Chapter 7

Infinite Series of Functions

7.1 Uniform Convergence of Sequences

Exercise 7.1.2

a) *Proof.* Consider

$$f_n(x) = \frac{nx^{99}+5}{x^3+nx^{66}} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = x^{33}.$$

Since f_n is continuous for each n on $[1,3]$, then so does integrable. For any $x \in [1,3]$, we have

$$|f_n(x) - f(x)| = \left|\frac{5-x^{36}}{x^3+nx^{66}}\right| \le \frac{5+3^{36}}{x^3+nx^{66}} \le \frac{5+3^{36}}{n} \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[1,3]$ implies f is integrable on $[1,3]$. Finally, we conclude

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}}\,dx = \int_1^3 \left(\lim_{n\to\infty} \frac{nx^{99}+5}{x^3+nx^{66}}\right) dx = \int_1^3 x^{33} dx = \left.\frac{x^{34}}{34}\right|_1^3 = \frac{3^{34}-1}{34}.$$

□

b) *Proof.* Consider

$$f_n(x) = e^{\frac{x^2}{n}} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = 1.$$

Since f_n is continuous for each n on $[0,2]$, then so does integrable. For any $x \in [0,2]$, we have

$$|f_n(x) - f(x)| = \left|e^{\frac{x^2}{n}} - 1\right| \le e^{\frac{4}{n}} - 1 \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \ge N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[0,2]$ implies f is integrable on $[0,2]$. Finally, we conclude

$$\lim_{n\to\infty} \int_0^2 e^{\frac{x^2}{n}}\,dx = \int_0^2 \lim_{n\to\infty} \left(e^{\frac{x^2}{n}}\right) dx = \int_0^2 dx = x|_0^2 = 2.$$

□

c) *Proof.* Consider

$$f_n(x) = \sqrt{\sin\frac{x}{n} + x + 1}, \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = \sqrt{x+1}.$$

Since f_n is continuous for each n on $[0,3]$, then so does integrable. For any $x \in [0,3]$, we have

$$|f_n(x) - f(x)| = \frac{\left|\sin\frac{x}{n}\right|}{\sqrt{\sin\frac{x}{n} + x + 1} + \sqrt{x+1}} \le \sin\frac{3}{n} \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $[0,3]$ implies f is integrable on $[0,3]$. Finally, we conclude

$$\begin{aligned}
\lim_{n\to\infty} \int_0^3 \sqrt{\sin\frac{x}{n} + x + 1}\, dx &= \int_0^3 \lim_{n\to\infty} \left(\sqrt{\sin\frac{x}{n} + x + 1}\right) dx \\
&= \int_0^3 \sqrt{x+1}\, dx \\
&= \frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^3 \\
&= \frac{14}{3}.
\end{aligned}$$

□

Exercise 7.1.3

Proof. Since $f_n \to f$ uniformly on E, pick $\epsilon = 1$ and choose $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < 1 \text{ and } |f_n(x) - f_N(x)| < 1 \text{ for all } x \in E \text{ and } n \geq N.$$

Since each f_n is bounded, $\exists M_n > 0$ such that

$$|f_n(x)| \leq M_n \text{ for all } x \in E.$$

Therefore,

$$|f(x)| \leq |f_N(x)| + 1 \leq M_n + 1 \text{ and } |f_n(x)| \leq |f_N(x)| + 1 \leq M_n + 1.$$

Set $M := \max\{M_1, M_2, ..., M_N\} + 1$, we have

$$|f_n(x)| \leq M \text{ for all } x \in E \text{ and } n \in \mathbb{N}$$

which implies f_n is uniformly bounded on E and f is a bounded function on E. □

Exercise 7.1.6

Proof. Since $f_n \to f$ uniformly on E, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \frac{\epsilon}{3} \text{ for all } x \in E$$

Consider f_N and since each f_n is uniformly continuous on E, for the same ϵ, $\exists \delta > 0$ such that

$$|x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \text{ for all } x, y \in E.$$

Hence, for any $x, y \in E$ with $|x-y| < \delta$, we have

$$\begin{aligned}|f(x)-f(y)| &\leq |f(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

So, f is uniformly continuous on E. □

Exercise 7.1.8

Proof. Consider

$$f_n(x) = \left(1+\frac{x}{n}\right)^n e^{-x} \text{ and } f(x) = \lim_{n\to\infty} f_n(x) = 1.$$

Since each f_n is continuous on $\mathbb{R}$, then so does integrable. For any $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| = \left| e^{-x} \left(\left(1+\frac{x}{n}\right)^n - e^x \right) \right| \leq \left| \left(1+\frac{x}{n}\right)^n - e^x \right| \to 0 \text{ as } n \to \infty.$$

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon,$$

so $f_n \to f$ uniformly on $\mathbb{R}$ implies f is integrable on $\mathbb{R}$. Finally, we conclude

$$\lim_{n\to\infty} \int_a^b \left(1+\frac{x}{n}\right)^n e^{-x}\, dx = \int_a^b \lim_{n\to\infty} \left(\left(1+\frac{x}{n}\right)^n e^{-x} \right)\, dx = \int_a^b dx = b-a.$$

□