Derivation of the Damped and Driven Harmonic Oscillator

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1 Damped Harmonic Oscillator

Governing Equation

We begin with Newton's second law, including a velocity-dependent damping force $fr = -b\dot{u}$ and the spring force $-ku$. For a **spring–damper system**, damping force is proportional to velocity and for a **spring-mass system**, restoring force (spring force) is proportional to the displacement. Strain e form is used in continuum mechanic, not in the simple spring–mass ODE.

$$m\ddot{u} = F_{restoring} + F_{friction}\ (F_{damping})$$

$$m\ddot{u} = -ku - b\dot{u}$$

N:B Negative sign means force always acts opposite to displacement and b is the damping coefficient

Rearranging all terms to one side and dividing by the mass m:

$$\ddot{u} + \frac{b}{m}\dot{u} + \frac{k}{m}u = 0$$

Recall that $\omega_0^2 = \frac{k}{m}$ is the square of the natural frequency. Define the damping coefficient $\gamma = \frac{b}{m}$. This yields the standard form:

$$\ddot{u} + \gamma\dot{u} + \omega_0^2 u = 0 \tag{1}$$

General Solution via Complex Exponential Ansatz

We assume a solution of the form $u(t) = Ae^{i\nu t}$, where ν is a complex frequency to account for damping. Calculating the derivatives:

$$\dot{u}(t) = \frac{d}{dt}(Ae^{i\nu t}) = i\nu Ae^{i\nu t}$$
$$\ddot{u}(t) = \frac{d}{dt}(i\nu Ae^{i\nu t}) = (i\nu)^2 Ae^{i\nu t} = -\nu^2 Ae^{i\nu t}$$

Substituting u, $\dot{u}$, and $\ddot{u}$ into equation (1):

$$(-\nu^2 Ae^{i\nu t}) + \gamma(i\nu Ae^{i\nu t}) + \omega_0^2(Ae^{i\nu t}) = 0$$

Factoring out the common term $Ae^{i\nu t}$:

$$Ae^{i\nu t}\left(-\nu^2 + i\gamma\nu + \omega_0^2\right) = 0$$

For this to hold for all t, the expression in parentheses must be zero:

$$-\nu^2 + i\gamma\nu + \omega_0^2 = 0 \quad \Rightarrow \quad \nu^2 - i\gamma\nu - \omega_0^2 = 0$$

This is the characteristic equation. Solving for ν using the quadratic formula:

$$\nu = \frac{i\gamma \pm \sqrt{(-i\gamma)^2 + 4\omega_0^2}}{2} = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

$$\nu = \frac{i\gamma}{2} \pm \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}$$

We express the complex frequency as $\nu = \omega + i\sigma$. Identifying real and imaginary parts:

$$\text{Real part (Oscillation frequency):} \quad \omega = \frac{1}{2}\sqrt{4\omega_0^2 - \gamma^2}$$
$$\text{Imaginary part (Damping rate):} \quad \sigma = \frac{\gamma}{2}$$

Thus, $\nu = \omega + i\gamma/2$. Substituting back into the trial solution:

$$u(t) = Ae^{i\nu t} = Ae^{i(\omega + i\frac{\gamma}{2})t} = Ae^{i\omega t}e^{-(\gamma/2)t}$$

Taking the real part (or constructing a real solution from two complex conjugates), the general *real* solution is:

$$\boxed{u(t) = e^{-(\gamma/2)t} \left[C_1 \cos(\omega t) + C_2 \sin(\omega t)\right]}$$

where C_1 and C_2 are real constants determined by initial conditions.

Energy Dissipation

The total mechanical energy is $E = \frac{1}{2}m\dot{u}^2 + \frac{1}{2}ku^2$. Its rate of change is:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{u}^2 + \frac{1}{2}ku^2\right) = m\dot{u}\ddot{u} + ku\dot{u}$$

Factoring out $\dot{u}$:

$$\frac{dE}{dt} = \dot{u}(m\ddot{u} + ku)$$

From the equation of motion (1), $m\ddot{u} = -b\dot{u} - ku$, so $m\ddot{u} + ku = -b\dot{u}$. Substituting:

$$\frac{dE}{dt} = \dot{u}(-b\dot{u}) = -b(\dot{u})^2$$

Since $b > 0$ and $(\dot{u})^2 \geq 0$, we conclude:

$$\frac{dE}{dt} = -b(\dot{u})^2 \leq 0$$

Q.E.D. The rate of change of total energy is always negative or zero (only zero when the velocity is zero). This means the total energy of the system is monotonically decreasing (dissipating) over time, which is exactly what we expect from a damped system. The energy is lost as heat due to friction.

2 Driven Damped Harmonic Oscillator

Governing Equation

Adding an external driving force $F(t)$ to the equation:

$$m\ddot{u} = -ku - b\dot{u} + F(t)$$

Rearranging and dividing by m:

$$\ddot{u} + \gamma\dot{u} + \omega_0^2 u = \frac{F(t)}{m}$$

For a sinusoidal drive, we use $a_f(t) = a_f e^{i\omega_f t}$, where a_f is a complex amplitude representing the drive strength and phase:

$$\ddot{u} + \gamma\dot{u} + \omega_0^2 u = a_f e^{i\omega_f t} \tag{2}$$

Steady-State Solution

The general solution is $u(t) = u_h(t) + u_p(t)$, where $u_h(t)$ is the transient homogeneous solution (which decays as $e^{-(\gamma/2)t}$) and $u_p(t)$ is the particular solution representing the steady state. For the steady state, we assume oscillation at the drive frequency:

$$u_p(t) = Ae^{i\omega_f t}$$

where A is a complex constant to be determined. Calculating derivatives:

$$\begin{aligned}
\dot{u}_p(t) &= i\omega_f A e^{i\omega_f t} \\
\ddot{u}_p(t) &= (i\omega_f)^2 A e^{i\omega_f t} = -\omega_f^2 A e^{i\omega_f t}
\end{aligned}$$

Substituting into equation (2):

$$\left(-\omega_f^2 A e^{i\omega_f t}\right) + \gamma\left(i\omega_f A e^{i\omega_f t}\right) + \omega_0^2\left(A e^{i\omega_f t}\right) = a_f e^{i\omega_f t}$$

Factoring out $Ae^{i\omega_f t}$ on the left:

$$Ae^{i\omega_f t}\left(-\omega_f^2 + i\gamma\omega_f + \omega_0^2\right) = a_f e^{i\omega_f t}$$

Dividing both sides by $e^{i\omega_f t}$:

$$A\left(\omega_0^2 - \omega_f^2 + i\gamma\omega_f\right) = a_f$$

Solving for the complex amplitude A:

$$A = \frac{a_f}{\omega_0^2 - \omega_f^2 + i\gamma\omega_f} \tag{3}$$

Magnitude and Phase

The complex number A describes both the amplitude and phase of the oscillation relative to the drive. To find the magnitude and argument of A, we express the denominator in polar form. Let $D = (\omega_0^2 - \omega_f^2) + i(\gamma\omega_f)$.

$$|D| = \sqrt{(\omega_0^2 - \omega_f^2)^2 + (\gamma\omega_f)^2}, \quad \arg(D) = \tan^{-1}\left(\frac{\gamma\omega_f}{\omega_0^2 - \omega_f^2}\right)$$

Therefore,

$$A = \frac{a_f}{|D|}e^{-i\arg(D)} = \frac{a_f}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + (\gamma\omega_f)^2}}\, e^{-i\tan^{-1}\left(\frac{\gamma\omega_f}{\omega_0^2 - \omega_f^2}\right)}$$

From this, we extract:

$$|A| = \frac{|a_f|}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + (\gamma\omega_f)^2}}, \quad \arg(A) = -\tan^{-1}\left(\frac{\gamma\omega_f}{\omega_0^2 - \omega_f^2}\right) \tag{4}$$

The steady-state solution is $u_p(t) = Ae^{i\omega_f t} = |A|e^{i(\omega_f t + \arg(A))}$.

Resonance

Resonance occurs when the amplitude $|A|$ is maximized. This happens when the denominator $S(\omega_f) = (\omega_0^2 - \omega_f^2)^2 + (\gamma\omega_f)^2$ is minimized. To find the minimum, treat S as a function of ω_f^2:

$$S = (\omega_0^2 - x)^2 + \gamma^2 x, \quad \text{where } x = \omega_f^2$$

Differentiating with respect to x and setting the derivative to zero:

$$\frac{dS}{dx} = -2(\omega_0^2 - x) + \gamma^2 = 0$$

$$2(\omega_0^2 - x) = \gamma^2 \quad \Rightarrow \quad x = \omega_0^2 - \frac{\gamma^2}{2}$$

Thus, the resonant frequency is:

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}$$

For the undamped case ($\gamma = 0$), $\omega_{\text{res}} = \omega_0$. Substituting into equation (4):

$$|A| = \frac{|a_f|}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + 0}} = \frac{|a_f|}{0} \to \infty$$

This confirms the amplitude becomes unbounded at resonance for an undamped oscillator.