Linear Optimization - Homework 2

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Problem 1. (1^{st} Bullet)

Formally, following the argument on pg. 83, if $\mathbf{d}$ is a feasible direction at $\mathbf{x}$, that means $\exists\theta > 0$ s.t. $\mathbf{x} + \theta\mathbf{d} \in$ the feasible polyhedron, P. Since we're only looking for feasible solutions, $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{b}$. Also, since x is feasible, $\mathbf{Ax} = \mathbf{b}$. Since, $\theta > 0$ and subtracting b from both sides, it is apparent $Ad = 0$.

Now for the second part of the statement, we're moving in a feasible direction in the polyhedron, so $\mathbf{x} + \theta\mathbf{d} \geq 0$. That means for a given index, i, $x_i + \theta d_i \geq 0$. Since $\theta > 0$, $d_i \geq 0$ for any index where $x_i = 0$.

(2^{nd} Bullet)

We should follow the same analysis as above.

If we can move in the d direction then $\mathbf{A}(x + \theta\mathbf{d}) = \mathbf{b}$. Since $\mathbf{Ax} = \mathbf{b}$, those terms cancel leaving $\theta\mathbf{Ad} = 0$. We divide out θ leaving $\mathbf{Ad} = 0$. This proves the first statement of the set.

For, $\mathbf{Dd} \leq 0$, we again start with the definition of moving in a feasible direction. This gives us $\mathbf{D}(\mathbf{x} + \theta\mathbf{d}) \leq \mathbf{f}$. From the problem statement, $\mathbf{Dx} \leq \mathbf{f}$, so those terms cancel. That leaves us with $\theta\mathbf{Dd} \leq 0$. We divide out θ, leaving us with the second statement defining the set, $\mathbf{Dd} \leq 0$.

Problem 2. (1^{st} Bullet) The extreme points will be on the axes of the polyhedron. That means the extreme points will be $\{(0, 0, 1/3), (0, 1/2, 0), (1, 0, 0), (0, 0, 0)\}$.

(2^{nd} Bullet) Assuming cost can not be negative,

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix} \implies \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \forall c_1, c_2$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \implies \mathbf{c} = \begin{pmatrix} c_1 \\ 0 \\ c_3 \end{pmatrix} \forall c_1, c_3$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \implies \mathbf{c} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix} \forall c_2, c_3$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \forall c_2, c_3$$

Problem 3.12. (a) In standard form, our problem is described as

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2 \\ \text{st} \quad & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

BFS of $(x_1, x_2) = (0, 0)$ yields a cost of 0 and $x_3 = 2$ and $x_4 = 6$.

(b)

$$\begin{array}{c} \\ \\ x_3 \\ x_4 \end{array} \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 0 & -2 & -1 & 0 & 0 \\ \hline 2 & 1^* & -1 & 1 & 0 \\ 6 & 1 & 1 & 0 & 1 \end{array} \right] \tag{1}$$

$$\begin{array}{c} \\ \\ x_1 \\ x_4 \end{array} \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 4 & 0 & -3 & 2 & 0 \\ \hline 2 & 1 & -1 & 1 & 0 \\ 4 & 0 & 2^* & -1 & 1 \end{array} \right] \tag{2}$$

$$\begin{array}{c} \\ \\ x_1 \\ x_2 \end{array} \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline 10 & 0 & 0 & 0.5 & 1.5 \\ \hline 4 & 1 & 0 & .5 & .5 \\ 2 & 0 & 1 & -0.5 & 0.5 \end{array} \right] \tag{3}$$

That means we have an optimal solution at $x_1 = 4, x_2 = 2$.

(c) Graph goes here.

Problem 4.

$$\begin{array}{c} \\ \\ y_1 \\ y_2 \\ y_3 \end{array}\left[\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 \\ \hline -5 & -1 & -1 & -3 & -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 3 & 0 & 4 & 1^* & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & -3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -4 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

$$\begin{array}{c} \\ \\ x_5 \\ y_2 \\ y_3 \end{array}\left[\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 \\ \hline -1 & 1 & 5 & -3 & 7 & 0 & 2 & 0 & 0 \\ \hline 2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -7 & 0 & -1 & 1 & 0 \\ 1 & -1 & -4 & 3^* & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{5}$$

$$\begin{array}{c} \\ \\ x_5 \\ y_2 \\ x_3 \end{array}\left[\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 \\ \hline 0 & 0 & 1 & 0 & 7 & 0 & 2 & 0 & 1 \\ \hline 2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1^* & 0 & -7 & 0 & -1 & 1 & 0 \\ 1/3 & -1/3 & -4/3 & 1 & 0 & 0 & 0 & 0 & 1/3 \end{array}\right] \tag{6}$$

We still have y_2 in the basis which we must drive out. We apply a change of basis to drive it out.

$$\begin{array}{c} \\ \\ x_5 \\ x_2 \\ x_3 \end{array}\left[\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 2 & 1 & 0 & 0 & -17 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 7 & 0 & 1 & 1 & 0 \\ 1/3 & -1/3 & 0 & 1 & 28/3 & 0 & 0 & 0 & 1/3 \end{array}\right] \tag{7}$$

This means our Phase I BFS is $(x_5, x_2, x_3) = (2, 0, 1/3)$. We now re-introduce the original objective function.

$$\begin{array}{c} \\ \\ x_2 \\ x_3 \\ x_5 \end{array}\left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 6 & 1 & 0 & 0 & -82 & 0 \\ \hline 0 & 0 & 1 & 0 & 7 & 0 \\ 2 & -1/3 & 0 & 1 & 28/3^* & 0 \\ 1/3 & 1 & 0 & 0 & -17 & 1 \end{array}\right] \tag{8}$$

$$
\begin{array}{c}
\\
\\
x_2\\
x_4\\
x_5
\end{array}
\left[\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
660/28 & -218/28 & 0 & 246/28 & 0 & 0 \\
\hline
-3/2 & 1/4^* & 1 & -3/4 & 0 & 0 \\
3/14 & -1/28 & 0 & 3/28 & 1 & 0 \\
167/42 & 11/28 & 0 & 51/28 & 0 & 1
\end{array}\right] \tag{9}
$$

$$
\begin{array}{c}
\\
\\
x_1\\
x_4\\
x_5
\end{array}
\left[\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
-162/7 & 0 & 218/7 & -102/7 & 0 & 0 \\
\hline
-6 & 1 & 4 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
19/3 & 0 & -11/7 & 3^* & 0 & 1
\end{array}\right] \tag{10}
$$

$$
\begin{array}{c}
\\
\\
x_1\\
x_4\\
x_3
\end{array}
\left[\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
160/21 & 0 & 778/49 & 0 & 0 & 34/7 \\
\hline
1/3 & 1 & -4/7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
19/9 & 0 & -11/21 & 1 & 0 & 1/3
\end{array}\right] \tag{11}
$$

Problem 5. (a)

$$
\begin{array}{c}
\\
\\
x_1\\
x_2\\
x_3
\end{array}
\left[\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
339,000,000 & 0 & 30,000,000 & 309,000,000 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
500,000 & 0 & 1 & 0 & 0 & 0 \\
1,500,000 & 0 & 0 & 1 & 0 & 0
\end{array}\right] \tag{12}
$$

(b) The price of gasoline can go arbitrarily high because that does not change the geometry of the solution.

Problem 6. We formulate this as a transportation problem. The supplier nodes $\mathcal{S}$ will be the teams with "supply" or number of wins $\mathbf{x} = (x_1, x_2, ..., x_n)$. The consumer nodes, $\mathcal{C}$, will be games between S_i and S_j. That means there will be a total number of $\frac{n(n-1)k}{2}$ which is equal to $\sum_{i=1}^{n} x_i$. The arcs are between nodes in $\mathcal{S}$ and $\mathcal{C}$ and they represent the number of games won by i against team j. We then solve the transportation problem for a our test vector $\mathbf{x}$. If the problem for $\mathbf{x}$ is feasible then $\mathbf{x}$ is a valid solution.