

Table of Contents

- **0. Preliminaries**
 - Resources Used
 - Notation
- **1. Rings**
 - Def. Ring
 - Thm. Basic Ring Properties
 - Thm. Basic Ring with Unity Properties
 - Def. Zero-Divisor
 - Def. Nilpotent Element
 - Def. Idempotent Element
 - Thm. (Binomial Theorem)
 - Def. Integral Domain
 -
 - Def. Division Ring
 - Def. Field
 - Thm. On Integral Domains, Division Rings, and Fields
 - Thm. Basic Idempotent Properties
- **2. Subrings**
 - Def. Subring
 - Remarks
 - Def. Maximal Subring
 - Def. Opposite Ring
 - Thm. Self-Opposite iff Commutative
- **3. Ring Examples**
 - Def. Ring of Continuous Functions
 - Def. Matrix Ring
 - Def. Ring of Polynomials
 - Thm. Integral Domain Polynomials Properties
 - Def. Power Series Ring
 - Def. Laurent Ring and Series
 - Def. Boolean Ring
 - Thm. Structure Theorem for Boolean Rings
 - Def. Group Rings
 - Thm. Integers Modulo n
 - Cartesian Two Product
 - Remarks
- **4. Ring Characteristic**
 - Def. Characteristic
 - Thm. Basic Characteristic Properties
 - Example. Characteristic Examples
 - Thm. Characteristic of Cartesian Product Ring
 - Thm. '

- **5. Ideals**
 - Def. Ideal
 - Notation. Ideal
 - Thm. On Improper Ideals
 - Thm. Division Ring and Ideals
 - Def. Maximal Ideal
 - Def. Minimal Ideal
 - Thm. Existence of Maximal Ideal
 - Def. Prime Ideal
 - Thm. Nilpotents of a Commutative Ring
 - Thm. Prime Avoidance Lemma
- **6. Ideals and Generators**
 - Def. Ideal Generator
 - Def. Principal Ideal
 - Def. Principal Ideal Ring
 - Def. Principal Ideal Domain
 - Def. Finitely Generated Ideal
 - Thm. Basic Ideal Properties
 - Corollary
 - Thm. Ideals of Division Ring
 - Ideals of Matrices
 - Simple Ring
- **7. Algebra of Ideals**
 - Remark
 - Def. Nilpotent Ideal
 - Def. Nil Ideal
 - Thm. Nilpotent and Nil Ideal
- **8. Quotient Rings**
 - Def. Quotient

0. Preliminaries

Resources Used

- **Group Theory** notes by me, howion,
- **Introduction to Rings and Modules**, 2nd Revised Ed., by C. Musili,
- **Algebra** by Thomas W. Hungerford,
- **Abstract Algebra**, 3rd Ed., by David S. Dummit and Richard M. Foote.

Notation

- $0 \in \mathbb{N}$ and $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$.
- (m, n) denotes the **greatest common divisor** of m and n .
- lcm denotes the least common multiple function.
- $a \mid b$ denotes that the integer a divides the integer b .
- $M_n(R)$ denotes $n \times n$ matrices over R .
- $R[X]$ denotes the set of finite polynomials with the constants in R in the variable X .
- $R[X_1, \dots, X_n]$... in several variables.

1. Rings

From now on, fundamental knowledge of Group Theory (notes) are assumed.

Def. Ring

A set R with two binary operations $+$ and \cdot , respectively called **addition** and **multiplication**, is called a **ring** if:

- $(R, +, 0)$ is an abelian group,
- (R, \cdot) is a semigroup, and
- Distribute laws hold for $+$ and \cdot .

Notice R is necessarily non-empty as the additive identity $0 \in R$.

A ring is said to be **commutative** (but not abelian) if the semigroup is commutative.

If the semigroup has an identity (that is, if the multiplication is a monoid) then its identity, denoted with 1 or 1_R , is called the **identity element** or the **unity** (of the ring). Such identity is always unique (exercise).

If the ring is with unity, then an element $u \in R$ is said to be **unit** or **invertible** if there exists $v \in R$ such that $uv = vu = 1$. Such v is unique and is called **multiplicative inverse** (or simply **inverse**) of u and is denoted with u^{-1} .

Do not mistake unit with unity. The unity is the **trivial unit**.

The **set of all units** in the ring R is denoted by $\mathcal{U}(R)$.

The **set of all non-zero elements** of R is denoted by R^* .

The multiplication is called **trivial** if for all $a, b \in R$ we have $ab = 0$.

Thm. Basic Ring Properties

Let R be a ring, then for all $a, b \in R$

1. $0a = 0 = a0$.
2. $-(a \cdot b) = (-a)b = a(-b)$.
3. $(-a)(-b) = ab$

For all $m, n \in \mathbb{Z}$

4. $n(ab) = (na)b = a(nb)$.
5. $(mn)a = m(na) = n(ma)$.

Exercise

Thm. Basic Ring with Unity Properties

Let R be a ring with unity. Then

1. The 0 is never an unit unless $0 = 1$.
2. $0 = 1$ only if $R = \{0 = 1\}$, the **trivial ring** or the **zero ring**.
3. If u and v are units in R , then so is uv and $(uv)^{-1} = v^{-1}u^{-1}$.
4. $\mathcal{U}(R)$ is a group under multiplication, called the **group of units of R** .

Def. Zero-Divisor

Let R be a ring and $a \in R$. Then a is called a **left zero-divisor** if there exists $0 \neq b \in R$ such that $ab = 0$. It is defined analogously for the **right zero-divisor**.

If a is either left or right zero-divisor, then it is said to be a **zero divisor**.

Def. Nilpotent Element

Let R be a ring and $a \in R$. Then a is said to be **nilpotent** if there exists an positive integer n such that $a^n = 0$.

| Note that in any ring 0 is nilpotent which is called the **trivial nilpotent element**.

Def. Idempotent Element

Let R be a ring and $a \in R$. Then a is said to be **idempotent** if $a^2 = a$.

| Similarly, in any ring 0 and, if exists, 1 are idempotent which are called **trivial idempotents**.

We say two idempotent elements are **orthogonal** to each other if $ab = ba = 0$.

Thm. (Binomial Theorem)

Let R be a ring with identity, $n \in \mathbb{N}^+$ and for $a, b \in R$ we have $ab = ba$, then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Def. Integral Domain

A non-zero ring R is called an **integral domain** if it has no non-trivial zero-divisors.

Therefore, let R be an integral domain, and $a, b, c \in R$. If $ab = ac$, then either $a = 0$ or $b = c$.

Def. Division Ring

A ring $(R, +, \cdot)$ is called a **division ring** (or a **skew-field**) if, equivalently

1. Every non-zero element of R , denoted R^* , has a multiplicative inverse, or
2. (R^*, \cdot) forms a group.

Def. Field

A ring $(R, +, \cdot)$ is called a **field** if, equivalently (exercise)

1. It is a commutative division ring, or
2. R^* is abelian under multiplication.
3. It is a finite integral domain.

Thm. On Integral Domains, Division Rings, and Fields

Let R be a ring. Then

1. If R is a field, then it is a division ring.
2. If R is a division ring, then it is an integral domain.

Moreover,

3. If R is a division ring, then multiplicative cancellation holds for non-zero elements.
4. If R is an integral domain with unit, then only idempotent elements are 0 and 1.

| **Exercise**

Thm. Basic Idempotent Properties

Let R be a ring, and $a \in R$ idempotent. Then

1. $1 - a$ is idempotent as well.
2. If a is non-trivial, it is a zero-divisor as well. This shows that integral domains and division rings do not have such idempotents.

| **Exercise**

2. Subrings

Def. Subring

Let $(R, +, \cdot)$ be a ring and S a non-empty subset of R . Then $(S, +, \cdot)$ is called a subring if:

- $(S, +)$ is a subgroup of $(R, +)$, and
- (S, \cdot) is a sub-semigroup of (R, \cdot) .

| $\{0\}$ and R are called the **trivial subrings**.

The **center of R** is, similar to groups, defined as

$$Z(R) = \{r \in R \mid rx = xr \text{ for all } x \in R\}$$

is a subring, and any subring of $Z(R)$ is called a **central subring**.

Beware that existence of unity in subring or the ring does not imply existence of unity in the other. Indeed, if they both have unity, they are not necessarily equal.

Same issue is also true for the units. Remember, for multiplication operation, we are assuming sub-semigroup not subgroup.

Remarks

Let R be a ring and $S \leq R$ its subring. Then

1. S may be commutative even if R is not.
2. S may not have unity even if R does. Moreover, S may have unity even if R does not.
3. S and R may have different unities.

| **Exercise**, show this is also the case for zero-divisors and units.

Def. Maximal Subring

Let R a ring and S a subring of R , then S is said to be **maximal subring** if $S \neq R$ and for any subring T of R we have

$$S \subseteq T \subseteq R \implies T = S \vee T = R$$

| Notice how we exclude the ring itself to be called maximal subring in itself.

Def. Opposite Ring

Given a ring R , the **opposite ring** is the ring with the same set of elements and same additive operation but multiplication reversed.

Thm. Self-Opposite iff Commutative

A ring R is it's **self-opposite** (isomorphic to it's opposite) if and only if R is commutative.

3. Ring Examples

Def. Ring of Continuous Functions

Let R be the set of real valued continuous functions from the topological space X to \mathbb{R} . For $f, g \in R$, define the pointwise operations for all $x \in X$ as

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (fg)(x) &= f(x)g(x)\end{aligned}$$

Then R is a commutative ring with unity where the additive identity is the constant map $\mathbf{0}$ and the unity is the constant map $\mathbf{1}$.

Def. Matrix Ring

| TODO

Def. Ring of Polynomials

Let R be a ring and X an *indeterminate* or *variable* over R . Define the set called **ring of polynomials over R** as

$$R[x] = \{ a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \mid a_i \in R, n \in \mathbb{N}^+ \}$$

| We could have actually wrote $n \in \mathbb{N}$ since $X^0 = 1$, but we don't know if R is with identity.

then $R[X]$ is a ring where addition and multiplication defined as expected over polynomials. Notice how elements of $R[X]$ are of finite length, so this a set of **finite polynomials**.

Let $a_0 + a_1X + \cdots + a_nX^n = p(X) \in R[X]$. Then

- n is called the **degree** of $p(X)$ denoted with $d(p)$. If $p(X)$ is the zero polynomial it is defined to be 0,
- a_n is called the **leading coefficient** of $p(X)$,
- $p(X)$ is said to be **monic** if $a_n = 1$.

Thm. Integral Domain Polynomials Properties

Let R be an integral domain and $p(X), q(X) \in R[X]$, then

- $d(p(X)q(X)) = d(p(X)) + d(q(X))$,
- Units of $R[X]$ are the units of R ,
- $R[X]$ is also an integral domain.

► **Proof**

Def. Power Series Ring

If we extend the definition of $R[X]$ to infinite polynomials we have

$$R[[X]] = \{ a_0 + a_1X + a_2X^2 + \cdots \mid a_i \in R \}$$

which is called the **power series over R** and is also a ring (exercise).

Def. Laurent Ring and Series

Similar to ring of polynomials, Let R be a ring then a **Laurent polynomial** denoted with $R[X^{\pm 1}]$ or $R[X, X^{-1}]$ is the set

$$R[X^{\pm 1}] = \{ a_{-n}X^{-n} + \cdots + a_{-1}X^{-1} + a_0 + a_1X + \cdots + a_mX^m \mid m, n \in \mathbb{Z}^*, a_i \in R \}.$$

If we further extend this definition for the positive part as we did in power series the resulting set is called the **Laurent series** denoted with $R\langle X \rangle$. Moreover, the bounded negative side is called the **principal part** and the other is called the **power series** part.

Def. Boolean Ring

A ring R in which every element is idempotent is called a **boolean ring**.

Thm. Structure Theorem for Boolean Rings

Every boolean ring is a subring of $(\mathcal{P}(X), \triangle, \cap)$, **the universal boolean ring**, for some set X where the addition operation \triangle is the *symmetric difference* of two sets.

Notice that under these operations $\mathcal{P}(X)$ is a commutative ring with unity. Moreover, a boolean ring need not to have unity. Take X infinite, then the subring which consists of all finite subsets of X has no unity.

Def. Group Rings

Let $(R, +, \cdot)$ be a commutative ring with identity $1 \neq 0$, and $(G, *) = \{ g_1, g_2, \dots, g_n \}$ a finite group. Define the **group ring RG** of G with coefficients in R as the set

$$RG = \{ a_1g_1 + a_2g_2 + \cdots + a_ng_n \mid a_i \in R \text{ and } 1 \leq i \leq n \}$$

Notice $a_i g_i$ multiplication is not defined.

If g_1 is the identity of G , then instead of a_1g_1 , simply, a_1 will be written.

Addition and multiplication in RG is defined componentwise on coefficients canonically. This makes RG a ring (exercise).

Thm. Integers Modulo n

Let $n \in \mathbb{Z}^+$ so that $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$. Then \mathbb{Z}_n is a ring and for $\bar{x} \in \mathbb{Z}_n$

1. \bar{x} is a unit $\iff \bar{x}$ is not a zero-divisor $\iff x$ is coprime to n .
2. \mathbb{Z}_n is an integral domain $\iff n$ prime $\iff \mathbb{Z}_n$ is a field.
3. \mathbb{Z}_n^* forms an additive subgroup $\iff n$ is a power of a prime.
4. If n prime, then $x^n = x$ for all $x \in \mathbb{Z}_n$.

Exercise

Cartesian Two Product

Let $\{ R_i \}$ be a non-empty family of rings, then $\prod \{ R_i \}$ is a ring called the **direct product** of $\{ R_i \}$'s under component-wise addition and multiplication.

Remarks

Let $T = \prod \{ R_i \}$ be the direct product of family of rings $\{ R_i \}$ with at least two rings.

1. T is a ring with identity if and only if each R_i is with identity.
2. T is commutative if and only if each R_i is commutative.
3. Even if each R_i is a field, T is not even an integral domain.

4. Ring Characteristic

Def. Characteristic

Let R be any ring. The **characteristic** of R , denoted by $\text{Char}(R)$ is the least positive integer n such that $na = 0$ for all $a \in R$. If such n does not exist, then it is defined to be 0.

Thm. Basic Characteristic Properties

1. $\text{Char}(R) = 1$ if and only if $R = \langle 0 \rangle$.
2. $\text{Char}(R) = 0$ if and only if the additive order $|1|$ is infinite.
3. $\text{Char}(R) = n \neq 0$ if and only if the additive order $|1|$ is equal to n .

Example. Characteristic Examples

1. $\text{Char}(\mathbb{Z}) = \text{Char}(\mathbb{Q}) = \text{Char}(\mathbb{R}) = \text{Char}(\mathbb{C}) = \text{Char}(\mathbb{H}) = 0$,
2. $\text{Char}(M_n(R)) = \text{Char}(R)$,
3. $\text{Char}(R) = \text{Char}(R[x]) = \text{Char}(R[[x]])$,
4. $\text{Char}(\mathbb{Z}_n) = n$

► **Proof**

Thm. Characteristic of Cartesian Product Ring

Let R and S be rings, then their characteristic is

1. 0 if either R or S has characteristic 0, or
2. $\text{lcm}(\text{Char}(R), \text{Char}(S))$.

Thm. '

Suppose R is a ring with 1 whose non-units forms a subgroup under addition. Then either,

1. $\text{Char}(R) = 0$, or
2. $\text{Char}(R) = p^n$ where p prime and n positive integer.

► **Proof**

5. Ideals

Def. Ideal

Let R be a ring. A subset I of R is called a **left (respectively right) ideal** of R if

1. $I \leq (R, +)$, and
2. for all $a \in R$ we have $aI \subseteq I$ (respectively $Ia \subseteq I$), under the ring multiplication.

If I is both a left and a right ideal, then it is called a **two-sided ideal** or a **2-sided ideal**. Notice that in this case we have $Ia = aI = I$.

Noting a ring R is an ideal of itself, such ideal R is called the **unit ideal**. $(0) = \{0\}$ is also an ideal in R called the **zero ideal**. These two ideals are called the **trivial ideals** of R .

| Notice how the concept of an ideal is similar to the concept of a coset in group theory.

Notation. Ideal

Let R be a ring and $x \in R$, then

- (x) denotes the 2-sided ideal generated by x .
- $(x)_l$ denotes the left-ideal generated by x .
- $(x)_r$ denotes the right-ideal generated by x .

| More on generations later.

Thm. On Improper Ideals

Let R be a ring and I a left (resp. right) ideal, then the following are equivalent

1. $I = R$,
2. $1 \in I$,
3. I has an unit, or just
4. I has an element which has an left (resp. right) inverse.

► **Proof**

Thm. Division Ring and Ideals

Let R be a ring with 1, then R is a division ring if and only if (0) and R are the only left ideals (or the only right ideals) in R .

This is also true if R , instead of a ring with identity, is a non-zero ring with non-trivial multiplication. For the proof of this theorem check out Musli pp. 43-44.

► Proof

Def. Maximal Ideal

A left (resp. right or 2-sided) ideal I of a ring R is called **maximal ideal** in R if for any left (resp. right or 2-sided) ideal J of R we have

$$I \subseteq J \subseteq R \implies J = I \vee J = R$$

where $I \neq R$. Thus, we exclude unit ideal to be called maximal ideal.

Def. Minimal Ideal

Similar to maximal ideal, a left (resp. right or 2-sided) ideal of R is called a **minimal ideal** in R if for any left (resp. right or 2-sided) ideal J of R we have

$$(0) \subseteq J \subseteq I \implies J = (0) \vee J = I$$

where $I \neq (0)$. Thus, we exclude zero ideal to be called minimal ideal.

Thm. Existence of Maximal Ideal

Let R be a ring with 1 and I its left (resp. right or 2-sided) ideal such that $I \neq R$. Then there exists left (resp. right or 2-sided) maximal ideal M such that $I \subseteq M$.

This theorem need not to be true for minimal ideals even if the ring is commutative. For example, take the ring \mathbb{Z} and its ideal $2\mathbb{Z}$.

► Proof

Def. Prime Ideal

Also see **Wiki**: Prime Ideal.

Let R be a commutative ring and I its ideal. I is called a **prime ideal** if $I \neq R$ and for all $x, y \in R$

$$xy \in I \implies x \in I \vee y \in I.$$

Clearly non-trivial R is a commutative integral domain if and only if (0) is a prime ideal in R .

Thm. Nilpotents of a Commutative Ring

The set of all nilpotent elements in a commutative ring R with 1 is the intersection of all prime ideals.

Thm. Prime Avoidance Lemma

Let R be a commutative ring, $A \leq R$, and $I_1, I_2, \dots, I_n \leq R$ such that I_i is prime for $i \geq 3$ (that is at most two ideals are not prime). Then

If $A \not\subseteq I_j$ for any one j , then $A \not\subseteq \bigcup_{1 \leq k \leq n} I_k$. So that if A is not contained in any of the ideals, it is also not contained in their union.

6. Ideals and Generators

Def. Ideal Generator

Let R a ring and $X \subseteq R$. Then the **left ideal generated by X** is the smallest left ideal in R which contain X , or equivalently intersection of all left ideals which contain X .

In particular, if $X = \emptyset$, then the left ideal generated by X is (0) .

If $X = \{x\}$, then the left ideal generated by X is

$$\{ nx + rx \mid n \in \mathbb{Z}, r \in R \}$$

Notice that if the ring is with identity, then the set is equal to $\{ sx \mid s \in R \}$. This set is denoted by $(x)_l$ called the **left ideal generated by x** .

Def. Principal Ideal

A left ideal I in R is called a **principal left ideal** if $I = (x)_l$ for some $x \in I$.

| Definition of right principal ideal is similar which is denoted by $(x)_r$.

(x) denotes the 2-sided ideal generated by x .

Def. Principal Ideal Ring

A ring R is called a **principal ideal ring (P.I.R.)** if

- R is commutative, and
- Every ideal of R is principal.

Def. Principal Ideal Domain

A principal ideal ring R which is an integral domain is called a **principal ideal domain (P.I.D.)**.

Def. Finitely Generated Ideal

We say an ideal I in a ring R is **finitely generated** if there exists a finite $X \subseteq R$ which generates I . Notice that in this case, every element of I can be expressed as a R -linear combinations of $x_1, \dots, x_n \in X$.

Thm. Basic Ideal Properties

Let R be a ring with 1 and I a left ideal of R , then the following are equivalent

1. $I = R$,
2. $1 \in I$,
3. I contains an unit, or in particular
4. I contains an element which has a left inverse.

By symmetry, of course, this also holds for right ideals where (4) is right inverse.

For the 2-sided ideal case (4) is left inverse OR right inverse.

Corollary

Let $I = (x)_l$, then $I = R$ if and only if x has a left inverse.

Thm. Ideals of Division Ring

Let R be a ring with 1, then the following are equivalent

- R is a division ring,
- (0) and R are the only left ideals in R ,
- (0) and R are the only right ideals in R .

Notice that if R is commutative ring with 1, then it is a field in this case.

This theorem also holds if R is a non-zero ring with non-trivial multiplication.

Ideals of Matrices

Let R be a ring with 1 and $S = M_n(R)$, then if I is a left (resp. right or 2-sided) ideal in R , then $M_n(I)$ is a left (right or 2-sided) ideal in S .

Simple Ring

A non-zero ring R is said to be a **simple ring** if only 2-sided ideals of R are (0) and R .

This implies that $M_n(R)$ is simple if R is simple. Similarly, division rings are therefore simple.

7. Algebra of Ideals

Let R be a ring and I and J its ideals of the same kind. Then the **addition of ideals** I and J is defined as

$$I + J = \{ i + j \mid i \in I, j \in J \} \subseteq R$$

which is an ideal of the same kind in R . Such addition is commutative and associative.

The **multiplication of ideals** I and H is defined as

$$IJ = \{ i_1 j_1 + \cdots + i_n j_n \mid i_i \in I, j_i \in J \}$$

that is the finite sums of products of pairs from I and J . This product is again an ideal in R . Such multiplication is associative but not necessarily commutative.

Notice how the multiplication in Ring Ideals differs from the subset/subring/coset multiplication in Group Theory.

Remark

Let R be a ring and $I, J \subseteq R$.

1. IJ is a left ideal if I is an ideal.
2. IJ is a right ideal if J is an ideal.
3. IJ is a 2-sided ideal if I is a left ideal and J is a right ideal.
 1. $IR \subseteq I$ and $RI \subseteq I$.
 2. Moreover, if R is with 1, then $IR = I = RI$.

Def. Nilpotent Ideal

Let I be an ideal. If for some $n \geq 1$ we have $I^n = (0)$, then I is called a **nilpotent ideal**.

Def. Nil Ideal

Let I be an ideal such that every element of I is nilpotent, then I is called a **nil ideal**.

Thm. Nilpotent and Nil Ideal

Every nilpotent ideal I is a nil ideal. Converse is not necessarily true unless I is finitely generated.

8. Quotient Rings

Def. Quotient

Let R be a ring and I