Table of Contents

Universal Algebra

- **Introduction to Lattices and Order**, 2nd Ed. by B.A. Davey and H.A. Priestley.
- **A Course in Universal Algebra**, Millenium Ed. by Stanley Burris and H.P. Sankappanavar.

Preliminaries

Fundamental set-theoretic notation.

Notation

- $x \nleqslant y$ means "not $x \leqslant y$".
- $\phi : P \to Q$ denotes a map (function) ϕ from P to Q.
- $\phi : P \hookrightarrow Q$ denotes the *injective (one-to-one)* map ϕ from P *into* Q.
- $\phi : P \twoheadrightarrow Q$ denotes the *surjective* map ϕ from P *onto* Q.
- $\phi : P \xrightarrow{\sim} Q$ denotes the *bijective* map ϕ from P *into* and *onto* Q.
- $P \multimap\!\!\to Q$ denotes the *set of all partial maps* from P to Q.

Ordered Sets

Def. Partial Order

A **partial order** (relation) or simply an **order** (relation) on some set P is a binary relation $\leqslant$ on P such that, for all $x, y, z \in P$, it is

1. **Reflexive:** $x \leqslant x$.
2. **Antisymmetric:** $x \leqslant y$ and $y \leqslant x$ implies $x = y$.
3. **Transitive:** $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$.

We say x and y are **comparable** if either $x \leqslant y$ or $y \leqslant x$.

The set P with such order relation $\leqslant$ is said to be a (partially) **ordered set**, or simply a **poset** denoted $\langle P; \leqslant \rangle$.

On any set, $=$ is an order called the **discrete order**.

A binary relation $\leqslant$ that satisfies (1) and (3) but not necessarily (3) is called a **quasi-order** or **pre-order**.

Let $\langle P; \leqslant_P \rangle$ and $Q \subseteq P$. Then Q inherits an order relation $\leqslant_Q$ from P such that for all $x, y \in Q$ we have $x \leqslant_Q y \iff x \leqslant_P y$ called the **induced order** or the **order inherited from** P.

Def. Chains

Let $\langle P; \leqslant \rangle$ be a poset. Then P is said to be a **chain** (or **linearly ordered set** or **totally ordered set**) if any two elements of P are comparable.

Similarly, P is said to be an **antichain** if, for all $x, y \in P$, we have $x \leqslant y$ implies $x = y$.

Notice that any subset of a chain (an antichain) is a chain (an antichain).

Notation

We will utilize the symbol $\mathbf{n}$ to denote the finite n-element linearly ordered set $\{0, 1, ..., n-1\}$ with the natural linear order. Similarly, $\bar{\mathbf{n}}$ will denote the n-element antichain.

Def. Maps on Orders

Let P and Q be two ordered sets. We say a map $\phi : P \to Q$ is:

- **order-preserving** if $x \leqslant_P y$ implies $\phi(x) \leqslant_Q \phi(y)$.
- **order-embedding** if it is order-preserving and $\phi(x) \leqslant_Q \phi(y)$ implies $x \leqslant_P y$.

- **order-isomorphism** if it is order-embedding and ϕ is surjective.

Notice that:

- Order-embeddings are injective. Therefore, order-isomorphisms are bijective.
- Not every bijective map between P and Q is an order-isomorphism.
- Finite composition of order-preserving maps is again order-preserving.

Example. Social Choice Function

See **Wikipedia**: Arrow's impossibility theorem.

Def. Cover Relation

> TODO: Check if this definition is equivalent to the one in the main book.

Let P be an ordered set and $x, y \in P$. We say x is **covered by** y denoted with $x \prec y$ if $x \neq y$ and there is no $z \in P$ distinct from x and y such that

$$x \leqslant z \leqslant y$$

Def. Hasse Diagrams

See **Wikipedia:** Hasse Diagram.

Thm. TFAE

Let P and Q be finite ordered sets and $\phi : P \to Q$ a bijective map. Then TFAE:

- ϕ is an order-isomorphism
- $x <_P y$ if and only if $\phi(x) <_Q \phi(y)$.
- $x \prec_P y$ if and only if $\phi(x) \prec_Q \phi(y)$.
- They can be drawn with identical Hasse Diagrams.

Def. Dual

Let P be an ordered set with the order relation $\leqslant$. The **dual** of P denoted with P^∂ is the set ordered with $\leqslant_\partial$ where, for all $x, y \in P$:

$$x \leqslant_\partial y \iff y \leqslant x$$

Def. Bottom and Top

For an ordered set P, we say P has a **bottom** $\bot \in P$ if for all $x \in P$ we have $\bot \leqslant x$. Similarly, we say P has a **top** $\top \in P$ if for all $x \in P$ we have $x \leqslant \top$.

Notice that $\top$ and $\bot$ are unique when they exist due to antisymmetry, and they are comparable with any element.

For example, for $\langle \mathcal{P}(X); \subseteq \rangle$, we have $\bot = \varnothing$ and $\top = X$.

A finite chain always has bottom and top element.

Def. Min-Max(imal)

Let P be an ordered set and $p \in P$. We say $a \in P$ is:

- **maximal element of** P if $a \leqslant x \implies a = x$. We denote the **set of maximal elements** with $\mathrm{Max}\ P$.
- **minimal element of** P if $x \leqslant a \implies a = x$. Similarly, we denote **the set of minimal elements** with $\mathrm{Min}\ P$.
- **maximum** (or **greatest**) if a is the (unique) top element.
- **minimum** (or **least**) if a is the (unique) bottom element.

Notice that if P has a top element $\top$, then $\mathrm{Max}\ P = \{\top\}$.

Def. Sums

Suppose P and Q are two disjoint ordered sets. The **disjoint union** denoted $P \sqcup Q$ is the ordered set $P \cup Q$ ordered by $\leqslant$ where $x \leqslant y$ if and only if either:

- For $x, y \in P$ we have $x \leqslant_P y$,
- For $x, y \in Q$ we have $x \leqslant_Q y$.

Def. Linear Sum

For two disjoint ordered sets P and Q, the **linear sum** denoted $P \oplus Q$ is the ordered set $P \cup Q$ ordered by $\leqslant$ where $x \leqslant y$ if and only if either:

- For $x, y \in P$ we have $x \leqslant_P y$,
- For $x, y \in Q$ we have $x \leqslant_Q y$.
- $x \in \mathrm{Max}\ P$ and $y \in \mathrm{Min}\ Q$.

Obviously, $\oplus$ is not necessarily commutative.

Note that both $\sqcup$ and $\oplus$ are associative (up to isomorphism).

For example, $\mathbf{2} \oplus \mathbf{3} = \mathbf{5}$.

Def. Product

Let $P_1, ..., P_n$ be ordered sets. The (Cartesian) product $P_1 \times ... \times P_n$ can be (coordinatewise) ordered with $\leqslant$ where $(x_1, ..., x_n) \leqslant (y_1, ..., y_n)$ if and only if, for all i, we have $x_i \leqslant_{P_i} y_i$. As a shorthand we will use P^n to denote the n-fold cartesian product ordered with such order.

Example. '

Let $X = \{1, 2, ..., n\}$ and $\phi : \mathcal{P}(X) \to \mathbf{2}^n$ such that $\phi(A) = (\varepsilon_1, ..., \varepsilon_n)$ where

$$\varepsilon_i = \begin{cases} 1 & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

Then ϕ is an order-isomorphism.

Def. Ups and Downs

Let P be an ordered set and $Q \subseteq P$.

Then, we say Q is a **down-set** (or **order ideal**) if for all $x \in Q$ and $y \in P$:

$$y \leqslant x \implies y \in Q$$

Dually, we say Q is a **up-set** (or **order filter**) if for all $x \in Q$ and $y \in P$:

$$y \geqslant x \implies y \in Q$$

You may think of them as a subset “closed” under increase or decrase.

For an arbitrary subset Q of ordered P, define the unary operators $\downarrow$ called **down** and $\uparrow$ **up** on the subset as:

$$\begin{aligned} \downarrow Q &:= \{\, y \in P \mid (\exists x \in Q)\ y \leqslant x \,\} \\ \uparrow Q &:= \{\, y \in P \mid (\exists x \in Q)\ y \geqslant x \,\} \end{aligned}$$

and for $x \in P$:

$$\begin{aligned} \downarrow x &:= \{\, y \in P \mid y \leqslant x \,\} \\ \uparrow x &:= \{\, y \in P \mid y \geqslant x \,\} \end{aligned}$$

Notice that:

- $\downarrow Q$ is the smallest down-set that contains Q.
- Q is a **down-set** if and only if $Q = \downarrow Q$.
- $\downarrow \{x\} = \downarrow x$.
- Q is a down-set of P if and only if $P \setminus Q$ is an up-set of P (or equivalently, a down-set of P^∂).

Down-sets (dually up-sets) of the form $\downarrow x$ (dually $\uparrow x$) are called **principal**.

Def. Ordered Set of Down-sets

The family of all down-sets of the ordered set P is denoted by $\mathcal{O}(P)$. Under the inclusion order, $\mathcal{O}(P)$ is an ordered set.

When P is finite, every non-empty down-set Q of P is expressible in the form

$$\bigcup_{i=1}^{k} \downarrow x_i$$

where $\{x_1, ..., x_k\} = \mathrm{Max}\ Q$ is an antichain.

Notice that $\mathcal{O}(P)^\partial \cong \mathcal{O}(P^\partial)$ as $A \subseteq$ iff $P \setminus A \supseteq P \setminus B$.

Thm. ′

Let P, P_1, P_2 be ordered sets. Then

- $\mathcal{O}(P \oplus \mathbf{1}) \cong \mathcal{O}(P) \oplus \mathbf{1}$
- $\mathcal{O}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{O}(P)$
- $\mathcal{O}(P_1 \sqcup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$

Lattices

Def. Bounds

Let P be an ordered set and $S \subseteq P$. Then, $x \in P$ is called an **upper bound of** S if $s \leqslant x$ for all $s \in S$. **Lower bound** is defined dually.

The set of all upper bounds of S is denoted by S^+ and the set of all lower bounds denoted by S^-.

The least element of S^+, if exists, is called the **supremum** (or **least upper bound**) of S denoted $\sup S$. Dually, the greatest element of S^- is called **infimum** of S denoted $\inf S$.

Notice that:

- Since $\leqslant$ is transitive, S^+ is always an up-set and S^- is a down-set.

Notation. Join and Meet

If exists, we will denote $\sup\{x, y\}$ with $x \vee y$ read as x **join** y. Similarly, we will denote $\inf\{x, y\}$ with $x \wedge y$ read as x **meet** y.

Similarly, we will also utilize $\bigvee S$ and $\bigwedge S$ for $\sup S$ and $\inf S$.

Def. Lattice and Complete Lattice

Let P be a non-empty (partially) ordered set.

- If join and meet exist for all $x, y \in P$, then P is called a **lattice**.
- If join of and meet of exist for all $S \subseteq P$, then P is called a **complete lattice**.

Def. Axiomatic Definition

> From **A Course in Universal Algebra**.

A non-empty set L with two binary operations $\vee$ and $\wedge$ on L is called a **lattice** if it satisfies:

- **(commutative laws)**
  $x \vee y \approx y \vee x$
  $x \wedge y \approx y \wedge x$

- **(associative laws)**
  $x \vee (y \vee z) \approx x \vee (y \vee z)$
  $x \wedge (y \wedge z) \approx x \wedge (y \wedge z)$

- **(idempotent laws)**
  $x \vee x \approx x$
  $x \wedge x \approx x$

- **(absorption laws)**
  $x \approx x \vee (x \wedge y)$
  $x \approx x \wedge (x \vee y)$