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0. Preliminaries

Resources Used

- **Group Theory** notes by me, howion,
- **Introduction to Rings and Modules**, 2nd Revised Ed., by C. Musili,
- **Algebra** by Thomas W. Hungerford,
- **Abstract Algebra**, 3rd Ed., by David S. Dummit and Richard M. Foote.

Notation

- $0 \in \mathbb{N}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.
- (m, n) denotes the **greatest common divisor** of m and n.

1. Rings

> From now on, knowledge of the Group Theory notes are assumed.

Def. Ring

A set R with two binary operations $+$ and $\cdot$, respectively called **addition** and **multiplication**, is called a **ring** if:

- $(R, +, 0)$ is an abelian group,
- $(R, \cdot)$ is a semigroup, and
- Distribute laws hold for $+$ and $\cdot$.

> Notice R is necessarily non-empty as the additive identity $0 \in R$.

A ring is said to be **commutative** (but not abelian) if the semigroup is commutative.

If the semigroup has an identity (that is, if the multiplication is a monoid) then its identity, denoted with 1 or 1_R, is called the **identity element** or the **unity** (of the ring). Such identity is always unique (exercise).

If the ring is with unity, then an element $u \in R$ is said to be **unit** or **invertible** if there exists $v \in R$ such that $uv = vu = 1$. Such v is unique and is called **multiplicative inverse** (or simply **inverse**) of u and is denoted with u^{-1}.

> Do not mistake unity with unit. There may be one unique unit and if there is there may be many units.

The **set of all units** in the ring R is denoted by $\mathcal{U}(R)$.

The **set of all non-zero elements** of R is denoted by R^*.

The multiplication is called **trivial** if for all $a, b \in R$ we have $ab = 0$.

Thm. Basic Ring Properties

Let R be a ring, then

- The 0 is never an unit unless $0 = 1$.
- $0 = 1$ only if $R = \{0 = 1\}$, the **trivial ring** or the **zero ring**.

For all $a, b \in R$

- $0a = 0 = a0$.
- $-(a \cdot b) = (-a)b = a(-b)$.
- $(-a)(-b) = ab$

For all $m, n \in \mathbb{Z}$

- $n(ab) = (na)b = a(nb)$.
- $(mn)a = m(na) = n(ma)$.

▶ **Proof**

Thm. Basic Ring with Unity Properties

Let R be a ring with unity. Then

1. If u and v are units in R, then so is uv and $(uv)^{-1} = v^{-1}u^{-1}$.
2. $\mathcal{U}(R)$ is a group under multiplication, called the **group of units of** R.
3. Unless the ring is trivial, 0 is never a unit.

Def. Zero-Divisor

Let R be a ring and $a \in R$. Then a is called a ***left* zero-divisor** if there exists $0 \neq b \in R$ such that $ab = 0$. It is defined analogously for the ***right* zero-divisior**.

If a is either left or right zero-divisor, then it is said to be a **zero divisor**.

Def. Nilpotent Element

Let R be a ring and $a \in R$. Then a is said to be **nilpotent** if there exists an positive integer n such that $a^n = 0$.

> Note that in any ring 0 is nilpotent which is called the **trivial nilpotent element**.

Def. Idempotent Element

Let R be a ring and $a \in R$. Then a is said to be **idempotent** if $a^2 = a$.

> Similarly, in any ring 0 and, if exists, 1 are idempotent which are called **trivial idempotents**.

We say two idempotent elements are **orthogonal** to each other if $ab = ba = 0$.

Thm. (Binomial Theorem)

Let R be a ring with identity, $n \in \mathbb{N}^+$ and for $a, b \in R$ we have $ab = ba$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Def. Integral Domain

A non-zero ring R is called an **integral domain** if it has no non-trivial zero-divisors.

Thm. ′

Let R be an integral domain, and $a, b, c \in R$. If $ab = ac$, then either $a = 0$ or $b = c$.

Def. Division Ring

A ring $(R, +, \cdot)$ is called an **division ring** (or a **skew-field**) if, equivalently

1. $(R \setminus \{0\}, \cdot)$ forms a group, or
2. Every non-zero element of R, denoted R^*, has a multiplicative inverse.

Def. Field

A ring $(R, +, \cdot)$ is called a **field** if, equivalently (exercise)

1. It is a commutative division ring, or
2. R^* is abelian under multiplication.
3. It is a finite integral domain.

Thm. On Integral Domains, Division Rings, and Fields

Let R be a ring. Then

1. If R is a field, then it is a division ring.
2. If R is a division ring, then it is an integral domain.

3. If R is a division ring, then multiplicative cancellation holds for non-zero elements.
4. If R is an integral domain with unit, then only idempotent elements are 0 and 1.

Thm. Basic Idempotent Properties

Let R be a ring, and $a \in R$ idempotent. Then

1. $1 - a$ is idempotent as well.
2. If a is non-trivial, it is a zero-divisor as well. This shows that integral domains and division rings do not have such idempotents.

2. Subrings

Def. Subring

Let $(R, +, \cdot)$ be a ring and S a non-empty subset of R. Then $(S, +, \cdot)$ is called a subring if:

- $(S, +)$ is a subgroup of $(R, +)$, and
- $(S, \cdot)$ is a sub-semigroup of $(R, \cdot)$.

> $\{0\}$ and R are called the **trivial subrings**.

The **center of** R is, similar to groups, defined as

$$Z(R) = \{r \in R \mid rx = xr \text{ for all } x \in R\}$$

is a subring, and any subring of $Z(R)$ is called a **central subring**.

> **Beware** that existence of unity in subring or the ring does not imply existence of unity in the other. Indeed, if they both have unity, they are not necessarily equal.
>
> Same issue is also true for the units. Remember, for multiplication operation, we are assumming sub-semigroup not subgroup.

Def. Maximal Subring

Let R a ring and S a subring of R, then S is said to be **maximal subring** if $S \neq R$ and for any subring T of R we have

$$S \subseteq T \subseteq R \implies T = S \ \vee \ T = R$$

> Notice how we exclude the ring itself to be called maximal subring in itself.

Def. Opposite Ring

Given a ring R, the **opposite ring** is the ring with the same set of elements and same additive operation but multiplication reversed.

Thm. Self-Opposite iff Commutative

A ring R is it’s **self-opposite** (isomorphic to it’s opposite) if and only if R is commutative.

3. Ring Examples

Def. Ring of Continious Functions

Let R be the set of real valued continious functions from the topological space X to $\mathbb{R}$. For $f, g \in R$, define the pointwise operations for all $x \in X$ as

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (fg)(x) &= f(x)g(x) \end{aligned}$$

Then R is a commutative ring with unity where the additive identity is the constant map $\mathbf{0}$ and the unity is the constant map $\mathbf{1}$.

Example

Def. Matrix Ring

Def. Ring of Polynomials

Let R be a ring and x an *indeterminate* or *variable* over R. Define the set called **ring of polynomials over** R as

$$R[x] = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \;\middle|\; a_i \in R, n \in \mathbb{N}^* \right\}$$

> We could have actually wrote $n \in \mathbb{N}$ since $X^0 = 1$, but we don't know if R is with identity.

then $R[x]$ is a ring where addition and multiplication defined as expected over polynomials. Notice how elements of $R[x]$ of finite length, so this a set of **finite polynomials**.

Let $a_0 + a_1 x + \cdots + a_n x^n = p(x) \in R[x]$. Then

- n is called the **degree** of $p(x)$ denoted with $\mathrm{d}(p)$. If $p(x)$ is the zero polynomial it is defined to be 0,
- a_n is called the **leading coefficient** of $p(x)$,
- $p(x)$ is said to be **monic** if $a_n = 1$.

Thm. Integral Domain Polynomials Properties

Let R be an integral domain and $p(x), q(x) \in R[x]$, then

- $\mathrm{d}(p(x)q(x)) = \mathrm{d}(p(x)) + \mathrm{d}(q(x))$,
- Units of $R[x]$ are the units of R,
- $R[x]$ is also an integral domain.

▶ **Proof**

Def. Power Series Ring

If we extend the definition of $R[[x]]$ to infinite polynomials, that is the set

$$F[[x]] = \{\, a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in R \,\}$$

is called the **power series over** R and is also a ring (exercise).

Def. Boolean Ring

A ring R in which every element is idempotent is called a **boolean ring**.

Thm. Structure Theorem for Boolean Rings

Every boolean ring is a subring of $\mathcal{P}(X)$, **the universal boolean ring**, for some set X.

Def. Group Rings

Let $(R, +, \cdot)$ be a commutative ring with identity $1 \neq 0$, and $(G, *) = \{\, g_1, g_2, \ldots, g_n \,\}$ a finite group. Define the **group ring** RG of G with coefficients in R as the set

$$RG = \{\, a_1 g_1 + a_2 g_2 + \cdots + a_n g_n \mid a_i \in R \ \text{ and } \ 1 \leq i \leq n \,\}$$

> Notice $a_1 g_1$ multiplication is not defined.

If g_1 is the identity of G, then instead of $a_1 g_1$, simply, a_1 will be written.

Addition and multiplication in RG is defined componentwise on coefficients canonically. This makes RG a ring (exercise).

> **(Exercise)** define addition (obvious) and multiplication.

5. Ideals

Def. Ideal

Let R be a ring. A subset I of R is called a **left (respectively right) ideal** of R if

1. $I \leq (R, +)$, and
2. for all $a \in R$ we have $aI \subseteq I$ (respectively $Ia \subseteq I$), under the ring multiplication.

If I is both a left and a right ideal, then it is called a **two-sided ideal** or a **2-sided ideal**. Notice that in this case we have $Ia = aI = I$.

Noting a ring R is an ideal of itself, such ideal R is called the **unit ideal**. $(0) = \{0\}$ is also an ideal in R called the **zero ideal**. These two ideals are called the **trivial ideals** of R.

> Notice how the concept of an ideal is similar to the concept of a coset in group theory.

Thm. On Improper Ideals

Let R be a ring and I a left (resp. right) ideal, then the following are equivalent

1. $I = R$,
2. $1 \in I$,
3. I has an unit, or just
4. I has an element which has an left (resp. right) inverse.

▶ **Proof**

Thm. Division Ring and Ideals

Let R be a ring with 1, then R is a division ring if and only if (0) and R are the only left ideals (or the only right ideals) in R.

> This is also true if R, instead of a ring with identity, is a non-zero ring with non-trivial multiplication. For the proof of this theorem check out Musli pp. 43-44.

▶ **Proof**

Def. Maximal Ideal

A left (resp. right or 2-sided) ideal I of a ring R is called **maximal ideal** in R if for any left (resp. right or 2-sided) ideal J of R we have

$$I \subseteq J \subseteq R \implies J = I \ \vee \ J = R$$

where $I \neq R$. Thus, we exclude unit ideal to be called maximal ideal.

Def. Minimal Ideal

Similar to maximal ideal, a left (resp. right or 2-sided) ideal of R is called a **minimal ideal** in R if for any left (resp. right or 2-sided) ideal J of R we have

$$(0) \subseteq J \subseteq I \implies J = (0) \ \vee \ J = I$$

where $I \neq (0)$. Thus, we exclude zero ideal to be called minimal ideal.

Thm. Existence of Maximal Ideal

Let R be a ring with 1 and I its left (resp. right or 2-sided) ideal such that $I \neq R$. Then there exists left (resp. right or 2-sided) maximal ideal M such that $I \subseteq M$.

> This theorem need not to be true for minimal ideals even if the ring is commutative. For example, take the ring $\mathbb{Z}$ and its ideal $2\mathbb{Z}$.

▶ **Proof**

Def. Prime Ideal

> Also see **Wiki:** Prime Ideal.

Let R be a commuative ring and I its ideal. I is called a **prime ideal** if $I \neq R$ and for all $x, y \in R$

$$xy \in I \implies x \in I \ \vee \ y \in I.$$

Thm. Nilpotents of a Commutative Ring

The set of all nilpotent elements in a commutative ring R with 1 is the intersection of all prime ideals.

Thm. Prime Avoidance Lemma

Let R be a commutative ring, $A \leq R$, and $I_1, I_2, ..., I_n \trianglelefteq R$ such that I_i is prime for $i \geq 3$ (that is at most two ideals are not prime). Then

If $A \nsubseteq I_j$ for any one j, then $A \nsubseteq \bigcup_{1 \leq k \leq n} I_k$. So that if A is not contained in any of the ideals, it is also not contained in their union.

Def. Simple Ring

A non-zero ring R is called **simple** if R has non 2-sided ideals other than (0) and R.

Therefore, division rings are simple.

4. Ring Characteristic

Def. Characteristic

Let R be any ring. The **characteristic** of R, denoted by $\mathrm{Char}(R)$ is the least positive integer n such that $na = 0$ for all $a \in R$. If such n does not exists, then it is defined to be 0.

Thm. Basic Characteristic Properties

1. $\mathrm{Char}(R) = 1$ if and only if $R = \langle 0 \rangle$.
2. $\mathrm{Char}(R) = 0$ if and only if the additive order $|1|$ is infinite.
3. $\mathrm{Char}(R) = n \neq 0$ if and only if the additive order $|1|$ is equal to n.

Example. Characteristic Examples

1. $\mathrm{Char}(\mathbb{Z}) = \mathrm{Char}(\mathbb{Q}) = \mathrm{Char}(\mathbb{R}) = \mathrm{Char}(\mathbb{C}) = \mathrm{Char}(\mathbb{H}) = 0$,
2. $\mathrm{Char}(M_n(R)) = \mathrm{Char}(R)$,
3. $\mathrm{Char}(R) = \mathrm{Char}(R[x]) = \mathrm{Char}(R[[x]])$,
4. $\mathrm{Char}(\mathbb{Z}_n) = n$

▶ **Proof**

Thm. Characteristic of Cartesian Product Ring

Let R and S be rings, then their characteristic is

1. 0 if either R or S has characteristic 0, or
2. $\mathrm{lcm}(\mathrm{Char}(R), \mathrm{Char}(S))$.

Thm. ′

Suppose R is a ring with 1 whose non-units forms a subgroup under addition. Then either,

1. $\mathrm{Char}(R) = 0$, or
2. $\mathrm{Char}(R) = p^n$ where p prime and n positive integer.

▶ **Proof**