Isomorphism Theorems

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Def. Projection

Let $N \trianglelefteq G$. Then

$$\begin{array}{rccc} \pi: & G & \to & G/N \\ & a & \mapsto & aN \end{array}$$

is an epimorphism and $\mathrm{Ker}\ \pi = N$. Such π is called the **canonical epimorphism** or **(natural) projection**. Therefore, unless otherwise stated, $G \to G/N$ always denotes the cannonical epimorphism.

▶ **Proof**

Thm. Fundamental Theorem on Homomorphisms

Let $\varphi : G \to H$ be a group homomorphism and $N \trianglelefteq \mathrm{Ker}\ f \trianglelefteq G$. Then there exists an unique homomorphism $\bar{\varphi}$ where

$$\begin{array}{rccc} \bar{\varphi}: & G/N & \to & H \\ & aN & \mapsto & \varphi(a) \end{array}$$

and

- $\varphi(G) = \bar{\varphi}(G/N)$,
- $\mathrm{Ker}\ \bar{\varphi} = (\mathrm{Ker}\ \varphi)/N$

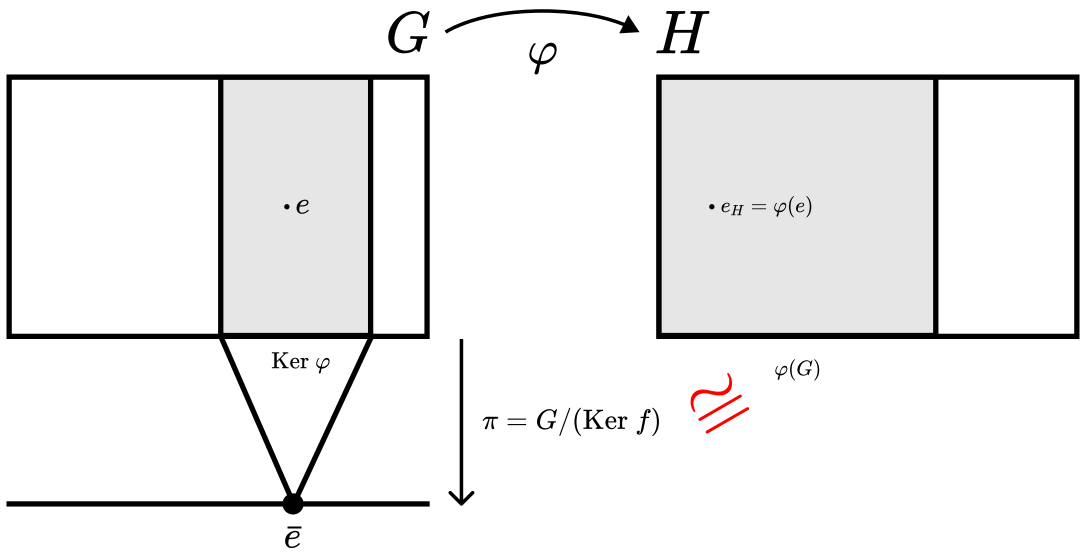
▶ **Proof**

Thm. First Isomorphism Theorem

Let $\varphi : G \to H$ be a group homomorphism. Then

1. $\mathrm{Ker}\ \varphi \trianglelefteq G$, so kernel of any group homomorphism is normal,
2. $\varphi(G) \leq H$, so image of any group homomorphism is a subgroup,
3. $\varphi(G) \cong G/(\mathrm{Ker}\ \varphi)$, so if φ is an epimorphism, then $H \cong G/(\mathrm{Ker}\ \varphi)$.

▶ **Proof**



(Figure 1) First Isomorphism Theorem

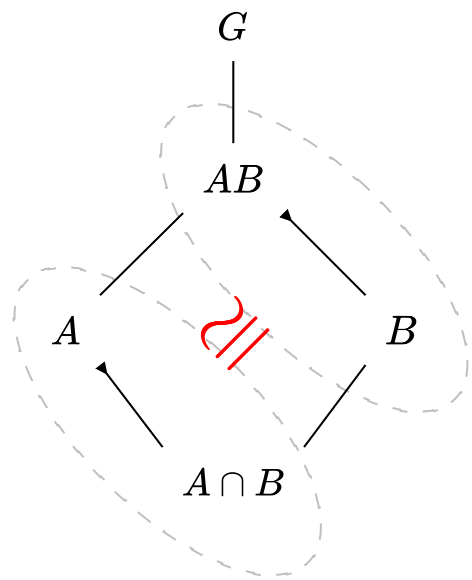
Thm. Second Isomorphism Theorem

> This theorem is also called the **Diamond Isomorphism Theorem** or **Parallelogram Theorem** due to lattice it draws.

Let $B \leq G$ and $A \leq N_G(B)$, so that A is a subgroup of the *normalized* B. Then, noting A is normal

1. $AB \leq G$,
2. $B \trianglelefteq AB$,
3. $A \cap B \trianglelefteq A$, and
4. $AB/B \cong A/A \cap B$.

▶ **Proof**



(Figure 2) Second Isomorphism Theorem

Thm. Third Isomorphism Theorem

Let $K \trianglelefteq H \trianglelefteq G$, then

1. $K/H \trianglelefteq G/H$, and
2. $(G/K)/(H/K) \cong G/H$.

▶ **Proof**