Time Domain Complex-valued ViscoAcoustic Wave Equation [2]

OYEKAN, Hammed

October, 2025

1 Starting Point: Standard Acoustic Wave Equation

The general inhomogeneous acoustic wave equation in the time domain is:

$$\frac{1}{\rho(x)v^2(x)}\frac{\partial^2 p(x,t)}{\partial t^2} - \nabla \cdot \left[\frac{1}{\rho(x)}\nabla p(x,t)\right] = f(x_s,t)\delta(x - x_s) \tag{1}$$

where $p(x,t)$ is the pressure wavefield, $\rho(x)$ is density, $v(x)$ is velocity, and $f(x_s,t)$ is the source function.

Transform to Frequency Domain

Apply the Fourier transform to equation 1:

$$\mathcal{F}\left\{\frac{\partial^2 p(x,t)}{\partial t^2}\right\} = -\omega^2 P(x,\omega) \tag{2}$$

The frequency-domain acoustic wave equation becomes:

$$-\frac{\omega^2}{\rho(x)v^2(x)}P(x,\omega) - \nabla \cdot \left[\frac{1}{\rho(x)}\nabla P(x,\omega)\right] = F(x_s,\omega)\delta(x - x_s) \tag{3}$$

2 Introduce Visco-Acoustic Effects

2.1 Complex-Valued Velocity Model

For visco-acoustic media, the velocity is replaced by a complex-valued, frequency-dependent velocity which describe the viscous attenuation effect in frequency domain (Aki & Richards, 1980) [1].

The complex-valued velocity $v(\mathbf{x}, \omega)$ that models both dispersion and dissipation is given by:

$$v(x,\omega) = v_0(x)\left(1 + \underbrace{\frac{1}{\pi Q(x)}\ln\frac{\omega}{\omega_0}}_{\textbf{Velocity Dispersion}} - \underbrace{i\frac{\mathrm{sgn}(\omega)}{2Q(x)}}_{\textbf{Amplitude Dissipation}}\right) \tag{4}$$

This expression has three parts:

- $v_0(\mathbf{x})$: The reference velocity at a reference frequency ω_0.
- $\frac{1}{\pi Q(x)}\ln\frac{\omega}{\omega_0}$: The term responsible for **velocity dispersion** (frequency-dependent phase speed).
- $Q(x)$ is the quality factor (frequency-independent)
- $-\frac{i\,sgn(\omega)}{2Q(x)}$: The term responsible for **amplitude dissipation** (frequency-dependent energy loss). It characterizes amplitude loss.
- **sgn** is the the called sign or signum function sgn(x) = (1 if $x > 0$, 0 if $x = 0$, -1 if $x < 0$)

2.2 Frequency-Domain Visco-Acoustic Wave Equation

Substituting the complex velocity into equation 4:

$$-\frac{\omega^2}{\rho(x)v_0^2(x)\left(1 + \frac{1}{\pi Q(x)}\ln\frac{\omega}{\omega_0} - i\frac{\mathrm{sgn}(\omega)}{2Q(x)}\right)^2}P(x,\omega) - \nabla\cdot\left[\frac{1}{\rho(x)}\nabla P(x,\omega)\right] = F(x_s,\omega)\delta(x - x_s) \tag{5}$$

For weak attenuation (high Q, $1/Q$ is small for weak attenuation), we can expand the denominator and neglect higher-order terms in $1/Q$ (using first-order expansion). This leads to:

$$\left(1 + \frac{1}{\pi Q}\ln\frac{\omega}{\omega_0} - i\frac{\mathrm{sgn}(\omega)}{2Q}\right)^{-2} \approx 1 - \frac{2}{\pi Q}\ln\frac{\omega}{\omega_0} + i\frac{\mathrm{sgn}(\omega)}{Q} \tag{6}$$

Equation (7) then becomes:

$$\boxed{-\frac{\omega^2}{\rho(x)v_0^2(x)}\left[1 - \frac{2}{\pi Q(x)}\ln\frac{\omega}{\omega_0} + i\frac{\mathrm{sgn}(\omega)}{Q(x)}\right]P(x,\omega) - \nabla\cdot\left[\frac{1}{\rho(x)}\nabla P(x,\omega)\right] = F(x_s,\omega)\delta(x - x_s)} \tag{7}$$

The terms $\omega^2 \ln \frac{\omega}{\omega_0}$ and $i\omega^2\, sgn(\omega)$ are still difficult to transform directly back to the time domain. So, we need to approximate these two terms as well.

3 Approximations for Time-Domain Transformation

The challenge is that the **logarithmic** and **the signum functions** make direct inverse Fourier transformation difficult. Two approximations are needed.

3.1 Dispersion Approximation

The authors propose approximating the logarithmic dispersion term with a second-order polynomial in ω

$$\omega^2 \ln \frac{\omega}{\omega_0} \approx a\omega^2 + b\omega + c \tag{8}$$

where a, b, and c are coefficients determined by least-squares fitting over the frequency band of interest (e.g., 1-150 Hz).

Substituting this approximation into the dispersion term in Equation 7:

$$\begin{aligned} &-\frac{\omega^2}{\rho(x)v_0^2(x)}\left[1 - \frac{2}{\pi Q(x)} \ln \frac{\omega}{\omega_0}\right] P(x,\omega) \\ &\qquad\approx -\frac{1}{\rho(x)v_0^2(x)}\left[\omega^2 - \frac{2}{\pi Q(x)}\left(a\omega^2 + b\omega + c\right)\right] P(x,\omega) \end{aligned} \tag{9}$$

Rearranging:

$$\boxed{-\frac{1}{\rho(x)v_0^2(x)}\left[\left(1 - \frac{2a}{\pi Q(x)}\right)\omega^2 - \frac{2b}{\pi Q(x)}\omega - \frac{2c}{\pi Q(x)}\right] P(x,\omega)} \tag{10}$$

This is equation (5) from the paper and it represents the approximation for the **dispersion term**.

This is a crucial step. The frequency-domain operator is now a simple polynomial in ω, which is easy to transform to the time domain using the properties of the Fourier transform.

3.2 Dissipation Approximation

The dissipation term in Eq. (7) is:

$$\frac{i\omega^2\, sgn(\omega)}{\rho v_0^2 Q}$$

The presence of $sgn(\omega)$ makes direct transformation difficult. The authors use an approximation from Zhu & Harris (2014) [3], which is valid for weak attenuation. The key insight is to relate the frequency ω to the wavenumber k.

$$k \approx k_r \quad \text{and} \quad k_r \approx \frac{\omega}{v_0} \tag{11}$$

where k is the complex wavenumber and k_r is its real part.

For a propagating wave, $\omega \approx v_0|k|$. Therefore:

$$|\omega| \approx v_0|k| \approx k_r v_0$$

Using this, they approximate the term as follows:

$$\begin{aligned}\frac{i\omega^2\, sgn(\omega)}{\rho v_0^2 Q} = \frac{i\omega^2}{\rho v_0^2 Q} \cdot \frac{|\omega|}{\omega} &\approx \frac{i\omega^2}{\rho v_0^2 Q} \cdot \frac{v_0|k|}{\omega} \\ &= \frac{i\omega|k|}{\rho v_0 Q}\end{aligned} \tag{12}$$

$$i\frac{\omega^2 \mathrm{sgn}(\omega)}{\rho v_0^2 Q} \approx i\frac{\omega|k|}{\rho v_0 Q} \tag{13}$$

In the space domain:

The wavenumber k corresponds to the spatial derivative operator $-i\nabla$, so $|k|$ corresponds to the square root of the negative Laplacian, $\sqrt{-\nabla^2}$. This gives the final approximation for the dissipation term:

$$\boxed{-i\frac{\omega^2 \mathrm{sgn}(\omega)}{\rho(x) v_0^2(x) Q(x)} P(x,\omega) \approx -i\frac{\omega|k|}{\rho(x) v_0(x) Q(x)} P(x,\omega) \approx -i\frac{\omega\sqrt{-\nabla^2}}{\rho(x) v_0(x) Q(x)} P(x,\omega)} \tag{14}$$

This is equation (8) from the paper.

Where:

- $|\cdot|$ denotes the absolute value.

- $\sqrt{-\nabla^2}$ is known as the **fractional Laplacian**, a pseudo-differential operator

4 Assembling the Approximated Frequency-Domain Equation

Now, we plug the two approximations (Eq. (10) and Eq. (14)) back into the original equation (Eq. (7)).

Original Eq. (7)**:**

$$-\frac{\omega^2}{\rho v_0^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+i\frac{sgn(\omega)}{Q}\right]P-\nabla\cdot\left[\frac{1}{\rho}\nabla P\right]=F\delta$$

Substitute with Approximations:

1. Substitute the **dispersion part** from **Eq.** (10).
2. Substitute the **dissipation part** from **Eq.** (14).

This yields the approximated **frequency-domain visco-acoustic equation**:

$$\boxed{-\frac{1}{\rho v_0^2}\left[\left(1-\frac{2a}{\pi Q}\right)\omega^2-\frac{2b}{\pi Q}\,\omega-\frac{2c}{\pi Q}\right]P-i\,\frac{\omega\sqrt{-\nabla^2}}{\rho v_0 Q}\,P-\nabla\cdot\left(\frac{1}{\rho}\nabla P\right)=F\,\delta} \tag{15}$$

$$\begin{aligned}-\frac{1}{\rho(x)v_0^2(x)}\left[\left(1-\frac{2a}{\pi Q(x)}\right)\omega^2-\frac{2b}{\pi Q(x)}\omega-\frac{2c}{\pi Q(x)}\right]P(x,\omega)\\ -\frac{i\omega\sqrt{-\nabla^2}}{\rho(x)v_0(x)Q(x)}P(x,\omega)-\nabla\cdot\left[\frac{1}{\rho(x)}\nabla P(x,\omega)\right]=F(x_s,\omega)\delta(x-x_s)\end{aligned} \tag{16}$$

5 Inverse Fourier Transform to the Time Domain

Finally, we apply the **inverse Fourier transform** to the entire equation (15).

Fourier Transform Pairs

- $i\omega\rightarrow\frac{\partial}{\partial t}$ (First time derivative)
- $-\omega^2\rightarrow\frac{\partial^2}{\partial t^2}$ (Second time derivative)

- $P(\mathbf{x}, \omega) \rightarrow p(\mathbf{x}, t)$ (Wavefield)
- $F(\mathbf{x}_s, \omega) \rightarrow f(\mathbf{x}_s, t)$ (Source)
- The spatial operators ∇, $\nabla\cdot$, and $\sqrt{-\nabla^2}$ remain unchanged.

Let's transform Eq. (15) term-by-term.

1. **First Term (Dispersion Approximation):**

$$-\frac{1}{\rho v_0^2}\left[\left(1-\frac{2a}{\pi Q}\right)\omega^2 P - \frac{2b}{\pi Q}\omega P - \frac{2c}{\pi Q}P\right]$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad \frac{1}{\rho v_0^2}\left[\left(1-\frac{2a}{\pi Q}\right)\frac{\partial^2 p}{\partial t^2} - i\frac{2b}{\pi Q}\frac{\partial p}{\partial t} + \frac{2c}{\pi Q}p\right]$$

Note: The sign changes because $-\omega^2 \rightarrow \frac{\partial^2}{\partial t^2}$ *and* $i\omega \rightarrow \frac{\partial}{\partial t}$.

2. **Second Term (Dissipation Approximation):**

$$-\frac{i\omega\sqrt{-\nabla^2}}{\rho v_0 Q}P \quad \xrightarrow{\mathcal{F}^{-1}} \quad -\frac{\sqrt{-\nabla^2}}{\rho v_0 Q}\frac{\partial p}{\partial t}$$

Note: $i\omega P \rightarrow \frac{\partial p}{\partial t}$.

3. **Third Term (Laplacian):**

$$-\nabla\cdot\left[\frac{1}{\rho}\nabla P\right] \quad \xrightarrow{\mathcal{F}^{-1}} \quad -\nabla\cdot\left[\frac{1}{\rho}\nabla p\right]$$

4. **Source Term:**

$$F(x_s, \omega)\delta(x - x_s) \xrightarrow{\mathcal{F}^{-1}} f(x_s, t)\delta(x - x_s)$$

Now, let us assmble all these transformed terms into the final time-domain equation. Let's also define the coefficients C_1, C_2, C_3, C_4 for clarity.

The final **Time-Domain Complex-Valued Visco-Acoustic Wave Equation** is:

$$\boxed{\underbrace{\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})}\left[C_1\frac{\partial^2 p}{\partial t^2} - iC_2\frac{\partial p}{\partial t} + C_3 p\right]}_{\textbf{Dispersion-dominated}} - \underbrace{C_4\sqrt{-\nabla^2}\frac{\partial p}{\partial t}}_{\textbf{Dissipation-dominated}} - \nabla\cdot\left[\frac{1}{\rho(\mathbf{x})}\nabla p\right] = f(\mathbf{x}_s,t)\delta(\mathbf{x}-\mathbf{x}_s)} \tag{17}$$

Where the coefficients are:

$$\begin{aligned} C_1 &= \left(1 - \frac{2a}{\pi Q(\mathbf{x})}\right), \qquad & C_2 &= \frac{2b}{\pi Q(\mathbf{x})}, \\ C_3 &= \frac{2c}{\pi Q(\mathbf{x})}, & C_4 &= \frac{1}{\rho(\mathbf{x})v_0(\mathbf{x})Q(\mathbf{x})} \end{aligned} \tag{18}$$

Note on the imaginary sign of C_2:

Note that in equation (17), we have $-i\frac{2b}{\pi Q}$. The presence of the imaginary unit i in the C_2 term is what makes the wavefield solution **complex-valued**.

6 Features of the Derived Equation

1. **Complex-valued:** The imaginary unit i in the C_2 term makes the wavefield complex-valued, coupling the real and imaginary parts during propagation.

2. **Dispersion:** The three terms with coefficients C_1, C_2, C_3 represent the polynomial approximation to the logarithmic dispersion.

3. **Dissipation:** The fractional Laplacian term $C_4\sqrt{-\nabla^2}\frac{\partial p(x,t)}{\partial t}$ represents amplitude attenuation.

4. **Flexibility:** By setting different coefficients to zero, we can obtain:
   - Dispersion-dominated equation: Set $C_4 = 0$
   - Dissipation-dominated equation: Set $C_1 = 1$, $C_2 = C_3 = 0$

7 Physical Interpretation

- The standard wave equation only has the second-derivative term $\frac{\partial^2 p}{\partial t^2}$
- Visco-acoustic effects add:
  - Modified second derivative ($C_1 \neq 1$): Velocity dispersion
  - First derivative with i ($-iC_2\frac{\partial p}{\partial t}$): Phase coupling due to dispersion

- Zeroth-order term ($C_3 p$): Frequency-dependent baseline
- Fractional derivative ($C_4\sqrt{-\nabla^2}\frac{\partial p}{\partial t}$): Non-local dissipation

8 Discretization of the Time-Domain Complex-Valued Visco-Acoustic Wave Equation

The general time-domain visco-acoustic wave equation is written as

$$\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})}\left[C_1\frac{\partial^2 p(\mathbf{x},t)}{\partial t^2}-iC_2\frac{\partial p(\mathbf{x},t)}{\partial t}+C_3 p(\mathbf{x},t)\right]-C_4\sqrt{-\nabla^2}\frac{\partial p(\mathbf{x},t)}{\partial t}-\nabla\cdot\left[\frac{1}{\rho(\mathbf{x})}\nabla p(\mathbf{x},t)\right]=f(\mathbf{x}_s,t)\delta(\mathbf{x}-\mathbf{x}_s). \tag{19}$$

We introduce the shorthand

$$m(\mathbf{x})=\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})},\qquad \mathcal{S}=\sqrt{-\nabla^2},\qquad s(\mathbf{x},t)=f(\mathbf{x}_s,t)\delta(\mathbf{x}-\mathbf{x}_s),$$

so that Eq. (19) can be written compactly as

$$m(\mathbf{x})\Big[C_1\,\partial_{tt}p-iC_2\,\partial_t p+C_3\,p\Big]-C_4\,\mathcal{S}\,\partial_t p-\nabla\cdot\left[\frac{1}{\rho}\nabla p\right]=s. \tag{20}$$

Time discretization

Let $p^n(\mathbf{x})=p(\mathbf{x},t_n)$ where $t_n=n\Delta t$. We use centered finite differences for the time derivatives:

$$\partial_{tt}p|_{t_n}\approx\frac{p^{n+1}-2p^n+p^{n-1}}{\Delta t^2}, \tag{21}$$

$$\partial_t p|_{t_n}\approx\frac{p^{n+1}-p^{n-1}}{2\Delta t}. \tag{22}$$

The fractional operator acting on the time derivative is also centered:

$$\mathcal{S}\,\partial_t p|_{t_n}\approx\mathcal{S}\left(\frac{p^{n+1}-p^{n-1}}{2\Delta t}\right). \tag{23}$$

The spatial Laplacian term is evaluated explicitly at time n:

$$\mathrm{Div}^n=\nabla\cdot\left(\frac{1}{\rho}\nabla p^n\right). \tag{24}$$

With pseudospectral method $\mathcal{S}$ and Div are computed via FFTs: $\mathcal{S}\leftrightarrow|\mathbf{k}|$, $-\nabla^2\leftrightarrow|\mathbf{k}|^2$, etc.

Fully discrete equation

Substituting these approximations into Eq. (20), we obtain

$$m\left[C_1\frac{p^{n+1}-2p^n+p^{n-1}}{\Delta t^2}-iC_2\frac{p^{n+1}-p^{n-1}}{2\Delta t}+C_3p^n\right]-C_4\mathcal{S}\frac{p^{n+1}-p^{n-1}}{2\Delta t}-\mathrm{Div}^n=s^n. \tag{25}$$

We collect terms multiplying p^{n+1}, p^n, and p^{n-1}:

$$A(\mathbf{x})=m\left(\frac{C_1}{\Delta t^2}-\frac{iC_2}{2\Delta t}\right)-\frac{C_4}{2\Delta t}\mathcal{S}, \tag{26}$$

$$B(\mathbf{x})=m\left(-\frac{2C_1}{\Delta t^2}+C_3\right), \tag{27}$$

$$C(\mathbf{x})=m\left(\frac{C_1}{\Delta t^2}+\frac{iC_2}{2\Delta t}\right)+\frac{C_4}{2\Delta t}\mathcal{S}. \tag{28}$$

Equation (25) becomes

$$A\,p^{n+1}+B\,p^n+C\,p^{n-1}-\mathrm{Div}^n=s^n. \tag{29}$$

Rearranging for p^{n+1}:

$$A\,p^{n+1}=s^n+\mathrm{Div}^n-B\,p^n-C\,p^{n-1}. \tag{30}$$

To emphasize positive coefficients, define

$$D=m\left(\frac{2C_1}{\Delta t^2}-C_3\right),\qquad E=m\left(\frac{C_1}{\Delta t^2}+\frac{iC_2}{2\Delta t}\right)+\frac{C_4}{2\Delta t}\mathcal{S},$$

so that the discrete update equation becomes

$$A\,p^{n+1}=s^n+\mathrm{Div}^n+D\,p^n-E\,p^{n-1}. \tag{31}$$

Explicit update for p^{n+1}

Finally, the update formula for the new wavefield p^{n+1} is

$$\boxed{p^{n+1}=\frac{s^n+\mathrm{Div}^n+D\,p^n-E\,p^{n-1}}{A}.} \tag{32}$$

The coefficients are summarized as

$$A = m\left(\frac{C_1}{\Delta t^2} - \frac{iC_2}{2\Delta t}\right) - \frac{C_4}{2\Delta t}\mathcal{S}, \tag{33}$$

$$D = m\left(\frac{2C_1}{\Delta t^2} - C_3\right), \tag{34}$$

$$E = m\left(\frac{C_1}{\Delta t^2} + \frac{iC_2}{2\Delta t}\right) + \frac{C_4}{2\Delta t}\mathcal{S}. \tag{35}$$

Here all coefficients are generally complex-valued, and both the real and imaginary parts of p evolve simultaneously during propagation.

Implementation notes

In a pseudospectral framework:

- The operator $\mathcal{S} = \sqrt{-\nabla^2}$ is applied in the wavenumber domain:

  $$\widehat{\mathcal{S}p}(\mathbf{k}) = |\mathbf{k}|\, \widehat{p}(\mathbf{k}).$$

- The Laplacian term is computed as

  $$\mathrm{Div}^n = \nabla \cdot \left(\frac{1}{\rho}\nabla p^n\right),$$

  using FFT-based gradients and divergence operators.

- p must be treated as a complex-valued field since coefficients C_2 and C_4 introduce complex coupling between real and imaginary parts.

Equation (32) thus provides a fully discrete time-stepping scheme for the complex-valued visco-acoustic wave equation.

References

[1] Keiiti Aki and Paul G Richards. "Quantative seismology: Theory and methods". In: *Quantative Seismology: Theory and Methods. by K. Aki and PG Richards. San Francisco: Freeman* (1980).

[2] Jidong Yang and Hejun Zhu. "A time-domain complex-valued wave equation for modelling visco-acoustic wave propagation". In: *Geophysical journal international* 215.2 (2018), pp. 1064–1079.

[3] Tieyuan Zhu and Jerry M Harris. “Modeling acoustic wave propagation in heterogeneous attenuating media using decoupled fractional Laplacians”. In: *Geophysics* 79.3 (2014), T105–T116.