SICP: Exercise 1.13

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Problem) Prove that $\text{Fib}(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$.
Hint: Let $\psi = (1-\sqrt{(5)})/2$. Use induction and the definition of the Fibonacci numbers to prove that $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.

For reference,

$$\text{Fib}(n) = \begin{cases} 0, n = 0 \\ 1, n = 1 \\ \text{Fib}(n-1) + \text{Fib}(n-2), \text{else} \end{cases}$$

Also note that $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$.

As the hint suggests, we begin by proving by mathematical induction that $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.
Base case (n = 0): $(\phi^0 - \psi^0)/\sqrt{5} = (1-1)/\sqrt{5} = 0 \equiv \text{Fib}(0)$.
Base case (n = 1): $(\phi^1 - \psi^1)/\sqrt{5} = (\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2})/\sqrt{5} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \equiv \text{Fib}(1)$.
Inductive Step: Assume this relationship holds up to some n value. We want to show that $\text{Fib}(n+1) = (\phi^{n+1} - \psi^{n+1})/\sqrt{5}$. By definition, $\text{Fib}(n+1) = \text{Fib}(n) + \text{Fib}(n-1)$. By inductive assumption, therefore, we have:

$$\begin{aligned} Fib(n+1) &= Fib(n) + Fib(n-1) \\ &= \frac{\phi^n - \psi^n}{\sqrt{5}} + \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\ &= \frac{\phi^n + \phi^{n-1} - (\psi^n + \psi^{n-1})}{\sqrt{5}} \stackrel{?}{=} \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} \\ &\Leftrightarrow \phi^n + \phi^{n-1} - (\psi^n + \psi^{n-1}) = \phi^{n+1} - \psi^{n+1} \end{aligned}$$

To make this work, note that $\phi^{n+1} = \phi^2 \cdot \phi^{n-1} = (\phi + 1) \cdot \phi^{n-1} = \phi^n + \phi^{n-1}$. Similarly, $\psi^{n+1} = \psi^n + \psi^{n-1}$. So indeed, our inductive step holds, establishing that $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.
To show that this implies that $\text{Fib}(n)$ is the closest integer to $\phi^n/\sqrt{5}$, we essentially need the ψ term in our expression for $\text{Fib}(n)$ to be insignificant. And as luck would have it, $|\psi| \approx 0.618034 < 1$, so that $\lim_{n\to\infty} \psi^n = 0$.

Of course, both ϕ and ψ are irrational numbers, and yet the difference of their powers must always be an integer (since Fibonacci numbers are integers). This limit demonstrates that ψ^n essentially accounts for the irrational component of ϕ^n without really changing the magnitude of ϕ^n. So indeed, the closest integer to ϕ^n is $\mathrm{Fib}(n)$.