Homework 4
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Answers given in 5 s.f. where required.

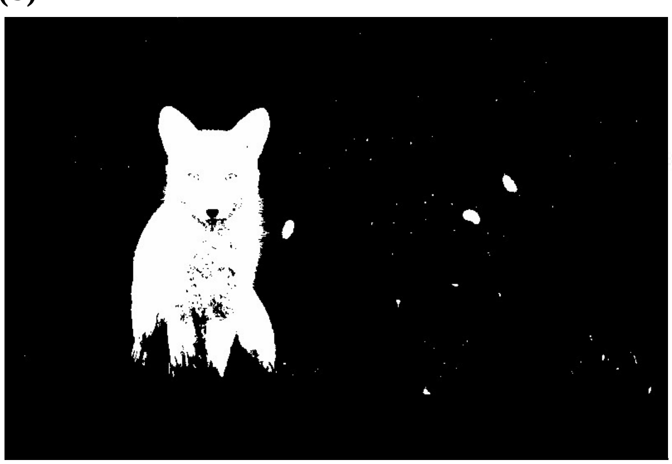
Q1
(a)
Package versions:
- Python: 3.10.10
- Jupyter: 6.5.4
- Numpy: 1.24.2
- OpenCV: 4.7.0

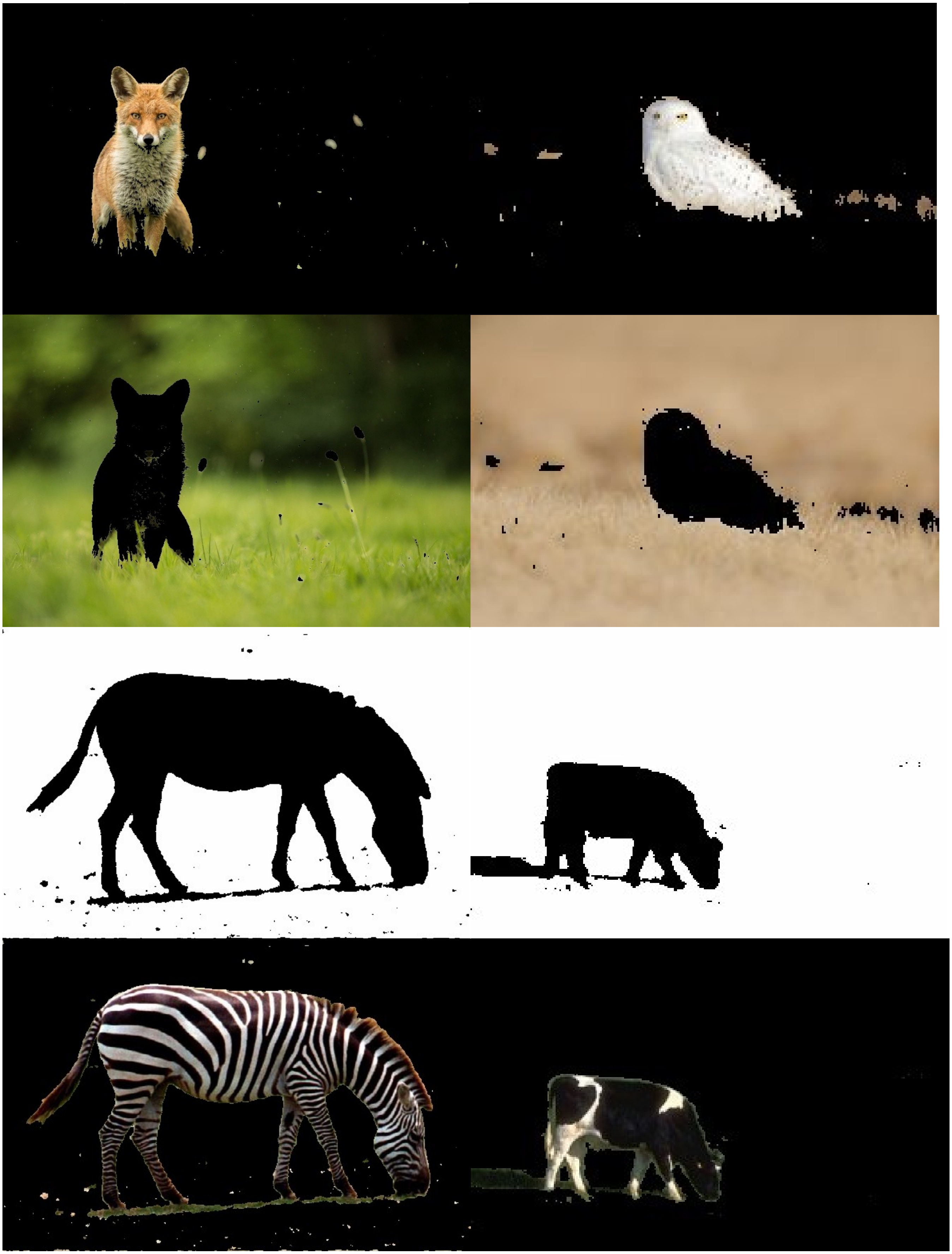
Steps:
1. Unzip `Homework4.zip` to extract `HW4.ipynb`.
2. Download `HW4_Q1_supp.zip` from eDimension.
3. Unzip `HW4_Q1_supp.zip`.
4. Copy `HW4_Q1_supp/data` and `HW4_Q1_supp/functions/io_data.py` to the directory containing `HW4.ipynb`.
5. Open `HW4.ipynb` with Jupyter notebook.
6. Run all cells.

Note:
Due to the random initialisation of the means (as required by the question) and the low contrast in `owl.jpg`, there is a good chance that my algorithm will fail to distinguish the owl from the background, resulting in the entire image being treated as the foreground or background.

Simply rerun the cell for `owl.jpg` until an image mask is produced.

If not for the question's restriction, a better approach would be to use k-means algorithm to initialise the means, so that the EM algorithm won't assign all of the pixels to one cluster.

(b)


(c)
For the EM algorithm, my main point of reference was slide 18 of the Gaussian mixture model slides. The algorithm takes in a file path to a text file containing the image data, and reads it using the provided `io_data.py`. It then extracts the Lab colour data into a Numpy array, which is used as the data set to run the EM algorithm on. In the E-step, it computes the responsibilities, then adjust the weights, means and covariance in the M-step. Once it is done with all of the iterations, it returns the responsibilities.

There is also the `gaussian()` function which is used to calculate the probability based on the image data and model parameters. I based it off of the probability density function formula of multivariate Gaussian distribution from Wikipedia.

I also do a rudimentary handling of NaN and 0 before returning, by replacing those values with 0.01. NaN is problematic as it will propagate, affecting the results of future iterations, whereas 0 could lead to a division by 0 error (which would also produce NaN).

The responsibilities produced by the EM algorithm is used by the `generate_images()` function to assign each pixel to either the foreground or the background layer. Then, an image mask is generated by making a deep copy of the image data, followed by replacing all of the pixels in one layer with black and the other layer with white.

To generate the foreground and background images, I use the same process as the for the image mask, but I preserve the pixels of the desired layer and replace the others with black. Note that I did not code out a way for the function to determine which layer is actually the foreground or the background, so the labelling in the code and output file names is actually arbitrary.

Q2

$$P(x_1|x_5)=\frac{P(x_1,x_5)}{P(x_5)}$$

The probability of $X_5=x_5$ depends on X_2 and X_3, but they are not observed, so we need to marginalise over them.

$$P(x_1|x_5)=\frac{P(x_1)\sum_{X_2}\sum_{X_3}P(X_2)P(X_3)P(x_5|X_2,X_3)}{\sum_{X_2}\sum_{X_3}P(X_2)P(X_3)P(x_5|X_2,X_3)}$$
$$=P(x_1)$$

Hence, $P(x_1|x_5)$ does not depend on the observation of X_5.

X_5	$p(x_1|x_5)$
T	0.2
F	0.2

$$P(x_2|x_4)=\frac{P(x_2,x_4)}{P(x_4)}$$
$$=\frac{P(x_2)\sum_{X_1}P(X_1)P(x_4|X_1,x_2)}{\sum_{X_1}\sum_{X_2}P(X_1)P(X_2)P(x_4|X_1,X_2)}$$

$$P(X_2=T|X_4=T)=\frac{0.44*(0.2*0.35+0.8*0.01)}{0.44*(0.2*0.35+0.8*0.01)+0.56*(0.2*0.6+0.8*0.95)}$$
$$=\frac{39}{599}$$

$$P(X_2=T|X_4=F)=\frac{0.44*(0.2*0.65+0.8*0.99)}{0.44*(0.2*0.65+0.8*0.99)+0.56*(0.2*0.4+0.8*0.05)}$$
$$=\frac{5071}{5911}$$

X_4	$p(x_2|x_4)$
T	0.065109
F	0.85789

$$
\begin{aligned}
P(x_3|x_2) &= \frac{P(x_3, x_2)}{P(x_2)} \\
&= \frac{P(x_3)P(x_2)}{P(x_2)} \\
&= 0.2
\end{aligned}
$$

$P(x_3|x_2)$ does not depend on the observation of X_2.

X_2	$P(x_3\|x_2)$
T	0.2
F	0.2

$$
\begin{aligned}
P(x_4|x_3) &= \frac{P(x_4, x_3)}{P(x_3)} \\
&= \frac{P(x_3)\sum_{X_1}\sum_{X_2} P(X_1)P(X_2)P(x_4|X_1, X_2)}{P(x_3)} \\
&= \sum_{X_1}\sum_{X_2} P(X_1)P(X_2)P(x_4|X_1, X_2) \\
&= P(x_4)
\end{aligned}
$$

$P(x_4|x_3)$ does not depend on the observation of X_3.

$$
\begin{aligned}
P(x_4) &= 0.44*(0.2*0.65+0.8*0.99)+0.56*(0.2*0.4+0.8*0.05) \\
&= 0.52712
\end{aligned}
$$

X_3	$P(x_4\|x_3)$
T	0.52712
F	0.52712

$$
\begin{aligned}
P(x_5) &= \sum_{X_2}\sum_{X_3} P(X_2)P(X_3)P(x_5|X_2, X_3) \\
&= 0.44*0.2*0.35+0.44*0.8*0.6+0.56*0.2*0.01+0.56*0.8*0.95 \\
&= 0.66872
\end{aligned}
$$

$P(x_5)$
0.66872