5

Proof. By Abel's formula, consider
$$\sum_{k=1}^{n} a_k b_k = A_{n,1}b_n - \sum_{k=1}^{n-1} A_{k,1}(b_{k+1}-b_k) = A_{n,1}b_n + \sum_{k=1}^{n-1} A_{k,1}(b_k - b_{k+1}).$$
Notice $A_{n,1} = \sum_{k=1}^{n} a_k = s_n$ and s_n is bounded by the hypotheses.

Since $b_n \to 0$ as $n \to \infty$, then $s_n b_n \to 0$ as $n \to \infty$. Now taking the limit on both sides as $n \to \infty$, we conclude
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} s_k(b_k - b_{k+1}).$$
□

Exercise 6.4.6

Proof. Let
$$B_{n,m} := \sum_{k=m}^{n} b_k.$$

By Abel's formula, consider
$$\sum_{k=1}^{n} a_k b_k = B_{n,1} a_n + \sum_{k=1}^{n-1} B_{k,1}(a_k - a_{k+1}). \tag{6.1}$$

Since $a_n \to 0$ as $n \to \infty$ and $B_{n,1}$ is bounded by the hypotheses, then
$$B_{n,1} a_n \to 0 \text{ as } n \to \infty.$$

On the other hand,
$$\sum_{k=1}^{n-1} \left| B_{k,1}(a_k - a_{k+1}) \right| \leq M \sum_{k=1}^{n-1} \left| a_k - a_{k+1} \right|.$$

Since
$$\sum_{k=1}^{\infty} |a_k - a_{k+1}| = \sum_{k=1}^{\infty} |a_{k+1} - a_k|$$

converges by the hypotheses. By the Comparison Test, we know
$$\sum_{k=1}^{\infty} \left| B_{k,1}(a_k - a_{k+1}) \right|$$

converges so does
$$\sum_{k=1}^{\infty} B_{k,1}(a_k - a_{k+1}).$$

Finally, taking the limit of both sides on the equation (6.1), we conclude
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} B_{k,1}(a_k - a_{k+1})$$

converges as promised. □

Chapter 7

Infinite Series of Functions

7.1 Uniform Convergence of Sequences

Exercise 7.1.2

a) *Proof.* Consider

$$f_n(x) = \frac{nx^{99}+5}{x^3+nx^{66}} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = x^{33}.$$

Since f_n is continuous for each n on $[1,3]$, then so does integrable. For any $x \in [1,3]$, we have

$$|f_n(x) - f(x)| = \left|\frac{5-x^{36}}{x^3+nx^{66}}\right| \le \frac{5+3^{36}}{x^3+nx^{66}} \le \frac{5+3^{36}}{n} \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[1,3]$ implies f is integrable on $[1,3]$. Finally, we conclude

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}}\,dx = \int_1^3 \left(\lim_{n\to\infty} \frac{nx^{99}+5}{x^3+nx^{66}}\right) dx = \int_1^3 x^{33}dx = \left.\frac{x^{34}}{34}\right|_1^3 = \frac{3^{34}-1}{34}.$$

□

b) *Proof.* Consider

$$f_n(x) = e^{\frac{x^2}{n}} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = 1.$$

Since f_n is continuous for each n on $[0,2]$, then so does integrable. For any $x \in [0,2]$, we have

$$|f_n(x) - f(x)| = \left|e^{\frac{x^2}{n}} - 1\right| \le e^{\frac{4}{n}} - 1 \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[0,2]$ implies f is integrable on $[0,2]$. Finally, we conclude

$$\lim_{n\to\infty} \int_0^2 e^{\frac{x^2}{n}}\,dx = \int_0^2 \lim_{n\to\infty} \left(e^{\frac{x^2}{n}}\right) dx = \int_0^2 dx = x|_0^2 = 2.$$

□

c) *Proof.* Consider

$$f_n(x) = \sqrt{\sin\frac{x}{n} + x + 1}, \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = \sqrt{x+1}.$$

Since f_n is continuous for each n on $[0,3]$, then so does integrable. For any $x \in [0,3]$, we have

$$|f_n(x) - f(x)| = \frac{\left|\sin\frac{x}{n}\right|}{\sqrt{\sin\frac{x}{n} + x + 1} + \sqrt{x+1}} \le \sin\frac{3}{n} \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[0,3]$ implies f is integrable on $[0,3]$. Finally, we conclude

$$\begin{aligned}
\lim_{n\to\infty} \int_0^3 \sqrt{\sin\frac{x}{n} + x + 1}\, dx &= \int_0^3 \lim_{n\to\infty} \left(\sqrt{\sin\frac{x}{n} + x + 1}\right) dx \\
&= \int_0^3 \sqrt{x+1}\, dx \\
&= \frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^3 \\
&= \frac{14}{3}.
\end{aligned}$$

□

Exercise 7.1.3

Proof. Since $f_n \to f$ uniformly on E, pick $\epsilon = 1$ and choose $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < 1 \text{ and } |f_n(x) - f_N(x)| < 1 \text{ for all } x \in E \text{ and } n \geq N.$$

Since each f_n is bounded, $\exists M_n > 0$ such that

$$|f_n(x)| \leq M_n \text{ for all } x \in E.$$

Therefore,

$$|f(x)| \leq |f_N(x)| + 1 \leq M_n + 1 \text{ and } |f_n(x)| \leq |f_N(x)| + 1 \leq M_n + 1.$$

Set $M := \max\{M_1, M_2, ..., M_N\} + 1$, we have

$$|f_n(x)| \leq M \text{ for all } x \in E \text{ and } n \in \mathbb{N}$$

which implies f_n is uniformly bounded on E and f is a bounded function on E. □

Exercise 7.1.6

Proof. Since $f_n \to f$ uniformly on E, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \frac{\epsilon}{3} \text{ for all } x \in E$$

Consider f_N and since each f_n is uniformly continuous on E, for the same ϵ, $\exists \delta > 0$ such that

$$|x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \text{ for all } x, y \in E.$$

Hence, for any $x, y \in E$ with $|x - y| < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So, f is uniformly continuous on E. □

Exercise 7.1.8

Proof. Consider

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-x} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = 1.$$

Since each f_n is continuous on $\mathbb{R}$, then so does integrable. For any $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| = \left|e^{-x}\left(\left(1 + \frac{x}{n}\right)^n - e^x\right)\right| \le \left|\left(1 + \frac{x}{n}\right)^n - e^x\right| \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $\mathbb{R}$ implies f is integrable on $\mathbb{R}$. Finally, we conclude

$$\lim_{n\to\infty} \int_a^b \left(1 + \frac{x}{n}\right)^n e^{-x}\, dx = \int_a^b \lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^n e^{-x}\right)\, dx = \int_a^b dx = b - a.$$

□

Exercise 7.1.10

Proof. Since $f_n \to f$ uniformly on E, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon$$

for all $x \in E$; and then since each f_n is bounded and $f_n \to f$ uniformly on E, we know f is bounded. So $\exists M_n > 0$ such that

$$|f_n(x) - f(x)| \le M_n \text{ for } n = 1, 2, ..., N - 1.$$

Hence

$$\begin{aligned} \left|\frac{f_1(x) + ... + f_n(x)}{n} - f(x)\right| &\le \frac{\sum_{k=1}^{N-1} |f_k(x) - f(x)|}{n} + \frac{\sum_{k=N}^{n} |f_k(x) - f(x)|}{n} \\ &< \frac{(N-1)M}{n} + \left(1 - \frac{N-1}{n}\right)\epsilon \end{aligned}$$

where

$$M = \max\{M_1, M_2, ..., M_{N-1}\} + 1.$$

Taking the limit of both sides as $n \to \infty$, we have

$$\limsup_{n\to\infty} \left| \frac{f_1(x) + ... + f_n(x)}{n} - f(x) \right| \le \epsilon.$$

This holds for any $\epsilon > 0$, hence

$$\frac{f_1(x) + ... + f_n(x)}{n} \to f(x)$$

uniformly on E as $n \to \infty$. □

Exercise 7.1.11

Proof. Since f_n is integrable on $[0,1]$ and $f_n \to f$ uniformly on $[0,1]$, we know f is integrable and hence is bounded. Then $\exists M > 0, N_1 \in \mathbb{N}$ such that

$$n \ge N_1 \implies |f(x)| \le M \text{ for } x \in [0,1].$$

Moreover, given $\epsilon > 0$, $\exists N_2 \in \mathbb{N}$ such that

$$n \ge N_2 \implies \left| \int_0^d f_n(x)dx - \int_0^d f(x)dx \right| < \frac{\epsilon}{3} \text{ for } d \in [0,1].$$

Also, by the Uniform Cauchy Criterion, $\exists N_3 \in \mathbb{N}$ such that

$$n, m \ge N_3 \implies \left| \int_0^d f_n(x)dx - \int_0^d f_m(x)dx \right| < \frac{\epsilon}{3} \text{ for } d \in [0,1].$$

Notice $b_n \uparrow 1$ as $n \to \infty$, hence $\exists N_4 \in \mathbb{N}$ such that

$$n \ge N_4 \implies 1 - b_n < \frac{\epsilon}{3M}.$$

Finally, pick $N = \max\{N_1, N_2, N_3, N_4\}$, for any $n \ge N$, we have

$$\begin{aligned}
&\left| \int_0^{b_n} f_n(x)dx - \int_0^1 f(x)dx \right| \\
&\le \left| \int_0^{b_n} f_n(x)dx - \int_0^{b_n} f_N(x)dx \right| + \left| \int_0^{b_n} f_N(x)dx - \int_0^{b_n} f(x)dx \right| + \left| \int_{b_n}^1 f(x)dx \right| \\
&\le \int_0^{b_n} |f_n(x) - f_N(x)|\, dx + \int_0^{b_n} |f_N(x) - f(x)|\, dx + \int_{b_n}^1 |f(x)| dx \\
&< b_n \left(\frac{\epsilon}{3}\right) + b_n \left(\frac{\epsilon}{3}\right) + (1 - b_n)\, M \\
&< 1 \times \frac{\epsilon}{3} + 1 \times \frac{\epsilon}{3} + \frac{\epsilon}{3M} \times M \\
&= \epsilon.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \int_0^{b_n} f_n(x)dx = \int_0^1 f(x)dx.$$

□

7.2 Uniform Convergence of Series

Exercise 7.2.1

a) *Proof.* Since

$$\left|\sin\left(\frac{x}{k^2}\right)\right| \leq \frac{1}{k^2}$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, by the way ,$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. By the Comparison Test, we know $\sum_{k=1}^{\infty} \sin\left(\frac{x}{k^2}\right) < \infty$. Then by the Weierstrass M-Test, we know

$$\sum_{k=1}^{\infty} \sin\left(\frac{x}{k^2}\right)$$

converges uniformly on any bounded interval in $\mathbb{R}$. □

b) *Proof.* Consider

$$\left|e^{-kx}\right| = e^{-kx} \leq e^{-ka}$$

for all $k > 0$ where $a > 0$. Since $\sum_{k=1}^{\infty} e^{-ka}$ converges, then $\sum_{k=1}^{\infty} e^{-kx}$ converges absolutely. By the Weierstrass M-Test, we conclude

$$\sum_{k=1}^{\infty} e^{-kx}$$

converges uniformly on any closed subinterval of $(0, \infty)$. □

Exercise 7.2.4

Proof. Since

$$\sum_{k=1}^{\infty} \left|\frac{cos(kx)}{k^2}\right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges, then we know

$$\sum_{k=1}^{\infty} \frac{cos(kx)}{k^2}$$

converges absolutely. By the Weierstrass M-Test,

$$\sum_{k=1}^{\infty} \frac{cos(kx)}{k^2}$$

converges uniformly on $\mathbb{R}$. Hence,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} f(x)dx &= \int_0^{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{cos(kx)}{k^2} dx \\
&= \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{cos(kx)}{k^2} dx \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\frac{\pi}{2}} cos(kx)dx \\
&= \sum_{k=1}^{\infty} \frac{1}{k^3} \sin(kx)|_0^{\frac{\pi}{2}} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^3} \sin(\frac{k\pi}{2}) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}
\end{aligned}$$

□

Exercise 7.2.7

Proof. Since g_1 is bounded on E and $g_k(x) \geq g_{k+1}(x) \geq 0$ for all $x \in E$ and $k \in \mathbb{N}$, we know there is a number $M > 0$ such that

$$|g_n(x)| \leq M$$

for all $x \in E$ and $n \in \mathbb{N}$. Set

$$S_n(x) := \sum_{k=1}^{n} f_k(x) \text{ and } S(x) := \sum_{k=1}^{\infty} f_k(x).$$

Since f converges uniformly on E, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n > m > N \implies |S_n(x) - S_{m-1}(x)| \leq \frac{\epsilon}{3M} \text{ and } |S_{m-1}(x) - S(x)| \leq \frac{\epsilon}{3M}.$$

Hence,

$$
\begin{aligned}
&\left|\sum_{k=m}^{n} f_k(x)g_k(x)\right| \\
&= \left|S_n(x)g_n(x) - S_{m-1}g_m(x) + \sum_{k=m}^{n-1} S_k(x)\left(g_k(x) - g_{k+1}(x)\right)\right| \\
&= \left|\left[S_n(x) - S_{m-1}(x)\right]g_n(x) + S_{m-1}(x)\left[g_n(x) - g_m(x)\right] + \sum_{k=m}^{n-1} S_k(x)\left(g_k(x) - g_{k+1}(x)\right)\right| \\
&\le \left|\left(S_n(x) - S_{m-1}(x)\right)g_n(x) + \left(S_{m-1}(x) - S(x)\right)\left(g_n(x) - g_m(x)\right)\right| \\
&\quad + \left|\sum_{k=m}^{n-1} \left(S_k(x) - S(x)\right)\left(g_k(x) - g_{k+1}(x)\right)\right| \\
&< \frac{\epsilon}{3M} \times M + \frac{\epsilon}{3M} \times M + \frac{\epsilon}{3M} \times M \\
&= \epsilon.
\end{aligned}
$$

By the Uniform Cauchy Criterion, $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E. □

Exercise 7.2.10

Proof. Let

$$g_n(x) := \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}}.$$

Consider for all $x \in [a, b]$, since $0 \le f_k(x) \le f_{k+1}(x) \implies f_k^n(x) \le f_{k+1}^n(x)$. Hence

$$g_n(x) \le \left(n f_n^n(x)\right)^{\frac{1}{n}} = n^{\frac{1}{n}} f_n(x) \le 2 f_n(x)$$

for n sufficiently large. Notice $f_n \to f$ uniformly on $[a, b]$, hence g_n converges uniformly on $[a, b]$. We know given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies n^{\frac{1}{n}} < 1 + \epsilon,$$

hence

$$
\begin{aligned}
& g_n(x) \le (1+\epsilon) f_n(x) \\
\implies & \int_a^b g_n(x)dx \le (1+\epsilon)\int_a^b f_n(x)dx \\
\implies \lim_{n\to\infty} & \int_a^b g_n(x)dx \le \lim_{n\to\infty}\left((1+\epsilon)\int_a^b f_n(x)dx\right) \\
& = (1+\epsilon)\int_a^b \lim_{n\to\infty} f_n(x)dx \\
& = (1+\epsilon)\int_a^b f(x)dx
\end{aligned}
$$

This holds for any $\epsilon > 0$, hence

$$\lim_{n\to\infty} \int_a^b g_n(x)dx \leq \int_a^b f(x)dx. \tag{7.1}$$

On the other hand, since $f_{k+1}(x) \geq f_k(x) \geq 0$ on $[a,b]$,

$$\begin{aligned} g_n(x) &= \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}} \\ &= \left(\sum_{k=1}^{N-1} f_k^n(x) + \sum_{k=N}^{n} f_k^n(x)\right)^{\frac{1}{n}} \\ &\geq \left(\sum_{k=N}^{n} f_k^n(x)\right)^{\frac{1}{n}} \\ &\geq (n-N)^{\frac{1}{n}} f_N(x). \end{aligned}$$

Then,

$$\int_a^b \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}} dx \geq \int_a^b (n-N)^{\frac{1}{n}} f_N(x)dx.$$

Taking the limit of both sides as $n \to \infty$,

$$\lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}} dx \geq \int_a^b f_N(x)dx.$$

Taking the limit of both sides as $N \to \infty$,

$$\lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}} dx \geq \int_a^b f(x)dx. \tag{7.2}$$

From equation (7.1) and (7.2), we conclude

$$\lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} f_k^n(x)\right)^{\frac{1}{n}} dx = \int_a^b f(x)dx.$$

□

7.3 Power Series

Exercise 7.3.2

a) *Solution.* Let $a_k = \frac{1}{2^k}$, then

$$R = \frac{1}{\limsup_{k\to\infty} |a^k|^{\frac{1}{k}}} = 2.$$

For $x = -2$,

$$\sum_{k=0}^{\infty}(-1)^k$$

diverges. For $x = 2$,

$$\sum_{k=0}^{\infty} 1$$

diverges. Hence the interval of convergence is $(-2, 2)$. □

b) *Solution.* Let $a_k = \left((-1)^k + 3\right)^k$, then

$$R = \frac{1}{\limsup_{k\to\infty} |a^k|^{\frac{1}{k}}} = \frac{1}{4}.$$

For $x = \frac{3}{4}$,

$$\sum_{k=0}^{\infty}\left(\frac{-(-1)^k - 3}{4}\right)^k = \sum_{k=0}^{\infty}\left(1 + \left(-\frac{1}{2}\right)^{2k+1}\right).$$

diverges. For $x = \frac{5}{4}$,

$$\sum_{k=0}^{\infty}\left(\frac{(-1)^k + 3}{4}\right)^k = \sum_{k=0}^{\infty}\left(1 + \left(\frac{1}{2}\right)^{2k+1}\right)$$

diverges. Hence the interval of convergence is $(\frac{3}{4}, \frac{5}{4})$. □

c) *Solution.* Let $a_k = \log\left(\frac{k+1}{k}\right)$, then

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{\log\left(\frac{k+1}{k}\right)}{\log\left(\frac{k+2}{k+1}\right)}\right| = \lim_{k\to\infty}\left(\frac{k+2}{k}\right) = 1.$$

For $x = -1$,

$$\sum_{k=1}^{\infty}(-1)^k \log\left(1 + \frac{1}{k}\right)$$

converges by the Alternating Series Test. For $x = 1$,

$$\sum_{k=1}^{\infty}\log\left(1 + \frac{1}{k}\right) \geq \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$$

diverges since

$$\log(1 + \frac{1}{k}) \geq \frac{1}{2k}$$

for all $k \in \mathbb{N}$. Hence the interval of convergence is $[-1, 1)$. □

d) *Solution.* Let $a_k = \frac{1\cdot 3\cdots(2k-1)}{(k+1)!}$, then

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{k+2}{2k+1}\right| = \frac{1}{2}.$$

Now set $b_k = \frac{1\cdot 3\cdots(2k-1)}{(k+1)!}\left(\frac{1}{4}\right)^k$, since

$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left|\frac{2k+1}{k+2}\cdot\frac{1}{4}\right| = \frac{1}{2} < 1,$$

we know $\sum_{k=1}^{\infty} b_k$ converges by the Ratio Test. Hence the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$. □

Exercise 7.3.5

Proof. Since a_k is bounded, there is a number $M > 0$ such that $|a_k| \le M$ for all $k \in \mathbb{N}\bigcup\{0\}$. Consider

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|} \le \limsup_{k\to\infty}\sqrt[k]{M} = 1.$$

Hence,

$$R = \frac{1}{\limsup_{k\to\infty}\sqrt[k]{|a_k|}} \ge 1 > 0.$$

We conclude

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

has a positive radius of convergence. □

Exercise 7.3.10

Proof. Set

$$f(x) = \sum_{k=0}^{\infty}(-1)^k a_k x^k.$$

For any $x \in [0,1]$, since $a_k \downarrow 0$ as $k \to \infty$, we know $f(x)$ converges on $[0,1]$ by the Alternating Series Test.

Moreover, by the Abel's Theorem, $f(x)$ is continuous and converges uniformly on $[0,1]$; hence, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \forall x, y \in [0,1].$$

which implies

$$\left|\sum_{k=0}^{\infty}(-1)^k a_k(x^k - y^k)\right| < \epsilon.$$

□

7.4 Analytic Functions

Exercise 7.4.1

a) *Proof.* Set $f(x) = x^2 + \cos(2x)$. Then

$$\begin{aligned} f'(x) &= 2x - 2\sin(2x); & f''(x) &= 2 - 4\cos(2x); \\ f^{(4j+3)}(x) &= 2^{4j+3}\sin(2x); & f^{(4j+4)}(x) &= 2^{4j+4}\cos(2x); \\ f^{(4j+5)}(x) &= -2^{4j+5}\sin(2x); & f^{(4j+6)}(x) &= -2^{4j+6}\cos(2x) \end{aligned}$$

for $j = 0, 1, \cdots$. For any $C > 0$, we know

$$|f^{(n)}(x)| \leq \begin{cases} 2C+2 & \text{for } n = 0,\ 1 \\ 3^n & \text{for } n \geq 2 \end{cases}.$$

□

Since $C > 0$ is arbitrary, f is analytic on $\mathbb{R}$. And then

$$\cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!}.$$

We conclude the Maclaurin expansion is

$$f(x) = x^2 + \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - x^2 + \sum_{k=2}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}.$$

c) *Proof.* Set $f(x) = \sin^2(x) + \cos^2(x) = \cos(2x)$. Then

$$\begin{aligned} f^{(4j)}(x) &= 2^{4j}\cos(2x); & f^{(4j+1)}(x) &= -2^{4j+1}\sin(2x); \\ f^{(4j+2)}(x) &= -2^{4j+2}\cos(2x); & f^{(4j+3)}(x) &= 2^{4j+3}\sin(2x) \end{aligned}$$

for $j = 0, 1, \cdots$. Since $|f^{(n)}(x)| \leq 3^n$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then f is analytic on $\mathbb{R}$. We conclude the Maclaurin expansion is

$$f(x) = \cos(2x) = \sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}.$$

□

Exercise 7.4.4

Proof. By hypothesis, we suppose

$$P(x) = f(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Since $P \in \mathbb{C}^{\infty}(-\infty, \infty)$, by uniqueness, we set

$$\beta_k = \begin{cases} \frac{f^{(k)}(x_0)}{k!} & \text{for } k = 0, 1, \cdots, n \\ 0 & \text{for } k \geq n+1 \end{cases}.$$

Hence, the Taylor expansion of P centered at x_0 is

$$P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \sum_{k=0}^{\infty} \beta_k (x-x_0)^k = \beta_0 + \beta_1(x-x_0) + \cdots + \beta_n(x-x_0)^n$$

for all $x \in \mathbb{R}$ as promised. □

Exercise 7.4.6

Proof. Since $f \in \mathbb{C}^{\infty}(-\infty, \infty)$, by the Lagrange Theorem, we know

$$R_n(x) = \frac{1}{(n-1)!} \int_{x_0}^{x} (x-t)^{n-1} f^{(n)}(t)\, dt$$

for all x, x_0 on $\mathbb{R}$. Then we set $x_0 := 0,\ x := a,\ t := x,\ n := n+1$, by change of variables, we have

$$R_{n+1}(x) = \frac{1}{n!} \int_0^a x^n f^{(n+1)}(x)\, dx.$$

Since $\lim_{n\to\infty} R_{n+1}(x) = 0$ for all $a \in \mathbb{R}$, then the Taylor series of f centered at 0 converges to f for all $x \in \mathbb{R}$. So

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

for all $x \in \mathbb{R}$. By definition of analytic function, we conclude f is analytic on $(-\infty, \infty)$. □

Chapter 8

Euclidean Spaces

8.1 Algebraic Structure

Exercise 8.1.2

a) *Proof.* Consider

$$\begin{aligned}\|v\| &= \left\| \frac{(a \cdot b)c + (a \cdot c)b + (c \cdot b)a}{3} \right\| \\ &= \frac{1}{3} \left(\|a \cdot b\| \|c\| + \|a \cdot c\| \|b\| + \|c \cdot b\| \|a\| \right) \\ &\leq \|a\| \|b\| \|c\| \\ &\leq 1,\end{aligned}$$

hence $v \in B$ as promised. ☐

b) *Proof.* For all $c, d \in \mathbb{R}^n$,

$$\begin{aligned}|a \cdot c - b \cdot d| &= |a \cdot c - a \cdot b + a \cdot b - b \cdot d| \\ &= |a(c - b) + b(a - d)| \\ &\leq |a(c - b)| + |b(a - d)| \\ &= \|a\| \|b - c\| + \|b\| \|a - d\| \\ &\leq \|b - c\| + \|a - d\|\end{aligned}$$

☐

c) *Proof.*

$$\begin{aligned}\sqrt{|a \cdot (b \times c)|^2 + |a \cdot b|^2} &= \sqrt{|(a \times b) \cdot c|^2 + |a \cdot b|^2} \\ &= \sqrt{\|a \times b\|^2 \|c\|^2 + |a \cdot b|^2} \\ &\leq \sqrt{\|a \times b\|^2 + |a \cdot b|^2} \\ &= \sqrt{(a \cdot a)(b \cdot b) - (a \cdot b)^2 + |a \cdot b|^2} \\ &= \sqrt{(\|a\|^2 \|b\|^2 - |a \cdot b|^2 + |a \cdot b|^2} \\ &= \sqrt{\|a\|^2 \|b\|^2} \\ &\leq 1\end{aligned}$$

☐

Exercise 8.1.9

Proof. Since $|a_k|, |b_k| \geq 0$, by the Arithmetic-Geometric mean Inequality, we consider

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} \frac{a_k^2 + b_k^2}{2} = \frac{1}{2} \left(\sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 \right).$$

Since the rightest hand converges by hypotheses. Moreover, by the Comparison Test, we conclude

$$\sum_{k=1}^{\infty} a_k b_k$$

converges absolutely. $\square$

8.2 Planes and Linear Transformation

Exercise 8.2.4

a) *Solution.*

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

$\square$

b) *Solution.*

$$T = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}.$$

$\square$

c) *Solution.*

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2\times n}.$$

$\square$

Exercise 8.2.11

a) *Proof.* Let $x \in \mathbb{R}^n$ with $\|x\| = 1$, then

$$\|T(x)\| \le \|T\|\|x\| = \|T\|.$$

By definition of M_1, we conclude $M_1 \le \|T\|$. $\square$

b) *Proof.* Since T is linear, for $x \neq 0$,

$$\frac{\|T(x)\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right) \right\| \le M_1.$$

where $\|\frac{x}{\|x\|}\| = 1$. $\square$

c) *Proof.* From part a), we know $M_1 \leq \|T\|$. And from part b), taking the supremum of left hand, then by definition of $\|T\|$, we also know

$$\sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \|T\| \leq M_1.$$

Hence we have $M_1 = \|T\|$ at first.

For any $C > 0$, considering

$$\|T(x)\| \leq C\|x\|$$

, we have $\|T\| \leq C$. Then taking the infimum of right hand, we know $\|T\| \leq M_2$. On the other hand, $\|T\|$ is finite and

$$\|T(x)\| \leq \|T\|\|x\|.$$

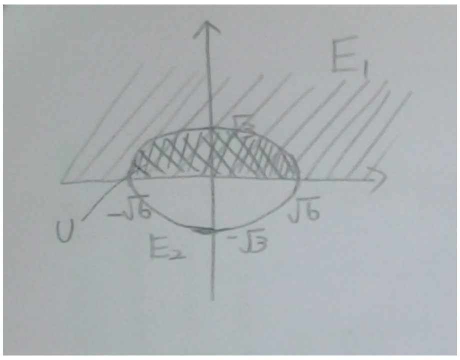
Hence $M_2 \leq \|T\|$, then we have $M_2 = \|T\|$.

Finally, together all the discussions, we conclude $M_1 = M_2 = \|T\|$ as promised. □

8.3 Topology of $\mathbb{R}^n$

Exercise 8.3.4

a) *Sketch.* See the handwriting.



□

b) *Solution.* Consider E_1 and E_2 are subsets of $\mathbb{R}^2$. Since E_1 is closed, hence $U = E_1 \cap E_2$ is **relatively open** in E_1. □

c) *Solution.* Since E_2 is open, hence $U = E_1 \cap E_2$ is **relatively closed** in E_2. □

Exercise 8.3.6

a) *Proof.* ($\Longrightarrow$) Since C is relatively closed in E, then by definition, there exists a closed set B such that $C = E \cap B$. Because E and B are closed, we conclude C is closed.

($\Longleftarrow$) Since $C \subseteq E$ implies $C = C \cap E$, and we know C and E are closed by hypotheses. By definition, we conclude C is relatively closed in E. □

b) *Proof.* ($\Longrightarrow$) Since C is relatively closed in E, there exists a closed set B such that $C = E \cap B$. Consider

$$C \backslash E = C \cap E^c = E \cap (E \cap B)^c = E \cap (E^c \cup B^c) = (E \cap E^c) \cup (E \cap B^c) = E \cap B^c.$$

Since B^c is open, we conclude $E \backslash C$ is relatively open in E.

($\Longleftarrow$) Since $E \backslash C$ is relatively open in E, there exists an open set A such that $E \backslash C = E \cap A$. Consider

$$\begin{aligned} C &= E \cap C = (E \cap E^c) \cup (E \cap C) = E \cap (E^c \cup C) \\ &= E \cap (E \cap C^c)^c = E \cap (E \cap A)^c = E \cap (E^c \cup A^c) \\ &= E \cap A^c. \end{aligned}$$

Since A^c is closed, we conclude C is relatively closed in E. □

Exercise 8.3.7

a) *Proof.* Suppose $A \cup B$ is **not** connected. i.e., there exists nonempty sets U, V are relatively open in $A \cup B$ such that $U \cap V = \phi$ and $U \cup V = A \cup B$.

Case 1. $A \cap U \neq \phi$ and $B \cap V \neq \phi$.

Set $U' := A \cap U$ and $V' := A \cap V$. We claim U' and V' separate $A \cup B$.
Consider $U' \cap V' = (A \cap U) \cap (A \cap V) = A \cap (U \cap V) = A \cap \phi = \phi$. Then since U is relatively open in $A \cup B$, there exists an open set G such that $U = (A \cup B) \cap G$. Moreover,

$$U' = A \cap U = A((A \cup B) \cap G) = A \cap G.$$

Hence U' is relatively open in A so is V' with a similar argument. Finally, we verify

$$U' \cup V' = (A \cap U) \cup (A \cap V) = A \cap (U \cup V) = A \cap (A \cup B) = A.$$

By definition, we conclude U' and V' separate A which implies A is not connected. This leads to a contradiction with hypothesis that A is connected.

Case 2. $A \cap U = \phi$ or $B \cap V = \phi$.

W.L.O.G., we assume $A \cap U = \phi$. Consider

$$\begin{aligned} & A \cup B = U \cup V \\ \Longrightarrow \quad & U \cap (A \cup B) = U \cap (U \cup V) \\ \Longrightarrow \quad & (U \cap A) \cup (U \cap B) = U \\ \Longrightarrow \quad & \phi \cup (U \cap) = U \\ \Longrightarrow \quad & U \cap B = U \\ \Longrightarrow \quad & U \subseteq B. \end{aligned}$$

Set $B_1 := B \cap U$ and $B_2 := B \cap V$. We consider

$$\begin{aligned} B_1 \cup B_2 &= (B \cap U) \cup (B \cap V) \\ &= B \cap (U \cup U) \\ &= B \cap (A \cup B) \\ &= B. \end{aligned}$$

Moreover, with a similar argument from **Case 1.**, we also know B_1 and B_2 are relatively open in B. By the way, $U = U \cap B = B_1 \neq \phi$. Then we should discuss whether $B_2 = \phi$ or not.

If $B_2 \neq \phi$, then since $B_2 = B \cap V$, we claim B_1 and B_2 separate B by definition. Otherwise $B_2 = \phi$, then continue to consider

$$\begin{aligned} & A \cup B = U \cup V \\ \Longrightarrow\ & A \cap (A \cup B) = A \cap (U \cup V) \\ \Longrightarrow\ & A = (A \cap U) \cup (A \cap V) \\ \Longrightarrow\ & A = A \cap V \\ \Longrightarrow\ & A \subseteq V. \end{aligned}$$

Since $B \cap V = \phi$ implies $V \subseteq B^c$, so $A \subseteq B^c$. We conclude $A \cap B = \phi$ which leads to a contradiction with hypothesis that $A \cap B \neq \phi$.

A similar argument proves the condition $B \cap V = \phi$.

Finally, we conclude $A \cap B$ is connected under the hypothesis as promised. □

b) *Proof.* Denote

$$E_k := \bigcup_{i=1}^{k} E_{n_i}.$$

where $n_i \in A$ for $i = 1, 2, \cdots, k$. We pick $n_1 \in A$ arbitrarily, then $E_1 = E_{n_1}$ is connected by hypothesis.

Assume for $n = k$, E_k is connected. Then for $n = k+1$, we pick $n_{k+1} \in A$ arbitrarily, then we know E_k and $E_{n_{k+1}}$ are connected sets. Since $\cap_{\alpha \in A} E_\alpha \neq \phi$, it follows that $E_k \cap E_{n_{k+1}} \neq \phi$. From part a), we have $E_k \cup E_{n_{k+1}} = E_{k+1}$ is also connected. By induction, we know E_k is connected for all $k \in \mathbb{N}$ which implies

$$E = \bigcup_{\alpha \in A} E_\alpha$$

is connected. □

c) *Proof.* Since a subset of $\mathbb{R}$ is connected if and only if E is an interval, we suppose $A = [a_1, b_1]$ and $B = [a_2, b_2]$. Since $A \cap B \neq \phi$, we discuss all conditions which are fulfilled.

Case 1. $a_1 \leq b_1$ and $b_1 \leq a_2 < b_2$

where $A \cap B = [b1, a2] \neq \phi$. Then $A \cup B = [a_1, b_2]$ is an interval, so is connected.

Case 2. $a_1 \leq b_1$ and $a_2 \geq b_2$.

where $A \cap B = [b1, b2] \neq \phi$. Then $A \cup B = [a_1, a_2]$ is an interval, so is connected.

Case 3. $b_1 \le a_1$ and $a_1 \le b_2 < a_2$.
where $A \cap B = [a1, b2] \neq \phi$. Then $A \cup B = [b_1, a_2]$ is an interval, so is connected.

Case 4. $b_1 \le a_1$ and $b_2 \ge a_2$.
where $A \cap B = [a1, a2] \neq \phi$. Then $A \cup B = [b_1, b_2]$ is an interval, so is connected.

Hence, $A \cap B$ is always connected as promised. □

d) *Proof.* The counter-example is that two subsets of $\mathbb{R}^2$ are supposed as following,

$$A := \{(x, y) : -1 \le x \le 1, -|x| \le y \le |x|\}$$
$$B := \{(x, y) : -1 \le x \le 1, -1 \le y \le 1\} \backslash \{(0, 0)\}$$

where $A \cap B = \{(x, y) : -1 \le x \le 1, -|x| \le y \le |x|\} \backslash \{(0, 0)\} \neq \phi$.

We can pick $U := \{(x, y) : x < 0\}$ and $V := \{(x, y) : x > 0\}$ to separate $A \cap B$, so $A \cap B$ is not connected. □

Exercise 8.3.8

a) *Proof.* ($\Longrightarrow$)

For any $x \in V$, since V is open, there exists $r(x) > 0$ such that $B_{r(x)}(x) \subset V$. Then

$$V \subset \bigcup_{x \in V} B_{r(x)}(x) \subset \bigcup_{x \in V} V = V.$$

Hence $V = \bigcup_{\alpha \in A} B_\alpha$.

($\Longleftarrow$)

We replace the notation B_α with V_α. Let $x \in \bigcup_{\alpha \in A} V_\alpha$. Then $x \in V_\alpha$ for some $\alpha \in A$. Since V_α is open, it follows that there is an $r > 0$ such that $B_r(x) \subseteq V_\alpha$. Thus $B_r(x) \subseteq \bigcup_{\alpha \in A} V_\alpha = V$ is open. □

b) *Solution.* We interpret the statement as "Prove that V is **closed** if and only if there is a collection of **closed** balls $\{B_\alpha : \alpha \in A\}$ such that $V = \bigcup_{\alpha \in A} B_\alpha$".

So let $B_k = [\frac{1}{k+1}, \frac{k}{k+1}]$ is closed for each $k \in \mathbb{N}$. Then we know

$$\bigcup_{k \in \mathbb{N}} B_k = (0, 1)$$

is open. Hence the result will fail. □

Exercise 8.3.9

a) *Proof.* Since E is closed, then E^c is open. So for any $a \in E^c$, there exists $r > 0$ such that $B_r(a) \subseteq E^c$. For any $x \in E$, since $x \notin B_r(a)$, then

$$\|x - a\| \geq r > 0.$$

It follows that

$$\inf_{x \in E} \|x - a\| > 0.$$

□

8.4 Interior, Closure, and Boundary

Exercise 8.4.2

a) *Sketch.* See the Figure 8.1.

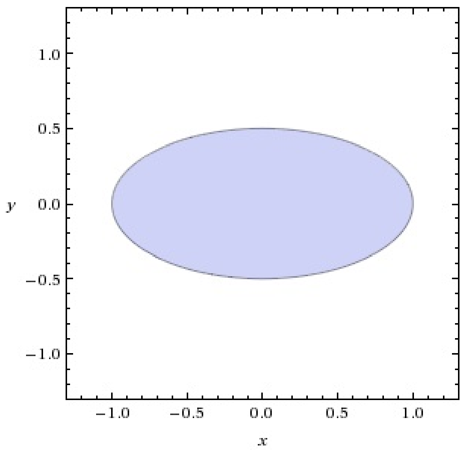


Figure 8.1

$E^\circ = \{(x,y) : x^2 + 4y^2 < 1\}$.
$\overline{E} = E = \{(x,y) : x^2 + 4y^2 \leq 1\}$.
$\partial E = \{(x,y) : x^2 + 4y^2 = 1\}$. □

b) *Sketch.* See the Figure 8.2.

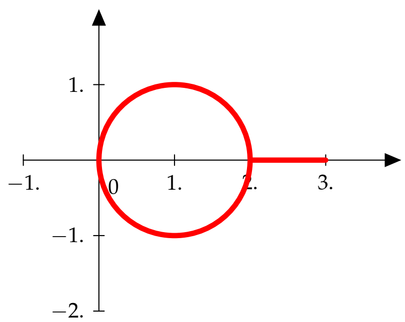


Figure 8.2

$E^\circ = \phi$.
$\overline{E} = E = \{(x,y) : x^2 - 2x + y^2 = 0\} \cup \{(x,0) : x \in [2,3]\}$.
$\partial E = E = \{(x,y) : x^2 - 2x + y^2 = 0\} \cup \{(x,0) : x \in [2,3]\}$. $\square$

c) *Sketch.* See the Figure 8.3.

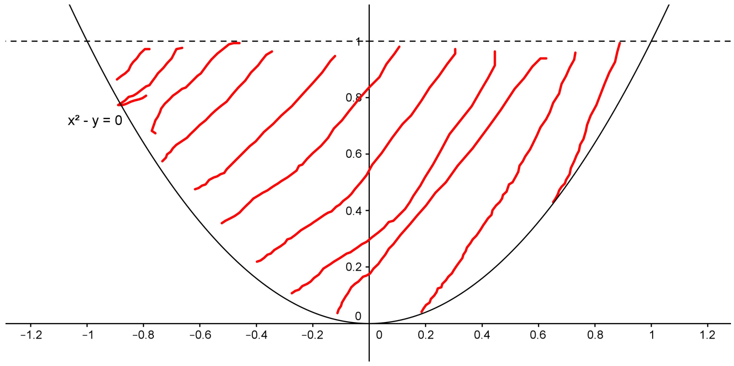


Figure 8.3

$E^\circ = \{(x,y) : y > x^2, 0 < y < 1\}$.
$\overline{E} = \{(x,y) : y \geq x^2, 0 \leq y \leq 1\}$.
$\partial E = \{(x,y) : y = x^2, 0 \leq y \leq 1\} \cup \{(x,1) : -1 \leq x \leq 1\}$. $\square$

d) *Sketch.* See the Figure 8.4.

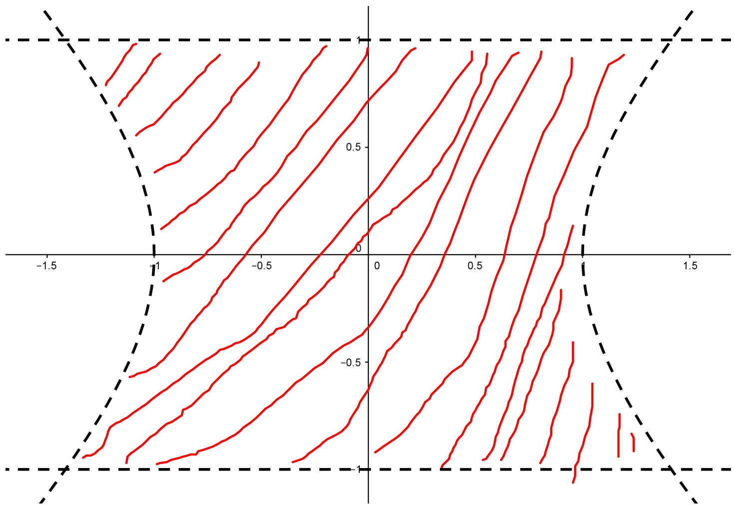


Figure 8.4

$E^\circ = E = \{(x,y) : x^2 - y^2 < 1, -1 < y < 1\}$.
$\overline{E} = \{(x,y) : x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$.
$\partial E = \{(x,y) : x^2 - y^2 = 1, -1 \leq y \leq 1\} \cup \{(x,1) : -\sqrt{2} \leq x \leq \sqrt{2}\} \cup \{(x,-1) : -\sqrt{2} \leq x \leq \sqrt{2}\}$. $\square$

Exercise 8.4.4

a) *Proof.* Let
$$B = \bigcup \{V : V \subseteq A, V \text{ is relatively open in } E\}.$$
It's clear that B is relatively open in E. □

b) *Proof.* Let
$$B = \bigcap \{V : V \subseteq A, V \text{ is relatively closed in } E\}.$$
It's clear that B is relatively closed in E. □

Exercise 8.4.6

Proof. Since E is connected, then we can suppose a, b are extended real numbers, and discuss four situations as following.

Case 1. $E = (a, b)$

Case 2. $E = [a, b)$

Case 3. $E = (a, b]$

Case 4. $E = [a, b]$

Then no matter the case is, we always conclude $E^\circ = (a, b)$ is an interval, so is connected.

If $\mathbb{R}$ is replaced by $\mathbb{R}^2$, we make
$$E = \{(x, y) : -1 \leq x \leq 1, -|x| \leq y \leq |x|\},$$
then
$$E^\circ = \{(x, y) : -1 < x < 1, -|x| < y < |x|\} \backslash \{(0, 0)\}$$
is not connected. Hence the proposition is false. □

Exercise 8.4.9

a) *Solution.* Let $A = [0, 1]$ and $B = (1, 2]$, then
$$(A \cup B)^\circ = (0, 2)$$
$$A^\circ \cup B^\circ = (0, 1) \cup (1, 2).$$
Hence $(A \cup B)^\circ \neq A^\circ \cup B^\circ$. □

b) *Solution.* Let $A = [0, 1]$ and $B = (1, 2]$, then
$$\overline{\mathrm{A} \cap \mathrm{B}} = \phi$$
$$\overline{\mathrm{A}} \cap \overline{\mathrm{B}} = \{1\}$$
Hence $\overline{\mathrm{A} \cap \mathrm{B}} \neq \overline{\mathrm{A}} \cap \overline{\mathrm{B}}$. □

c) *Solution.* Let $A = [0,1]$ and $B = (1,2]$, then

$$\begin{aligned} \partial(A \cup B) &= \{0,2\} \\ \partial A \cup \partial B &= \{0,1,2\} \end{aligned}$$

Hence $\partial(A \cup B) \neq \partial A \cup \partial B$. On the other proposition,

$$\begin{aligned} \partial(A \cap B) &= \phi \\ \partial A \cup \partial B &= \{0,1,2\} \end{aligned}$$

Hence $\partial(A \cap B) \neq \partial A \cup \partial B$. □

Chapter 10

Metric Spaces

10.1 Introduction

Exercise 10.1.2

Proof. ($\Longrightarrow$) There exists a number $M > 0$ and $b \in X$ such that $\rho(x_k, b) \leq M$ for all $k \in \mathbb{N}$. Notice $\rho(b, a)$ is finite since $a, b \in X$. Hence

$$\rho(x_k, a) \leq \rho(x_k, b) + \rho(b, a) \leq M + \rho(b, a) < \infty$$

for all $k \in \mathbb{N}$ which implies

$$\sup_{k \in \mathbb{N}} \rho(x_k, a) < \infty.$$

($\Longleftarrow$) Since

$$\sup_{k \in \mathbb{N}} \rho(x_k, a) < \infty,$$

we choose $M > 0$ such that $\rho(x_k, a) \leq M$ for all $k \in \mathbb{N}$. Then by definition, since $a \in X$, we conclude $\{x_k\}$ is bounded in X. □

Exercise 10.1.5

a) *Proof.* We suppose $\{x_n\}, \{y_n\}$ converges to a point $a \in X$. Given $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies \rho(x_n, a) < \frac{\epsilon}{2}. \\ n \geq N_2 &\implies \rho(y_n, a) < \frac{\epsilon}{2}. \end{aligned}$$

Set $N = \max\{N_1, N_2\}$. For any $n \geq N$,

$$\begin{aligned} \rho\{x_n, y_n\} &\leq \rho(x_n, a) + \rho(a, y_n) \\ &= \rho(x_n, a) + \rho(y_n, a) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $\rho(x_n, y_n) \to 0$ as $n \to \infty$ as promised. □

b) *Solution.* Let X be Euclidean space. Set $x_n = y_n = n$. We observe $\lim_{n\to\infty} \rho(x_n, y_n) = 0$, but both x_n and y_n diverge. □

Exercise 10.1.7

Proof. Pick $\epsilon = \frac{1}{2}$. There exists $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \rho(x_n, x_m) < \epsilon = \frac{1}{2}.$$

By definition of discrete space, $x_n = x_m$ for all $n, m \geq N$. Hence for all $k \geq N$, we have $x_k = a \in \mathbb{R}$. This proves every $\{x_n\}$ converges to a as $n \to \infty$. We conclude the discrete space is complete. $\square$

Exercise 10.1.8

a) *Proof.* Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \epsilon.$$

Taking $m \to \infty$ on both sides of the second inequality, we have

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon.$$

Hence $f_n \to f$ uniformly converges on $[a, b]$. Moreover, since $f_n \in C[a, b]$, we have $f \in C[a, b]$. We conclude the metric space $C[a, b]$ with supremum norm is complete. $\square$

b) *Proof.* We change the notation "$dist(a, b)$" as "$\rho(a, b)$". Suppose $f, g, h \in C[a, b]$.

Positive Definite

$$\rho(f, g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)| dx \geq \int_a^b 0 \, dx = 0.$$

The equality holds if and only if $f(x) = g(x)$ for all $x \in [a, b]$.

Symmetric

$$\rho(f, g) = \|f - g\|_1 = \int_a^b |f(x) - g(x)| dx = \int_a^b |g(x) - f(x)| dx = \|g - f\|_1 = \rho(g, f).$$

Triangle Inequality

$$\begin{aligned}
\rho(f, g) = \|f - g\|_1 &= \int_a^b |f(x) - g(x)| dx \\
&= \int_a^b |f(x) - h(x) + h(x) - g(x)| dx \\
&\leq \int_a^b |f(x) - h(x)| dx + \int_a^b |h(x) - g(x)| dx \\
&= \|f - h\|_1 + \|h - g\|_1 \\
&= \rho(f, h) + \rho(h, g).
\end{aligned}$$

Hence the dist function makes $C[a, b]$ a metric space. $\square$

c) *Proof.* Assume $C[a,b]$ is complete in the metric ρ. Then $\{f_n\}$ converges to some functions which belongs to $C[a,b]$.

Set $f_n = x^n$ and $[a,b] = [0,1]$ and

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}.$$

We know $\rho(f_n, g) \to 0$ as $n \to \infty$ which implies $g \in C[0,1]$. However g is discontinuous at $x = 1$. i.e., $g \notin C[0,1]$. This leads to a contradiction to $g \in C[0,1]$.

Hence we conclude $C[a,b]$ is not complete. □

Exercise 10.1.10

a) *Proof.* Suppose E is sequentially compact and $x_n \in E$ is a convergent sequence.

E is closed
Since E is sequentially compact, then there exists subsequence x_{n_k} of x_n converges to some point in E. Since a sequence can have at most one limit, we know they converge to the same limit called $a \in E$. Hence E is closed.

E is bounded
By contradiction, we suppose E is not bounded. Let $a \in E$ such that $\rho(x_n, a) > n$ for all $n \in \mathbb{N}$. Since E is sequentially compact, we have subsequence x_{n_k} of x_n which converges to a point $b \in E$. i.e., given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n_k \ge N \implies \rho(x_{n_k}, b) < \epsilon.$$

Then for any $n_k \ge N$,

$$\rho(x_{n_k}, a) \le \rho(x_{n_k}, b) + \rho(b, a) < \epsilon + \rho(b, a) < \infty.$$

However $\rho(x_{n_k}, a) > n_k$ which implies $\rho(x_{n_k}, a) \to \infty$ as $k \to \infty$. This leads to a contradiction. Hence E is bounded. Finally, we conclude E is closed and bounded. □

b) *Proof.*

$\mathbb{R}$ is closed
Notice $\mathbb{R}^c = \phi$. Since the empty set contains no points, so every point $x \in \phi$ implies $B_\epsilon(x) \subseteq \phi$ vacuously where $\epsilon > 0$. Then by definition, we know ϕ is open and hence $\mathbb{R}$ is closed.

$\mathbb{R}$ is not sequentially compact
Let $x_n = n$ be a sequence in $\mathbb{R}$. However

$$x_n \to \infty \notin \mathbb{R} \text{ as } n \to \infty.$$

Hence $\mathbb{R}$ is not sequentially compact. □

c) *Proof.* Let $E \subseteq \mathbb{R}$ and E is closed and bounded. Set x_n is a bounded sequence in E.

By the Bolzano-Weierstrass Theorem, there exists a subsequence x_{n_k} of x_n converges to $x_0 \in \mathbb{R}$. i.e., $\lim_{k\to\infty} x_{n_k} = x_0$. Since $x_{n_k} \in E$ and E is closed, we know $x_0 \in E$. Hence E is sequentially compact.

Since E is arbitrary, then we conclude every closed bounded subset of $\mathbb{R}$ is sequentially compact. $\square$

10.2 Limits of Functions

Exercise 10.2.3

Proof. ($\Longrightarrow$) Since a is a cluster point, for any $\frac{1}{n} > 0$, we know $B_{\frac{1}{n}}(a)$ contains infinitely many points. Pick a sequence $x_n \neq a$ for all $n \in \mathbb{N}$ such that $|x_n - a| < \frac{1}{n}$ which implies

$$\lim_{n\to\infty} x_n = a.$$

($\Longleftarrow$) Since $\lim_{n\to\infty} x_n = a$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n - a| < \epsilon.$$

Then $\{x_n\}_{n\geq N} \subseteq B_\epsilon(a)$. The sequence have infinitely many elements. Hence a is a cluster point. $\square$

Exercise 10.2.8

Proof. ($\Longrightarrow$) Since $f_n \to f$ uniformly on $[a, b]$. Hence $f \in C[a, b]$ and there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

for all $x \in [a, b]$. Then for any $n \geq N$,

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \epsilon$$

implies

$$\rho(f_n, f) \leq \epsilon.$$

So $f_n \to f$ in the metric of $C[a, b]$.

($\Longleftarrow$) Since $f_n \to f$ in the metric of $C[a, b]$, we know $f \in C[a, b]$ and there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \sup_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon.$$

implies

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in [a, b]$. Then we conclude $f_n \to f$ uniformly on $[a, b]$. $\square$

Exercise 10.2.9

a) *Proof.* Since x_n is bounded, there exists a subsequence x_{n_k} of x_n which converges to $a \in X$. In particular, since $x_{n_k} \in E$ and E is closed, we have $a \in E$. $\square$

b) *Proof.* By contradiction, we suppose f is not bounded on E. There exists $N \in \mathbb{N}$, for any $n \geq N$, there a sequence $x_n \in E$ such that
$$|f(x_n)| > n.$$
Moreover, since E is closed and bounded in X, from part a), we know there is a subsequence x_{n_k} of x_n such that $x_{n_k} \to a$ as $n \to \infty$ where $a \in E$. Since f is continuous on E, we know $f(x_n) \to f(a)$ as $n \to \infty$. Consider
$$\left|f(x_{n_k})\right| > n_k.$$
Taking the limit of both sides as $k \to \infty$, we have
$$|f(a)| = \infty.$$
However $a \in E$ implies $f(a) \in \mathbb{R}$. This leads a contradiction to $|f(a)| = \infty \notin \mathbb{R}$. Hence f is bounded on E as promised. $\square$

c) *Proof.* Suppose to the contrary $f(x) < M := \sup_{x \in E} f(x)$ for all $x \in E$. Set
$$g(x) = \frac{1}{M - f(x)} > 0$$
is continuous and hence bounded on E. In particular, there is $C > 0$ such that $|g(x)| = g(x) \leq C$. Then
$$f(x) = M - \frac{1}{g(x)} \leq M - \frac{1}{C}.$$
Taking the supremum of both sides over E, we get
$$M \leq M - \frac{1}{C} < M$$
which is a contradiction. Hence there is $x_M \in E$ such that
$$f(x_M) = M = \sup_{x \in E} f(x).$$
A similar argument proves there is $x_m \in E$ such that $f(x_m) = \inf_{x \in E} f(x)$. $\square$

10.3 Interior, Closure, and Boundary

Exercise 10.3.5

Proof. Since E is closed, then E^c is open. So for any $a \in E^c$, there exists $r > 0$ such that $B_r(a) \subseteq E^c$. For any $x \in E$, since $x \notin B_r(a)$, then
$$\rho(x, a) \geq r > 0.$$
It follows that
$$\inf_{x \in E} \rho(x, a) > 0.$$
$\square$

Exercise 10.3.8

a) *Proof.* ($\Longrightarrow$) For any $a \in V$, there exists $r(a) > 0$ such that $B^Y_{r(a)}(a) \subseteq V$. Hence

$$\begin{aligned} V &= \bigcup_{a\in V} B^Y_{r(a)}(a) \\ &= \bigcup_{a\in V} \left(B_{r(a)}(a) \cap Y\right) \\ &= \left(\bigcup_{a\in V} B_{r(a)}(a)\right) \cap Y \end{aligned}$$

where $\bigcup_{a\in V} B_{r(a)}(a)$ is open in X. Set $U := \bigcup_{a\in V} B_{r(a)}(a)$.
We conclude there is an open set U in X such that $V = U \cap V$.

($\Longleftarrow$) For any $a \in V$, since $V = U \cap Y$, then $a \in U$. Moreover, U is open by hypothesis, hence there exists $r > 0$ such that

$$B_r(a) \subseteq U.$$

Then

$$B_r(a) \cap Y = B^Y_r(a) \subseteq V.$$

We conclude V is open in Y. □

b) *Proof.* ($\Longrightarrow$) Since E is closed in Y, then E^c is open in Y. From part a), there is an open set U such that $E^c = U \cap V$. We consider

$$\begin{aligned} E &= E \cap Y \\ &= (U \cap V)^c \cap Y \\ &= (U^c \cup V^c) \cap Y \\ &= U^c \cap Y \end{aligned}$$

We know U^c is closed and hence set $A := U^c$ as desired.

($\Longleftarrow$) Considering $E = A \cap Y$ where A is closed by hypothesis, we know $E \subseteq Y$. Moreover,

$$E^c = A^c \cup Y^c$$

implies

$$E^c \cap Y = A^c \cap Y.$$

Then $E^c \cap Y$ is open in Y. With $E \subseteq Y$, we conclude E is closed in Y. □

10.4 Compact Sets

Exercise 10.4.1

a) *Solution.* E is bounded by 1 obviously. Every subsequence of E converges to $0 \in E$, so E is closed. Since this is Euclidean space, then we conclude E is **compact**. □

b) *Solution.* Sketch the graph. So we know E is bounded by b and closed. Since the space is $\mathbb{R}^2$, we conclude E is **compact**. $\square$

c) *Solution.* Set $(x_n, y_n) = \left(\frac{1}{2n\pi}, \sin(2n\pi)\right)$. Then

$$(x_n, y_n) \to (0,0) \notin E \text{ as } n \to \infty.$$

Regardless of the boundedness, it is not closed. Hence E is **not compact**.

We can find $H = E \cup \{(0,y) : -1 \leq y \leq 1\}$ which is compact. $\square$

d) *Solution.* E is not bounded, so is **not compact**. Moreover, we can not find H as well. $\square$

Exercise 10.4.3

Proof. Since E is compact, then E is bounded and closed. Since E is bounded and nonempty, then $\sup E$ and $\inf E$ exist. There exists a sequence x_n such that $x_n \to \sup E$ as $n \to \infty$. Since E is closed, then

$$\lim_{n\to\infty} x_n = \sup E \in E.$$

A similar argument proves that $\inf E \in E$. $\square$

Exercise 10.4.5

Proof.

- **The First Proposition**

  Since V is open, for any $a \in V$, pick $\delta_a > 0$ such that $B_\delta(a) \subseteq V$. Moreover,

  $$V = \bigcup_{a\in V} B_{\delta_a}(a).$$

  Since $\{B_{\delta_a}(a)\}_{a\in V}$ is a collection of open sets, by Lindelöf's Theorem, there exists a countable subset V_0 of V such that

  $$\bigcup_{a\in V} B_{\delta_a}(a) \subseteq \bigcup_{a\in V_0} B_{\delta_a}(a).$$

  A similar argument proves that the reverse containment relation holds. So

  $$\bigcup_{a\in V} B_{\delta_a}(a) = \bigcup_{a\in V_0} B_{\delta_a}(a)$$

  which implies that there are open balls $B_1, B_2, \cdots$ such that

  $$V = \bigcup_{j\in\mathbb{N}} B_j. \tag{10.1}$$

- **The Second Proposition**

  Since $\mathbb{R}$ is a separable metric space. Let V be an open set in $\mathbb{R}$, then (10.1) holds. We know B_j for each $j \in \mathbb{N}$ is an open ball in $\mathbb{R}$ and there exists $x_j \in V$ and $r_j > 0$ such that
  $$B_j = (x_j - r_j, x_j + r_j)$$
  which is an open interval.

  Since V is arbitrary, we conclude every open set in $\mathbb{R}$ is a countable union of open intervals.

□

Exercise 10.4.7

Proof. We denote
$$f(x) := \text{dist}(x, B) = \{\rho(x, y) : y \in B\}.$$

Since A, B are compact and X has Bolzano-Weierstrass Property, then A and B are closed and bounded.

- **Claim f is continuous on A**

  Given $\epsilon > 0$, for any $y \in B$, there exists $\delta = \epsilon > 0$ and $x_1, x_2 \in A$ such that $\rho(x_1, x_2) < \delta$. Then
  $$\begin{array}{rlcl} & \rho(x_1, y) \leq \rho(x_1, x_2) + \rho(x_2, y) & \text{and} & \rho(x_2, y) \leq \rho(x_2, x_1) + \rho(x_1, y) \\ \Longrightarrow & f(x_1) \leq \rho(x_1, x_2) + f(x_2) & \text{and} & f(x_2) \leq \rho(x_1, x_2) + f(x_1) \\ \Longrightarrow & f(x_1) - f(x_2) \leq \rho(x_1, x_2) & \text{and} & f(x_2) - f(x_1) \leq \rho(x_1, x_2) \end{array}$$
  Hence,
  $$|f(x_1) - f(x_2)| \leq \rho(x_1, x_2) < \delta = \epsilon.$$
  It follows that f is continuous on A.

Since A is bounded and closed, and f is continuous on A, by the Extreme Value Theorem, there exists $x_0 \in A$ such that $f(x_0) = \inf_{x \in A} f(x)$.
Since $A \cap B = \phi$, for any $x \in A$ implies $x \notin B$.
Since B is closed, from exercise 10.3.5, we know $f(x) > 0$ for all $x \in A$.
For any $x \in A$, we know
$$\rho(x, y) \geq f(x) \geq f(x_0) > 0$$
for all $y \in B$. Hence $\rho(x, y) > 0$ for all $x \in A$ and $y \in B$.
By definition of dist function, we conclude
$$\text{dist}(A, B) > 0.$$

- **If $\mathbf{A}, \mathbf{B}$ are only closed**

  Let
  $$A := \{(x, 1) : x \geq 0\}$$
  $$B := \left\{(x, y) : y = \frac{x}{1 + x},\ x \geq 0\right\}.$$
  i.e., A, B are not bounded. We know $A \cap B = \phi$ as desired. However, set $(x, y) \in B$. $y \to 1$ as $x \to \infty$. Pick $(x, 1) \in A$ where x is the same. In this view, $\text{dist}(A, B) = 0$.

□

Exercise 10.4.8

a) *Proof.* W.L.O.G., we assume
$$H_1 \subset H_2 \subset \cdots .$$
Suppose to contrary that
$$\bigcap_{k=1}^{\infty} H_k = \phi. \tag{10.2}$$
For each $k \in \mathbb{N}$, we set $V_k = X \backslash H_k$. Taking the complement of both sides on (10.2), we have
$$X = \bigcup_{k=1}^{\infty} V_k.$$
Then $\{V_k\}$ is an open covering of X. Since X is compact, we extract a finite subcover, say $V_1 \supset V_2 \supset \cdots$ and $V_i = X$ for some $i \in \mathbb{N}$. It follows that $H_i = X \backslash V_i = \phi$ which is a contradiction to H_i is nonempty by hypothesis.

Hence, we conclude
$$\bigcap_{k=1}^{\infty} H_k \neq \phi.$$
□

b) *Proof.*
- $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ **is closed and bounded**

  $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is bounded trivially. Let $x_n \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ and $x_n \to a \in \mathbb{Q}$, then by the limit comparison theorem,
  $$\sqrt{2} \leq a \leq \sqrt{3}.$$
  Since $a \in \mathbb{Q}$, then
  $$\sqrt{2} < a < \sqrt{3}$$
  which implies $a \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ and hence $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is closed.
- $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ **is not compact**

  Let $\{x_n\} \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ with $x_1 < x_2 < \cdots$. Set $x_0 := 0$. Let
  $$r_k = \min\left\{\frac{x_{k+1} - x_k}{2}, \frac{x_k - x_{k-1}}{2}\right\}.$$
  At first, we know
  $$(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q} \subset \bigcup_{k=1}^{\infty} B_{r_k}(x_k).$$
  Secondly, we show that open balls are disjoint to each other. Suppose to contrary that they are not disjoint to each other, W.L.O.G, we assume $i < j$, then there is a point $x \in B_{r_i}(x_i) \cap B_{r_j}(x_j)$. Hence,
  $$|x - x_i| < r_i \text{ and } |x - x_j| < r_j.$$

which implies

$$x < x_i + r_i \leq x_i + \frac{x_{i-1} - x_i}{2} = \frac{x_i + x_{i-1}}{2};$$
$$x > x_j - r_j \geq x_j - \frac{x_j - x_{j-1}}{2} = \frac{x_j + x_{j-1}}{2}.$$

Since the index $i < j$, and then

$$x < \frac{x_i + x_{i-1}}{2} < \frac{x_j + x_{j-1}}{2} < x.$$

which is a contradiction.
If $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is compact, then there exists $N \in \mathbb{N}$ such that

$$(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q} \subseteq \bigcup_{k=1}^{N} B_{r_k}(x_k).$$

But the open balls are disjoint, so there is a point $x \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ such that

$$x \notin \bigcup_{k=1}^{N} B_{r_k}(x_k).$$

We conclude $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is not compact.

□

c) *Solution.* Set $H_k := \left(\sqrt{2} - \frac{1}{k}, \sqrt{2} + \frac{1}{k}\right) \cap \mathbb{Q}$ for all $k \in \mathbb{N}$. A similar argument in part b) can get H_k is closed and bounded for each $k \in \mathbb{N}$. Notice that

$$H_1 \subset H_2 \subset \cdots.$$

We know

$$\bigcap_{k=1}^{\infty} H_k = \left\{\sqrt{2}\right\} \text{ and } \bigcap_{k=1}^{\infty} H_k \subseteq \mathbb{Q}.$$

Since $\sqrt{2} \notin \mathbb{Q}$, this leads to a contradiction. Hence,

$$\bigcap_{k=1}^{\infty} H_k = \phi$$

might hold. □

10.5 Connected Sets

Exercise 10.5.4

Proof. Since E is connected, then we can suppose a, b are extended real numbers, and discuss four situations as following.

Case 1. $E = (a, b)$

Case 2. $E = [a, b)$

Case 3. $E = (a, b]$

Case 4. $E = [a, b]$

Then no matter the case is, we always conclude $E^\circ = (a, b)$ is an interval, so is connected.

If $\mathbb{R}$ is replaced by $\mathbb{R}^2$, we make

$$E = \{(x, y) : -1 \leq x \leq 1, -|x| \leq y \leq |x|\},$$

then

$$E^\circ = \{(x, y) : -1 < x < 1, -|x| < y < |x|\} \backslash \{(0, 0)\}$$

is not connected. Hence the proposition is false. □

Exercise 10.5.6

Proof. Since X is compact, there exists $N \in \mathbb{N}$ and points $x_k \in X$ for all $k \in \{1, 2, \cdots, N\}$ such that

$$\bigcup_{k=1}^{N} B_{x_k} = X. \tag{10.3}$$

Suppose to contrary that f is not constant on X. W.L.O.G, we pick $a \in Y$, and assume $f(x_1) \neq a$ but $f(x_2) = f(x_3) = \cdots = f(x_N) = a$. Since $f(x_1) \neq f(x_j)$ for all $j \in \{2, 3, \cdots, N\}$. Consider

$$B_{x_1} \cap \left(\bigcup_{j=2}^{N} B_{x_j} \right) = \phi.$$

Let $U := B_{x_1}$ and $V := \bigcup_{j=2}^{N} B_{x_j}$. We know both U and V are nonempty and open in X. Also, $U \cap V = \phi$, and $U \cup V = X$ by (10.3). This implies X is not connected which is a contradiction to X is connected by hypothesis.

Hence, we conclude f is constant on X. □

Exercise 10.5.7

Proof. $(\Longrightarrow)$

By Exercise 10.3.8, we know for each $\{E_\alpha\}$, there is a open set F_α in X such that

$$E_\alpha = F_\alpha \cap H.$$

Notice that $\{F_\alpha\}_{\alpha \in A}$ is an open covering of H. Since H is compact, there exists finite subset A_0 of A such that

$$H \subseteq \bigcup_{\alpha \in A_0} F_\alpha.$$

Hence,

$$H \subseteq H \cap \bigcup_{\alpha \in A_0} F_\alpha = \bigcup_{\alpha \in A_0} F_\alpha \cap H = \bigcup_{\alpha \in A_0} E_\alpha.$$

which means H has a finite subcover $\{E_\alpha\}_{\alpha \in A_0}$.

($\Longleftarrow$) Let $\{F_\beta\}_{\beta \in B}$ be an open covering of H, Then

$$H \subseteq \bigcup_{\beta \in B} F_\beta.$$

Hence,

$$H \subseteq H \cap \bigcup_{\beta \in B} F_\beta = \bigcup_{\beta \in B} F_\beta \cap H$$

which implies $F_\beta \cap H$ is an open covering of H. Since $F_\beta \cap H$ are relatively open in X for all $\beta \in B$, by hypothesis, we know there is a subset B_0 of B such that

$$H \subseteq \bigcup_{\beta \in B_0} F_\beta \cap H \subseteq \bigcup_{\beta \in B_0} F_\beta.$$

This conclude every cover of H has a finite subcover, which means H is compact. □

10.6 Continuous Functions

Exercise 10.6.3

Proof. ($\Longrightarrow$)

For any closed set C in Y, C^c is open. By Theorem 10.58, $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$ is open, so $f^{-1}(C)$ is closed. Since C is arbitrary, we have $f^{-1}(C)$ is closed in Y for every closed C in Y.

($\Longleftarrow$)

Let V be an open set in Y, then V^c is closed. By hypothesis, $f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$ is closed in X. Hence $f^{-1}(V)$ is open in X. Since V is arbitrary, we have $f^{-1}(V)$ is open in X for every open V in Y. By Theorem 10.58, f is continuous. □

Exercise 10.6.5

Proof. Since E is connected and $f : E \to \mathbb{R}$ is continuous, then $f(E)$ is connected. Since $f(a) \neq f(b)$, W.L.O.G., we assume $f(a) < f(b)$.

Suppose to contrary that for all $x \in E$, there is a number $y \in (f(a), f(b))$ such that $f(x) \neq y$. Then $(f(a), y) \cup (y, f(b)) \subset f(E)$ is not connected which is a contradiction to that $f(E)$ is connected.

Therefore, there is an $x \in E$ such that $f(x) = y$. □

Exercise 10.6.6

a) *Proof.* Since f is continuous and H is compact, then Y is compact. Moreover by hypothesis, Y is Euclidean space, then Y is bounded and closed.

Since Y is bounded, then
$$\sup_{x \in H} \|f(x)\|_Y$$
is finite.

Y has a finite supremum, hence, there exists a sequence $x_n \in Y$ such that
$$x_n \to \sup Y = \sup_{x \in H} \|f(x)\|_Y \text{ as } n \to \infty.$$
Because Y is closed, $\sup Y \in Y$. It follows that there exists $x_0 \in H$ such that
$$\|f(x_0)\|_Y = \sup_{x \in H} \|f(x)\|_Y = \|f\|_H.$$
□

b) *Proof.* ($\Longrightarrow$)

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n \geq N \implies \|f_n - f\|_H < \epsilon.$$
Let $k \geq N$ and $x \in H$, Then
$$\|f_k(x) - f(x)\|_Y \leq \sup_{x \in H} \|f_k(x) - f(x)\|_Y = \|f_k - f\|_H < \epsilon.$$
By definition, $f_k \to f$ uniformly on H as $k \to \infty$.

($\Longleftarrow$)

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$k \geq N \text{ and } x \in H \implies \|f_k - f\| < \frac{\epsilon}{2}.$$
Let $n \geq N$, $\sup_{x \in H} \|f_k - f\|_Y \leq \frac{\epsilon}{2}$. Then
$$\|f_k - f\|_H = \sup_{x \in H} \|f_k - f\|_Y \leq \frac{\epsilon}{2} < \epsilon$$
which means $\|f_k - f\|_H \to 0$ as $k \to \infty$. □

c) *Proof.* ($\Longrightarrow$)

Given $\epsilon > 0$,
$$\exists N_1 \in \mathbb{N} \text{ such that } k \geq N_1 \text{ and } x \in H \implies \|f_k(x) - f(x)\|_Y < \frac{\epsilon}{4};$$
$$\exists N_2 \in \mathbb{N} \text{ such that } j \geq N_2 \text{ and } x \in H \implies \|f_j(x) - f(x)\|_Y < \frac{\epsilon}{4}.$$

They follow that

$$\sup_{x\in H} \|f_k(x) - f(x)\|_Y \le \frac{\epsilon}{4} < \frac{\epsilon}{2};$$
$$\sup_{x\in H} \|f_j(x) - f(x)\|_Y \le \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. We know for $k, j \ge N$ and $x \in H$, then

$$\begin{aligned}\|f_k - f_j\|_H &\le \|f_k - f\|_H + \|f_j - f\|_H \\ &= \sup_{x\in H} \|f_k - f\|_Y + \sup_{x\in H} \|f_j - f\|_Y \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

($\Longleftarrow$)
Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$k, j \ge N \implies \|f_k - f_j\|_H < \epsilon.$$

Consider

$$\|f_k - f_j\|_Y \le \sup_{x\in H} \|f_k - f_j\|_Y = \|f_k - f_j\|_H < \epsilon.$$

Hence, $\{f_k(x)\}$ is Cauchy for all $x \in H$. Since Euclidean space is complete, then

$$f(x) = \lim_{k\to\infty} f_k(x).$$

It follows that f_k converges uniformly on H. □

Exercise 10.6.9

Proof. Suppose to contrary that X has most countably many points.

Since f is continuous, then $f(X)$ also has most countably many points. Also f is not constant, i.e., $f(X)$ has at least two points. Since X is connected and f is continuous, then $f(X)$ is connected on $\mathbb{R}$.

Now we know $f(X)$ is one of following forms,

$$(a, b) \text{ or } [a, b) \text{ or } (a, b] \text{ or } [a, b]$$

where $a < b$ and a, b are extended real numbers. However all of these forms have uncountably many points which is a contradiction to the supposition.

We conclude X has uncountably many points. □

Chapter 11

Differentiability on $\mathbb{R}^n$

11.1 Partial Derivatives and Partial Integrals

Exercise 11.1.2

a) *Proof.* For $(x,y) \neq (0,0)$, we have

$$f_x(x,y) = \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2}.$$

f_x is continuous on $\mathbb{R}^2$ except $(0,0)$ obviously, then we check for the continuity of $(0,0)$.
For $(x,y) = (0,0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 - 0}{h} = 0.$$

For $(x,y) \neq (0,0)$, consider

$$\begin{aligned}
|f_x(x,y) - f_x(0,0)| &= \left| \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2} \right| \\
&= \left| \frac{4x^3y^2}{(x^2+y^2)^2} + \frac{2x(x^4 - y^4)}{(x^2+y^2)^2} \right| \\
&\leq \left(\frac{4|x^3y^2|}{(x^2+y^2)^2} \right) + \left(\frac{2|x|(|x^4| + |y^4|)}{(x^2+y^2)^2} \right) \\
&\leq |x| + |x| \\
&= 2|x| \to 0 \text{ as } (x,y) \to (0,0).
\end{aligned}$$

Hence, f_x is continuous at $(0,0)$. Then we conclude f_x is continuous on $\mathbb{R}^2$. □

b) *Proof.* For $(x,y) \neq (0,0)$, we have

$$f_x(x,y) = \frac{4x^3 + 8xy^2}{3(x^2+y^2)^{\frac{4}{3}}}.$$

f_x is continuous on $\mathbb{R}^2$ except $(0,0)$ obviously, then we check for the continuity of $(0,0)$.
For $(x,y) = (0,0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^{\frac{4}{3}} - 0}{h} = 0.$$

For $(x,y) \neq (0,0)$, consider

$$\begin{aligned}
|f_x(x,y) - f_x(0,0)| &\leq \frac{8x(x^2+y^2)}{3(x^2+y^2)^{\frac{4}{3}}} \\
&= \frac{8|x|}{3(x^2+y^2)^{\frac{1}{3}}} \\
&\leq \frac{8}{3} \left| \frac{x}{(x^2)^{\frac{1}{3}}} \right| \\
&= \frac{8}{3}|x|^{\frac{1}{3}} \to 0 \text{ as } (x,y) \to (0,0).
\end{aligned}$$

Hence, f_x is continuous at $(0,0)$. Then we conclude f_x is continuous on $\mathbb{R}^2$. $\square$

Exercise 11.1.4

Proof. For each $y \in [c,d]$, $f(\cdot,y)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$, so is fg by the hypothesis that g is integrable on $[a,b]$. Hence, $F(y)$ exists for all $y \in [c,d]$.

$|g|$ is integrable and then set

$$M := \int_a^b |g(x)|\,dx < \infty.$$

Fix $y_0 \in [c,d]$, given $\epsilon > 0$, since H is compact and f is continuous on H, then f is uniformly continuous on H. Hence there is $\delta > 0$ such that

$$\|(x_1,y_1)-(x_2,y_2)\| < \delta \text{ and } (x_1,y_1),\ (x_2,y_2) \in H \implies |f(x_1,y_1)-f(x_2,y_2)| < \frac{\epsilon}{M+1}.$$

Because $|(y,y_0)| = \|(x,y)-(x,y_0)\|$, for all $y \in [c,d]$ with $|y-y_0| < \delta$,

$$\begin{aligned} |F(y)-F(y_0)| &\le \int_a^b |g(x)||f(x,y)-f(x,y_0)|\,dx \\ &< \frac{M}{M+1}\epsilon \\ &< \epsilon. \end{aligned}$$

Hence, F is continuous on $[c,d]$. Moreover, $[c,d]$ is compact, then we conclude F is uniformly continuous on $[c,d]$. $\square$

Exercise 11.1.5

a) *Solution.* Let $f(x,y) = e^{x^3y^2+x}$ and f is continuous on $H = [0,1]\times[0,1]$.

By Theorem 11.4, we know

$$\lim_{y\to 0}\int_0^1 e^{x^3y^2+x}\,dx = \int_0^1 \lim_{y\to 0} e^{x^3y^2+x}\,dx = \int_0^1 e^x\,dx = e-1.$$

$\square$

b) *Solution.* Let $f(x,y) = \sin(e^x y - y^3 + \pi - e^x)$ and $H = [0,1]\times[0,1]$.

For each $y \in [0,1]$, $f(\cdot,y)$ is continuous on $[0,1]$, and hence integrable. We calculate $f_y(x,y) = \cos(\pi-1)(e^x-3)$, then $f_y(x,\cdot)$ exists on $[c,d]$, and $f_y(x,y)$ is continuous on H.

By Theorem 11.5, we have

$$\begin{aligned} \left.\frac{d}{dy}\right|_{y=1}\int_0^1 \sin(e^x y - y^3+\pi-e^x)\,dx &= \int_0^1 \left.\frac{\partial f}{\partial y}\right|_{y=1}\sin(e^x y - y^3+\pi-e^x)\,dx \\ &= \int_0^1 \cos(e^x y - y^3+\pi-e^x)(e^x-3)\Big|_{y=1}\,dx \\ &= \int_0^1 \cos(\pi-1)(e^x-3)\,dx \\ &= \cos(\pi-1)(e-4). \end{aligned}$$

$\square$

c) *Solution.* Let $f(x,y,z) = \sqrt{x^3+y^3+z^3-2}$ and $H = [1,3]\times[1,3]\times[1,3]$.
Since

$$\begin{aligned} f_x(x,y,z) &= \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}; \\ f_y(x,y,z) &= \frac{3y^2}{2\sqrt{x^3+y^3+z^3-2}}; \\ f_z(x,y,z) &= \frac{3z^2}{2\sqrt{x^3+y^3+z^3-2}}, \end{aligned}$$

we know $f \in C^1(H)$.
Therefore, by Theorem 11.5, we have

$$\begin{aligned} \frac{\partial}{\partial x}\bigg|_{(x,y)=(1,1)} \int_1^3 \sqrt{x^3+y^3+z^3-2}\, dz &= \int_1^3 \frac{\partial}{\partial x}\bigg|_{(x,y)=(1,1)} \sqrt{x^3+y^3+z^3-2}\, dz \\ &= \int_1^3 \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}\bigg|_{(x,y)=(1,1)} dz \\ &= \frac{3}{2}\int_1^3 z^{-\frac{3}{2}}\, dz \\ &= -3\, z^{-\frac{1}{2}}\bigg|_1^3 \\ &= 3-\sqrt{3}. \end{aligned}$$

$\square$

11.2 The Definition of Differentiability

Exercise 11.2.2

Proof. Consider

$$\lim_{x\to a}\frac{\|f(x)\|}{\|g(x)\|} = \lim_{x\to a}\frac{\frac{\|f(x)\|}{x-a}}{\frac{\|g(x)\|}{x-a}}.$$

Then

$$\begin{aligned}
\lim_{x\to a}\frac{\|f(x)\|}{x-a} &= \lim_{x\to a}\left\|\frac{f(x)}{x-a}\right\| \\
&= \lim_{x\to a}\left\|\frac{f(x)-f(a)}{x-a}\right\| \\
&= \lim_{x\to a}\left\|\frac{f(x)-f(a)-Df(a)(x-a)}{x-a}+Df(a)\right\| \\
&= \left\|\lim_{x\to a}\left(\frac{f(x)-f(a)-Df(a)(x-a)}{x-a}+Df(a)\right)\right\| \\
&= \|Df(a)\|.
\end{aligned}$$

A similar argument establishes

$$\lim_{x\to a}\frac{\|g(x)\|}{x-a} = \|Dg(a)\|.$$

We conclude

$$\lim_{x\to a}\frac{\|f(x)\|}{\|g(x)\|} = \frac{\|Df(a)\|}{\|Dg(a)\|}.$$

□

Exercise 11.2.3

Proof. Calculate

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Similarly, $f_y(0,0) = 0$. We know $Df(0,0) = (0,0)$, then we calculate

$$\frac{f(h,k)-f(0,0)-Df(0,0)\cdot(h,k)}{\|(h,k)\|} = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} = \frac{\sqrt{t}}{\sqrt{1+t^2}}$$

with $k = ht$. This depends on t. Hence, f is not differentiable at $(0,0)$. □

Exercise 11.2.4

Proof. Calculate

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h}{\sin|h|}.$$

Moreover,

$$\begin{aligned}
\lim_{h\to 0+}\frac{h}{\sin|h|} &= \lim_{h\to 0+}\frac{1}{\cos h} = 1; \\
\lim_{h\to 0-}\frac{h}{\sin|h|} &= \lim_{h\to 0-}\frac{1}{-\cos h} = -1.
\end{aligned}$$

Since the left limit doesn't equal to the right limit, hence, the limit doesn't exist.

Since the partial derivative of f at $(0,0)$ doesn't exist, so doesn't the total derivative. □

Exercise 11.2.5

Proof. Notice that $\alpha < \frac{3}{2}$, we calculate

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} h^{3-2\alpha} = 0.$$

Similarly, $f_y(0,0) = 0$. We know $Df(0,0) = (0,0)$, then we calculate

$$\frac{f(h,k) - f(0,0) - Df(0,0)\cdot(h,k)}{\|(h,k)\|} = \left(h^4 + k^4\right)\left(h^2 + k^2\right)^{\frac{1}{2}-\alpha} \to 0 \text{ as } (h,k) \to (0,0).$$

Hence, f is differentiable at $(0,0)$.

For $(x,y) = \mathbb{R}^2 \backslash (0,0)$, we calculate

$$\begin{aligned} f_x(x,y) &= \frac{4x^3(x^2+y^2) - 2\alpha x(x^4+y^4)}{(x^2+y^2)^{\alpha+1}}; \\ f_y(x,y) &= \frac{4y^3(x^2+y^2) - 2\alpha y(x^4+y^4)}{(x^2+y^2)^{\alpha+1}}, \end{aligned}$$

are continuous on $\mathbb{R}^2\backslash(0,0)$, and hence f is differentiable on $\mathbb{R}^2\backslash(0,0)$.

We conclude f is differentiable on $\mathbb{R}^2$ for all $\alpha < \frac{3}{2}$. □

Exercise 11.2.6

Proof. Calculate

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Similarly, $f_y(0,0) = 0$. We know $Df(0,0) = (0,0)$, then we calculate

$$\begin{aligned} \left|\frac{f(h,k) - f(0,0) - Df(0,0)\cdot(h,k)}{\|(h,k)\|}\right| &= \frac{|hk|^\alpha \left|\log(h^2+k^2)\right|}{\sqrt{h^2+k^2}} \\ &\le \frac{|\frac{h^2+k^2}{2}|^\alpha \left|\log(h^2+k^2)\right|}{\sqrt{h^2+k^2}} \\ &= \frac{1}{2^\alpha} \frac{|h^2+k^2|^\alpha \left|\log(h^2+k^2)\right|}{\sqrt{h^2+k^2}} \\ &\le \frac{1}{2^\alpha} \frac{|h^2+k^2|^\alpha \left|(h^2+k^2)\right|}{\sqrt{h^2+k^2}} \\ &= \frac{1}{2^\alpha}(h^2+k^2)^{\alpha+\frac{1}{2}} \to 0 \text{ as } (h,k) \to (0,0). \end{aligned}$$

where $\alpha > \frac{1}{2}$. Hence, f is differentiable at $(0,0)$. □

Exercise 11.2.8

Proof. Let $f(x) = Tx$, by hypothesis, we have

$$\begin{aligned}\epsilon(h) &= f(a+h) - f(a) - DT(a)h \\ &= T(a+h) - T(a) - Th \\ &= Ta + Th - Ta - Th \\ &= 0.\end{aligned}$$

Therefore, $\frac{\epsilon(h)}{\|h\|} \to 0$ as $h \to 0$. By definition, T is differentiable on $\mathbb{R}^n$. $\square$

Exercise 11.2.9

Proof. Since $|f(x)| \leq \|x\|^\alpha$, then $f(0) = 0$. Consider

$$\begin{aligned}\left|\frac{(f(h) - f(0) - \nabla f(0) \cdot h}{\|h\|}\right| &= \frac{|f(h)|}{\|h\|} \\ &\leq \frac{\|h\|^\alpha}{\|h\|} \\ &= \|h\|^{\alpha-1} \to 0 \text{ as } h \to 0.\end{aligned}$$

Hence, f is differentiable at 0.

- When $\alpha = 1$
  Let $f(x) = \|x\|$, then f is not differentiable at 0. Hence f might not be differentiable.

$\square$

Exercise 11.2.11

a) *Proof.* Since f_x, f_y exist in $B_r(a,b)$, we set $|h| < \frac{r}{\sqrt{2}}$.

Moreover, since f_y is differentiable at (a,b), by the Mean Value Theorem, there is $k \in (0,h)$ such that

$$\begin{aligned}f(a+h, b+h) - f(a+h, b) &= f_y(a+h, b+k)((b+h) - b); \\ f(a, b+h) - f(a,b) &= f_y(a, b+k)((b+h) - b).\end{aligned}$$

Then substituting k with th where $t \in (0,1)$, we have

$$\begin{aligned}\Delta(h) &= (f(a+h, b+h) - f(a+h, b)) - (f(a, b+h) - f(a,b)) \\ &= f_y(a+h, b+k)h - f_y(a, b+k)h \\ &= h\left(f_y(a+h, b+th) - f_y(a, b+th)\right).\end{aligned}$$

We know $\nabla f_y(a,b) = (f_{yx}(a,b), f_{yy}(a,b))$ exists by hypothesis, hence

$$\nabla f_y(a,b)\cdot(h,th) - \nabla f_y(a,b)\cdot(0,th) = hf_{yx}(a,b).$$

We conclude

$$\begin{aligned}
\frac{\Delta(h)}{h} &= f_y(a+h,b+th) - f_y(a,b+th) \\
&= f_y(a+h,b+th) - f_y(a,b+th) - f_y(a,b) + f_y(a,b) \\
&\quad + hf_{yx}(a,b) - \big(\nabla f_y(a,b)\cdot(h,th) - \nabla f_y(a,b)\cdot(0,th)\big) \\
&= f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th) \\
&\quad - \big(f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th)\big) + hf_{yx}(a,b)
\end{aligned}$$

for some $t \in (0,1)$. □

b) *Proof.* Let

$$\begin{aligned}
\epsilon_1(h) &:= f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th); \\
\epsilon_2(h) &:= f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th),
\end{aligned}$$

imply

$$\begin{aligned}
\left|\frac{\epsilon_1(h)}{h}\right| &= \left|\frac{f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th)}{\|(h,th)\|} \times \frac{\|(h,th)\|}{h}\right| \\
&\le \left|\frac{f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th)}{\|(h,th)\|}\right|; \qquad (11.1) \\
\left|\frac{\epsilon_2(h)}{h}\right| &= \left|\frac{f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th)}{\|(0,th)\|} \times \frac{\|(0,th)\|}{h}\right| \\
&\le \left|\frac{f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th)}{\|(0,th)\|}\right|. \qquad (11.2)
\end{aligned}$$

Since f_y is differentiable at (a,b), we have

$$\begin{aligned}
&\lim_{h\to 0}\frac{f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th)}{\|(h,th)\|} = 0; \qquad (11.3) \\
&\lim_{h\to 0}\frac{f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th)}{\|(0,th)\|} = 0. \qquad (11.4)
\end{aligned}$$

Taking limit of both sides on (11.1) and (11.2) as $h \to 0$, by the Squeeze Theorem from (11.3) and (11.4), we have

$$\lim_{h\to 0}\frac{\epsilon_1(h)}{h} = 0, \text{ and } \lim_{h\to 0}\frac{\epsilon_2(h)}{h} = 0.$$

Finally, we conclude

$$\lim_{h\to 0}\frac{\Delta(h)}{h^2} = \lim_{h\to 0}\left(\frac{\epsilon_1(h)}{h} - \frac{\epsilon_2(h)}{h} + f_{yx}(a,b)\right) = f_{yx}(a,b).$$

□

c) *Proof.* By a similar argument from part a), we have

$$\begin{aligned}\frac{\Delta(h)}{h} &= f_x(a+th, b+h) - f_x(a,b) - \nabla f_x(a,b)\cdot(th,h) \\ &\quad - (f_x(a+th, b+) - f_x(a,b) - \nabla f_x(a,b)\cdot(th,0)) + hf_{xy}(a,b)\end{aligned}$$

for some $t \in (0,1)$.

Moreover, by a similar argument from part b), we have

$$\lim_{h\to 0}\frac{\Delta(h)}{h^2} = f_{xy}(a,b).$$

Hence,

$$f_{yx}(a,b) = \lim_{h\to 0}\frac{\Delta(h)}{h^2} = f_{xy}(a,b).$$

We conclude

$$\frac{\partial^2 f}{\partial x\,\partial y}(a,b) = \frac{\partial^2 f}{\partial y\,\partial x}(a,b).$$

□

11.3 Derivatives, Differentials, and Tangent Planes

Exercise 11.3.1

a) *Proof.* Consider

$$Df(x,y) = (1,-1) \text{ and } Dg(x,y) = (2x, 2y).$$

We know f_x, f_y, g_x, g_y exist and are continuous on $\mathbb{R}^2$, hence f and g are differentiable on $\mathbb{R}^2$.

Moreover,

$$D(f+g)(x,y) = Df(x,y) + Dg(x,y) = (2x+1, 2y-1).$$

And

$$\begin{aligned}D(f\cdot g)(x,y) &= f(x,y)Dg(x,y) + g(x,y)Df(x,y) \\ &= (x-y)\cdot(2x,2y) + (x^2+y^2)\cdot(1,-1) \\ &= (3x^2 - 2xy + y^2, -x^2 + 2xy - 3y^2).\end{aligned}$$

□

b) *Proof.* Consider

$$Df(x,y) = (y,x) \text{ and } Dg(x,y) = (\sin x + x\cos x, \sin y).$$

We know f_x, f_y, g_x, g_y exist and are continuous on $\mathbb{R}^2$, hence f and g are differentiable on $\mathbb{R}^2$.

Moreover,

$$D(f+g)(x,y) = Df(x,y) + Dg(x,y) = (\sin x + x\cos x + y, x + \sin y).$$

And

$$\begin{aligned} D(f \cdot g)(x,y) &= f(x,y)Dg(x,y) + g(x,y)Df(x,y) \\ &= xy \cdot (\sin x + x\cos x, \sin y) + (x \sin x - \cos y) \cdot (y, x) \\ &= (x^2 y \cos x + 2xy \sin x - y\cos y, x^2 \sin x + xy \sin y - x \cos y). \end{aligned}$$

□

Exercise 11.3.2

a) *Solution.* Consider

$$\nabla f(x,y) = (2x, 2y),$$

then

$$\nabla f(1,-1) = (2,-2).$$

Hence,

$$n = (2,-2,-1).$$

We conclude the tangent plane $\Pi_n(c)$ is

$$2(x-1) - 2(y-1) - z = 2$$

which implies

$$2x - 2y - z = 2.$$

□

b) *Solution.* Consider

$$\nabla f(x,y) = (3x^2 y - y^3, x^3 - 3xy^2),$$

then

$$\nabla f(1,1) = (2,-2).$$

Hence,

$$n = (2,-2,-1).$$

We conclude the tangent plane $\Pi_n(c)$ is

$$2(x-1) - 2(y-1) - z = 0$$

which implies

$$2x - 2y - z = 0.$$

□

c) *Solution.* Consider

$$\nabla f(x,y,z) = (y, x, \cos z),$$

then

$$\nabla f(1,0,\frac{\pi}{2}) = (0,1,0).$$

Hence,

$$n = (0,1,0,-1).$$

We conclude the tangent plane $\Pi_n(c)$ is

$$y - w = -1.$$

□

11.4 The Chain Rule

Exercise 11.4.1

Solution. Use Chain Rule directly, then

$$w_p = \frac{\partial F}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial p},$$

and

$$w_q = \frac{\partial F}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial q}.$$

Also, doing carefully and patiently, we have

$$\begin{aligned}
w_{pp} &= \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial x}\right)\frac{\partial x}{\partial p} + \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial p^2} \\
&\quad + \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial y}\right)\frac{\partial y}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial p^2} \\
&\quad + \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial z}\right)\frac{\partial z}{\partial p} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial p^2} \\
&= \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial p^2} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial p^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial p^2} \\
&\quad + \left(\frac{\partial^2 F}{\partial x^2}\frac{\partial x}{\partial p} + \frac{\partial^2 F}{\partial x \partial y}\frac{\partial y}{\partial p} + \frac{\partial^2 F}{\partial x \partial z}\frac{\partial z}{\partial p}\right)\frac{\partial x}{\partial p} \\
&\quad + \left(\frac{\partial^2 F}{\partial y \partial x}\frac{\partial x}{\partial p} + \frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial p} + \frac{\partial^2 F}{\partial y \partial z}\frac{\partial z}{\partial p}\right)\frac{\partial y}{\partial p} \\
&\quad + \left(\frac{\partial^2 F}{\partial z \partial x}\frac{\partial x}{\partial p} + \frac{\partial^2 F}{\partial z \partial y}\frac{\partial y}{\partial p} + \frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial p}\right)\frac{\partial z}{\partial p} \\
&= \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial p^2} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial p^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial p^2} \\
&\quad + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 x}{\partial p^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 y}{\partial p^2} + \frac{\partial^2 F}{\partial z^2}\frac{\partial^2 z}{\partial p^2} \\
&\quad + 2\frac{\partial^2 F}{\partial x \partial y}\left(\frac{\partial x}{\partial p}\right)\left(\frac{\partial y}{\partial p}\right) + 2\frac{\partial^2 F}{\partial x \partial z}\left(\frac{\partial x}{\partial p}\right)\left(\frac{\partial z}{\partial p}\right) + 2\frac{\partial^2 F}{\partial y \partial z}\left(\frac{\partial y}{\partial p}\right)\left(\frac{\partial z}{\partial p}\right).
\end{aligned}$$

□

Exercise 11.4.3

Proof. Consider

$$\rho^k f(x) = f(\rho x).$$

Since f is differentiable on $\mathbb{R}^n$, we differentiate both sides in ρ, then

$$k\rho^{k-1}f(x) = \nabla f(\rho x)\frac{\partial}{\partial\rho}\rho x = \nabla f(\rho x)\cdot x.$$

Since it holds for all $\rho \in \mathbb{R}$, we pick $\rho = 1$, then

$$kf(x) = \nabla f(x)\cdot x$$

which implies

$$x_1\frac{\partial f}{\partial x_1}(x) + \cdots + x_n\frac{\partial f}{\partial x_n}(x) = kf(x)$$

for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. □

Exercise 11.4.4

Proof. Consider

$$\frac{\partial u}{\partial x} = yf'(xy) \text{ and } \frac{\partial u}{\partial y} = xf'(xy),$$

we have

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$$

On the other hand,

$$\begin{aligned}
\frac{\partial v}{\partial x} &= f'(x-y) + g'(x+y);\\
\frac{\partial^2 v}{\partial x^2} &= f''(x-y) + g''(x+y),\\
\frac{\partial v}{\partial y} &= -f'(x-y) + g'(x+y);\\
\frac{\partial^2 v}{\partial y^2} &= f''(x-y) + g''(x+y).
\end{aligned}$$

Hence,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

□

Exercise 11.4.5

Proof. At first, we claim

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Consider

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$

and

$$\frac{\partial v}{\partial \theta} = \frac{\partial g}{\partial x}(-r\sin\theta) + \frac{\partial g}{\partial y}(r\cos\theta).$$

By hypothesis that f, g satisfy the Cauchy-Riemann equations, we have

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ &= \frac{\partial g}{\partial y}\cos\theta - \frac{\partial g}{\partial x}\sin\theta \\ &= \frac{1}{r}\left(\frac{\partial g}{\partial x}(-r\sin\theta) + \frac{\partial g}{\partial y}(r\sin\theta)\right) \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta}.\end{aligned}$$

Secondly, we claim

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Consider

$$\frac{\partial v}{\partial r} = \frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta,$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta).$$

By hypothesis that f, g satisfy the Cauchy-Riemann equations, we have

$$\begin{aligned}\frac{\partial v}{\partial r} &= \frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}(r\cos\theta) \\ &= -\frac{\partial f}{\partial y}\cos\theta + \frac{\partial f}{\partial x}\sin\theta \\ &= -\frac{1}{r}\left(\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)\right) \\ &= -\frac{1}{r}\frac{\partial u}{\partial \theta}.\end{aligned}$$

We make two claims hold, hence we complete the proof. $\square$

Exercise 11.4.6

Proof. Consider
$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{11.5}$$
And
$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 f}{\partial x^2}(\cos^2\theta) + \frac{\partial^2 f}{\partial x\partial y}(\cos\theta\sin\theta) + \frac{\partial^2 f}{\partial y^2}(\sin^2\theta) + \frac{\partial^2 f}{\partial y\partial x}(\sin\theta\cos\theta)\\ &= \frac{\partial^2 f}{\partial x^2}(\cos^2\theta) + \frac{\partial^2 f}{\partial y^2}(\sin^2\theta) + 2\frac{\partial^2 f}{\partial x\partial y}(\sin\theta\cos\theta). \end{aligned}\tag{11.6}$$
On the other hand,
$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta).$$
And
$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= \left(\frac{\partial^2 f}{\partial x^2}(r^2\sin^2\theta) + \frac{\partial^2 f}{\partial x\partial y}(-r^2\sin\theta\cos\theta)\right) + \frac{\partial f}{\partial x}(-r\cos\theta)\\ &\quad + \left(\frac{\partial^2 f}{\partial y\partial x}(-r^2\sin\theta\cos\theta) + \frac{\partial^2 f}{\partial y^2}(r^2\cos^2\theta)\right) + \frac{\partial f}{\partial y}(-r\sin\theta)\\ &= r^2\left(\frac{\partial^2 f}{\partial x^2}(\sin^2\theta) + \frac{\partial^2 f}{\partial y^2}(\cos^2\theta) - 2\frac{\partial^2 f}{\partial x\partial y}(\sin\theta\cos\theta)\right) - r\frac{\partial f}{\partial x}\cos\theta - r\frac{\partial f}{\partial y}\sin\theta. \end{aligned}\tag{11.7}$$
By (11.5), (11.6), (11.7) and hypothesis that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$
we conclude
$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$
□

Exercise 11.4.10

Proof. Since $f(I) \subseteq \partial B_r(0)$, we know
$$\|f(t)\|^2 = r^2$$
for all $t \in I$.

Then for all $t \in I$, consider
$$f(t)\cdot f(t) = \|f(t)\|^2 = r^2.$$
Differentiating both sides in t, we have
$$2f(t)\cdot f'(t) = 0$$
which implies
$$f(t)\cdot f'(t) = 0.$$
Hence, we conclude $f(t)$ is orthogonal to $f'(t)$ for all $t \in I$. □

Exercise 11.4.11

a) *Proof.* Notice that
$$D_u f(a) := \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t}$$
where $\|u\| = 1$.
For each $u \in \mathbb{R}^n$ with $\|u\| = 1$, let $g(t) = f(a+tu)$, then differentiate both sides in t, we have $\nabla g(t) = \nabla f(a+tu)\cdot u$.
Then we consider
$$\begin{aligned} D_u f(a) &= \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t} \\ &= \lim_{t\to 0} \frac{g(t)-g(0)}{t} \\ &= \nabla g(0) \\ &= \nabla f(a)\cdot u. \end{aligned}$$
□

b) *Proof.* By hypothesis, we have
$$\nabla f(a)\cdot u = \|\nabla f(a)\| \|u\| \cos\theta.$$
From part a) and $\|u\| = 1$, we conclude
$$D_u f(a) = \nabla f(a)\cdot u = \|\nabla f(a)\| \|u\| \cos\theta = \|\nabla f(a)\| \cos\theta.$$
□

c) *Solution.* From part b), we know $\theta \in [0, \frac{\pi}{2}]$ which implies
$$0 \le D_u f(a) \le \|\nabla f(a)\|.$$
The maximum of $D_u f(a)$ is $\|\nabla f(a)\|$ if and only if $\theta = 0$. This occurs when u is parallel to $\nabla f(a)$. □

11.5 The Mean Value Theorem and Taylor's Formula

Exercise 11.5.1

a) *Solution.* Notice that
$$\begin{aligned} &f_x = 2x + y; \quad f_y = x + 2y; \\ &f_{xx} = 2; \quad f_{xy} = f_{yx} = 1; \quad f_{yy} = 2. \end{aligned}$$

Apply Taylor's Formula, by setting $a = (-1,1), h = (x+1, y-1)$, then

$$f(x,y) = f(-1,1) + D^{(1)} f\left((-1,1);(x+1,y-1)\right) + \frac{1}{2!} D^{(2)} f\left((-1,1);(x+1,y-1)\right).$$

Compute

$$\begin{aligned} D^{(1)} f\left((-1,1);(x+1,y-1)\right) &= \nabla f(-1,1) \cdot (x+1, y-1) \\ &= -(x+1) + (y-1), \end{aligned}$$

and

$$\begin{aligned} D^{(2)} f\left((-1,1);(x+1,y-1)\right) &= (x+1)^2 f_{xx}(-1,1) + (x+1)(y-1) f_{xy}(-1,1) + (y-1)^2 f_{yy}(-1,1) \\ &= 2(x+1)^2 + 2(x+1)(y-1) + 2(y-1)^2. \end{aligned}$$

Hence,

$$f(x,y) = 1 - (x+1) + (y-1) + (x+1)^2 + (x+1)(y-1) + (y-1)^2.$$

□

c) *Solution.* Notice that

$$\begin{aligned} f_x &= ye^{xy}; & f_y &= xe^{xy}; & & \\ f_{xx} &= y^2 e^{xy}; & f_{xy} &= (1+xy)e^{xy}; & f_{yy} &= x^2 e^{xy}; \\ f_{xxx} &= y^3 e^{xy}; & f_{xxy} &= (2y + xy^2)e^{xy}; & f_{xyy} &= (2x + x^2 y)e^{xy}; \\ f_{yyy} &= x^3 e^{xy}; & f_{xxxx} &= y^4 e^{xy}; & f_{xxxy} &= (3y^2 + xy^3)e^{xy}; \\ f_{xxyy} &= (2 + 4xy + x^2y^2)e^{xy}; & f_{xyyy} &= (2x^2 + x^2 + x^3 y)e^{xy}; & f_{yyyy} &= x^4 e^{xy}. \end{aligned}$$

Apply Taylor's Formula, there is $(c,d) \in L\left((x,y);(0,0)\right)$ such that

$$f(x,y) = f(0,0) + \sum_{k=1}^{3} D^{(k)} \frac{1}{k!} f\left((0,0);(x,y)\right) + D^{(4)} \frac{1}{4!} f\left((c,d);(x,y)\right).$$

Computing patiently, we have

$$\begin{aligned} & D^{(1)} f\left((0,0);(x,y)\right) = 0; \\ & D^{(2)} f\left((0,0);(x,y)\right) = 2xy; \\ & D^{(3)} f\left((0,0);(x,y)\right) = 0; \\ & D^{(4)} f\left((c,d);(x,y)\right) = \left((dx+cy)^4 + 12(dx+cy)^2 xy + 12x^2y^2\right) e^{cd}. \end{aligned}$$

Hence,

$$e^{xy} = 1 + xy + \frac{1}{4!}\left((dx+cy)^4 + 12(dx+cy)^2 xy + 12x^2y^2\right) e^{cd}$$

for some $(c,d) \in L\left((x,y);(0,0)\right)$. □

Exercise 11.5.2

Proof. Since f is C^p and $B_r(x_0, y_0)$ is convex, it follows from Theorem 11.37 for $n = 2$ such that

$$\sum_{i_1=1}^{2} \cdots \sum_{i_j=1}^{2} \frac{\partial^j f}{\partial x_{i_1} \cdots \partial x_{i_j}} h_{i_1} \cdots h_{i_j} = \sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \, \partial y^{k-j}} (x_0, y_0).$$

Hence, we conclude there is $(c, d) \in L\left((x, y); (0, 0)\right)$ such that

$$\begin{aligned} f(x, y) = f(x_0, y_0) &+ \sum_{k=1}^{p-1} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \, \partial y^{k-j}} (x_0, y_0) \right) \\ &+ \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \, \partial y^{p-j}} (c, d). \end{aligned}$$

□

Exercise 11.5.5

Proof. When $p = 1$, we know

$$f(x) - f(a) = \int_0^1 D^{(1)} f(a + th; h) dt.$$

Since f is C^p on V and $L(x; a) \subset V$, by the Mean Value Theorem, there is $c \in L(x; a)$ such that

$$\int_0^1 D^{(1)} f(a + th; h) dt = D^{(1)} f(c; h).$$

Moreover, applying Taylor's Formula, we have

$$f(x) - f(a) = D^{(1)} f(c; h).$$

Hence, the formula holds for $p = 1$.

Suppose when $p = q$, the formula holds. We set

$$u = D^{(q)} f(a + th; h) \text{ and } dv = (1 - t)^{p-1} dt,$$

then

$$du = D^{(p+1)} f(a + th; h) dt \text{ and } v = -\frac{1}{p} (1 - t)^p.$$

By Integration by Part, we have

$$\begin{aligned} \frac{1}{(q-1)!} \int_0^1 (1 - t)^{q-1} D^{(q)} f(a + th; h) dt &= -\frac{1}{p} (1 - t)^q D^{(q)} f(a + th; h) \bigg|_0^1 \\ &\quad + \int_0^1 \frac{1}{q} (1 - t)^q D^{(q+1)} f(a + th; h) dt \\ &= \frac{1}{q!} D^{(q)} f(a; h) + \frac{1}{q!} \int_0^1 (1 - t)^q D^{(q+1)} f(a + th; h) dt. \end{aligned}$$

i.e,

$$\begin{aligned} f(x) - f(a) &= \sum_{k=1}^{q-1} \frac{1}{k!} D^{(k)} f(a;h) + \frac{1}{(q-1)!} \int_0^1 (1-t)^{q-1} D^{(q)} f(a+th;h) dt \\ &= \sum_{k=1}^{q} \frac{1}{k!} D^{(k)} f(a;h) + \frac{1}{q!} \int_0^1 (1-t)^{q} D^{(q+1)} f(a+th;h) dt. \end{aligned}$$

Hence, we know the formula also holds for $p = q + 1$.

By induction, we conclude the formula holds for all $p \in \mathbb{N}$. □

Exercise 11.5.6

a) *Solution.*

$$\begin{aligned} g'(t) = &\frac{\partial}{\partial x} f(tx + (1-t)a, y) \cdot \frac{\partial}{\partial t}(tx + (1-t)a) \\ &+ \frac{\partial}{\partial y} f(tx + (1-t)a, y) \cdot \frac{\partial}{\partial t}(y) \\ &+ \frac{\partial}{\partial x} f(a, ty + (1-t)b) \cdot \frac{\partial}{\partial t}(a) \\ &+ \frac{\partial}{\partial y} f(a, ty + (1-t)b) \cdot \frac{\partial}{\partial t}(ty + (1-t)b) \\ = &(x-a) \frac{\partial}{\partial x} f(tx + (1-t)a, y) + (y-b) \frac{\partial}{\partial y} f(a, ty + (1-t)b). \end{aligned}$$

□

b) *Proof.* By hypothesis which satisfy the Mean Value Theorem, there is $t \in (0,1)$ such that

$$g(1) - g(0) = g'(t).$$

which implies there are numbers $c = tx + (1-t)a$ and $d = ty + (1-t)b$ so that c is between a and x, and d is between b and y, such that

$$\begin{aligned} &g(1) - g(0) = g'(t) \\ \Longrightarrow\ &f(x,y) - f(a,b) = (x-a) \frac{\partial}{\partial x} f(tx + (1-t)a, y) + (y-b) \frac{\partial}{\partial y} f(a, ty + (1-t)b) \\ \Longrightarrow\ &f(x,y) - f(a,b) = (x-a) f_x(c,y) + (y-b) f_y(a,d). \end{aligned}$$

□

Exercise 11.5.8

Proof. We pick $h \in \mathbb{R}^n$ such that $x = a + h \in H$, and also fix n.

Applying Taylor's Formula, we have

$$f(x) = f(a) + D^{(1)}f(a;h) + \frac{1}{2!}D^{(2)}f(a;h). \tag{11.8}$$

Since $f_{x_j}(a) = 0$ for some $a \in H$ and all $j = 1, \cdots, n$, then $D^{(1)}f(a;h) = 0$.

For $1 \leq i, j \leq n$, since f is C^2 on V, then $\frac{\partial^2 f}{\partial x_i\,\partial x_j}$ is continuous on V. Moreover, H is compact convex subset of V and $a \in H$, then $\frac{\partial^2}{\partial x_i \partial x_j} f(a)$ is compact and hence bounded. So there is $M_{ij} > 0$ such that

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} f(a) \right| \leq M_{ij}.$$

It follows that

$$\begin{aligned}
\left| D^{(2)}f(a;h) \right| &= \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(a)(x_i - a_i)(x_j - a_j) \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{\partial^2}{\partial x_i \partial x_j} f(a) \cdot (x_i - a_i)(x_j - a_j) \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \cdot \frac{1}{2} \left((x_i - a_i)^2 + (x_j - a_j)^2 \right) \\
&\leq n^2 M' \left(\frac{1}{2} n \|x - a\|^2 \right) \\
&= \frac{1}{2} n^3 M' \|x - a\|^2
\end{aligned}$$

where $M' = \max_{1 \leq i,j \leq n}\{M_{ij}\}$.

From (11.8), for all $x \in H$, we have

$$\begin{aligned}
|f(x) - f(a)| &= \left| D^{(1)})f(a;h) + \frac{1}{2!} D^{(2)}f(a;h) \right| \\
&= \left| \frac{1}{2!} D^{(2)}f(a;h) \right| \\
&\leq \frac{1}{4} n^3 M' \|x - a\|^2 \\
&= M\|x - a\|^2
\end{aligned}$$

where $M = \frac{1}{4}n^3 M'$ is constant as promised. □

Exercise 11.5.11

a) *Proof.* Since u is C^2 on V, then u is continuous on V.

Because $H \subset V$ and H is closed, and notice that H is a subset of Euclidean space, we know H is compact. Then u is uniformly continuous on H.

Given $\epsilon > 0$, for all $(x,t), (x_0, t_0) \in H$, there is $\delta > 0$ with $2\delta < \min\{b-a, c\}$ such that

$$\|(x,t) - (x_0, t_0)\| < \delta \implies |u(x,t) - u(x_0,t_0)| < \epsilon. \tag{11.9}$$

Let $K = \left[a + \frac{\delta}{2}, b - \frac{\delta}{2}\right] \times \left[\frac{\delta}{2}, c - \frac{\delta}{2}\right]$, then we know $K \subset H^\circ$ and K is closed and hence compact.

Finally, for all $(x,t) \in H \backslash K$, we can pick $(x_0, t_0) \in \partial H$ such that

$$\|(x,t) - (x_0, t_0)\| < \delta.$$

By (11.9) and hypothesis, we conclude

$$u(x,t) > u(x_0,t_0) - \epsilon \geq -\epsilon.$$

□

b) *Proof.* We know w is continuous on H and H is compact subset of Euclidean space, hence H is bounded and closed.

By the Extreme Value Theorem, w is bounded on H. Moreover, there is $(x_2, t_2) \in H$ such that $w(x_2, t_2)$ is minimum on H.

Suppose to contrary that $(x_2, t_2) \notin K$, that is, $(x_2, t_2) \in H \backslash K$. Notice that $t_2 > 0$. Then by hypothesis and part a), we have

$$\begin{aligned} w(x_2,t_2) &= u(x_2,t_2) + r(t_2 - t_1) \\ &\geq -\epsilon + r(t_2 - t_1) \\ &= rt_2 - \frac{l}{2} \\ &> rt_2 - l \\ &> -l \\ &= u(x_1,t_1) + r(t_1 - t_1) \\ &= w(x_1,t_1). \end{aligned}$$

This leads to a contradiction that $w(x_2, t_2)$ is minimum on H.

Hence, we conclude the minimum of $w(x,t)$ on H occurs at some $(x_2, t_2) \in K$ as promised. □

c) *Proof.* Suppose to contrary that u attains the minimum on H°, then $u(x_1, t_1) < 0$ for some $(x_1, t_1) \in H^\circ$.

Moreover from part b), we know w attains the minimum on H only occurs at some $(x_2, t_2) \in K \subset H^\circ$. Therefore, $w_{xx}(x_2, t_2) \geq 0$ and $w_t(x_2, t_2) = 0$.

By hypothesis that $u_{xx} - u_t = 0$, we consider

$$0 = w_t(x_2,t_2) = u_t(x_2,t_2) + r = u_{xx}(x_2,t_2) + r \geq r > 0.$$

This leads to a contradiction. Hence u attains the minimum on ∂H. Since the minimum of $u(x,t)$ on ∂H is 0, then we conclude $u(x,t) \geq 0$ for all $(x,t) \in H$. □

Exercise 11.5.12

a) *Proof.* Denote E is a convex set in $\mathbb{R}^n$.

Suppose to contrary that E is not connected, then there are two nonempty and relatively open sets U and V in E such that $U \cap V = \phi$ and $U \cup V = E$.

Since U, V are nonempty and E is convex, then there are $a \in U$ and $b \in V$ such that $L(a; b) \subset E$.

We denote $L(a; b)$ clearly, that is,

$$L(a; b) := \{ta + (1-t)b : 0 \le t \le 1\}.$$

Set $f(t) = ta + (1-t)b$. Since $f : [0,1] \to \mathbb{R}^n$ is continuous on $[0,1]$ and $[0,1]$ is connected, then L is connected.

However, we claim L is not connected under the supposition.

We know $(L \cap U) \cup (L \cap V) = L$ and $(L \cap U) \cap (L \cap V) = \phi$. Since U is relatively open in E, then there is an open set $A \subset E$ such that

$$U = E \cap A$$

which implies that

$$L \cap U = L \cap (E \cap A) = L \cap A.$$

Hence, $L \cap U$ is relatively open in L. Similarly, $L \cap V$ is also relatively open in L.

Now we know L is connected which is a contradiction to that L is connected.

We get E is connected. Moreover, since E is arbitrary, then we conclude every convex set in $\mathbb{R}^n$ is connected. □

b) *Solution.* We give a counter-example.

For $n = 2$, we set

$$E = \{(x, y) : -1 \le x \le 1, -|x| \le y \le |x|\},$$

then E is connected.

Pick $a = (-1, 1)$ and $b = (1, 1)$, we know $(0, 1) \in L(a; b)$ but $(0, 1) \notin E$. So E is not convex. □

11.6 The Inverse Function Theorem

Exercise 11.6.1

a) *Proof.* For any (a, b), consider

$$\begin{cases} 3u - v = a \\ 2u + 5v = b \end{cases}.$$

Compute

$$\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 17\,,\ \Delta u = \begin{vmatrix} 3 & a \\ 2 & b \end{vmatrix} = 3b - 2a, \text{ and } \Delta v = \begin{vmatrix} a & -1 \\ b & 5 \end{vmatrix} = 5a + b.$$

Use Cramer's Rule, we know

$$u = \frac{3b - 2a}{17} \text{ and } v = \frac{5a + b}{17}.$$

Hence, u and v are well defined. Then f^{-1} exists and is differentiable in some nonempty, open set containing (a, b). Moreover,

$$Df(u,v) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix},$$

by the Inverse Function Theorem with $f(u,v) = (a,b)$, then

$$D(f^{-1})(a,b) = (Df(u,v))^{-1} = \frac{1}{17}\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}.$$

□

b) *Proof.* Consider

$$\begin{aligned}
&\begin{cases} u + v = 0 \\ \sin u + \cos v = 1 \end{cases} \\
\Longrightarrow\ & \sin u + \cos u = 1 \\
\Longrightarrow\ & \frac{\sqrt{2}}{2}\sin u + \frac{\sqrt{2}}{2}\cos u = \frac{\sqrt{2}}{2} \\
\Longrightarrow\ & \cos\frac{\pi}{4}\sin u + \sin\frac{\pi}{4}\cos u = \sin\frac{\pi}{4} \\
\Longrightarrow\ & \sin(u + \frac{\pi}{4}) = \sin\frac{\pi}{4} \\
\Longrightarrow\ & u + \frac{\pi}{4} = 2k\pi + \frac{\pi}{4} \textbf{ or } u + \frac{\pi}{4} = \pi - \left(2k\pi + \frac{\pi}{4}\right)
\end{aligned}$$

for $k \in \mathbb{Z}$. Hence we have

$$\begin{cases} u = 2k\pi \\ v = -2k\pi \end{cases} \quad \text{or} \quad \begin{cases} u = -2k\pi + \frac{\pi}{2} \\ v = 2k\pi - \frac{\pi}{2} \end{cases}.$$

Now u and v are well defined. Then f^{-1} exists and is differentiable in some nonempty, open set containing $(0, 1)$. Moreover,

$$Df(2k\pi, -2k\pi) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } Df(-2k\pi + \frac{\pi}{2}, 2k\pi - \frac{\pi}{2}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Apply the Inverse Function Theorem with $f(u,v) = (0,1)$, then

$$D(f^{-1})(0,1) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

□

Exercise 11.6.2

a) *Proof.* Let
$$F(x,y,z) = xyz - \sin(x+y+z).$$
Then
$$F_z(x,y,z) = xy - \cos(x+y+z).$$
Therefore
$$F(0,0,0) = 0 \text{ and } F_z(0,0,0) = 1 \neq 0.$$
By the Implicit Function Theorem, we know there is a nonempty, open set $V \subset \mathbb{R}^2$ containing $(0,0,0)$ and a unique continuously differentiable function g such that $z = g(0,0)$. So the solution is differentiable near $(0,0)$. □

b) *Proof.* Let
$$F(x,y,z) = x^2 + y^2 + z^2 + \sqrt{\sin(x^2+y^2) + 3z + 4} - 2.$$
Then
$$F_z(x,y,z) = 2z + \frac{3}{2\sqrt{\sin(x^2+y^2)+3z+4}}.$$
Therefore
$$F(0,0,0) = 0 \text{ and } F_z(0,0,0) = \frac{4}{3} \neq 0.$$
By the Implicit Function Theorem, we know there is a nonempty, open set $V \subset \mathbb{R}^2$ containing $(0,0,0)$ and a unique continuously differentiable function g such that $z = g(0,0)$. So the solution is differentiable near $(0,0)$. □

Exercise 11.6.3

Proof. Set
$$\begin{aligned}
F_1(u,v,w,x,y) &= u^5 + xv^2 - y + w; \\
F_2(u,v,w,x,y) &= v^5 + yu^2 - x + w; \\
F_3(u,v,w,x,y) &= w^4 + y^5 - x^4 - 1; \\
F(u,v,w,x,y) &= (F_1, F_2, F_3).
\end{aligned}$$
We know $F(1,1,-1,1,1) = (0,0,0)$. Also,
$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \det \begin{vmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{vmatrix} = 4w^3(25u^4v^4 - 4xyuv) = -84 \neq 0$$
when $(u,v,w,x,y) = (1,1,-1,1,1)$.

Hence, by the Implicit Function Theorem, there is $r > 0$ such that $B_r(1,1)$ containing $(1,1)$ and a unique continuously differentiable function g such that
$$g(x,y) = (u(x,y), v(x,y), w(x,y)) \text{ with } g(1,1) = (1,1,-1).$$
□

Exercise 11.6.7

Proof. By hypothesis, we use the Implicit Function Theorem. Since $F(a) = 0$, and furthermore, $F_{x_j}(a) \neq 0$ implies

$$\frac{\partial(F_1, \cdots, F_n)}{\partial(x_1, \cdots, x_n)}(a) \neq 0,$$

then there exists open sets W_j containing $(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)$, and a unique continuously differentiable function $g_j(u^{(j)})$ such that $F(x_1, \cdots, x_{j-1}, g_j(u^{(j)}), x_{j+1}, c \ldots, x_n) = 0$ for all $u^{(j)} \in W^j$.

Notice that $g_j(u^{(j)}$ is continuously differentiable on W^j, then $g_j(u^{(j)}$ is C^1 on W^j.

For any $j = 1, \cdots, n$, we know the open set W^j contains $(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)$, so there is r_k^j with $1 \leq k \leq n$ and $k \neq j$ such that the Cartesian product $(a_1 - r_1^j, a_1 + r_1^j) \times \cdots \times (a_{j-1} - r_{j-1}^j, a_{j-1} + r_{j-1}^j) \times (a_{j+1} - rj + 1^j, a_{j+1} + r_{j+1}^j) \times \cdots \times (a_n - r_n^j, a_n + r_n^j)$ is a subset of W^j. Hence for a_j, we pick

$$r_j = \min_{1 \leq k \leq n, k \neq j} \{r_k^j\},$$

then the Cartesian product $(a_1 - r_1, a_1 + r_1) \times \cdots \times (a_{j-1} - r_{j-1}, a_{j-1} + r_{j-1}) \times (a_{j+1} - rj + 1, a_{j+1} + r_{j+1}) \times \cdots \times (a_n - r_n, a_n + r_n)$ contains $a = (a_1, \cdots, a_j, \cdots, a_n)$.

Fix a_j, pick $r > 0$ with $r = \min_{1 \leq j \leq n}\{r_j\}$ such that $(x_1, \cdots, a_j, \cdots, x_n) \in B_r(a)$. Since $F_{x_j}(a) \neq 0$, then $F_{x_j}(x) \neq 0$ for all $x \in B_r(a)$.

Moreover, for all $x \in B_r(a)$, we consider $F_j := F(x_1, \cdots, x_{j-1}, g_j(u^{(j)}), x_{j+1}, c \ldots, x_n) = 0$ on W^j, then we differentiate both sides on F_j over x_{j-1} for $2 \leq j \leq n$, and F_1 over x_n, we obtain

$$\frac{\partial F}{\partial x_n} + \frac{\partial F}{\partial x_1}\frac{\partial g_1}{\partial x_n} = 0, \text{ and } \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial x_{j-1}}\frac{\partial g_j}{\partial x_{j-1}} = 0 \text{ for } 2 \leq j \leq n$$

which implies

$$\frac{\partial g_1}{\partial x_n} = -\frac{\frac{\partial F}{\partial x_n}}{\frac{\partial F}{\partial x_1}}, \text{ and } \frac{\partial g_j}{\partial x_{j-1}} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_{j-1}}} \text{ for } 2 \leq j \leq n.$$

Finally, by multiplying these n equations, we conclude

$$\frac{\partial g_1}{\partial x_n}\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_2}\cdots\frac{\partial g_n}{\partial x_{n-1}} = (-1)^n$$

on $B_r(a)$ as promised. □

Exercise 11.6.10

a) *Proof.* Consider

$$f'(t_0) = \begin{bmatrix} u'(t_0) \\ v'(t_0) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $u'(t_0)$ and $v'(t_0)$ cannot both be zero by definition of a matrix clearly. □

b) *Proof.* Since $f'(t_0) \neq 0$, and f is C^2 on $\mathbb{R}$ so is C^1 on $\mathbb{R}$.

Suppose $u'(t_0) \neq 0$, by the Inverse Function Theorem, there is an open set W containing t_0 and a unique continuously differentiable 1-1 function g on $f(W)$ containing x_0 such that $g(x) = t$ for all $x \in f(W)$. Notice that g is C^1 on $f(W)$.

Since $x_0 \in f(W)$, then $g(x_0) = t_0$. For x near x_0 which means $x \in f(W)$, we have $g(x) = t$. Hence, $u(g(x)) = u(t) = x$.

Suppose $v'(t_0) \neq 0$, then a similar argument establishes there is a C^1 function h such that $h(y_0) = t_0$ and $v(h(y)) = y$ for y near y_0.

From part a), $u'(t_0)$ and $v'(t_0)$ cannot both be zero. Therefore, we conclude either one of two statements will hold. $\square$