Topology and Computational Performance of Attractor Neural Networks

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Abstract

To explore the relation between network structure and function, we studied the computational performance of Hopfield-type attractor neural nets with regular lattice, random, small-world and scale-free topologies. The random configuration is the most efficient for storage and retrieval of patterns by the network as a whole. However, in the scale-free case retrieval errors are not distributed uniformly: the portion of a pattern encoded by the subset of highly connected nodes is more robust and efficiently recognized than the rest of the pattern. The scale-free network thus achieves a very strong partial recognition. Implications for brain function and social dynamics are suggestive.

While intense research activity is centered on structural and topological properties of social, biological and technological networks[1], the consequences of network structure for the dynamics of cooperative processes have been addressed to a lesser extent. Topology is known to affect the ordering and disordering of the Ising model[2]-[4] and the synchronization of coupled oscillators [5][6]. Another area of burning interest is the relation between structure and function in the organization of brains. [7][8][9]

The goal of this letter is to study the effect of structure on the dynamics of sparsely connected Hopfield-type[11]-[13] attractor neural networks. It is known that randomly pruning the connections of a Hopfield net (HN) increases the storage capacity per synapse.[14] Amongst other questions we ask whether there is an optimal topology given a fixed number of nodes and connections. The HN is of interest because it provides a tractable toy model of collective computation and can also be viewed as an extension of the Ising model with limited amounts of frustration and quenched disorder.[15] We hope therefore that our results lead to further insights into collective computation as well as ordering and disordering processes occuring on networks.

Our computations involve Hopfield nets with asynchronous updating in random order[11][12], p random stored binary pattern vectors ξ^μ and Hebbian[16] connection strengths

$$w_{ij} = a_{ij} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \qquad (1)$$

where a_{ij} is the adjacency matrix ($a_{ij} = 1$ if i and j are connected, $a_{ij} = 0$ otherwise). The degree of node i is $k_i = \sum_{j=1}^{N} a_{ij}$. We always compare networks with the same number of nodes $N = 5000$ and average degree $\langle k \rangle = 50$, varying only the arrangement of connections. Each node is connected on average to 1% of the other nodes, compared to $\approx 0.1\%$ in the mouse cortex.[7] The networks compared are: (1) a regular one-dimensional ring of nodes, each connected to its 50 nearest neighbors, (2) a random (Erdős-Renyi[17]) network, (3) a small world (Watts-Strogatz[10]) net constructed from a regular lattice by randomly rewiring local links with probability r, and (4) a scale-free network with degree distribution $P(k) \sim k^{-3}$ with a lower cutoff of 25 generated by the Barabasi-Albert algorithm of prefential attachment[18].

We measured two aspects of the performance of the associative memory networks: the *stability* of the memorized patterns (inversely related to the number of errors induced by crosstalk) and the nework's *ability to recognize* one of the patterns from a state with a large number of errors. These two features of an associative memory are related but not identical: a pattern can be stable but nonetheless have a small basin of attraction, while on the other hand it is possible for an attractor to have a large basin but nonetheless be imperfectly correlated with the memorized pattern. To quantify pattern retrieval we used the overlap order parameters

$$m^\mu \equiv \frac{1}{N} \sum_{i=1}^{N} x_i \xi_i^\mu \qquad (2)$$

where $x_i = \pm 1$ denotes the output of the i-th node, as well as partial overlaps $m^\mu(k_{\min}) = m^\mu(k > k_{\min})$, defined as in (2) except that the sum runs only over those nodes whose degree exceeds $k_{\min}$ and is normalized appropriately. $k_{\min}$ partitions the network into hubs and non-hubs, and $m^\mu(k_{\min})$ measures recognition of the portion of the pattern encoded in the
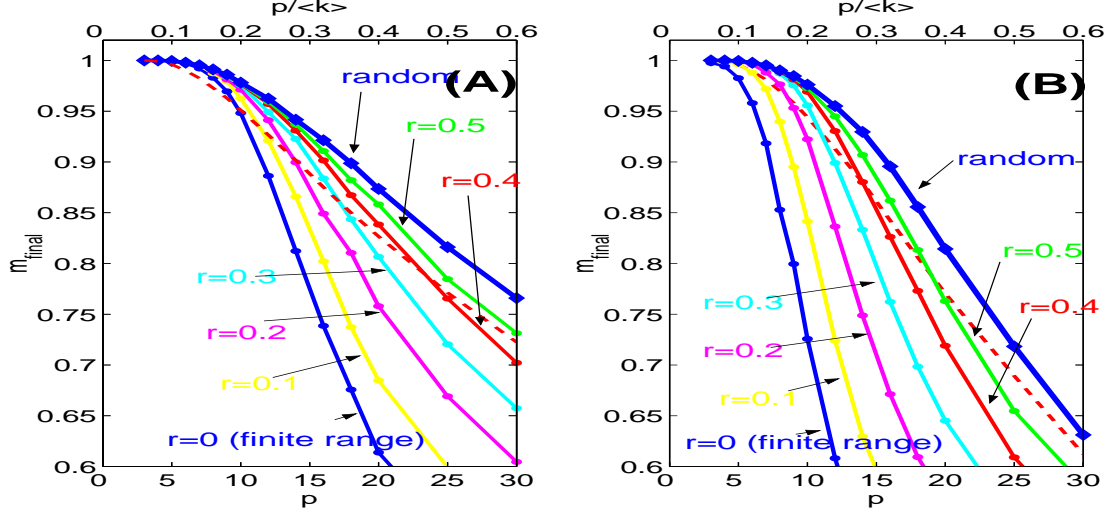
FIG. 1: Performance of different networks as a function of the number of patterns p and loading ratio $p/\langle k \rangle$. All have $\langle k \rangle = 50$ and $N = 5000$. Solid lines: random net and small-world net ranging from $r = 0$ (i.e., a locally connected lattice) to $r = 0.5$. Dotted line: scale-free net (see also figure 2). Data were averaged over patterns and network realizations, for a total of 80 trials per point. (A) m_{final} when $m_{init} = 1$, a measure of the stability of the memorized patterns against crosstalk-induced errors. (B) m_{final} when $m_{init} = 0.5$, a measure of how successfully a pattern is retrieved from a corrupted version.

hubs. The stability of the memory patterns was measured by initializing the network to a memory state ($x_i = \xi_i^\mu$) and measuring m_{final}^μ after the dynamics converged. The departure of m_{final}^μ from 1 reflects the number of errors induced by crosstalk. As an indicator of the network's ability to retrieve a pattern from a randomly corrupted version, we measured m_{final}^μ when the initial overlap was $m_{init}^\mu = 0.5$. We averaged these quantities over several realizations of the topology and patterns, varying the number of patterns p to see how the performance degrades with increasing loading.

With $\langle k \rangle = 50$ and $p \lesssim 50$, the networks studied are far from the commonly studied limit where both p and N simultaneously approach infinity[11]. Therefore no discontinuous overloading phase transition is apparent, but comparisons are still possible at finite p and N. Figure 1 shows results for the networks as a whole. The most rapid degradation in both stability and retrievability occurs in the regular lattice, and the slowest in the random net. Not surprisingly, the addition of shortcuts to a regular lattice enhances pattern stability and retrieval. The performance of small-world nets is intermediate between that of a regular and a random net. The variation with rewiring probability r is not linear, however. A network with $r = 0.5$ behaves almost as a random ($r = 1$) net.

The performance of the scale-free net as a whole degrades slightly faster than that of a random net, but the errors are not distributed evenly as is apparent in figure 2 where we examine $m^\mu(k_{\min})$ for $k_{\min} = 50, 100, 200,$ and 400. These represent subsets having average sizes of 1235, 333, 97 and 29 nodes, respectively. From figure 2A, it is evident that the frequency of crosstalk-induced errors decreases with increasing degree. For example, the nodes with $k > k_{\min} = 200$ have very few errors even when $p = \langle k \rangle = 50$. The nodes with $k > 200$ form a subset of approximately 100 nodes. A fully connected net of 100 nodes
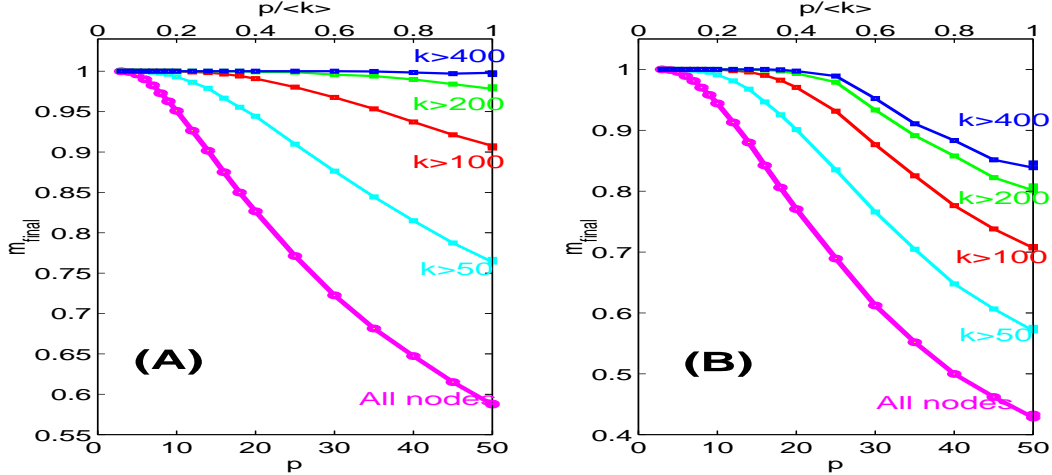
FIG. 2: Performance of the whole network compared to "hub" subsets of the scale-free network, averaged over 200 trials per data point. (A) Pattern stability ($m_{init} = 1$). (B) Pattern retrievability ($m_{init} = 0.5$).

alone would be able to store only ≈ 14 patterns.[11][19] Thus, the performance of the hubs in the original network as pattern recognizers is much higher than it would be if the nodes with $k < k_{\min}$ were pruned away. Even though the less connected nodes are more prone to errors, they nonetheless assist the hubs in retaining the patterns. The enhanced performance of the well-connected subset manifests itself not only in the stability of the patterns but also in their retrieval, as seen in figure 2B. The hubs are able to distinguish clearly among a large number of patterns even if the pattern reconstruction is incomplete (i.e. limited to the hubs.)

The lower rate of errors among the hubs is not surprising in view of the fact that their input comes from a larger number of nodes. It can be shown using arguments as in [11] and [12] that if the state of the network is initially set to one of the patterns ($\mathbf{x} = \xi^\mu$) then an individual node with degree k_i experiences a crosstalk-induced noise-to-signal ratio $(N/S)_i \approx \sqrt{(p-1)/k_i}$. Hence the probability of a crosstalk-induced error in the i-th node decreases with increasing degree. The presence of one error reduces the strength of the signal and may increase the likelihood of additional errors, resulting in a cascade of the type responsible for the abrupt overloading phase transition that occurs in the fully connected Hopfield network[11][19]. Cascades in the opposite direction may also play a role in the reconstruction of patterns from noisy input. The differences between differently connected networks thus lie not just in the initial signal-to noise ratio but in the dynamics of the spread of error cascades. This dynamics differs from ordinary percolation or epidemic propagation, since the susceptibility of a node is inversely correlated with its degree. In most models of epidemic propagation only one infected neighbor suffices to infect a node, regardless of its degree.

Our results are reminiscent of those for the simple Ising model: a one-dimensional lattice is easily disordered by thermal noise[20] while even a few long range connections can restore order at a finite temperature[2], and in a scale-free network the nodes with high degree are more strongly magnetized than those with low degree[3]. The difference is that in the present case, the disorder is induced by interference among the stored patterns and not by

stochastic noise. It is a quenched rather than thermal disorder.

While we found that the most efficient arrangement for storage and retrieval of patterns by the network as a whole is a random network, connections in real brains do not appear to be fully random. One reason may lie in the economy of wiring length.[8] The majority of connections in brains of higher animals as well as in *C. elegans* [10] appears to occur between nearby neurons, while fewer paths connect more distant regions, suggesting a small-world topology. Our results imply that small-world networks with a moderate number of shortcuts can be almost as computationally efficient as a random network while saving considerably on wiring costs. The suitability of small-world networks for complex computations was also suggested on the basis of other models.[9] To what extent scale-free structures play a role in real brains remains to be seen, but our results suggest a mechanism by which information can be centralized in the more connected nodes while the remaining nodes, although noisy, are nonetheless indispensable for the computation. It will be of interest to study the implications of these notions for the formation of knowledge, opinions and power structures in scientific and social networks.[21]

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