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# Universal Algebra

- **Introduction to Lattices and Order**, 2nd Ed. by B.A. Davey and H.A. Priestley.
- **A Course in Universal Algebra**, Millenium Ed. by Stanley Burris and H.P. Sankappanavar.

## Preliminaries

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Fundamental set-theoric notation.

## Notation

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- $x \not\leq y$  means "not  $x \leq y$ ".
- $\phi : P \rightarrow Q$  denotes a map (function)  $\phi$  from  $P$  to  $Q$ .
- $\phi : P \hookrightarrow Q$  denotes the *injective (one-to-one)* map  $\phi$  from  $P$  into  $Q$ .
- $\phi : P \twoheadrightarrow Q$  denotes the *surjective* map  $\phi$  from  $P$  onto  $Q$ .
- $\phi : P \xrightarrow{\sim} Q$  denotes the *bijective* map  $\phi$  from  $P$  into and onto  $Q$ .
- $P \multimap Q$  denotes the *set of all partial maps* from  $P$  to  $Q$ .

# Ordered Sets

## Def. Partial Order

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A **partial order** (relation) or simply an **order** (relation) on some set  $P$  is a binary relation  $\leq$  on  $P$  such that, for all  $x, y, z \in P$ , it is

1. **Reflexive:**  $x \leq x$ .
2. **Antisymmetric:**  $x \leq y$  and  $y \leq x$  implies  $x = y$ .
3. **Transitive:**  $x \leq y$  and  $y \leq z$  implies  $x \leq z$ .

We say  $x$  and  $y$  are **comparable** if either  $x \leq y$  or  $y \leq x$ .

The set  $P$  with such order relation  $\leq$  is said to be a (partially) **ordered set**, or simply a **poset** denoted  $\langle P; \leq \rangle$ .

On any set,  $=$  is an order called the **discrete order**.

A binary relation  $\leq$  that satisfies (1) and (3) but not necessarily (2) is called a **quasi-order** or **pre-order**.

Let  $\langle P; \leq_P \rangle$  and  $Q \subseteq P$ . Then  $Q$  inherits an order relation  $\leq_Q$  from  $P$  such that for all  $x, y \in Q$  we have  $x \leq_Q y \iff x \leq_P y$  called the **induced order** or the **order inherited from  $P$** .

## Def. Chains

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Let  $\langle P; \leq \rangle$  be a poset. Then  $P$  is said to be a **chain** (or **linearly ordered set** or **totally ordered set**) if any two elements of  $P$  are comparable.

Similarly,  $P$  is said to be an **antichain** if, for all  $x, y \in P$ , we have  $x \leq y$  implies  $x = y$ .

Notice that any subset of a chain (an antichain) is a chain (an antichain).

## Notation

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We will utilize the symbol  $\mathbf{n}$  to denote the finite  $n$ -element linearly ordered set  $\{0, 1, \dots, n-1\}$  with the natural linear order. Similarly,  $\bar{\mathbf{n}}$  will denote the  $n$ -element antichain.

## Def. Maps on Orders

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Let  $P$  and  $Q$  be two ordered sets. We say a map  $\phi : P \rightarrow Q$  is:

- **order-preserving** if  $x \leq_P y$  implies  $\phi(x) \leq_Q \phi(y)$ .
- **order-embedding** if it is order-preserving and  $\phi(x) \leq_Q \phi(y)$  implies  $x \leq_P y$ .

- **order-isomorphism** if it is order-embedding and  $\phi$  is surjective.

Notice that:

- Order-embeddings are injective. Therefore, order-isomorphisms are bijective.
- Not every bijective map between  $P$  and  $Q$  is an order-isomorphism.
- Finite composition of order-preserving maps is again order-preserving.

## Example. Social Choice Function

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See **Wikipedia**: Arrow's impossibility theorem.

## Def. Cover Relation

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TODO: Check if this definition is equivalent to the one in the main book.

Let  $P$  be an ordered set and  $x, y \in P$ . We say  $x$  is **covered by**  $y$  denoted with  $x \prec y$  if  $x \neq y$  and there is no  $z \in P$  distinct from  $x$  and  $y$  such that

$$x \leq z \leq y$$

## Def. Hasse Diagrams

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See **Wikipedia**: Hasse Diagram.

## Thm. TFAE

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Let  $P$  and  $Q$  be finite ordered sets and  $\phi : P \rightarrow Q$  a bijective map. Then TFAE:

- $\phi$  is an order-isomorphism
- $x <_P y$  if and only if  $\phi(x) <_Q \phi(y)$ .
- $x \prec_P y$  if and only if  $\phi(x) \prec_Q \phi(y)$ .
- They can be drawn with identical Hasse Diagrams.

## Def. Dual

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Let  $P$  be an ordered set with the order relation  $\leq$ . The **dual** of  $P$  denoted with  $P^\partial$  is the set ordered with  $\leq_\partial$  where, for all  $x, y \in P$ :

$$x \leq_\partial y \iff y \leq x$$

## Def. Bottom and Top

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For an ordered set  $P$ , we say  $P$  has a **bottom**  $\perp \in P$  if for all  $x \in P$  we have  $\perp \leq x$ . Similarly, we say  $P$  has a **top**  $\top \in P$  if for all  $x \in P$  we have  $x \leq \top$ .

Notice that  $\top$  and  $\perp$  are unique when they exist due to antisymmetry, and they are comparable with any element.

For example, for  $\langle \mathcal{P}(X); \subseteq \rangle$ , we have  $\perp = \emptyset$  and  $\top = X$ .

A finite chain always has bottom and top element.

## Def. Min-Max(imal)

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Let  $P$  be an ordered set and  $p \in P$ . We say  $a \in P$  is:

- **maximal element of  $P$**  if  $a \leq x \implies a = x$ . We denote the **set of maximal elements** with  $\text{Max } P$ .
- **minimal element of  $P$**  if  $x \leq a \implies a = x$ . Similarly, we denote **the set of minimal elements** with  $\text{Min } P$ .
- **maximum** (or **greatest**) if  $a$  is the (unique) top element.
- **minimum** (or **least**) if  $a$  is the (unique) bottom element.

Notice that if  $P$  has a top element  $\top$ , then  $\text{Max } P = \{\top\}$ .

## Def. Sums

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Suppose  $P$  and  $Q$  are two disjoint ordered sets. The **disjoint union** denoted  $P \sqcup Q$  is the ordered set  $P \cup Q$  ordered by  $\leq$  where  $x \leq y$  if and only if either:

- For  $x, y \in P$  we have  $x \leq_P y$ ,
- For  $x, y \in Q$  we have  $x \leq_Q y$ .

## Def. Linear Sum

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For two disjoint ordered sets  $P$  and  $Q$ , the **linear sum** denoted  $P \oplus Q$  is the ordered set  $P \cup Q$  ordered by  $\leq$  where  $x \leq y$  if and only if either:

- For  $x, y \in P$  we have  $x \leq_P y$ ,
- For  $x, y \in Q$  we have  $x \leq_Q y$ .
- $x \in \text{Max } P$  and  $y \in \text{Min } Q$ .

Obviously,  $\oplus$  is not necessarily commutative.

Note that both  $\sqcup$  and  $\oplus$  are associative (up to isomorphism).

For example,  $\mathbf{2} \oplus \mathbf{3} = \mathbf{5}$ .

## Def. Product

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Let  $P_1, \dots, P_n$  be ordered sets. The (Cartesian) product  $P_1 \times \dots \times P_n$  can be (coordinatewise) ordered with  $\leq$  where  $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$  if and only if, for all  $i$ , we have  $x_i \leq_{P_i} y_i$ . As a shorthand we will use  $P^n$  to denote the  $n$ -fold cartesian product ordered with such order.

## Example. '

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Let  $X = \{1, 2, \dots, n\}$  and  $\phi : \mathcal{P}(X) \rightarrow \mathbf{2}^n$  such that  $\phi(A) = (\varepsilon_1, \dots, \varepsilon_n)$  where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

Then  $\phi$  is an order-isomorphism.

## Def. Ups and Downs

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Let  $P$  be an ordered set and  $Q \subseteq P$ .

Then, we say  $Q$  is a **down-set** (or **order ideal**) if for all  $x \in Q$  and  $y \in P$ :

$$y \leq x \implies y \in Q$$

Dually, we say  $Q$  is a **up-set** (or **order filter**) if for all  $x \in Q$  and  $y \in P$ :

$$y \geq x \implies y \in Q$$

You may think of them as a subset "closed" under increase or decrease.

For an arbitrary subset  $Q$  of ordered  $P$ , define the unary operators  $\downarrow$  called **down** and  $\uparrow$  **up** on the subset as:

$$\begin{aligned} \downarrow Q &:= \{ y \in P \mid (\exists x \in Q) y \leq x \} \\ \uparrow Q &:= \{ y \in P \mid (\exists x \in Q) y \geq x \} \end{aligned}$$

and for  $x \in P$ :

$$\begin{aligned} \downarrow x &:= \{ y \in P \mid y \leq x \} \\ \uparrow x &:= \{ y \in P \mid y \geq x \} \end{aligned}$$

Notice that:

- $\downarrow Q$  is the smallest down-set that contains  $Q$ .
- $Q$  is a **down-set** if and only if  $Q = \downarrow Q$ .
- $\downarrow \{x\} = \downarrow x$ .
- $Q$  is a down-set of  $P$  if and only if  $P \setminus Q$  is an up-set of  $P$  (or equivalently, a down-set of  $P^\partial$ ).

Down-sets (dually up-sets) of the form  $\downarrow x$  (dually  $\uparrow x$ ) are called **principal**.

## Def. Ordered Set of Down-sets

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The family of all down-sets of the ordered set  $P$  is denoted by  $\mathcal{O}(P)$ . Under the inclusion order,  $\mathcal{O}(P)$  is an ordered set.

When  $P$  is finite, every non-empty down-set  $Q$  of  $P$  is expressible in the form

$$\bigcup_{i=1}^k \downarrow x_i$$

where  $\{x_1, \dots, x_k\} = \text{Max } Q$  is an antichain.

Notice that  $\mathcal{O}(P)^\partial \cong \mathcal{O}(P^\partial)$  as  $A \subseteq$  iff  $P \setminus A \supseteq P \setminus B$ .

## Thm. '

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Let  $P, P_1, P_2$  be ordered sets. Then

- $\mathcal{O}(P \oplus \mathbf{1}) \cong \mathcal{O}(P) \oplus \mathbf{1}$
- $\mathcal{O}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{O}(P)$
- $\mathcal{O}(P_1 \sqcup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$

# Lattices

## Def. Bounds

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Let  $P$  be an ordered set and  $S \subseteq P$ . Then,  $x \in P$  is called an **upper bound of  $S$**  if  $s \leq x$  for all  $s \in S$ . **Lower bound** is defined dually.

The set of all upper bounds of  $S$  is denoted by  $S^+$  and the set of all lower bounds denoted by  $S^-$ .

The least element of  $S^+$ , if exists, is called the **supremum** (or **least upper bound**) of  $S$  denoted  $\sup S$ . Dually, the greatest element of  $S^-$  is called **infimum** of  $S$  denoted  $\inf S$ .

Notice that:

- Since  $\leq$  is transitive,  $S^+$  is always an up-set and  $S^-$  is a down-set.

## Notation. Join and Meet

---

If exists, we will denote  $\sup\{x, y\}$  with  $x \vee y$  read as  $x$  **join**  $y$ . Similarly, we will denote  $\inf\{x, y\}$  with  $x \wedge y$  read as  $x$  **meet**  $y$ .

Similarly, we will also utilize  $\bigvee S$  and  $\bigwedge S$  for  $\sup S$  and  $\inf S$ .

## Def. Lattice and Complete Lattice

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Let  $P$  be a non-empty (partially) ordered set.

- If join and meet exist for all  $x, y \in P$ , then  $P$  is called a **lattice**.
- If join and meet exist for all  $S \subseteq P$ , then  $P$  is called a **complete lattice**.

## Def. Axiomatic Definition

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From **A Course in Universal Algebra**.

A non-empty set  $L$  with two binary operations  $\vee$  and  $\wedge$  on  $L$  is called a **lattice** if it satisfies:

- **(commutative laws)**

$$x \vee y \approx y \vee x$$

$$x \wedge y \approx y \wedge x$$



- **(associative laws)**

$$x \vee (y \vee z) \approx x \vee (y \vee z)$$

$$x \wedge (y \wedge z) \approx x \wedge (y \wedge z)$$

- **(idempotent laws)**

$$x \vee x \approx x$$

$$x \wedge x \approx x$$

- **(absorption laws)**

$$x \approx x \vee (x \wedge y)$$

$$x \approx x \wedge (x \vee y)$$