Linear Algebra Done Right

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Chapter 1A

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Let $\alpha = a + bi$ and $\beta = c + di$, where $a, b, c, d \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
\alpha + \beta &= (a + bi) + (c + di) && \text{definition of } \alpha \text{ and } \beta \\
&= (a + c) + (b + d)i && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= (c + a) + (d + b)i && \text{commutativity of } \mathbb{R} \\
&= (c + di) + (a + bi) && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= \beta + \alpha && \text{definition of } \beta \text{ and } \alpha
\end{aligned}$$

■

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = f + gi$, where $a, b, c, d, f, g \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
(\alpha + \beta) + \lambda &= ((a + bi) + (c + di)) + (f + gi) && \text{definition of } \alpha, \beta, \lambda \\
&= ((a + c) + (b + d)i) + (f + gi) && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= ((a + c) + f) + ((b + d) + g)i && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= (a + (c + f)) + (b + (d + g))i && \text{associativity of addition on } \mathbb{R} \\
&= (a + bi) + ((c + f) + (d + g)i) && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= (a + bi) + ((c + fi) + (d + gi)) && \text{definition of addition on } \mathbb{C} \text{ (1.1)} \\
&= \alpha + (\beta + \lambda) && \text{definition of } \alpha, \beta, \lambda
\end{aligned}$$

■

3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = f + gi$, where $a, b, c, d, f, g \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
(\alpha\beta)\lambda &= ((a + bi)(c + di))(f + gi) && \text{def. of } \alpha, \beta, \lambda \\
&= ((ac - bd) + (ad + bc)i)(f + gi) && \text{def. of mult. on } \mathbb{C} \text{ (1.1)} \\
&= ((ac - bd)f - (ad + bc)g) + ((ac - bd)g + (ad + bc)f)i && \text{same} \\
&= (acf - bdf - adg - bcg) + (acg - bdg + adf + bcf)i && \text{distrib. of } \mathbb{R} \\
&= (acf - adg - bcg - bdf) + (acg + adf + bcf - bdg)i && \text{commut. of } \mathbb{R} \\
&= (a(cf - dg) - b(cg + df)) + (a(cg + df) + b(cf - dg))i && \text{distrib. of } \mathbb{R} \\
&= (a + bi)((cf - dg) + (cg + df)i) && \text{def. of mult. on } \mathbb{C} \text{ (1.1)} \\
&= (a + bi)((c + di)(f + gi)) && \text{def. of mult. on } \mathbb{C} \text{ (1.1)} \\
&= \alpha(\beta\lambda) && \text{def. of } \alpha, \beta, \lambda
\end{aligned}$$

■

4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Let $\lambda = a + bi$, $\alpha = c + di$, and $\beta = f + gi$, where $a, b, c, d, f, g \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
\lambda(\alpha + \beta) &= (a + bi)((c + di) + (f + gi)) && \text{def. of } \lambda, \alpha, \beta \\
&= (a + bi)((c + f) + (d + g)i) && \text{def. of addition on } \mathbb{C} \text{ (1.1)} \\
&= (a(c + f) - b(d + g)) + (a(d + g) + b(c + f))i && \text{def. of mult. on } \mathbb{C} \text{ (1.1)} \\
&= (ac + af - bd - bg) + (ad + ag + bc + bf)i && \text{distrib. of } \mathbb{R} \\
&= (ac - bd + af - bg) + (ad + bc + ag + bf)i && \text{commut. of } \mathbb{R} \\
&= (ac - bd) + (af - bg) + (ad + bc)i + (ag + bf)i && \text{distrib., assoc. of } \mathbb{R} \\
&= (ac - bd) + (ad + bc)i + ((af - bg) + (ag + bf)i) && \text{commut., assoc. of } \mathbb{R} \\
&= (a + bi)(c + di) + (a + bi)(f + gi) && \text{def. of mult. on } \mathbb{C} \text{ (1.1)} \\
&= \lambda\alpha + \lambda\beta && \text{def. of } \lambda, \alpha, \beta
\end{aligned}$$

■

5. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Let $\alpha = a + bi$, where $a, b \in \mathbb{R}$. If we take $\beta = (-a) + (-b)i$, then

$$\begin{aligned}
\alpha + \beta &= (a + bi) + ((-a) + (-b)i) && \text{definition of } \alpha, \beta \\
&= (a + (-a) + (b + (-b))i) && \text{definition of addition on } \mathbb{C} \\
&= (0) + (0)i && \text{additive inverse on } \mathbb{R} \\
&= 0 && \text{definition of 0 in } \mathbb{C}
\end{aligned}$$

So given $\alpha \in \mathbb{C}$, a $\beta \in \mathbb{C}$ definitely exists such that $\alpha + \beta = 0$.

To prove that this β is unique, suppose there is some other value $\lambda \in \mathbb{C}$ such that $\alpha + \lambda = 0$. Then $\lambda = c + di$, and $\alpha + \lambda = (a + bi) + (c + di) = (a + c) + (b + d)i$. Since $\alpha + \lambda = 0$, we must have $a + c = 0$ and $b + d = 0$. But solving these equations, we find that $c = -a$ and $d = -b$, which are exactly the values we found above for β. Thus, $\lambda = \beta$, i.e. this $\beta \in \mathbb{C}$ is unique. ■

6. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Let $\alpha = a + bi$. First, let's verify that at least one such $\beta \in \mathbb{C}$ exists. Let $\beta = c + di$. Then

$$\begin{aligned} \alpha\beta &= (a + bi)(c + di) && \text{def. of } \alpha, \beta \\ &= (ac - bd) + (ad + bc)i && \text{def. of mult. on } \mathbb{C} \ (1.1) \end{aligned}$$

For this product to equal $1 \in \mathbb{R}$, we must have:
- $ac - bd = 1$
- $ad + bc = 0$

Since $\alpha \neq 0$, it must be that $a \neq 0$ or $b \neq 0$ (or both). Consider these three cases separately:

Case 1: $a = 0, b \neq 0$

In this case, our system simplifies to:
- $-bd = 1$
- $bc = 0$

Since $b \neq 0$, c must be 0. And $d = \frac{-1}{b}$.

This tracks: $\alpha\beta = (bi)\left(\frac{-1}{b}i\right) = -1i^2 = (-1)(-1) = 1$, as desired.

Case 2: $a \neq 0, b = 0$

In this case, our system simplifies to:
- $ac = 1$
- $ad = 0$

Since $a \neq 0$, d must be 0. And $c = \frac{1}{a}$.

This also tracks: $\alpha\beta = (a + 0i)\left(\frac{1}{a} + 0i\right) = \frac{a}{a} = 1$, as desired.

Case 3: $a \neq 0, b \neq 0$

In this case, we have to solve this full system of two equations in two unknowns:

- $ac - bd = 1$
- $ad + bc = 0$

Using substitution, I obtained $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$.

We can see that this works:

$$\begin{aligned} \alpha\beta &= (ac - bd) + (ad + bc)i && \text{def. of mult. on } \mathbb{C} \ (1.1) \\ &= \left(a\frac{a}{a^2 + b^2} - b\frac{-b}{a^2 + b^2}\right) + \left(a\frac{-b}{a^2 + b^2} + b\frac{a}{a^2 + b^2}\right) && \text{values for c, d} \\ &= \left(\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}\right) + \left(\frac{-ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}\right) && \text{arithmetic, fractions} \\ &= \left(\frac{a^2 + b^2}{a^2 + b^2}\right) + \left(\frac{0}{a^2 + b^2}\right) && \text{more arithmetic} \\ &= 1 && \text{more fractions} \end{aligned}$$

Thus, for $\alpha \in \mathbb{C}, \alpha \neq 0$, we found a $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$. Furthermore, we can see that the values of a, b uniquely determine the values of c, d in all three Cases , so this $\beta \in \mathbb{C}$ is unique. ■

7. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

Well,

$$
\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} && \text{def. of cube} \\
&= \frac{\left(-1+\sqrt{3}i\right)\left(-1+\sqrt{3}i\right)\left(-1+\sqrt{3}i\right)}{8} && \text{multiply fractions across} \\
&= \frac{\left(-2-2\sqrt{3}i\right)\left(-1+\sqrt{3}i\right)}{8} && \text{def. of mult. on } \mathbb{C} \\
&= \frac{2-6i^2}{8} && \text{def. of mult. on } \mathbb{C} \\
&= \frac{2+6}{8} && i^2 = -1 \\
&= 1 && \text{fractions and arithmetic, yo}
\end{aligned}
$$

■

8. Find two distinct square roots of i.

Well, $x = \sqrt{i} \iff x^2 = i \iff (a+bi)^2 = i$, assuming $x \in \mathbb{C}$, which seems reasonable.

Then $i = (a+bi)(a+bi) = (a^2 - b^2) + (2ab)i$ by definition of complex multiplication.

Since $i = 0 + 1i$, we thus have a system of two equations emerge:
- $a^2 - b^2 = 0$
- $2ab = 1$

Just doing some algebra, we find that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ are the two solutions. Thus, our two distinct square roots of i are:
1. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$
2. $\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}i$,

as desired. ■

9. Find $x \in \mathbb{R}^4$ such that $(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$.

Well, let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Then $2x = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \\ 2x_4 \end{pmatrix}$, and $(4, -3, 1, 7) + 2x = \begin{pmatrix} 4+2x_1 \\ -3+2x_2 \\ 1+2x_3 \\ 7+2x_4 \end{pmatrix} := \begin{pmatrix} 5 \\ 9 \\ -6 \\ 8 \end{pmatrix}$. Since vectors are equal iff their components are equal, we can solve each of these four equations for the four components of x. Doing so, we obtain $\boldsymbol{x_1 = \frac{1}{2}, x_2 = 6, x_3 = -\frac{7}{2}, x_4 = \frac{1}{2}}$. ■

10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$$

By definition of scalar multiplication, λ will get multiplied by all three components of the left vector. By definition of vector equality, those vectors will be equal iff their components are all equal. By definition of complex number equality, both the real and imaginary parts must be equal for two complex numbers to be equal.

Consider the equation produced by the first component of these vectors: $\lambda(2-3i) = 12-5i$. If we let $\lambda = a + bi$, then $\lambda(2-3i) = (a+bi)(2-3i) = (2a+3b) + (-3a+2b)i$.

Setting this equal to its corresponding component in our target vector, $12-5i$, we get a system of equations:

- 2a + 3b = 12
- −3a + 2b = −5

Solving this system, we find $a = 3, b = 2$. Thus, $\lambda = 3 + 2i$ is the unique complex number that, when multiplied by $2-3i$, produces $12-5i$.

This value of λ actually works for the second vector component:

$$\lambda(5+4i) = (3+2i)(5+4i) = 7+22i$$

But unfortunately, when we move on to the third vector component,

$$\begin{aligned} \lambda(-6+7i) = (3+2i)(-6+7i) &= -32+9i \\ &\neq -32-9i. \end{aligned}$$

Thus, the only value that worked for the first and second components fails to work for the third component. This means that there is no value that will work for all three components, and thus, no such $\lambda \in \mathbb{C}$ exists, as desired. ■

11. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$.

Well,

$$\begin{aligned} (x+y)+z &= ((x_1, ..., x_n) + (y_1, ..., y_n)) + (z_1, ..., z_n) && (1.11) \\ &= (x_1+y_1, ..., x_n+y_n) + (z_1, ..., z_n) && (1.13) \\ &= ((x_1+y_1)+z_1, ..., (x_n+y_n)+z_n) && (1.13) \\ &= (x_1+(y_1+z_1), ..., x_n+(y_n+z_n))) && \text{assoc. of } \mathbb{R} \\ &= (x_1, ..., x_n) + (y_1+z_1, ..., y_n+z_n) && (1.13) \\ &= (x_1, ..., x_n) + ((y_1, ..., y_n) + (z_1, ..., z_n)) && (1.13) \\ &= x + (y+z) && (1.11) \end{aligned}$$

■

12. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

Well,

$$\begin{aligned}
(ab)x &= (ab)(x_1, ..., x_n) && \text{(1.11): def of } x \\
&= (abx_1, ..., abx_n) && \text{(1.18: scalar mult in } \mathbb{F}^n\text{)} \\
&= (a(bx_1), ..., a(bx_n)) && \text{assoc. of } \mathbb{F} \\
&= a(bx_1, ..., bx_n) && \text{(1.18)} \\
&= a(bx) && \text{(1.11)}
\end{aligned}$$

■

13. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Well,

$$\begin{aligned}
1x &= 1(x_1, ..., x_n) && \text{(1.11): def of } x \\
&= (1x_1, ..., 1x_n) && \text{(1.18): scalar mult in } \mathbb{F}^n \\
&= (x_1, ..., x_n) && \text{1 is mult. identity in } \mathbb{F} \\
&= x && \text{(1.11)}
\end{aligned}$$

■

14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Well,

$$\begin{aligned}
\lambda(x + y) &= \lambda((x_1, ..., x_n) + (y_1, ..., y_n)) && \text{(1.11): def of } x, y \\
&= \lambda(x_1 + y_1, ..., x_n + y_n) && \text{(1.13): vector addition} \\
&= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) && \text{(1.18): scalar mult in } \mathbb{F}^n \\
&= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) && \text{distrib. of } \mathbb{F} \\
&= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) && \text{(1.13)} \\
&= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) && \text{(1.18)} \\
&= \lambda x + \lambda y && \text{(1.11)}
\end{aligned}$$

■

15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

Well, since $(a + b) \in \mathbb{F}$ since $\mathbb{F}$ is closed under addition, we have:

$$\begin{aligned}
(a + b)x &= (a + b)(x_1, ..., x_n) && \text{(1.11): def of } x \\
&= ((a + b)x_1, ..., (a + b)x_n) && \text{(1.18): scalar mult. in } \mathbb{F}^n \\
&= (x_1(a + b), ..., x_n(a + b)) && \text{commut. of } \mathbb{F} \\
&= (x_1 a + x_1 b, ..., x_n a + x_n b) && \text{distrib. of } \mathbb{F} \\
&= (ax_1 + bx_1, ..., ax_n + bx_n) && \text{commut. of } \mathbb{F} \\
&= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) && \text{(1.13): vector addition} \\
&= a(x_1, ..., x_n) + b(x_1, ..., x_n) && \text{(1.18)} \\
&= ax + bx && \text{(1.11)}
\end{aligned}$$

■

Chapter 1B

1. Prove that $-(-v) = v$ for every $v \in V$.

According to (1.28), $-v$ denotes the additive inverse of v.

Thus, $-(-v)$ denotes the additive inverse of $-v$.

Then by definition of the additive inverse, $-(-v) + -v = 0$.

But $-v$ is the additive inverse of v, so $v + -v = 0$.

And since the additive inverse is unique, and since both v and $-(-v)$ are serving as additive inverses of $-v$, it must be the case that $v = -(-v)$. ∎

2. Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Suppose, to the contrary, that $a \in \mathbb{F}, v \in V$, $av = 0$, and yet it is not the case that $a = 0$ or $v = 0$. Thus, by De Morgan's Law, $a \neq 0$ and $v \neq 0$.

However, we can arrive at a contradiction by noting that, since $a \in \mathbb{F}$ and $a \neq 0$, a will have a multiplicative inverse $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = a^{-1}a = 1$. (Axler denotes this $\frac{1}{a}$, but I will use a^{-1}.)

Thus,

$$\begin{aligned} & av = 0 && \text{assumption} \\ \Leftrightarrow\ & a^{-1}(av) = a^{-1}0 && \text{left multiply by } a^{-1} \\ \Leftrightarrow\ & (a^{-1}a)v = 0 && \text{assoc. of } \mathbb{F}, \text{definition of } 0 \\ \Leftrightarrow\ & v = 0 && (1.5)\text{: } a^{-1}a = 1 \end{aligned}$$

But this contradicts our assumption that $v \neq 0$. Thus, it must be that $a = 0$ or $v = 0$, as desired. ∎

3. Suppose that $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Intuitively, we can "solve for x" in that equation using basic algebra to find that $x = \frac{1}{3}(w - v)$. To make this more rigorous, we can think about the $3x$ term as transforming, additively, v into w.

Let's simplify the problem slightly by finding a vector $y \in V$ such that $v + y = w$. (In other words, temporarily take the 3 out of the equation.) Letting $y = -v + w$, we can see that

$$v + y = v + (-v + w) = (v + -v) + w = 0 + w = w$$

by associativity and the definition of 0. (Note that $-v + w \in V$ by (1.19): def. of addition on V).

Finally, let $y = 3x$. Since $3 \neq 0$, it has a multiplicative inverse $\frac{1}{3}$. Using this inverse to solve for x, we obtain $x = \frac{1}{3}y = \frac{1}{3}(-v + w)$. Since $\frac{1}{3} \in \mathbb{R}$ is a scalar, $\frac{1}{3}(-v + w) \in V$. Thus, such a vector x exists.

To prove that x is unique, imagine to the contrary that there exist two different vectors $x_1, x_2 \in V$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$. This implies that $v + 3x_1 = v + 3x_2$. Adding $-v$ to both sides and multiplying both sides by $\frac{1}{3}$, we find that $x_1 = x_2$. This contradiction implies that x is unique, as desired. ∎

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Commutativity, associativity, the additive inverse, the multiplicative identity, and distributive properties are hold vacuously for the empty set. But there is no **additive identity** because there is no element in the empty set (it's empty!), which makes it impossible for there "to exist an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$."

5. Show that in the definition of the vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the LHS is the number 0, and the 0 on the RHS is the additive identity of V.

From a strategy perspective, it seems like the additive inverse condition can be replaced with this new condition if it can be derived from our new condition and the other, unchanged properties.

So assume that:
- all vector space properties other than the "additive inverse" property hold
- $0v = \mathbf{0}, \forall v \in V$.

We would like to show that there exists an element $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.

TODO: finish proof

6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ \infty \text{ if } t > 0 \end{cases} \qquad t(-\infty) = \begin{cases} \infty \text{ if } t < 0 \\ 0 \text{ if } t = 0 \\ -\infty \text{ if } t > 0 \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty = \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) = (-\infty) + \infty = 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

Well, let's check the various properties of a vector space to see if they all hold for $\mathbb{R} \cup \{\infty, -\infty\}$:

- **commutativity** holds because it holds for all reals, and it holds for ∞ and $-\infty$ per the addition rules defined above
- **associativity** doesn't hold, though, unfortunately. As a counter-example, note that

  $$(\infty + -\infty) + 1 \neq \infty + (-\infty + 1)$$

  On the left, the infinities cancel each other perfectly and a final sum of 1 emerges, whereas on the right, the $-\infty$ clobbers the 1, giving us a final sum of 0.

Thus, without associativity, $\mathbb{R} \cup \{\infty, -\infty\}$ is **not** a vector space. ■

Side Note: This reminds me of how floating-point operations aren't associative, especially at the edges and extremes. For example, Bryant and O'Hallaron point out on page 123 that

> Addition over real numbers also forms an abelian group, but we must consider what effect rounding has on these properties... The operation is commutative, with $x +^f y = y +^f x$ for all values of x and y. On the other hand, the operation is not associative. For example, with single-precision floating point the expression `(3.14+1e10)-1e10` evaluates to `0.0`–the value 3.14 is lost due to rounding. On the other hand, the expression `3.14+(1e10-1e10` evaluates to `3.14`... [1].

7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

This is similar to the example on pp. 13-14, except that example considered $\mathbb{F}^S$, the set of functions from S to $\mathbb{F}$, where $\mathbb{F}$ means $\mathbb{R}$ or $\mathbb{C}$. Now, we're considering V^S, the set of functions from S to V, where V denotes a vector space over $\mathbb{F}$ (1.29).

Despite that difference, it's still tempting to define the sum and product operations on V^S similarly:
- For $f, g \in V^S$, define *sum* $f + g \in V^S$ as $(f + g)(x) := f(x) + g(x), \forall x \in S$.
- For $f \in V^S$ and $\lambda \in \mathbb{F}$, define *product* $\lambda f \in V^S$ as $(\lambda f)(x) := \lambda f(x), \forall x \in S$.

Having defined these operations, we must now show that V^S is a vector space. Let's consider the required properties of a vector space in turn:

- **commutativity**. Take $u, v \in V^S$. Then $\forall x \in S$,

  $$(u + v)(x) = u(x) + v(x) = v(x) + u(x) = (v + u)(x),$$

  using the definition of our *sum* above and the fact that vector space V (of which $u(x)$ and $v(x)$ are elements) is itself commutative.

- **associativity, addition**. Take $u, v, w \in V^S$. Then $\forall x \in S$,

  $$\begin{aligned} ((u+v)+w)(x) &= (u+v)(x) + w(x) && \text{definition of } \textit{sum} \\ &= (u(x)+v(x)) + w(x) && \text{definition of } \textit{sum} \\ &= u(x) + (v(x)+w(x)) && \text{assoc. of } V, \text{since } u(x), v(x), w(x) \in V \\ &= u(x) + (v+w)(x) && \text{definition of } \textit{sum} \\ &= (u+(v+w))(x) && \text{definition of } \textit{sum} \end{aligned}$$

- **associativity, scalar multiplication**. Take $f \in V^S$ and $a, b \in \mathbb{F}$. Then $\forall x \in S$,

  $$\begin{aligned} ((ab)f)(x) &= (ab)f(x) && \text{definition of } \textit{product} \\ &= a(bf(x)) && \text{assoc. of vector space } V, \ \text{since } f(x) \in V \\ &= a(bf)(x) && \text{definition of } \textit{product} \end{aligned}$$

- **additive identity**. Define $0 \in V^S$ as the function from S to $\mathbb{F}$ where $0(x) = 0, \forall x \in S$, where the zero on the RHS is the additive identity of V. (This 0 is guaranteed to exist, since V is a vector space.) To demonstrate that this $0 \in V^S$ is really the additive identity for V^S, take $f \in V^S$. Then for all $x \in S$,

  $$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x),$$

  by definition of *sum*, our definition of the 0 function, and the fact that the last 0 is the additive identity of V. Thus, our 0 function is indeed the additive identity for V^S.

- **additive inverse**. Take $f \in V^S$. Then we can absolutely construct a $g \in V^S$ such that g is the additive inverse of f. In particular, define $g(x) := -f(x), \forall x \in S$. $-f(x)$ is guaranteed to exist, since $f(x) \in V$, a vector space that has an additive inverse. And for any $x \in S$,

  $$(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0,$$

  by definition of *sum*, our definition of g, and the definition of the additive identity for V. Thus, our additive inverse of f exists.

- **multiplicative identity**. Take $f \in V^S$. Then for all $x \in S$,

$$(1f)(x) = 1f(x) = f(x),$$

  by definition of *product* and the multiplicative identity of vector space V.

- **distributive properties**. Take $a, b \in \mathbb{F}$ and $f, g \in V^S$. Then $\forall x \in S$,

$$\begin{aligned} (a(f+g))(x) &= a((f+g)(x)) && \text{definition of } \textit{product} \\ &= a(f(x)+g(x)) && \text{definition of } \textit{sum} \\ &= af(x) + ag(x) && \text{distributive properties of V} \end{aligned}$$

and

$$\begin{aligned} ((a+b)f)(x) &= (a+b)f(x) && = \text{definition of } \textit{product} \\ &= af(x) + bf(x) && = \text{distributive properties of } V \end{aligned}$$

And thus, V^S is a vector space. Establishing these properties relied heavily on the fact that the co-domain of elements of V^S is V, a vector space, and thus the various vector space properties hold for the elements obtained by applying arbitrary functions in V^S to elements of S. ■

8. Suppose V is a real vector space.

- The *complexification* of V, denoted by V_C, equals $V \times V$ An element of V_C is an ordered pair (u, v) , where $u, v \in V$, but we write this as $u + iv$.
- Addition on V_C is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on V_C is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, V_C is a complex vector space.

Just like the previous problems, we can verify the properties of a vector space:

- **commutativity**. Take $a + ib, c + id \in V_C$, implying that $a, b, c, d \in V$. Then

$$\begin{aligned} (a + ib) + (c + id) &= (a + c) + i(b + d) && \text{definition of addition on } V_C \\ &= (c + a) + i(d + b) && \text{commutativity of } V \\ &= (c + id) + (a + ib) && \text{definition of addition on } V_C \end{aligned}$$

- **associativity, addition**. Take $a + ib, c + id, x + iy \in V_C$, implying that $a, b, c, d, x, y \in V$. Then

$$\begin{aligned} ((a + ib) + (c + id)) + (x + iy) &= ((a + c) + i(b + d)) + (x + iy) && \text{definition of addition on } V_C \\ &= ((a + c) + x) + i((b + d) + y) && \text{definition of addition on } V_C \\ &= (a + (c + x)) + i(b + (d + y)) && \text{assoc. of } V \\ &= (a + ib) + ((c + x) + i(d + y)) && \text{definition of addition on } V_C \\ &= (a + ib) + ((c + id) + (x + iy)) && \text{definition of addition on } V_C \end{aligned}$$

- **associativity, scalar multiplication**. Take $x + iy \in V_C$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} (ab)(x + iy) &= (ab + 0i)(x + iy) && \text{promote real to complex} \\ &= ((ab)x - 0y) + i((ab)y + 0x) && \text{def. of scalar multiplication} \\ &= ((ab)x) + i((ab)y) && \text{simplify, def. of 0} \\ &= a(bx) + i(a(by)) && \text{assoc. of } V \\ &= (a + 0i)(bx + i(by)) && \text{def. of scalar multiplication} \\ &= (a + 0i)((b + 0i)(x + iy)) && \text{def. of scalar multiplication} \\ &= (a)((b)(x + iy)) && \text{simplify, def. of 0} \\ &= a(b(x + iy)) && \text{drop extra parens} \end{aligned}$$

- **additive identity**. There absolutely exists an element $0 \in V_C$ such that $(a+ib)+0=(a+ib)$ for all $(a+ib) \in V_C$. In particular, let $0 = 0 + i0$, where both 0's on the RHS are the additive identity of V. Then

$$(a+ib)+0=(a+ib)+(0+i0)=(a+0)+i(b+0)=a+ib,$$

by our definition of 0, the definition of addition on V_C, and the fact that all but the first 0 are the additive identity of V. Thus, there exists an element $0 = 0 + i0$ that is an additive identity of V_C.

- **additive inverse**. Take $a + ib \in V_C$, implying $a, b \in V$. Then we can absolutely construct a $(c + id) \in V_C$ such that $(c + id)$ is the additive inverse of $(a + ib)$. In particular, let $c = -a$ and $d = -b$, which are guaranteed to exist by V's additive inverse property. Then

$$(a+ib)+(c+id)=(a+ib)+(-a+i(-d))=(a+(-a))+i(d+(-d))=0+i0=0,$$

  the additive identity we found previously. This confirms the existence of an additive inverse for all $(a + ib) \in V_C$.

- **multiplicative identity**. Take $(a + ib) \in V_C$. Then

$$1(a+ib)=(1+i0)(a+ib)=(1a-0b)+i(1b+0a)=(a-0)+i(b+0)=a+ib,$$

  by definition of multiplication on V_C, the fact that $0v = 0$, the fact that 1 is a multiplicative identity on V, and basic simplification. Thus, $1v = v, \forall v \in V_C$.

- **distributive properties**. Take $s, t \in \mathbb{R}$ and $(a + ib), (c + id) \in V_C$, implying $a, b, c, d \in V$.

Then

$$\begin{aligned}
s((a+ib)+(c+id)) &= s((a+c)+i(b+d)) && \text{def. of addition}\\
&= (s+0i)((a+c)+i(b+d)) && \text{promote to complex}\\
&= (s(a+c)-0(b+d))+i(s(b+d)+0(a+c)) && \text{def. of scalar mult.}\\
&= (sa+sc)+i(sb+sd) && \text{distrib. of } V, \text{simplify zeros}\\
&= (sa+i(sb))+(sc+i(sd)) && \text{def. of addition}\\
&= (s+0i)(a+ib)+(s+0i)(c+id) && \text{def. of scalar mult.}\\
&= s(a+ib)+s(c+id) && \text{simplify zeros}
\end{aligned}$$

and

$$\begin{aligned}
(s+t)(a+ib) &= ((s+t)+0i)(a+ib) && \text{promote to complex}\\
&= ((s+t)a-0b)+i((s+t)b+0a) && \text{def. of scalar mult.}\\
&= ((s+t)a)+i((s+t)b) && \text{simplify zeros}\\
&= (sa+ta)+i(sb+tb) && \text{distrib. of V}\\
&= (sa+i(sb))+(ta+i(tb)) && \text{def. of addition}\\
&= (s+i0)(a+ib)+(t+i0)(a+ib) && \text{def. of scalar mult.}\\
&= s(a+ib)+t(a+ib) && \text{simplify zeros}
\end{aligned}$$

Thus, V_C as constructed is a complex vector space. This essentially followed from the fact, as Axler pointed out, that our real scalars can be promoted to complex scalars: $a \to a + i0$. ■

Bibliography

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[2] R. Larson and B. Edwards, *Calculus*, 11th ed. Cengage Learning, 2014.

Chapter 1C

1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$.

a) $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

Let's check the three conditions required for a subset to be a subspace:

1. $0 \in U$, since $0 + 2 \cdot 0 + 3 \cdot 0 = 0$ ✓
2. Take $x, y \in U$. Write $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, and it is indeed true that $(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0$, since this can be rewritten as $(x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0$. ✓
3. Take $x \in U$ and $a \in \mathbb{F}$. Then $ax = (ax_1, ax_2, ax_3)$. Then it is indeed true that $(ax_1) + 2(ax_2) + 3(ax_3) = 0$, since we can factor out the a using associativity and distributivity to obtain $a(x_1 + 2x_2 + 3x_3) = a0 = 0$ ✓

Since all three conditions are satisfied, this is indeed a subspace of $\mathbb{F}^3$.

This probably also follows more directly from the fact that this subset describes a plane through the origin in $\mathbb{F}^3$, which reminds me of the facts Axler states on page 19 about the various subspaces of $\mathbb{R}^3$. ■

b) $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

This subset immediately fails the **additive identity** test: $0 + 2 \cdot 0 + 3 \cdot 0 \neq 4$, so $0 \notin U$. Therefore, it is not a subspace of $\mathbb{F}^3$. ■

c) $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$

1. $0 \in U$, since $0 \cdot 0 \cdot 0 = 0$ ✓
2. But this subset is not closed under addition. As a counter-example, consider $(0, a, b)$ and $(c, a, 0)$, where $a, b, c \neq 0$. Each vector has component-product zero, meaning it's in U, but the sum is $(c, 2a, b)$. Since no element is zero, it will not be the case that the component-product is zero, meaning that U is not closed under addition.

Therefore, U is not a subspace of $\mathbb{F}^3$. ■

d) $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$

1. $0 \in U$, since $0 = 5 \cdot 0$ ✓
2. **closed under addition**. Consider $(5b, a, b)$ and $(5d, c, d)$. Both are in U, and their sum is $(5b + 5d, a + c, b + d)$. But this list also has the desired property, since $5b + 5d = 5(b + d)$. Thus, U is closed under addition. ✓
3. **closed under scalar multiplication**. Consider $(5b, a, b) \in U$ and $\lambda \in \mathbb{F}$. Then $\lambda(5b, a, b) = (\lambda 5b, \lambda a, \lambda b)$, and by commutativity, the first component is indeed five times the third component. Thus, U is closed under scalar multiplication. ✓

Since it satisfies all three conditions, U is a subspace of $\mathbb{F}^3$. (And yes, this does seem to be a line through the origin of $\mathbb{F}^3$, which suggests directly that it is a valid subspace of $\mathbb{F}^3$.) ■

2. Verify all assertions about subspaces in Example 1.35.

(a) If $b \in \mathbb{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

($\Rightarrow$) Assume $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$. Then $0 = (0, 0, 0, 0)$ must be an element of U. This implies that $0 = 0 + b \Rightarrow b = 0$, as desired.

($\Leftarrow$) Assume $b = 0$. Then to prove that $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + 0]\}$ is a subspace of $\mathbb{F}^4$, we must verify the three properties. But this is essentially identical to Problem (1d). Thus, U is a subspace of $\mathbb{F}^4$, as desired. ■

(b) The set of continuous real-valued function on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Let's verify the three properties required for this to be true:

1. **zero**. Define $0 := 0(x) = 0, \forall x \in \mathbb{R}$ to be the zero function. Then 0 is continuous - it's a zero-degree polynomial imaginable, and polynomials are continuous ([2] p. 79).
2. **closed under addition**. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions. Then $f + g$ is a continuous function by properties of continuity ([2] p. 79).
3. **closed under scalar multiplication**. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and take $b \in \mathbb{R}$. Then bf is a continuous function by properties of continuity ([2] p. 79).

Thus, this subset is indeed a subspace of $\mathbb{R}^{[0,1]}$. ■

(c) The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

This proof is very similar to the previous situation, (2b):

1. The 0 function is differentiable by the The Constant Rule ([2] p. 110).
2. The sum of two differentiable functions is differentiable:
$$\frac{d}{dx}\{f + g\} = \frac{d}{dx}f + \frac{d}{dx}g$$
([2] p. 114).
3. Scalar multiples are handled appropriately by the Constant Multiple Rule:
$$\frac{d}{dx}\{cf\} = c\frac{d}{dx}f$$
([2] p. 113).

Thus, the set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. ■

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2)=b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b=0$.

($\Rightarrow$) Assume that set of functions (call it U) is a subspace of $\mathbb{R}^{(0,3)}$. Then it must contain the zero function and be closed under addition and scalar multiplication. Take $f, g \in U$. Then $f+g \in U$. Thus, $(f+g)'(2)=b$. But by the derivative property in (c.2) above, $(f+g)'(2)=f'(2)+g'(2)=b+b=2b$. Thus, we must have $b=2b$, which implies $b=0$ as desired.

($\Leftarrow$) Assume that $b=0$, i.e. that U is the set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2)=0$. To verify that this is a subspace of $\mathbb{R}^{(0,3)}$, we can check the three required properties:

1. $0 \in U$, since $0'(x)=0, \forall x$, so $0'(2)=0$.
2. For $f, g \in U$, $f+g \in U$ since $(f+g)'(2)=f'(2)+g'(2)=0+0=0$.
3. For $f \in U$ and $c \in \mathbb{R}$, $cf \in U$ since $(cf)'(2)=c \cdot f'(2)=c \cdot 0=0$.

Thus, U is a subspace of $\mathbb{R}^{(0,3)}$.

Since both directions of the implication are true, this iff holds. ■

(e) The set of all sequences of complex numbers with limit 0 is a subspace of C^∞.

Let's check the three properties for this subset (call it U):

1. **zero**. Define 0 as the sequence of all zeros. Then this has limit zero, so $0 \in U$.
2. **closed under addition**. If g and h are two sequences of complex numbers with limit zero, then $g+h$ will also be a sequence of complex numbers due to the fact that
$$\lim_{i\to\infty} g_i + h_i = \lim_{i\to\infty} g_i + \lim_{i\to\infty} h_i = 0+0=0$$
3. **closed under scalar multiplication**. For g a sequence of complex numbers with limit zero and $\lambda \in \mathbb{F}$, then
$$\lim_{i\to\infty} \lambda g_i = \lambda \lim_{i\to\infty} g_i = \lambda \cdot 0 = 0$$

Thus, U is a subspace of C^∞. ■

3. Show that U, the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$, is a subspace of $\mathbb{R}^{(-4,4)}$.

As usual, we'll demonstrate that the three required properties hold:

1. **zero**. As before, define $\mathbf{0}$ as the function such that $\mathbf{0}(x) = 0, \forall x \in (-4, 4)$. Then $\mathbf{0} \in U$ since $\mathbf{0}'(x) = 0$ and therefore $\mathbf{0}'(-1) = 3 \cdot \mathbf{0}(2)$, since $0 = 3 \cdot 0$. ✓
2. **closed under addition**. Take $f, g \in U$. Then $f + g \in U$, since

$$\begin{aligned}
(f+g)'(-1) &= f'(-1) + g'(-1) && \text{Derivative of Sum rule} \\
&= 3f(2) + 3g(2) && f, g \in U \\
&= 3(f(2) + g(2)) && \text{factor out the 3} \\
&= 3((f+g)(2)) && \text{definition of } f+g
\end{aligned}$$

3. **closed under scalar multiplication**. Take $f \in U$ and $\lambda \in F$. Then $\lambda f \in U$, since

$$\begin{aligned}
(\lambda f)'(-1) &= \lambda f'(-1) && \text{Constant Multiple Rule for derivatives} \\
&= \lambda 3 f(2) && f \in U \\
&= 3(\lambda f)(2) && \text{commut. and assoc. of scalars}
\end{aligned}$$

Thus, U is a subspace of $\mathbb{R}^{(-4,4)}$. ■

4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ (call this set U) is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

($\Rightarrow$) Assume that U is a subspace. Then function $\mathbf{0} : \mathbf{0}(x) = 0, \forall x \in [0, 1]$ is in U. This implies that $\int_0^1 \mathbf{0} = b$. But $\int_0^1 \mathbf{0} = 0$ using basic integral properties. Thus, $b = 0$.

($\Leftarrow$) Assume that $b = 0$, i.e. that U is the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = 0$. Then this satisfies all three properties required to be a subspace:

1. $\mathbf{0} \in U$ if $\mathbf{0}$ is defined as above, since $\int_0^1 \mathbf{0} = 0$. ✓
2. For $f, g \in U$, $\int_0^1 f + g = \int_0^1 f + \int_0^1 g = 0 + 0 = 0$, using the fact that the integral of a sum is the sum of the integrals. ✓
3. For $f \in U$ and $\lambda \in \mathbb{F}$, $\int_0^1 \lambda f = \lambda \int_0^1 f = \lambda \cdot 0 = 0$, using the fact that you can "pull" a constant multiple through an integral. ✓

Since both directions of the implication are true, this iff holds. ■

5. Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

   Well, $\mathbb{R}^2$ is definitely a subset of $\mathbb{C}^2$: any $(a, b) \in \mathbb{R}^2$ can also be written $(a + 0i, b + 0i) \in \mathbb{C}^2$.

   And $\mathbb{R}^2$ is definitely a vector space: addition will be commutative and associative, we have an additive identity $0 = (0, 0)$, every element has an additive inverse, we have a multiplicative identity 1, and distribution works perfectly.

   But I don't think that $\mathbb{R}^2$ is actually a subspace of complex vector space $\mathbb{C}^2$!

   Definition (1.33) tells us that "a subset U of V is called a subspace of V if U is also a vector space space with the same additive identity, addition, and scalar multiplication **as on V**" (emphasis mine). The additive identity of $\mathbb{C}^2$, $(0 + 0i, 0 + 0i)$ will also work as the additive identity of $\mathbb{R}^2$, since $0 + 0i = 0$. But $\mathbb{R}^2$ definitely isn't closed under scalar multiplication if the scalars are coming from $\mathbb{C}^2$: $i(1, 2) = (i, 2i) \notin \mathbb{R}^2$. Thus, $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$.

6.

(a) Is $U = \{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

   Yes. Note that for $a, b \in \mathbb{R}$, $a^3 = b^3 \Leftrightarrow a = b$. Thus,
   1. $\mathbf{0} = (0, 0, 0)$ respects the requirement $a = b$.
   2. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$, $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ will respect the requirement $a = b$, since $(x_1 + y_1) = (x_2 + y_2)$ follows directly from the fact that $x_1 = y_1$ and $x_2 = y_2$.
   3. For $x = (x_1, x_2, x_3) \in U$ and $\lambda \in \mathbb{R}$, $\lambda x = (\lambda x_1, \lambda x_2, \lambda x_3)$ will respect $a = b$, since $x_1 = x_2$.

   With these three properties established, U is indeed a subspace of $\mathbb{R}^3$.

(b) Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

   No. As described above in Q5, it won't be closed under scalar multiplication.

7. Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbb{R}^2$.

   This is *not* a subspace of $\mathbb{R}^2$.

   As a counterexample, consider $U = \{(z, 0, 0), z \in \mathbb{Z}\}$. This is certainly nonempty: U contains countably infinite elements. It's closed under addition, since the integers are closed under addition. And it's closed under additive inverses, since $-z \in \mathbb{Z}, \forall z \in \mathbb{Z}$ by def. of the integers.

   But U is not a subspace of $\mathbb{R}^2$. $0 \in U$ and U is closed under addition, but it's not closed under scalar multiplication: $(1, 0, 0) \in U$ and $\pi \in \mathbb{R}$, but $\pi(1, 0, 0) = (\pi, 0, 0) \notin U$, since π is clearly not an integer. Thus, U fails to satisfy the three conditions required to be a subspace. ■

8. Give an example of a nonempty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

   Well, if U is closed under scalar multiplication but not a subspace of $\mathbb{R}^2$, then it must
   - lack a zero element, or
   - not be closed under addition

   It cannot lack a zero because $0 \in \mathbb{R}$, U is nonempty, and U is closed under scalar multiplication. So for $u \in U$, $0 \cdot u \in U$, but $0 \cdot u = \mathbf{0}$. So U contains a zero element.

   Thus, it must not be closed under addition.

   Let $U = \{(a, b) : a, b \in \mathbb{R} \text{ with } ab = 0\}$. Then:
   - $(0, 0) \in U$, since $0 \cdot 0 = 0$
   - $\lambda(a, b) \in U$ for $\lambda \in \mathbb{R}$ and $(a, b) \in U$, since $\lambda(a, b) = (\lambda a, \lambda b)$ and $\lambda a \cdot \lambda b = \lambda^2 \cdot ab = 0$

   But U isn't closed under addition! For example, $(1, 0) \in \mathbb{R}^2$ and $(0, 1) \in \mathbb{R}^2$, but $(1, 0) + (0, 1) = (1, 1) \notin U$.

   Note that this counter-example corresponds geometrically to the coordinate axes. They contain the origin, and they go on forever (scalar multiplication), but by adding an x-coordinate vector with a y-coordinate vector, you escape the axes (and therefore U). ■

9. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

   Clearly the zero function $0(x) = 0$ satisfies the definition of *periodic*, so our subset contains a zero element. It will also be closed under scalar multiplication: geometrically, if a function is periodic, then a vertically stretched/compressed version of the function will also be periodic.

   But I don't think this will be closed under addition because, after giving it a bit of thought, my intuition suggested that it's possible to define two functions that are themselves periodic but whose periods will never align. I was imagining something like $f(x) = \sin(\pi x)$ and $g(x) = \cos(x)$. These have period $\frac{2\pi}{\pi} = 2$ and 2π, respectively, using basic trig properties. But there will never be an x-value that is divisible by both 2 and 2π. Therefore, $f(x) + g(x)$ is aperiodic, meaning we have left the world of periodic functions via addition. **Not a subspace.** ■

10. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

    I don't see any theorems regarding the intersections of subspaces, so I suppose we should just verify that the three required conditions hold.

    1. **zero**. Since V_1 and V_2 are subspaces of V, the zero element of V is in both V_1 and V_2. Therefore, it is also in the intersection, i.e. $0 \in V_1 \cap V_2$. ✓
    2. **closed under addition**. Take $x, y \in V_1 \cap V_2$. Then by definition of intesection, $x, y \in V_1$ and $x, y \in V_2$. Since both V_1 and V_2 are subspaces, they must be closed under addition, so $x + y \in V_1$ and $x + y \in V_2$. Thus, $x + y \in V_1 \cap V_2$. ✓
    3. **closed under scalar multiplication**. Take $x \in V_1 \cap V_2$ and $\lambda \in \mathbb{F}$. Then by definition of intersection, $x \in V_1$ and $x \in V_2$. Since V_1 and V_2 are subspaces, they are closed under scalar multiplication. Thus, $\lambda x \in V_1$ and $\lambda x \in V_2$, implying that $\lambda x \in V_1 \cap V_2$. ✓

    Since all three conditions hold, $V_1 \cap V_2$ is a subspace of V. ■

11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

    Initially, I tried to be cute and did this:

    > After Q10, this is a simple proof by induction.
    >
    > **Base case**: the intersection of a collection of one subspace U of V is just U, and U is a subspace, so the intersection is trivially a subspace of V.
    >
    > **Inductive step**: suppose this is true for every collection containing up to n subspaces $U_1, ..., U_n$ of V, i.e. that $\bigcap_{i=1}^{n} U_i$ is a subspace of V for all choices of subspaces $\{U_i\}$. Now consider another subspace U_{n+1} of V. Let $V_1 := \bigcap_{i=1}^{n} U_i$ and let $V_2 := U_{n+1}$. Then by Q10, $V_1 \cap V_2$ is a subspace of V, as desired. ■

    But this only establishes the desired fact for a collection of subspaces with finite size. To prove that this is true for *every* collection of subspaces of V, we would need to go back to first principles and establish the three required conditions, as we did in Q10.

    In particular, let $\{U_i\}$ be an arbitrary collection of subspaces of V. Then:
    1. **zero**. Since $0 \in U_i, \forall i$ by definition of a subspace, $0 \in \cap_i U_i$. ✓
    2. **closed under addition**. Take $x, y \in \cap_i U_i$. By definition of intersection, x and y are in all U_i. And since all U_i are subspaces of V, $x + y \in U_i, \forall i$. Thus, $x + y \in \cap_i U_i$. ✓
    3. **closed under scalar multiplication.** Take $x \in \cap_i U_i$ and $\lambda \in \mathbb{F}$. By definition of intersection, $x \in U_i, \forall i$. And since all U_i are subspaces of V, $\lambda x \in U_i, \forall i$. Thus, $\lambda x \in \cap_i U_i$. ✓

    Thus, the intesection of every collection of subspaces of V is a subspace of V. ■

12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

    Well, let X and Y be subspaces of V.

    Here was my first version of the forward direction:

    > ($\Rightarrow$, via a proof-by-contradiction of the contrapositive)
    >
    > Assume that it is not that case that one of the subspaces is contained in the other, i.e. that neither subspace contains the other. We want to show that $X \cup Y$ is not a subspace of V.
    >
    > Well, since neither subspace contains the other, there must be an $x \in X \setminus Y$ and a $y \in Y \setminus X$. Assume, to the contrary, that $X \cup Y$ *is* a subspace of V. Then by definition of a subspace, it must be closed under addition. And since $x \in X$ and $y \in Y$, it must be that $x, y \in X \cup Y$ by definition of the union. Thus, $x + y \in X \cup Y$. But $x + y$ cannot be in $X \cup Y$, because that would require $x + y$ to be in either X (impossible, since X is closed under addition and $y \notin X$) or Y (impossible, since Y is closed under addition and $x \notin Y$). This is a contradiction, implying that $X \cup Y$ is not a subspace of V.
    >
    > And thus, by the contrapositive, we can conclude that if the union of two subspaces of V is a subspace of V, then one of the subspaces is contained in the other. ✓

    But a proof-by-contraction of the contrapositive felt a bit roundabout, so I decided to try a direct proof instead. Here's what I came up with:

    > ($\Rightarrow$) Assume $X \cup Y$ is a subspace of V. We want to show that $X \subseteq Y$ or $Y \subseteq X$.
    >
    > Well, take $x \in X$. Then $x \in X \cup Y$ by definition of union. And therefore $x + y \in X \cup Y$ for all $y \in Y$, since subspace $X \cup Y$ is closed under addition. But if $x + y \in X \cup Y$ for all $y \in Y$, then every possible $x + y$ must be in either X or Y (or both). There are two interesting possibilities to consider:
    >
    > 1. $x + y \in X$ for all $y \in Y$. In this case, $Y \subseteq X$, since X is closed under addition.
    > 2. If $x + y \in Y$ for some $y \in Y$, then it must be that $x \in Y$, since subspace Y is closed under addition. But x was arbitrary, implying that $X \subseteq Y$.
    >
    > Thus, one of the subspaces must contain the other. ✓

    In either case, the backward direction is much simpler:

    ($\Leftarrow$) Assume, without loss of generality (due to symmetry in X and Y), that $X \subseteq Y$. Then using basic properties of union, $X \cup Y = Y$ which is a subspace of V! Thus, if one of the subspaces is contained in the other, then their union is a subspace of V. ✓

    Since both directions hold, the iff is true. ■

13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

    Well, let A, B, C be subspaces of V.

    ($\Rightarrow$) Assume that $A \cup B \cup C$ is a subspace of V. TODO: finish!

    ($\Leftarrow$) Assume, without loss of generality (due to symmetry of our subspaces), that A contains B and C. Then using basic properties of union, $A \cup B \cup C = A$, which is a subspace of V. Thus, if one of the subspaces contains the other two, then their union is a subspace of V. ✓

    Since both directions hold, the iff is true. ■

14. Suppose

    $$U = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\} \text{ and } W = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}.$$

    Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

    This is our first situation involving sums of subspaces! Just to refresh our memory in the context of this problem, $U + W = \{u + w : u \in U, w \in W\}$, i.e. the set of all possible sums of elements from U and W. Thus, based on the definitions of U and W,

    $$\begin{aligned} U + W &= \{(a, -a, 2a) + (b, b, 2b) \in \mathbb{F}^3 : a, b \in \mathbb{F}\} \\ &= \{(b + a, b - a, 2a + 2b) \in \mathbb{F}^3 : a, b \in \mathbb{F}\} \end{aligned}$$

    In words, $U + W$ is the set of elements of $\mathbb{F}^3$ such that:
    - the first component is the sum of two elements of $\mathbb{F}$
    - the second component is the difference of those same two elements of $\mathbb{F}$
    - the third component is twice the first component

    ■

15. Suppose U is a subspace of V. What is $U + U$?

    Well, by definition $U + U$ is the set of all possible sums of elements of U. In symbols, $U + U = \{u_1 + u_2 : u_1 \in U, u_2 \in U\}$. But since U is a subspace of V by assumption, it will be closed under addition, i.e. $\forall u_1, u_2 \in U, u_1 + u_2 \in U$. And any element of U can be written as a sum of two elements of U: $\forall u \in U, u = u + 0$ by definition of the additive identity. (Other sums may be possible too, of course.)

    Thus, $U + U = U$! ■

16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

    Yes, the operation of addition on the subspaces of V is commutative intuitively because of the commutatitivity of the subspaces involved. But in order to show that $U + W = W + U$ formally, we should show that an arbitrary element of one set is a member of the other.

    ($\subseteq$) Take $x \in U + W$. Then there exist $u \in U, w \in W$ such that $x = u + w$ by definition of $U + W$. But by commutativity of vector addition, $u + w = w + u$. Thus, $x = w + u$, which implies that $x \in W + U$ by definition of $W + U$. ✓

    ($\supseteq$) Take $x \in W + U$. Then there exist $w \in W, u \in U$ such that $x = w + u$ by definition of $W + U$. But by commutativity of vector addition, $w + u = u + w$. Thus, $x = u + w$, which implies that $x \in U + W$ by definition of $U + W$. ✓

    Note that, since U and W are both subspaces of V, they use the same additive identity, addition, and scalar multiplication as V (1.33). And by (1.40), $U + W$ and $W + U$ are themselves subspaces of V. Thus, the commutativity of $+$ on V will also apply to $U + W$ and $W + U$.

    Thus, the operation of addition on the subspaces of V is commutative. ■

17. Is the operation of addition on the subspaces of V associative? In other words, if A, B, C are subspaces of V, is

    $$(A + B) + C = A + (B + C)?$$

    Like Q17, the answer intuitively seems to be "yes" due to the associativity of addition. But to formally prove this, we can show that these two sets contain each other:

    ($\subseteq$) Take $x \in (A + B) + C$. Then there exist $a \in A, b \in B, c \in C$ such that $x = (a + b) + c$ by definition of $(A + B) + C$. But note that $\{A, B, C, A + B, B + C, (A + B) + C, A + (B + C)\}$ are all subspaces of V that use the same addition as V. Thus, by associativity of this addition operation, $(a + b) + c = a + (b + c)$. Thus, $x = a + (b + c)$, which implies that $x \in A + (B + C)$ by definition of $A + (B + C)$. ✓

    ($\supseteq$) In the same vein, for $x \in A + (B + C)$, there must exist $a \in A, b \in B, c \in C$ such that $x = a + (b + c)$. But by associativity of addition, $a + (b + c) = (a + b) + c$. Thus, $x = (a + b) + c$, which implies that $x \in (A + B) + C$. ✓

    Since these two sets are subsets of each other, they must be equal, implying that the operation of addition of the subspaces of V is associative. ■

18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Yes, it does have an additive identity. Let $Z = \{0\}$, i.e. the subset of V that contains only V's zero element. Then Z is a subspace of V: it obviously contains the zero element, it's closed under addition ($0 + 0 = 0$), and it's closed under scalar multiplication ($\forall \lambda \in \mathbb{F}, \lambda 0 = 0$). To show that this is the additive identity for the operation of addition on the subspaces of V, let A be an arbitrary subspace of V. Then $A + Z = \{a + 0 : a \in A\}$, since 0 is the only element of Z. But since $a + 0 = a$ by definition of the additive identity (which is the same for V and subspaces A and Z), $A + Z = A$. Thus, Z is the additive identity for this operation.

It's hard to imagine a non-trivial subspace B of V that would have an additive inverse, i.e. some other subspace B' such that $B + B' = Z$. Why? Because $B + B'$ is the set of *all* elements you can obtain by adding something from B and something from B'. So if B contains some element $x \neq 0$, then $x \in B + B'$ since $0 \in B'$ (by definition of a subspace) and $x = x + 0$. But $x \in B + B'$ implies that $x \in Z$, which is impossible, since Z only contains 0. Thus, I think that only Z has an additive inverse, namely itself. ■

19. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

This is clearly false - it's possible for U to do the heavy lifting with minimal, different contributions from $V_1 \neq V_2$. For example, define

$$V_1 = \{(a, 0, 0) \in \mathbb{R}^3 : a \in \mathbb{R}\}, V_2 = \{(0, b, 0) \in \mathbb{R}^3 : b \in \mathbb{R}\}, U = \{(a, b, 0) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}.$$

Then $V_1 + U = V_2 + U = U$, since all three of these subspaces contain all elements of $\mathbb{R}^3$ with a third component of zero. But $V_1 \neq V_2$. ■

20. Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Our first direct sum problem - exciting! Let's review the facts we have about direct sums:

1. (1.41) $V_1 + ... + V_m$ is a direct sum if each element of $V_1 + ... + V_m$ (which is a subspace of V) can be written in only one way as the sum $v_1 + ... + v_m$, where each $v_k \in V_k$.
2. (1.45) Given subspaces $V_1, ..., V_m$ of V, $V_1 + ... + V_h$ is a direct sum iff the only way to write 0 as a sum $v_1 + ... + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.
3. (1.46) Suppose U and W are subspaces of V. Then $U + W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$.

These last two facts could be useful for Q20. For example,

- we're stuck with vectors of the form (x, x, y, y) and we need to make sure that $0 = (0, 0, 0, 0)$ can only be written as $0 = 0 + 0$
- we could try to specify a subspace such that the only overlap with these (x, x, y, y)s is 0.

One way to satisfy this second point is $W = \{(a, b, c, d) : a \neq b, c \neq d, a, b, c, d \in \mathbb{F}\}$. Unfortunately, though, this doesn't seem to be closed under addition: $(1, 2, 3, 4) + (2, 1, 4, 3) = (3, 3, 7, 7)$. (And hey, this subset doesn't even contain 0!) So that won't work. ✘

After a bit more thought, I landed on $W = \{(a, b, a, b) : a, b, c, d \in \mathbb{F}\}$. This is a better attempt ($0 \in W$, for example), but unfortunately, it's a bit too general with respect to U. We can write $0 = (1, 1, 1, 1) + (-1, -1, -1, -1)$, where these two elements are both fair game in either subset.

So how can we exclude this possibility for W? Ah, consider $W = \{(a, -a, b, -b)\}$ and take $u \in U, w \in W$. Then when we consider the system

$$u + w = \begin{pmatrix} x + a \\ x - a \\ y + b \\ y - b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we find that $x = a$ and $x = -a$, which can be true iff $x = a = 0$. Likewise, $y = b = -b = 0$. And since the only way to write 0 is a sum $0 + 0$, it must be that $U \oplus W$, i.e. that they form a direct sum. But is this direct sum equal to $\mathbb{F}^4$?

For this to be true, an arbitrary $v \in \mathbb{F}^4$ must be reachable in exactly one way as a sum $u + w$. In other words, an arbitrary $v = (v_1, v_2, v_3, v_4)$ must be reachable and must uniquely determine our vectors (x, x, y, y) and (a, b, a, b). Thankfully, this is true too! Examining the system:

$$\begin{pmatrix} x + a \\ x - a \\ y + b \\ y - b \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

we see that we can let:

- x be the midpoint of v_1 and v_2, i.e. $x = \frac{v_1 + v_2}{2}$
- a be the distance from x to both v_1 and v_2, i.e. $a = \frac{v_1 - v_2}{2}$
- y be the midpoint of v_3 and v_4, i.e. $y = \frac{v_3 + v_4}{2}$
- b be the distance from y to both v_3 and v_4, i.e. $b = \frac{v_3 - v_4}{2}$

Thus, since U and W have trivial intersection and can reach any vector in $\mathbb{F}^4$, $U \oplus W = \mathbb{F}^4$. ■

21. Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$. Find $W \subset \mathbb{F}^5 : \mathbb{F}^5 = U \oplus W$.

Notice that U effectively has two degrees of freedom: once you pick the first two components, the remaining three components are determined. Since we're trying to span $\mathbb{F}^5$, it intuitively seems like we'll need three degrees of freedom in W to capture the five dimensions of our target.

Furthermore, W needs to accomplish two things simultaneously:
1. Different enough from U that their only intersection is 0.
2. Rich enough that together, you can reach any vector in $\mathbb{F}^5$.

After giving this an embarrassing amount of thought, let $W = \{(0, 0, c, d, e) \in \mathbb{F}^5 : c, d, e \in \mathbb{F}\}$. This is definitely a subspace. In considering

$$\begin{pmatrix} x \\ y \\ x+y \\ x-y \\ 2x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

note that the first two equations force $x = y = 0$, which makes the remaining components of the leftmost vector zero, which forces $c = d = e = 0$, ensuring that these two subsets form a direct sum.

And in considering

$$\begin{pmatrix} x \\ y \\ x+y \\ x-y \\ 2x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix},$$

note that we can let $x = v_1$ and $y = v_2$. This will determine the remaining three components of the leftmost vector, and then we can use c, d, e to make up the difference between these determined values and our target values v_3, v_4, v_5:
- $c = v_3 - (x + y) = v_3 - (v_1 + v_2)$
- $d = v_4 - (x - y) = v_4 - (v_1 - v_2)$
- $e = v_5 - 2x = v_5 - 2v_1$

Thus, we can reach an arbitrary element of $\mathbb{F}^5$. Thus, $U \oplus W = \mathbb{F}^5$. ■

20, Take 2. Let $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

I wanted to try the approach I used in Q21 to re-solve Q20. Using that same reasoning, we see that U has two degrees of freedom - we choose the first and third components, and then components two and four are determined. So what if we let $W = \{(0, a, 0, b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}$?

This is definitely a subspace: $0 \in W$, and it's closed under addition and scalar multiplication.

In considering

$$\begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

the first and third equations force $x = y = 0$, which then force $a = b = 0$, as needed to be a direct sum.

And in considering

$$\begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

we can let $x = v_1$ and $y = v_3$. Then $a = v_2 - v_1$ and $b = v_4 - v_3$. Thus, we can define our components in terms of an arbitrary element of $\mathbb{F}^4$, so all elements of $\mathbb{F}^4$ must be reachable. Thus, $U \oplus W = \mathbb{F}^4$, as desired. ■

22. Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$. Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Inspired by the technique we developed in **Q21** and **Q20 Take 2**, let:

- $W_1 = \{(0, 0, a, 0, 0) \in \mathbb{F}^5 : a \in \mathbb{F}\}$
- $W_2 = \{(0, 0, 0, b, 0) \in \mathbb{F}^5 : b \in \mathbb{F}\}$
- $W_3 = \{(0, 0, 0, 0, c) \in \mathbb{F}^5 : c \in \mathbb{F}\}$

Then

$$\begin{pmatrix} x \\ y \\ x + y \\ x - y \\ 2x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

implies that $x = y = 0$, making the rest of the left vector zero, making $a = b = c = 0$.

And if

$$\begin{pmatrix} x \\ y \\ x + y \\ x - y \\ 2x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix},$$

then $x = v_1$, $y = v_2$, a = $v_3 - (v_1 + v_2), b = v_4 - (v_1 - v_2), c = v_5 - 2v_1$ are the unique solutions to the problem of reaching an arbitrary $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{F}^5$.

Thus, $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$, as desired. ■

23. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{ and } \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

My two solutions to Q20 actually provide a counter-example:
- $V = \mathbb{F}^4$
- $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$
- $V_1 = \{(a, -a, b, -b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}$
- $V_2 = \{(0, a, 0, b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}$

Then I showed that $V = V_1 \oplus U$ and $V = V_2 \oplus U$, but clearly $V_1 \neq V_2$. For example, $(1, -1, 1, -1)$ is in V_1 but not V_2. Thus, this claim is false. ■

24. A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x), \forall x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(-x) = -f(x), \forall x \in \mathbb{R}$. Let V_e denote the set of real-valued even functions on $\mathbb{R}$ and let V_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

What an outrageous claim! I love it. And apparently it's even true, since we're asked to prove it.

Just to elaborate upon what we're being asked to prove, we must show that:
1. V_e and V_o form a direct sum $\Leftrightarrow V_e \cap V_o = \{0\}$ by (1.46).
2. Any function $f : R \to R$ can be decomposed into the sum of exactly two sub-functions: an even function and an odd function.

Proving #1 is fairly straightforward. Take $f \in V_e \cap V_o$. Then by definition of $\cap$, $f \in V_e$ and $f \in V_o$. Thus, it true that $f(-x) = f(x)$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. This implies that $f(x) = -f(x)$ for all $x \in \mathbb{R}$, which is true iff $f(x) = 0, \forall x \in \mathbb{R}$. Thus, $V_e \cap V_o = \{0\}$.

To prove #2, take $f \in \mathbb{R}^{\mathbb{R}}$ and do a bit of wishful thinking. Imagine for a moment that we can actually decompose f into $e + o$, the sum of an even and an odd function. Then we would have:

$$f(x) = e(x) + o(x)$$
$$f(-x) = e(x) - o(x)$$

Solving this system via:
- addition elimination, we find that $e(x) = \frac{f(x)+f(-x)}{2}$
- subtraction elimination, we find that $o(x) = \frac{f(x)-f(-x)}{2}$

But wait! We can't just assume that these nice even and odd functions exist. So let's verify that these functions actually have their stated properties...

Is $e(x)$ even? Well, $e(-x) = \frac{f(-x)+f(--x)}{2} = \frac{f(x)+f(-x)}{2} = e(x)$, as required.

Is $o(x)$ odd? Well, $o(-x) = \frac{f(-x)-f(--x)}{2} = \frac{f(-x)-f(x)}{2} = -\frac{f(x)-f(-x)}{2} = -o(x)$, as required.

Does the sum work? Well, $e(x) + o(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x)$, as required.

Thus, these functions $e(x)$ and $o(x)$ are even and odd respectively, can be defined uniquely in terms of arbitrary $f \in \mathbb{R}^{\mathbb{R}}$, and sum to f, meaning that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$, as desired. ■

Chapter 2A

1. Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}.$$

After thinking about this for a few minutes (I took a few weeks off for my honeymoon after finishing Chapter 1), it occurred to me that four distinct vectors in $\mathbb{F}^3$ that have the desired property (namely, that the sum of their elements is zero) should have a span equal to the vector space of all vectors with the desired property.

Thus, take: $(1, -1, 0), (1, 0, -1), (0, 1, -1), (2, -1, -1)$.

These four vectors are certainly distinct. And they certainly belong to $\mathbb{F}^3$, since $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ and these could be interpreted as either real numbers or as complex numbers of the form $a + 0i$.

To show that their span is as desired, take $(x, y, z) \in \mathbb{F}^3 : x + y + z = 0$. Then there must exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + a_4 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

This produces the following system, in which we want to show that the a_is must exist:

$$\begin{aligned} x &= a_1 + a_2 + 2a_4 \\ y &= -a_1 + a_3 - a_4 \\ z &= -a_2 - a_3 - a_4 \end{aligned}$$

But since we have four vectors in a space that feels two-dimensional ($\mathbb{F}^3$ has three dimensions, but once you've selected x and y, z is already determined if $x + y + z = 0$), I think we can safely set a_4 to be zero and still obtain a solution. This simplifies the system:

$$\begin{aligned} x &= a_1 + a_2 \\ y &= -a_1 + a_3 \\ z &= -a_2 - a_3 \end{aligned}$$

Summing both sides of these equations yields $x + y + z = 0$. And given x, y, z, we can solve for the a_is as follows:

- Choose $a_3 = 1$ (arbitrary) and set $a_4 = 0$
- $a_1 = 1 - y$
- $a_2 = -z - 1$

The arbitrariness of $a_3 = 1$ and my observation that our target subspace feels two-dimensional makes me tempted to set $a_3 = 0$ as well. In this case, our system simplifies even further:

$$\begin{aligned} x &= a_1 + a_2 \\ y &= -a_1 \\ z &= -a_2 \end{aligned}$$

This suggests that $a_1 = -y, a_2 = -z, a_3 = 0, a_4 = 0$ also works.

Thus, by definition of span, the span of these four vectors is the desired subspace of $\mathbb{F}^3$. ■

2. Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

At first glance, this doesn't seem like an unreasonable claim. For example, I notice that our four original vectors can be obtained as linear combinations of the new four vectors, thanks to the "telescoping" construction of these new vectors:

$$\begin{aligned}
v_1 &= 1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + 1v_4 \\
v_2 &= 0(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + 1v_4 \\
v_3 &= 0(v_1 - v_2) + 0(v_2 - v_3) + 1(v_3 - v_4) + 1v_4 \\
v_4 &= 0(v_1 - v_2) + 0(v_2 - v_3) + 0(v_3 - v_4) + 1v_4
\end{aligned}$$

This is hardly a conclusive proof. But it does suggest that for any $v \in V$, just as there exists a linear combination of $v_1 ... v_4$ that equals v (by definition of $v_1 ... v_4$ spanning V), there could also be a linear combination of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ that would equal v, making our new list a spanning list, as desired.

Let's formalize these observations into a more formal proof.

Take $v \in V$. By definition of v_1, v_2, v_3, v_4 spanning V, there exist coefficients $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = v$. We'd like to show that there exist coefficients b_1, b_2, b_3, b_4 such that

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = v.$$

The "telescoping" relationships above actually show us how to find $b_1 ... b_4$ as a function of $a_1 ... a_4$. By substitution, we have:

$$\begin{aligned}
a_1 v_1 &= a_1(1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + 1v_4) \\
a_2 v_2 &= a_2(0(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + 1v_4) \\
a_3 v_3 &= a_3(0(v_1 - v_2) + 0(v_2 - v_3) + 1(v_3 - v_4) + 1v_4) \\
a_4 v_4 &= a_4(0(v_1 - v_2) + 0(v_2 - v_3) + 0(v_3 - v_4) + 1v_4)
\end{aligned}$$

The sum of the LHSs equals v by assumption. Thus, the sum of the RHSs must also equal v. And by considering this RHS sum, we can see the coefficients that appear on our four new vectors:

- $v_1 - v_2$ has coefficient a_1, since only one $v_1 - v_2$ term has a non-zero coefficient
- $v_2 - v_3$ has coefficient $a_1 + a_2$, from the top two lines
- $v_3 - v_4$ has coefficient $a_1 + a_2 + a_3$, from the top three lines
- v_4 has coefficient $a_1 + a_2 + a_3 + a_4$, from all four lines

Thus, we have found the coefficients for our linear combination:

$$b_1 = a_1, b_2 = a_1 + a_2, b_3 = a_1 + a_2 + a_3, b_4 = a_1 + a_2 + a_3 + a_4.$$

And since arbitrary $v \in V$ can be written as a linear combination of this new list, this new list of vectors must span V. ■

3. Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$ let

$$w_k = v_1 + ... + v_k.$$

Show that $\text{span}(v_1, ..., v_m) = \text{span}(w_1, ..., w_m)$.

This is similar to #2, except we're constructing our new collection of vectors a bit differently, the size of the collection is now arbitrary, and we need to go in both directions.

Just to help me visualize the relationship between the v_is and w_is, we have:

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_1 + v_2 \\ &... \\ w_m &= v_1 + v_2 + ... + v_m \end{aligned}$$

Off we go...

$\text{span}(v_1, ..., v_m) \supseteq \text{span}(w_1, ..., w_m)$

Take $v \in \text{span}(w_1, ..., w_m)$. Then there exist coefficients $b_1 ... b_m$ such that $v = \sum_{i=1}^m b_i w_i$ by definition of *span*. By the same argument we used for #2, we thus have:

$$\begin{aligned} b_1 w_1 &= b_1 v_1 \\ b_2 w_2 &= b_2 (v_1 + v_2) \\ &... \\ b_m w_m &= b_m (v_1 + v_2 + ... + v_m) \end{aligned}$$

Since v equals the sum of the LHSs, it must equal the sum of the RHSs. And thus, by inspection, we can read off the coefficients a_i for each v_i such that $v = \sum_{i=1}^m a_i v_i$:

$$a_m = b_m, a_{m-1} = b_m + b_{m-1}, ..., a_1 = \sum_{i=1}^m b_i$$

In other words, taking $a_k = \sum_{i=k}^m b_i$, we will have $v = \sum_{i=1}^m a_i v_i$ as required for v to be in $\text{span}(v_1, ..., v_m)$, as desired.

$\text{span}(v_1, ..., v_m) \subseteq \text{span}(w_1, ..., w_m)$

Take $v \in \text{span}(v_1, ..., v_m)$. Then there exist coefficients $a_1 ... a_m$ such that $v = \sum_{i=1}^m a_i v_i$ by definition of *span*. To find the coefficients b_i for each w_i such that $v = \sum_{i=1}^m b_i w_i$, we must work a bit harder than we did in the other direction because we don't have v_i in terms of the w vectors. We can, however, find these relationships by doing some algebra from the top down:

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 - v_1 = w_2 - w_1 \\ v_3 &= w_3 - v_1 - v_2 = w_3 - w_1 - (w_2 - w_1) = w_3 - w_2 \\ &... \\ v_m &= w_m - w_{m-1} \end{aligned}$$

We are now back in familiar territory, and can use the same argument we used above:

$$\begin{aligned} a_1 v_1 &= a_1 w_1 \\ a_2 v_2 &= a_2 (w_2 - w_1) \\ &... \\ a_m v_m &= a_m (w_m - w_{m-1}) \end{aligned}$$

By summing the LHS and RHS and inspecting the coefficients that emerge for each w_i, we have:

$$b_1 = a_1 - a_2, b_2 = a_2 - a_3, ..., b_m = a_m$$

In other words, taking $b_k = a_k - a_{k+1}$ for $k \in \{1, ..., m-1\}$ and $b_m = a_m$ will mean that $v = \sum_{i=1}^{m} b_i w_i$, meaning that $v \in \operatorname{span}(w_1, ..., w_m)$, as desired.

Since both spans are subsets of each other, they must be equal, completing the proof. ■

4a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

($\rightarrow$) Let v form a linearly independent list of length one in vector space V. Assume, to the contrary, that $v = 0$. Then by 2.17, the list v would be linearly dependent because there exists an $a \in \mathbb{F}, a \neq 0$ such that $av = 0$... any a will do, by definition of 0! But this contradicts our assumption that the list is linearly independent, impling that $v \neq 0$.

($\leftarrow$) Let v be a non-zero vector in V, forming a list of length one. Assume, to the contrary, that this list is linearly dependent. Then by 2.17, there must exist an $a \in \mathbb{F}, a \neq 0$ such that $av = 0$. But since $a \neq 0$, by 1.5 there must exist a multiplicative inverse $\frac{1}{a}$ such that $\frac{1}{a}(a) = 1$. Multiplying both sides of $av = 0$ by this inverse, we have $\frac{1}{a}av = \frac{1}{a}0 \Rightarrow v = 0$. But this contradicts our assumption that $v \neq 0$. Thus, this list must be linearly independent, as desired. ■

4b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

($\rightarrow$) Let list of length two $u, v \in V$ be linearly independent. Assume, to the contrary, that one vector is a scalar multiple of the other, i.e. that there exists some $\alpha \in \mathbb{F}$ such that $\alpha u = v$. (Note that the direction here is arbitrary: $\alpha u = v \Leftrightarrow u = \frac{1}{\alpha}v$.)

Then for $a = -\alpha, b = 1$, we will have $au + bv = -\alpha u + 1v = -v + v = 0$. Thus, by 2.17, these two vectors would be linearly dependent, a contradiction. Thus, it must be that these two vectors are not scalar multiples of each other, as desired.

($\leftarrow$) Let $u, v \in V$ form a list of length two such that neither vector is a scalar multiple of the other. Assume, to the contrary, that this list is linearly dependent. Then by 2.17, there must exist $a, b \in \mathbb{F}$, not both 0, such that $au + bv = 0$. Since a, b are not both zero, there are three cases to consider:

1. $a \neq 0, b \neq 0$. Then $au = -bv \Leftrightarrow u = -\frac{b}{a}v$, i.e. u is a scalar multiple of v.
2. $a = 0, b \neq 0$. Since $0u = 0$, we must have $bv = 0$. And $b \neq 0$ implies $v = 0$, meaning that $0u = v$, i.e. u is a scalar multiple of v.
3. $a \neq 0, b = 0$. Since $0v = 0$, we must have $au = 0$. And $a \neq 0$ implies $u = 0$, meaning that $0v = u$, i.e. v is a scalar multiple of u.

All three cases lead to these two vectors being scalar multiples of each other, a contradiction. Thus, this list must be linearly independent, as desired. ■

5. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

If, for the proper choice of t, this list is not linearly independent, then it must be linearly dependent. By 2.19, this means that the last vector must be in the span of the first two, i.e.

$$a\begin{pmatrix}3\\1\\4\end{pmatrix} + b\begin{pmatrix}2\\-3\\5\end{pmatrix} = \begin{pmatrix}5\\9\\t\end{pmatrix}.$$

The first two components give us a system of two equations in two unknowns:

$$\begin{aligned}3a + 2b &= 5\\ 1a - 3b &= 9\end{aligned}$$

This system has unique solution $a = 3, b = -2$. We can use this solution and the third component to find t: $3(4) - 2(5) = t$, meaning $\boldsymbol{t = 2}$.

To verify that this list with $t = 2$ is linearly dependent, we can appeal to 2.17 with non-zero constants:

$$3\begin{pmatrix}3\\1\\4\end{pmatrix} - 2\begin{pmatrix}2\\-3\\5\end{pmatrix} - \begin{pmatrix}5\\9\\2\end{pmatrix} = 0.$$

■

6. Show that the list $(2,3,1),(1,-1,2),(7,3,c)$ is linearly dependent in $\mathbb{F}^3$ if and only if $c = 8$.

($\rightarrow$) Suppose this list is linearly dependent. Then by the same process we used in #5, we can find c by concluding that the third vector must be in the span of the first two. This produces the following system of equations:

$$\begin{aligned} 2a + b &= 7 \\ 3a - b &= 3 \end{aligned}$$

This system has unique solution $a = 2, b = 3$. We can use this solution to find c: $2(1) + 3(2) = c$, giving us $c = 8$, as desired.

($\leftarrow$) Let $c = 8$. Then the list $(2,3,1),(1,-1,2),(7,3,8)$ is linearly dependent by 2.17 because

$$2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 8 \end{pmatrix} = 0$$

and the scalars $2, 3, -1$ are not all zero. ■

7a. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $1+i, 1-i$ is linearly independent.

Thinking of $\mathbb{C}$ as a vector space over $\mathbb{R}$, we are essentially considering the list $(1,1),(1,-1) \in \mathbb{R}^2$. And if we consider the equation

$$a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0,$$

we obtain a system of two equations in two unknowns:

$$\begin{aligned} a + b &= 0 \\ a - b &= 0 \end{aligned}$$

Solving this system, we find a unique solution: $a = b = 0$. Thus, by 2.15, this list is linearly independent, as desired. ■

7b. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $1+i, 1-i$ is linearly dependent.

The key difference here, now that we're working over $\mathbb{C}$, is that now our scalars can be complex numbers. By 2.17, it would be sufficient to show that there exist $a, b \in \mathbb{C}$, not both zero, such that $a(1+i) + b(1-i) = 0$. That's straightforward to do: take $a = 1$ and then solve for $b = \frac{-(1+i)}{1-i} \in \mathbb{C}$. Thus, we have found the required $a, b \in \mathbb{C}$, meaning that this list is linearly dependent. ■

8. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

This looks familiar! Call the vectors in this new list u_1, u_2, u_3, u_4 respectively. Then note that

$$\begin{aligned} v_1 &= u_1 + u_2 + u_3 + u_4 \\ v_2 &= u_2 + u_3 + u_4 \\ v_3 &= u_3 + u_4 \\ v_4 &= u_4 \end{aligned}$$

Solving for these new vectors in terms of the old ones, we therefore get:

$$\begin{aligned} u_4 &= v_4 \\ u_3 &= v_3 - v_4 \\ u_2 &= v_2 - v_3 \\ u_1 &= v_1 - v_2 \end{aligned}$$

Since v_1, v_2, v_3, v_4 are linearly independent, the only solution to $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$ must be $a_1 = a_2 = a_3 = a_4 = 0$ by 2.15.

Now consider the equation $b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = 0$. By substitution, this is equivalent to

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = 0.$$

By rearranging terms, we have:

$$b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4 = 0.$$

Since we're back to working with v's, it must therefore be the case that $b_1 = 0, b_2 - b_1 = 0, b_3 - b_2 = 0, b_4 - b_3 = 0$ will provide the only solution to this equation. This implies that $b_1 = b_2 = b_3 = b_4 = 0$, establishing linear independence of u_1, u_2, u_3, u_4, as desired. ■

9. Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, ..., v_m$$

is linearly independent.

Intuitively, this feels as though it should be linearly independent. By the linear independence of the original list, v_1 cannot be written as a linear combination of $v_2, ..., v_m$. And I don't think that the construction $5v_1 - 4v_2$ can somehow introduce linear dependence. Let's prove that formally.

Let $v_1, ..., v_m$ be a linearly independent list of vectors in V. Let $u_1, ..., u_m$ denote our new list of vectors: $u_1 = 5v_1 - 4v_2$ and $u_j = v_j, j \in \{2, ..., m\}$. Consider the equation

$$a_1 u_1 + a_2 u_2 + ... + a_m u_m = 0.$$

By substitution, this can be rewritten

$$a_1(5v_1 - 4v_2) + a_2 v_2 + ... + a_m v_m = 0.$$

Distributing a_1 and combining like terms, we obtain

$$5a_1 v_1 + (a_2 - 4a_1)v_2 + ... + a_m v_m = 0.$$

By the linear independence of the v's and 2.15, the only choice of coefficients that will make this equation hold are all zeros. Thus, we must also have:

- $5a_1 = 0 \Rightarrow a_1 = 0$ from the v_1 term
- $\Rightarrow a_2 - 4(0) = 0 \Rightarrow a_2 = 0$ from the v_2 term
- $a_j = 0, j \in \{3, ..., m\}$

Thus, the only coefficients that will make our original linear combination of u's equal 0 are all zeros, satisfying the condition for linear independence of 2.15, as desired. ■

10. Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, ..., \lambda v_m$ is linearly independent.

    Intuitively, this is almost surely true - multiplying a collection of linearly independent vectors by a (nonzero) scalar shouldn't somehow introduce a linear dependence! Let's prove it.

    Let $v_1, ..., v_m$ be a linearly independent list of vectors in V. Consider the equation

    $$a_1 v_1 + a_2 v_2 + ... + a_m v_m = 0.$$

    By definition of linear independence, we know that all a's must be zero.

    Now consider the equation

    $$a_1 \lambda v_1 + a_2 \lambda v_2 + ... + a_m \lambda v_m = 0,$$

    with $\lambda \neq 0$, i.e. an arbitrary linear equation of the vectors we hope to prove are linearly independent. By factoring out λ and then multiplying both sides by $\frac{1}{\lambda}$ (guaranteed to exist since $\lambda \neq 0$), we obtain:

    $$\begin{aligned} a_1 \lambda v_1 + a_2 \lambda v_2 + ... + a_m \lambda v_m &= 0 \\ \Leftrightarrow \lambda(a_1 v_1 + a_2 v_2 + ... + a_m v_m) &= 0 \\ \Leftrightarrow a_1 v_1 + a_2 v_2 + ... + a_m v_m &= 0 \end{aligned}$$

    But this is our first equation, where we established that all the a's must be zero, implying that the coefficients must all be zero for our new λ vectors too. Thus, they must be linearly independent, as desired. ■

11. Prove or give a counterexample: If $v_1,, ..., v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

    This, intuitively, is false. I thought through an attempted proof in my head, in the style of the last few, and found myself working with $a_1(v_1 + w_1) + ... + a_m(v_m + w_m) = 0$. After distributing out the as and grouping the v and w terms together, we would have $(a_1 v_1 + ... + a_m v_m) + (a_1 w_1 + ... + a_m w_m) = 0$. And while it's true that either of these pieces equal to zero would force the coefficients to zero by linear independence of the two lists, there's no way to guarantee in this combined expression that they don't linearly combine into additive inverses that then cancel to give us the 0 we're interested in.

    With this seed of doubt in my mind, a counter-example presented itself immediately:

    $$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

    These two lists are each linearly dependent in $\mathbb{R}^2$, but they sum to a list of two copies of the 0 vector, which is not a linearly independent list in $\mathbb{R}^2$, as desired. ■

12. Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.

By linear independence of $v_1, ..., v_m$, we know that in the equation $a_1 v_1 + ... + a_m v_m = 0$, all the a's must be zero.

By the assumed linear *dependence* of $v_1 + w, ..., v_m + w$, in contrast, in the equation

$$b_1(v_1 + w) + ... + b_m(v_m + w) = 0,$$

there must exist b_i's, not all zero, that make this equation true. Distributing and rearranging, we have:

$$\begin{aligned} (b_1 v_1 + ... + b_m v_m) + (b_1 + ... + b_m)w = 0 \\ \Leftrightarrow b_1 v_1 + ... + b_m v_m &= -(b_1 + ... + b_m)w \\ \Leftrightarrow \frac{b_1}{\beta} v_1 + ... + \frac{b_m}{\beta} v_m &= w, \end{aligned}$$

where $\beta = -(b_1 + ... + b_m)$.

(Note that $\beta \neq 0$, and thus multiplicative inverse $\frac{1}{\beta}$ will exist. If, to the contrary, $\beta = 0$, then $b_1 v_1 + ... + b_m v_m = -0w = 0$, with linear indepedence implying that all b_is are zero. But this violates the assumption of linear independence of the v's!)

Since not all of our b's are zero, at least one of the LHS terms will survive and achieve a linear combination of w. Thus, $w \in \text{span}(v_1, ..., v_m)$, as desired. ■

13. Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent iff $w \notin \operatorname{span}(v_1, ..., v_m)$.

One direction is the converse of what we proved in #12, which unfortunately means that we don't get it for free. Instead, we'll just have to prove both directions...

($\rightarrow$) $v_1, ..., v_m, w$ linearly independent $\Rightarrow w \notin \operatorname{span}(v_1, ..., v_m)$

Let $v_1, ..., v_m, w \in V$ be linearly independent. Suppose, to the contrary, that $w \in \operatorname{span}(v_1, ..., v_m)$. Then by definition of span, there exist $a_1, ..., a_m \in \mathbb{F}$ such that $a_1 v_1 + ... + a_m v_m = w$. But this implies that $a_1 v_1 + ... + a_m v_m - w = 0$. Since $v_1, ..., v_m, w$ are linearly independent, the only coefficients that should make their linear combination 0 are all zeros, which we clearly see is not true - even if all the a's are zero, w has a coefficient of -1. This contradiction forces us to conclude that $w \notin \operatorname{span}(v_1, ..., v_m)$, as desired.

($\leftarrow$) $w \notin \operatorname{span}(v_1, ..., v_m) \Rightarrow v_1, ..., v_m, w$ linearly independent

Let $v_1, ..., v_m \in V$ be linearly independent, and take $w \in V$ such that $w \notin \operatorname{span}(v_1, ..., v_m)$. Consider the equation

$$b_1 v_1 + ... + b_m v_m + \beta w = 0.$$

Assume, to the contrary, that there were values of $b_1, ..., b_m, \beta$ (not all zero) such that this were true, i.e. that $v_1, ..., v_m, w$ are linearly dependent.

Though we don't know exactly which coefficients are nonzero, we can safely conclude that $\beta \neq 0$; if $\beta = 0$, then we would be left with $b_1 v_1 + ... + b_m v_m = 0$, and the linear independence of the v's would force all the b's to zero, meaning that all of the coefficients were zero after all.

But if $\beta \neq 0$, then w would be in the span of the v's:

$$-\frac{b_1}{\beta} v_1 + ... + -\frac{b_m}{\beta} v_m = w,$$

contradicting our assumption that $w \notin \operatorname{span}(v_1, ..., v_m)$. Thus, all $b_1, ..., b_m, \beta$, must be zero, making $v_1, ..., v_m, w$ linearly independent, as desired.

■

14. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \ldots + v_k.$$

Show that the list $v_1, \ldots, v_m$ is linearly independent if and only if the list $w_1, \ldots, w_m$ is linearly independent.

For reference, it is useful to see each set of vectors in terms of the others:

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_1 + v_2 \\ &\ldots \\ w_m &= v_1 + \ldots + v_m \end{aligned} \qquad\qquad \begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 - w_1 \\ &\ldots \\ v_m &= w_m - w_{m-1} \end{aligned}$$

($\rightarrow$) $v_1, \ldots, v_m$ linearly independent $\Rightarrow$ $w_1, \ldots, w_m$ linearly independent

Let $v_1, \ldots, v_m \in V$ be linearly independent. To establish the linear independence of $w_1, \ldots, w_m$, consider the equation $a_1 w_1 + a_2 w_2 + \ldots + a_m w_m = 0$. By substitution and the left column above, we obtain

$$\begin{aligned} a_1 v_1 + a_2(v_1 + v_2) + \ldots + a_m(v_1 + \ldots + v_m) &= 0 \\ \Leftrightarrow (a_1 + \ldots + a_m)v_1 + (a_2 + \ldots + a_m)v_2 + \ldots + a_m v_m &= 0 \qquad \text{distribute, regroup} \end{aligned}$$

Working from right to left, the linear independence of the v's forces all of these coefficients to zero:

- $a_m = 0$ directly from the rightmost term
- $a_{m-1} + a_m = 0 \Rightarrow a_{m-1} = 0$ from the penultimate term
- …
- $a_1 + \ldots + a_m = 0 \Rightarrow a_1 = 0$ if all the later coefficients are zero

The a's being zero satisfies the requirement for linear independence of $w_1, \ldots, w_m$, as desired.

($\leftarrow$) $w_1, \ldots, w_m$ linearly independent $\Rightarrow$ $v_1, \ldots, v_m$ linearly independent

Let $w_1, \ldots, w_m \in V$ be linearly independent. To establish the linear independence of $v_1, \ldots, v_m$, consider the equation $b_1 v_1 + b_2 v_2 + \ldots + b_m v_m = 0$. By substitution and the right column above, we obtain

$$\begin{aligned} b_1 w_1 + b_2(w_2 - w_1) + \ldots + b_m(w_m - w_{m-1}) &= 0 \\ \Leftrightarrow (b_1 - b_2)w_1 + (b_2 - b_3)w_2 + \ldots + b_m w_m &= 0 \qquad \text{distribute, regroup} \end{aligned}$$

Again working from right to left, the linear independence of the w's forces all of these coefficients to zero:

- $b_m = 0$ directly from the rightmost term
- $b_{m-1} - b_m = 0 \Rightarrow b_{m-1} = 0$ from the penultimate term, once $b_m = 0$ has been established
- …
- $b_1 - b_2 = 0 \Rightarrow b_1 = 0$ from the first term, once $b_2 = 0$ has been established

The b's being zero satisfies the requirement for linear independence of $v_1, \ldots, v_m$, as desired.

■

15. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathbb{P}_4(\mathbb{F})$.

    This conclusion essentially follows from the Pidgeonhole Principle. Suppose you start with an empty list and want to start adding polynomials to the list such that they remain linearly independent in $\mathbb{P}_4(\mathbb{F})$. Using the notation on page 31, you can safely add $1, z, z^2, z^3, z^4$ in this order, since no linear combination of $1, \ldots, z^m$ will equal z^{m+1}. But that only gives us five polynomials in our list, and we wanted six. When we go to "place" this sixth polynomial into one of the five "slots" afforded us by $\mathbb{P}_4(\mathbb{F})$, we must place into a slot that's already been used. But this duplicate will obviously be a scalar multiple of the polynomial already occupying that slot, and thus it will be in the span of these first five polynomials, making linear independence impossible.

    This conclusion could alternatively be drawn based on 2.22. Our list $1, z, z^2, z^3, z^4$ spans $\mathbb{P}_4(\mathbb{F})$, and thus every linearly independent list will have length at most 5, making a linearly independent list of six polynomials impossible in this case. ■

16. Explain why no list of four polynomials spans $\mathbb{P}_4(\mathbb{F})$.

    Continuing #15, we now come at $\mathbb{P}_4(\mathbb{F})$ from the other direction!

    2.22 is also quite useful in this case. Since we came up with a list of length five that is linearly independent in $\mathbb{P}_4(\mathbb{F})$ (1, z, z^2, z^3, z^4), we know from 2.22 that any spanning list must have length at least 5. Thus, no list of four polynomials will span $\mathbb{P}_4(\mathbb{F})$.

    Using the Pidgeonhole Principle approach from #15, we have the opposite problem now: we have five "slots" to fill and only four polynomials to place into slots. Thus, at least one slot will be empty, meaning there are certain polynomials in $\mathbb{P}_4(\mathbb{F})$ that our list will be unable to reach via its span. ■

17. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

    $(\rightarrow)$ V has infinite-dimension $\Rightarrow$ this infinite vector sequence exists

    Let V be an infinite-dimensional vector space. By 2.13, this means that V is not finite-dimensional. By 2.9, there must not be a (finite) list of vectors in V that spans the space.

    The desired sequence seems like it could be constructed quite naturally using mathematical induction, so let's proceed in this direction.

    Base case ($m = 1$): There must be some vector $v_1 \in V, v_1 \neq 0$; otherwise, 0 would be the only element of V, and finite list 0 would span V, a contradiction. By 2.16c, the list v_1 is linearly independent.

    Inductive step: Assume we have found a list of vectors $v_1, \ldots, v_m \in V$ such that $v_1, \ldots, v_m$ is linearly independent. This list is finite, and therefore $\operatorname{span}(v_1, \ldots, v_m) \neq V$.

    This implies that there must exist some $v_{m+1} \in V$ that is not in $\operatorname{span}(v_1, \ldots, v_m)$. By #13, v_{m+1} can be added to sequence $v_1, \ldots, v_m$ while still maintaining linear independence.

    Since this new vector v_{m+1} can be found for any finite $m > 0$, the desired sequence $v_1, v_2, \ldots$ therefore exists and can be constructed as above, as desired. ■

    $(\leftarrow)$ $\exists v_1, v_2, \ldots \in V : v_1, \ldots, v_m$ is linearly independent, $\forall m > 0 \Rightarrow$ V is infinite-dimensional

    Let $v_1, v_2, \ldots \in V$ be a sequence of vectors such that $v_1, \ldots, v_m$ is linearly independent for every integer $m > 0$. Assume, to the contrary, that V is finite-dimensional.

    By 2.9, there must be some (finite) list of vectors $v_1, \ldots, v_n \in V$ such that $V = \operatorname{span}(v_1, \ldots, v_n)$. By 2.22, this establishes a fixed upper bound n on the length of any linearly independent list. But this contradicts the premise that we can construct a list of linearly independent vectors of length m, $\forall m > 0$. Thus, V must be infinite-dimensional, as desired. ■

18. Prove that $\mathbb{F}^\infty$ is infinite-dimensional.

    Before we prove anything about $\mathbb{F}^\infty$, let's take a second to get clear on what $\mathbb{F}^\infty$ is. Back in Chapter 1, we fixed a positive integer n and then defined $\mathbb{F}^n$ to be the set of all lists of length n of elements of $\mathbb{F}$. For example, if we work with $\mathbb{R}$ for a second and consider $\mathbb{R}^2$, then we're essentially with all of the (infinitely many) points in the xy-plane.

    Then on page 13, we defined $\mathbb{F}^\infty$ to be the set of all sequences of elements of $\mathbb{F}$:

    $$\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_k \in \mathbb{F} \text{ for } k = 1, 2, \ldots\}$$

    We can use our result from #17 to easily establish the infinite-dimensionality of $\mathbb{F}^\infty$. For our sequence of vectors $v_1, v_2, \ldots \in \mathbb{F}^\infty$ such that $v_1, \ldots, v_m$ is linearly independent for all positive integers m, use sequence

    $$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \ldots \end{pmatrix}, \ldots$$

    For any finite $m > 0$, $v_1, \ldots, v_m$ will clearly be linearly independent: when considering the equation $a_1 v_1 + \ldots + a_m v_m = 0$, the fact that each non-zero element is in a different position will force the a's to zero is order to satisfy this equation. Thus, $\mathbb{F}^\infty$ is infinite-dimensional. ■

19. Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

We can again appeal to our result from #17 to easily prove this. In this case, we need an infinite sequence of continuous-real-valued functions on $[0, 1]$, call it $f_1, f_2, \ldots$, such that for any finite $m > 0$, the list $f_1, \ldots, f_m$ is linearly independent.

Polynomials, with their domain restricted to $[0, 1]$, will work perfectly: let $f_k(x) = x^k, x \in [0, 1]$. This sequence is clearly infinite, as it is indexed by the natural numbers. Polynomials are continuous. And for any $m > 0$, the list $f_1, \ldots, f_m$ will be linearly independent because in the equation $a_1 x + \ldots + a_m x^m = 0$, the unique powers of the polynomial terms will force all the a's to zero in order to make this true.

Thus, the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional, as desired. ■

20. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $P_m(\mathbb{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $P_m(\mathbb{F})$.

Let $p_0, p_1, \ldots, p_m$ be polynomials in $P_m(\mathbb{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Assume, to the contrary, that $p_0, \ldots, p_m$ *is* linearly independent in $P_m(\mathbb{F})$. Then in the equation

$$a_0 p_0 + \ldots + a_m p_m = 0,$$

it must be the case that the only set of coefficients that makes this true is $a_0 = \ldots = a_m = 0$, for any value $x \in \mathbb{F}$. But this is false: for $x = 2$, the coefficients are actually irrelevant because all of our polynomials are 0 at $x = 2$. Thus, $p_0, \ldots, p_m$ is not linearly independent, as desired. ■

Chapter 2B

1. Find all vector spaces that have exactly one basis.

Interesting. This isn't immediately obvious (I took about a month off between Chapters 1 and 2), but given that it's #1, I suspect the answer will emerge pretty naturally once we have the relevant definitions in front of us.

(1.20): A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that we have commutativity, associativity, an additive identity 0, an additive inverse, a multiplicative identity, and distributivity of both scalars and vectors.

(2.26): A **basis** of V is a list of vectors in V that is linearly independent and spans V.

The smallest vector space is $V_0 = \{0\}$, i.e. the set containing just the additive identity. And the empty list is actually a valid basis for V_0, since it's (vacuously) linear independent (by 2.15) and (vacuously) spans V_0 (by 2.4). Since the empty list is unique, V_0 has exactly one basis.

Any other vector space V will have multiple bases, I'm afraid, due to the requirement that the vector space be closed under scalar multiplication: if list of vectors $v_1, \ldots, v_m$ forms a basis of V (which is apparently m-dimensional), then $\lambda v_1, \ldots, \lambda v_m$ will also form a basis.

Thus, $V_0 = \{0\}$ is the only vector space that has exactly one basis.

2. Verify all assertions in Example 2.27.

a) Great, we all know that the *standard basis* is very useful.

b) The list $(1,2),(3,5)$ is linearly independent, since neither vector is a scalar multiple of the other in this list of length two. And these two vectors span $\mathbb{F}^2$, since any $v=(x,y)\in\mathbb{F}^2$ can be written uniquely in the form: $\binom{x}{y}=a_1\binom{1}{2}+a_2\binom{3}{5}$, where $a_1=3y-5x$ and $a_2=2x-y$ is the unique solution to this system of two equations in two unknowns.

c) The list $(1,2,-4),(7,-5,6)$ is definitely linearly independent in $\mathbb{F}^3$, since neither vector is a scalar multiple of the other in this list of length two. It definitely doesn't span $\mathbb{F}^3$ though - the length of the *standard basis* in $\mathbb{F}^3$, which is linearly independent, is 3, meaning that any spanning list of $\mathbb{F}^3$ must have length ≥ 3 (by 2.22).

d) The list $(1,2),(3,5),(4,13)$ definitely spans $\mathbb{F}^2$ because a subset of the list, namely the first two vectors, spans $\mathbb{F}^2$, as we showed in (b). And it's definitely not independent, since we found a spanning list of length 2, meaning that any linearly independent list must have length at most 2.

e) The list $(1,1,0),(0,0,1)$ is definitely linearly independent - neither is a scalar multiple of the other. And this list definitely spans $\{(x,x,y)\in\mathbb{F}^3 : x,y\in\mathbb{F}\}$, since any $v:=(x,x,y)$ can be written uniquely as $x(1,1,0)+y(0,0,1)$. Thus, this list is a basis for the given vector space.

f) The list $(1,-1,0),(1,0,-1)$ is definitely linearly independent: length two, neither is a scalar multiple of the other. As for spanning, (x,y,z) with $x+y+z=0$ can be formed uniquely as $\alpha(1,-1,0)+\beta(1,0,-1)$ by letting $\alpha=-y,\beta=-z$. Thus, this list forms a basis.

g) Great, we have a *standard polynomial basis* too.

3. Let U be the subspace of $\mathbb{R}^5$ defined by $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\}$.

   a) Find a basis of U.

   The list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ is a basis of U. It is clearly linearly independent, due to the non-overlapping indices of the non-zero elements. And it also spans U: for any vector $v = (x_1, x_2, x_3, x_4, x_5) \in U$, we will have unique coefficients that achieve v is a linear combination:

   $$v = x_2(3, 1, 0, 0, 0) + x_4(0, 0, 7, 1, 0) + x_5(0, 0, 0, 0, 1)$$

   b) Extend the basis in (a) to a basis of $\mathbb{R}^5$.

   This extension will involve adding two more vectors to the list, since $\mathbb{R}^5$'s standard basis has length 5. In order to give ourselves the flexibility to reach an arbitrary $v \in \mathbb{R}^5$, add $(1, 0, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$. This will give us the ability to decouple the first and second elements and the third and fourth elements, respectively.

   c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

   Clearly, this subspace will give us the flexibility we described in (b). In particular, let $W = \{(x, 0, y, 0, 0) \in \mathbb{R}^5\}$. Then $\mathbb{R}^5 = U \oplus W$, as desired.

4. Let U be the subspace of $\mathbb{C}^5$ defined by

   $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$.

   a) Find a basis of U.

   Well, vector $(1, 6, 0, 0, 0)$ will satisfy the first constraint. The second constraint means that vectors $(0, 0, -2, 1, 0)$ and $(0, 0, -3, 0, 1)$ should join the list.

   b) Extend the basis in (a) to a basis of $\mathbb{C}^5$.

   Clearly, we need two more vectors. $(0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$ will give us the extra flexibility needed to cover all of $\mathbb{C}^5$.

   c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

   Inspired by the flexibility we gained in (b), let $W = \{(0, z_2, z_3, 0, 0) \in \mathbb{C}^5 : z_2, z_3 \in \mathbb{C}\}$.

NOTE: I realize that I haven't really rigorously proved anything in #3 or #4. I appreciated these exercises for their ability to get me thinking about the relationships between dimension, bases, subspaces, etc. in real, concrete settings, which I believe was the point of them. Real proofs coming in later exercises!

5. Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

   Since V is finite-dimensional, it has a basis (2.31).

   By definition of *subspace sums*, $V = U + W = \{u + w : u \in U, w \in W\}$. In other words, V consists of a vector space of vectors that are formed via every possible sum of vectors from U and W. But this is why there's something interesting to prove here: $U \cup W$ consists of vectors that are in U or W (or both), but aren't necessarily sums of vectors from these two subspaces.

   I think we can prove this directly via construction. By 2.25, U and W are finite-dimensional. By 2.31, U and W each have a basis. Let $u_1, ..., u_m$ and $w_1, ..., w_n$ be a basis for U and W, respectively. Then the list $u_1, ..., u_m, w_1, ..., w_n$ must span V.

   To confirm this, note that any $v \in V$ must have been formed as the sum of some $u \in U, w \in W$ by our definition of subspace sums above. And by definition of basis, $u \in \text{span}(u_1, ..., u_m)$ and $w \in \text{span}(w_1, ..., w_n)$. The linear combinations of $u_1, ..., u_m$ and $w_1, ..., w_n$ that produce u and w, respectively, will sum to form v, since $v = u + w$.

   But with this spanning list for V consisting of vectors from U and W, we're done! Because by 2.30, this spanning list can be reduced to a basis of V. And since our list consists of vectors in $U \cup W$, removing vectors from it won't change the fact that it consists of vectors in $U \cup W$. ■

6. Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $P_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $P_3(\mathbb{F})$.

   This is actually false because *degree* refers to the *highest* power of the polynomial.

   Consider the list of polynomials 1, x, $x^2 + x^3$, x^3. This is a list in $P_3(\mathbb{F})$ such that none of the polynomials has degree 2; the degrees are 0, 1, 3, and 3, respectively.

   But this list *is* a basis of $P_3(\mathbb{F})$. Given $p \in P_3(\mathbb{F})$, $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, we will have

   $$p = a_0 1 + a_1 x + a_2(x^2 + x^3) + (a_3 - a_2)x^3.$$

   Thus, $p \in \text{span}(p_0, p_1, p_2, p_3)$. And this list is definitely linearly independent, due to the offsets of the powers. Thus, this list forms a basis of $P_3(\mathbb{F})$, completing the counter-example.

7. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

We know by definition of *basis* that v_1, v_2, v_3, v_4 are linearly independent and span V. And for any $v \in V$, there exist unique coefficients in $\mathbb{F}$ such that $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.

To make our lives easier, let us give names to these new vectors we wish to prove are a basis of V, and let us relate v_1, v_2, v_3, v_4 and our new vectors in both directions:

$$\begin{aligned} u_1 &= v_1 + v_2 \\ u_2 &= v_2 + v_3 \\ u_3 &= v_3 + v_4 \\ u_4 &= v_4 \end{aligned} \qquad \begin{aligned} v_1 &= u_1 - u_2 + u_3 - u_4 \\ v_2 &= u_2 - u_3 + u_4 \\ v_3 &= u_3 - u_4 \\ v_4 &= u_4 \end{aligned}$$

Are the u's linearly independent and do they span V?

To establish linearly independence, consider the equation $b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = 0$. By substition, this is equivalent to $b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4 = 0$. Distributing and regrouping, we obtain:

$$b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4 = 0.$$

By the linear independence of the v's, we know that these coefficients must all be zero. Working from left to right, this implies that $b_1 = 0 \Rightarrow b_2 = 0 \Rightarrow b_3 = 0 \Rightarrow b_4 = 0$, as required for linear independence.

To show that $V = \operatorname{span}(u_1, u_2, u_3, u_4)$, take $v \in V$. Then there exist unique coefficients in $\mathbb{F}$ such that $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$. By substitution, we obtain

$$\begin{aligned} v &= a_1(u_1 - u_2 + u_3 - u_4) + a_2(u_2 - u_3 + u_4) + a_3(u_3 - u_4) + a_4 u_4 \\ &= a_1 u_1 + (a_2 - a_1)u_2 + (a_3 - a_2 + a_1)u_3 + (a_4 - a_3 + a_2 - a_1)u_4. \end{aligned}$$

Thus, there are also unique coefficients such that v is a linear combination of the u's, meaning $V = \operatorname{span}(u_1, u_2, u_3, u_4)$. And thus, vectors u_1, u_2, u_3, u_4 form a basis of V, as desired. ■

8. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

   I didn't have a great intuition for this problem, going into it. On the one hand, it's not crazy to think that v_1, v_2 could form a basis of U. But the fear is that something in v_3 or v_4 "helps", and we're losing access to that help by excluding these last two vectors.

   Still, I tried to prove this by counterexample:

   Let v_1, v_2, v_3, v_4 be a basis of V. Let U be a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$.

   Suppose to the contrary that v_1, v_2 is *not* a basis of U.

   Then by definition of *basis*, this list is either linearly dependent or it doesn't span U. But since the larger list v_1, v_2, v_3, v_4 is a basis of V and is therefore linearly independent, the shorter list v_1, v_2 will also be linearly independent; thus, it must be the case that v_1, v_2 doesn't span U.

   This means there exists some $u \in U$ such that v_1, v_2 cannot "reach" u via linear combination. But $u \in V$ too, since U is a subspace of V. Thus, there exist unique coefficients such that

   $$u = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

   Since v_1, v_2 cannot reach u, we know that $a_3 \neq 0$ or $a_4 \neq 0$ (or both). This doesn't obviously produce a contradiction, and seems to suggest that sure enough, v_3 and/or v_4 are helping v_1 and v_2 with their spanning duties.

   Having hit that head end, let's try to find a counter-example. Since our basis has length four, let $V = \mathbb{R}^4$. For our basis, use the standard basis $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$. And for our subspace, let $U = \{(x, y, z, z) : x, y, z \in \mathbb{R}\}$. Then $v_1, v_2 \in U$, $v_3 \notin U$, $v_4 \notin U$. But v_1, v_2 definitely doesn't form a basis for U, since there's no way to achieve non-zero values for the last two coordinates with just the first two standard basis vectors. Thus, we have our counterexample! ■

9. Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + ... + v_k.$$

Show that $v_1, ..., v_m$ is a basis of V if and only if $w_1, ..., w_m$ is a basis of V.

As we often do in these problems, let's write out these vectors in terms of the others:

$$\begin{aligned} w_1 &= v_1 & v_1 &= w_1 \\ w_2 &= v_1 + v_2 & v_2 &= w_2 - w_1 \\ &\cdots & &\cdots \\ w_m &= v_1 + v_2 + ... + v_m & v_m &= w_m - w_{m-1} \end{aligned}$$

$(\rightarrow)$ $v_1, ..., v_m$ is a basis of $V \Rightarrow w_1, ..., w_m$ is a basis of V

Let $v_1, ..., v_m \in V$ be a basis of V. By definition, this list is linearly independent, and $V = \operatorname{span}(v_1, ..., v_m)$. We can use these facts to show that $w_1, ..., w_m$ is also a basis:

- Linearly independent: Consider the equation $a_1 w_1 + ... + a_m w_m = 0$. By substitution and the left column above, this is true iff $a_1 v_1 + a_2(v_1 + v_2) + ... + a_m(v_1 + ... + v_m) = 0$. Distributing and regrouping, we have $(a_1 + ... + a_m)v_1 + ... + a_m v_m = 0$. By the linear independence of the v's, these coefficients must all be zero. From right to left, as it always does, we get $a_m = 0 \Rightarrow a_{m-1} = 0 \Rightarrow ... \Rightarrow a_1 = 0$, as required for linear independence of the w's.
- Spans: For $v \in V$, there must exist unique coefficients such that $v = b_1 v_1 + ... + b_m v_m$, since the v's span V. By substition and the right column above, this is true iff

$$\begin{aligned} v &= b_1 w_1 + b_2(w_2 - w_1) + ... + b_m(w_m - w_{m-1}) \\ &= (b_1 - b_2)w_1 + (b_2 - b_3)w_2 + ... + b_m w_m. \end{aligned}$$

  Hooray, we've found unique coefficients for the w's to reach v, thus $V = \operatorname{span}(w_1, ..., w_m)$.

$(\leftarrow)$ $w_1, ..., w_m$ is a basis of $V \Rightarrow v_1, ..., v_m$ is a basis of V

Let $w_1, ..., w_m \in V$ be a basis of V. By definition, this list is linearly independent, and $V = \operatorname{span}(w_1, ..., w_m)$. We can use these facts to show that $v_1, ..., v_m$ is also a basis:

- Linearly independent: Consider the equation $a_1 v_1 + ... + a_m v_m = 0$. By substitution and the right column above, this is true iff $a_1 w_1 + a_2(w_2 - w_1) + ... + a_m(w_m - w_{m-1}) = 0$. Distributing and regrouping, we have $(a_1 - a_2)w_1 + ... + a_m w_m = 0$. By the linear independence of the w's, these coefficients must all be zero. From right to left, as it always does, we get $a_m = 0 \Rightarrow a_{m-1} = 0 \Rightarrow ... \Rightarrow a_1 = 0$, as required for linear independence of the v's.
- Spans: For $v \in V$, there must exist unique coefficients such that $v = b_1 w_1 + ... + b_m w_m$, since the w's span V. By substition and the left column above, this is true iff

$$\begin{aligned} v &= b_1 v_1 + b_2(v_1 + v_2) + ... + b_m(v_1 + ... + v_m) \\ &= (b_1 + ... + b_m)v_1 + (b_2 + ... + b_m)v_2 + ... + b_m v_m. \end{aligned}$$

  Hooray, we've found unique coefficients for the v's to reach v, thus $V = \operatorname{span}(v_1, ..., v_m)$. ■

10. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, ..., u_m$ is a basis of U and $w_1, ..., w_n$ is a basis of W. Prove that

$$u_1, ..., u_m, w_1, ..., w_n$$

is a basis of V.

Let U and W be subspaces of V such that $V = U \oplus W$.

Let $u_1, ..., u_m$ be a basis of U and $w_1, ..., w_n$ be a basis of W.

For $v \in V$, by definition of $\oplus$, there exist unique vectors $u \in U, w \in W : v = u + w$.

Since we have our basis vectors for U and W, there must exist unique coefficients such that $u = a_1 u_1 + ... + a_m u_m$ and $w = b_1 w_1 + ... + b_n w_n$. By substitution, we will therefore have

$$v = a_1 u_1 + ... + a_m u_m + b_1 w_1 + ... + b_n w_n.$$

Thus, v has a unique representation of the vectors $u_1, ..., u_m, w_1, ..., w_n$. And thus, by 2.28, $u_1, ..., u_m, w_1, ..., w_n$ is a basis for V, as desired. ■

11. Suppose V is a real vector space. Show that if $v_1, ..., v_n$ is a basis of V (as a real vector space), then $v_1, ..., v_n$ is also a basis of the complexification V_C (as a complex vector space).

Let V be a real vector space, and let $v_1, ..., v_n$ be a basis of V.

Referring back to the referenced problem for the definition of *complexification*, let $V_C = V \times V$. In other words, any element $z \in V_C$ is an ordered pair (u, v) with $u, v \in V$. (Recall that we often write this ordered pair as $u + vi$ when considering it a complex number.)

Take $z \in V_C$. We'd like to show that there exist unique coefficients s.t. $z = c_1 v_1 + ... + c_n v_n$.

By definition of V_C, $z = (u, v)$ with $u, v \in V$. Since $v_1, ..., v_n$ forms a basis for V, there must exist unique coefficients such that

$$u = a_1 v_1 + ... + a_n v_n$$
$$v = b_1 v_1 + ... + b_n v_n$$

Therefore, we have:

$$\begin{aligned} z &= u + vi \\ &= (a_1 v_1 + ... + a_n v_n) + (b_1 v_1 + ... + b_n v_n) i \\ &= (a_1 + b_1 i) v_1 + ... + (a_n + b_n i) v_n \\ &= c_1 v_1 + ... + c_n v_n \end{aligned}$$

Since the a's and b's were unique, the c's will be unique, completing the proof. ■

Chapter 2C

1. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin, and $\mathbb{R}^2$.

2. Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, all lines in $\mathbb{R}^3$ containing the origin, all planes in $\mathbb{R}^3$ containing the origin, and $\mathbb{R}^3$.

3.