Chapter 6

Infinite Series of Real Numbers

6.1 Introduction

Exercise 6.1.4

Solution.

$$\sum_{k=1}^{\infty}(a_{k+1} - 2a_k + a_{k-1}) = \lim_{n\to\infty}(a_0 - a_1 + a_{n+1} - a_n) = a_0 - a_1$$

□

Exercise 6.1.5

Solution.

$$\sum_{k=1}^{\infty}(x^k - x^{k-1})(x^k + x^{k-1}) = \sum_{k=1}^{\infty}(x^{2k} - x^{2k-2}) = \lim_{n\to\infty}(x^{2n} - 1) \text{ converges}$$

$\iff |x| \leq 1$

If $x = \pm 1$, then the value of the series is 0,
otherwise $x \in (-1, 1)$, the value of the series is -1. □

Exercise 6.1.7

a) *Proof.* Since $f'(x)$ exists for all $x \in \mathbb{R}$, we conclude $F'(x)$ exists.
By the Mean Value Theorem, there is a number $c \in I$ so that

$$|F(x) - F(y)| = F'(c)(x - y),\ \forall x, y \in I$$

where

$$F'(c) = 1 - \frac{f'(c)}{f'(a)}$$

Since $c \in I$, we know $\frac{f'(c)}{f'(a)} \in [1 - r, 1]$. Therefore

$$0 \leq F'(c) = 1 - \frac{f'(c)}{f'(a)} \leq r$$

As a result, we conclude that

$$|F(x) - F(y)| = |F'(c)||x - y| \leq r|x - y|,\ \forall x, y \in I$$

□

b) *Proof.* For $n = 1$, by definition of x_n, we get

$$|x_2 - x_1| = |F(x_1) - F(x_0)| \leq r|x_1 - x_0|$$

Assume for $n = k$,

$$|x_{k+1} - x_k| \leq r^k |x_1 - x_0|$$

holds, then for $n = k + 1$,

$$|x_{k+2} - x_{k+1}| = |F(x_{k+1}) - F(x_k)| \leq r|x_{k+1} - x_k| \leq r^{k+1}|x_1 - x_0|$$

also holds. By induction, we conclude

$$|x_{n+1} - x_n| \leq r^n |x_1 - x_0|, \ \forall n \in \mathbb{N}$$

□

c) *Proof.* Since $f(I) \subseteq I$ and $x_0 \in I$, we hold $x_n \in I$, $\forall n \in \mathbb{N}$. And I is a closed interval, by Bolzano-Weierstrass Theorem, there is a subsequence $\{x_{n_k}\}$ which converges to a fixed number $b \in I$.

Moreover, by part b), we get $\{x_n\}$ is Cauchy. Hence

$$\lim_{n \to \infty} x_n = b$$

Besides, F is differentiable on I, so F is continuous on I.
Consider the equation $x_n = F(x_{n-1})$ and notice F is continuous on I. Take the limit on both sides as $n \to \infty$, we have $b = F(b)$ which implies that

$$b = b - \frac{f(b)}{f'(a)}$$

Then, we conclude $f(b) = 0$, as promised. □

Exercise 6.1.9

a) *Proof.* For all $n > N$,

$$\left| nb - \sum_{k=1}^{n} b_k \right| = \left| \sum_{k=1}^{n} (b - b_k) \right| \leq \sum_{k=1}^{n} |b - b_k| \leq \sum_{k=1}^{N} |b - b_k| + \sum_{k=N+1}^{n} M = \sum_{k=1}^{N} |b_k - b| + M(n - N)$$

□

b) *Proof.* Since $\lim_{n \to \infty} b_n = b$, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |b_n - b| < \epsilon$. Then if $n > N$, by part a),

$$\left| \frac{b_1 + b_2 + ... + b_n}{n} - b \right| = \left| \frac{b_1 + b_2 + ... + b_n - nb}{n} \right| \leq \left| \frac{\sum_{k=1}^{n} |b_k - b|}{n} \right| + \epsilon(1 - \frac{N}{n})$$

Take the limit of both sides as $n \to \infty$, we have

$$\limsup_{n\to\infty} \left| \frac{b_1 + b_2 + ... + b_n}{n} - b \right| \le \epsilon$$

This hold for any $\epsilon > 0$, so

$$\lim_{n\to\infty} \left| \frac{b_1 + b_2 + ... + b_n}{n} - b \right| = 0$$

which means

$$\frac{b_1 + b_2 + ... + b_n}{n} \to b \text{ as } n \to \infty$$

□

c) *Solution.* The counter-example is $b_k = (-1)^{k-1}$. □

Exercise 6.1.12

Proof. Let

$$b_n := \sum_{k=1}^{n} k a_k = \frac{n+1}{n+2}$$

So

$$b_{n+1} - b_n = (n+1)a_{n+1} = \frac{1}{(n+2)(n+3)}$$

We have

$$a_1 = \frac{2}{3} \text{ and } a_n = \frac{1}{n(n+1)(n+2)} \text{ for } n > 1$$

Calculate the summation and we get

$$\sum_{k=1}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} a_k = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

□

6.2 Series with Nonnegative Terms

Exercise 6.2.1

a) *Proof.* Let

$$a_n := \frac{2n+5}{3n^3 + 2n - 1} \text{ and } b_n := \frac{1}{n^2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-Series Test, besides

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3} < \infty$$

By the Limit Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{2k+5}{3k^3+2k-1}$$

converges. □

d) *Proof.* Since $\log k < k$, $\forall k \in \mathbb{N}$, we have

$$\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k} \leq \sum_{k=1}^{\infty} \frac{k^5}{e^k}$$

Let

$$a_n := \frac{n^5}{e^n} \text{ and } b_n := \frac{1}{n^2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-Series Test, besides

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^7}{e^n} = 0$$

By the Limit Comparison Test and the Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k}$$

converges. □

Exercise 6.2.2

a) *Proof.* Let

$$a_n := \frac{3n^3+n-4}{5n^4-n^2+1} \text{ and } b_n := \frac{1}{n}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p-Series Test, besides

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{3}{5} < \infty$$

By the Limit Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{3k^3+k-4}{5k^4-k^2+1}$$

diverges. □

d) *Proof.* Consider

$$\int_2^{\infty} \frac{1}{x \log^p x} = \left|(\log x)^{1-p}\right|_2^{\infty} + p \int_2^{\infty} \frac{1}{x \log^p x}$$

Since $p \leq 1$, then $1 - p \geq 0$, we have

$$\int_2^\infty \frac{1}{x \log^p x} = \frac{\left|(\log x)^{1-p}\right|_2^\infty}{1-p} = \infty$$

By the Integral Test, we conclude

$$\int_2^\infty \frac{1}{x \log^p x} \quad for\ p \leq 1$$

diverges. □

Exercise 6.2.3

Proof. Assume $a_k \leq M$, $\forall k \in \mathbb{N}$ where $M > 0$. Therefore

$$\frac{a_k}{(k+1)^p} < \frac{M}{k^p}$$

Since $p > 1$, by the p-Series Test and the Limit Comparison Test, we know

$$\sum_{k=1}^{\infty} \frac{M}{k^p}$$

converges. Then by the Comparison Test, we conclude

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$$

converges. □

Exercise 6.2.5

Proof. Since for $p \geq 0$, we have

$$\frac{|a_k|}{k^p} \leq |a_k|$$

By the Comparison Test, we know

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$$

converges. When $p < 0$, the series might converge or diverge. e.g. set $p = -1$ and $a_k = \frac{1}{k}$, then

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p} = \sum_{k=1}^{\infty} 1$$

which diverges. For the same p, set $a_k = \frac{1}{k^3}$, then the series

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges by the p-Series Test. $\square$

Exercise 6.2.7

Proof. Since the series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \leq M$ for all k. It follows that $0 \leq a_k b_k \leq M b_k$ and the series $\sum_{k=1}^{\infty} b_k$ converges. By the Comparison Test, we conclude $\sum_{k=1}^{\infty} a_k b_k$ converges. $\square$

Exercise 6.2.9

Proof. ($\Longrightarrow$) Let $s_n = \sum_{k=1}^{\infty} a_k$. Since

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = S \in \mathbb{R}$$

converges, so does its partial sum. Then

$$\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) = \lim_{n\to\infty} (s_{2n+1} - a_1) = S - s_1 \in \mathbb{R}$$

converges.

($\Longleftarrow$) Since

$$\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) = \lim_{n\to\infty} (s_{2n+1} - a_1)$$

converges so does s_{2n+1}. Consider

$$s_{2n+2} = s_{2n+1} + a_{2n+2}$$

Taking the limit of both sides as $n \to \infty$, we have

$$\lim_{n\to\infty} s_{2n+2} = \lim_{n\to\infty} s_{2n+1}$$

which means s_{2n+2} also converges.

Combining above discussion, no matter the number of terms is either odd or even, the series always converges. We conclude

$$\sum_{k=1}^{\infty} a_k$$

converges. $\square$

6.3 Absolute Convergence

Exercise 6.3.3

a) *Solution.* Consider

$$\int_2^\infty \left|\frac{1}{x\log^p x}\right| = \int_2^\infty \frac{1}{x\log^p x} = \frac{\left|(\log x)^{1-p}\right|_2^\infty}{1-p}$$

converges by the Integral Test if and only if $1-p<0$. So

$$p \in (1,\infty)$$

□

c) *Solution.* Let

$$a_k = \frac{k^p}{p^k}.$$

Consider

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\left(\frac{k+1}{k}\right)^p\left(\frac{1}{p}\right)\right| = \frac{1}{|p|} < 1$$

which implies

$$|p| > 1.$$

So

$$p \in (-\infty,-1)\cup(1,\infty).$$

□

e) *Solution.* Let

$$a_k := \sqrt{k^{2p}+1} - k^p = \frac{1}{\sqrt{k^{2p}+1}+k^p} \text{ and } b_k := \frac{1}{k^p}.$$

Since $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if $p>1$ by the p-Series Test, besides

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \frac{1}{\sqrt{1+\frac{1}{k^{2p}}}+1} = \begin{cases} 0 & \text{if } p<0 \\ \frac{1}{2} & \text{if } p>0 \\ \frac{1}{\sqrt{2}+1} & \text{if } p=0 \end{cases}.$$

By the Limit Comparison Test, we have the series converges when $p>1$. For $p<0$, consider

$$\sum_{k=1}^\infty \frac{1}{\sqrt{k^{2p}+1}+k^p} \geq \sum_{k=1}^\infty \frac{1}{2\sqrt{k^{2p}+1}}.$$

Since

$$\lim_{k\to\infty}\frac{1}{2\sqrt{k^{2p}+1}} = \frac{1}{2},$$

by the Divergence Test and the Comparison Test, we know the series diverges when $p<0$. Together above discussion, we conclude

$$p \in (1,\infty).$$

□

Exercise 6.3.5

Proof. Since $0 < \frac{1}{k} \leq 1$ and $\sin \frac{1}{k} > 0$ for all $k \in \mathbb{N}$, we know

$$1 + k \sin \frac{1}{k} > 0.$$

Hence

$$\lim_{k \to \infty} \left| \frac{a_k}{a_{k-1}} \right| = \lim_{k \to \infty} \left(\frac{1}{1 + k \sin \frac{1}{k}} \right) = \frac{1}{2} < 1$$

where

$$\lim_{k \to \infty} \left(k \sin \frac{1}{k} \right) = \lim_{k \to \infty} \left(\frac{\sin \frac{1}{k}}{\frac{1}{k}} \right) = \lim_{k \to \infty} \left(\cos \frac{1}{k} \right) = 1.$$

By the Ratio Test, we conclude

$$\sum_{k=1}^{\infty} a_k$$

converges absolutely. □

Exercise 6.3.6

a) *Proof.* For $N \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{N} a_{kj} \right) = \sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} a_{kj} \right)$$

by the Limit Theorem. Then fix N, for any $K \in \mathbb{N}$,

$$\sum_{k=1}^{K} \left(\sum_{j=1}^{N} a_{kj} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right).$$

If right hand side is finite, by the Monotone Convergence Theorem, we have

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{N} a_{kj} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right).$$

On the other hand if it is infinite, then it holds trivially. Hence for any $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{\infty} a_{kj} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right).$$

Use the Monotone Convergence Theorem again, we conclude

$$\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kj} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right).$$

□

b) *Proof.* Set $B_j = \sum_{k=1}^{\infty} a_{kj}$. From part a),

$$\sum_{j=1}^{\infty} B_j = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty} A_k$$

converges so does $\sum_{j=1}^{\infty} B_j$ which means both

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) \text{ and } \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right) \leq \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right)$$

hold. So we conclude

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right).$$

□

c) *Proof.* Consider

$$a_{kj} = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } j = k+1 \\ 0 & \text{otherwise} \end{cases}.$$

Then the equation from part b) means

$$1 = 0$$

which is obviously a contradiction. □

Exercise 6.3.8

a) *Proof.* Let $s := \liminf_{k\to\infty} x_k$ and $s_n := \inf_{k>n} x_k$. We observe that $\lim_{n\to\infty} s_n = s$. If $s > x$ for some $x \in \mathbb{R}$,

$$\exists N \in \mathbb{N} \text{ such that } s_n > x.$$

i.e. $x_k > x$, $\forall k > N$, as promised. □

b) *Proof.* If x_k converges to x, given $\epsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } k \geq N \implies |x_k - x| < \epsilon.$$

i.e. for any $n \geq N$,

$$x_k < x - \epsilon, \ \forall k > n.$$

Taking the infimum of this last inequality over $k > n$, we see that

$$s_n \leq x + \epsilon \text{ for any } n \geq N.$$

Hence, the limit of the s_n's satisfies $s \leq x + \epsilon$. Thus $s \leq x$.

A similar argument proves that $s \geq x$, so

$$s = \liminf_{k\to\infty} x_k = x.$$

□

c) *Proof.* Let

$$b_n = \inf_{k>n} \frac{a_{k+1}}{a_k} \text{ and } b = \lim_{n\to\infty} b_n.$$

For any $N \in \mathbb{N}$ where $N < n$. By definition of b_n, we have

$$b_N \leq \frac{a_{k+1}}{a_k} \text{ for } k = N+1, N+2, ..., n-1.$$

Notice that $b_n > 0$, $\forall n \in \mathbb{N}$. Multiplying these $n - N + 1$ inequalities together, we have

$$b_N^{n-N-1} \leq \frac{a_{N+1}}{a_n}.$$

Then taking n-th root of both sides, we see that

$$b_N^{1-\frac{N+1}{n}} \times \sqrt[n]{a_{N+1}} \leq \sqrt[n]{a_n}.$$

Taking the limit infimum on both sides and since the limit exists on the left hand side,

$$b_N \leq \liminf_{n\to\infty} \sqrt[n]{a_n}.$$

Finally, taking the limit on both sides as $N \to \infty$, we get

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n}.$$

A similar argument proves that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

On the other hand, by the Limit Comparison Theorem,

$$\liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n}$$

holds trivially.

Combining all above discussions and replacing the notation n with k, we conclude

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \leq \liminf_{k\to\infty} \sqrt[k]{a_k} \leq \limsup_{k\to\infty} \sqrt[k]{a_k} \leq \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

□

d) *Proof.* Notice $|b_n| > 0$, $\forall n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = r \in \mathbb{R},$$

we have

$$\liminf_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \limsup_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = r.$$

From part c), it squeeze that

$$\liminf_{n\to\infty} \sqrt[n]{|b_n|} = \limsup_{n\to\infty} \sqrt[n]{|b_n|} = r.$$

which means

$$\lim_{n\to\infty} |b_n|^{\frac{1}{n}} = r.$$

□