Analytical Solution to 2D SH ViscoElastic Wave Equation

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To describe a viscoelastic medium, we modify the stress-strain relation because the conservation of momentum is independent of the material behavior. The viscoelastic stress-strain relation can be described by generalizing the purely elastic case by introducing frequency-dependent complex moduli (or quality factor, Q) or time-domain convolution integrals described by the Boltzmann superposition and causality principle:

$$\sigma(t) = \int_{-\infty}^{t} \Psi(t - t')\dot{\epsilon}(t')\, dt'$$

$\Psi(t)$ is the relaxation function.

1 Governing Wave Equation

2D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \tag{1}$$

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} \tag{2}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z}$$

1D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + f_y,$$

- ρ: Mass density (kg/m^3)

- $u_y(x, t)$: Transverse displacement field (m)
- σ_{yx}: Shear stress component (Pa)
- f_y: External force density ($\mathrm{N/m^3}$)
- x: Spatial coordinate along propagation direction
- t: Time coordinate

ViscoElastic Wave Equation

Kevin-Voigt Model:

which adds a velocity-dependent damping term to the stress-strain relation. The modified stress components become:

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} + \eta \frac{\partial}{\partial t} \left(\frac{\partial u_y}{\partial x} \right), \tag{3}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z} + \eta \frac{\partial}{\partial t} \left(\frac{\partial u_y}{\partial z} \right), \tag{4}$$

where η is the viscosity coefficient. Substituting these into the equation of motion:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \mu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + \eta \frac{\partial}{\partial t} \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + A\delta(x)\delta(z)\delta(t). \tag{5}$$

Let $\beta = \sqrt{\mu/\rho}$ and introduce the damping parameter $\alpha = \eta/(2\rho)$.

The equation becomes:

$$\boxed{\frac{\partial^2 u_y}{\partial t^2} = \beta^2 \nabla^2 u_y + 2\alpha \frac{\partial}{\partial t} \nabla^2 u_y + \frac{A}{\rho} \delta(x)\delta(z)\delta(t),} \tag{6}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

we seek to find the causal Green's function $u_y(x, z, t)$ for the wave equation in (6) i.e the response to the source $A\delta(x)\delta(z)\delta(t)$.

We define the Green's function $G(\mathbf{x}, \mathbf{z}, t)$ as the solution to:

$$\frac{\partial^2 G}{\partial t^2} - \beta^2 \nabla^2 G - 2\alpha \frac{\partial}{\partial t} \nabla^2 G = \frac{A}{\rho} \delta(x)\delta(z)\delta(t). \tag{7}$$

Spatial Fourier Transform

Apply the 2D spatial Fourier transform in x, z. Let

$$\tilde{G}(k_x, k_z, t) = \iint_{-\infty}^{\infty} G(x, z, t) e^{-i(k_x x + k_z z)}\, dx\, dz,$$

with inverse

$$G(x, z, t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}(k_x, k_z, t) e^{i(k_x x + k_z z)}\, \mathrm{d}k_x\, \mathrm{d}k_z$$

Under this transform, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} = -k^2$ with $k^2 = k_x^2 + k_z^2$, and $\delta(x)\delta(z) = 1$.

The wave equation in $\mathbf{k}$-space becomes

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + \beta^2 k^2 \tilde{G} - 2\alpha \frac{\partial}{\partial t}(-k^2)\tilde{G} = \frac{A}{\rho}\delta(t),$$

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial}{\partial t}\tilde{G} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t),$$

This is a **damped harmonic oscillator equation** with a Dirac delta forcing.

Solve the Transformed Equation

Homogeneous Solution (for $RHS = 0$)

The equation becomes:

$$\ddot{\tilde{G}} + 2\alpha k^2 \dot{\tilde{G}} + \beta^2 k^2 \tilde{G} = 0.$$

Assume solutions of the form $\tilde{G} \propto e^{st}$, leading to the characteristic equation:

$$s^2 + 2\alpha k^2 s + \beta^2 k^2 = 0.$$

Characteristic Roots and Damping Cases

Solving for the roots r of the characteristic equation:

$$s = -\alpha k^2 \pm \sqrt{\alpha^2 k^4 - \beta^2 k^2}$$

Three Damping Cases:

- **Overdamped**: $\alpha^2 k^2 > \beta^2$ (exponential decay, no oscillations)
- **Critically damped**: $\alpha^2 k^2 = \beta^2$ (fastest decay without oscillation)
- **Underdamped**: $\alpha^2 k^2 < \beta^2$ (oscillatory decay)

Underdamped Solution (Most Common in Wave Physics)

Define the natural frequency and damping coefficient:

$$\omega_0^2 = \beta^2 k^2, \quad \gamma = \alpha k^2$$

The roots become:

$$s = -\gamma \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}$$

The homogeneous solution is:

$$\tilde{G}_h(t) = e^{-\gamma t} \left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t) \right)$$

where ω_d is the damped natural frequency.

Particular Solution (Delta Forcing)

Now include the **impulse forcing term** $\delta(t)$. This introduces a discontinuity in $\frac{d\tilde{G}}{dt}$ at $t = 0$. The total solution is:

$$\tilde{G}(t) = \tilde{G}_h(t), \quad t > 0$$

Initial Conditions

- $\tilde{G}(0) = 0$ (continuity of displacement)
- Jump in derivative from delta function: Integrate the ODE around $t = 0$:

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{d^2\tilde{G}}{dt^2} + 2\alpha k^2 \frac{d\tilde{G}}{dt} + \beta^2 k^2 \tilde{G} \right) dt = \int_{-\epsilon}^{+\epsilon} \frac{A}{\rho} \delta(t) dt$$

Since $\tilde{G}(t)$ and $\frac{d\tilde{G}}{dt}$ are finite at $t = 0$, only $\frac{d^2\tilde{G}}{dt^2}$ contributes:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} - \left. \frac{d\tilde{G}}{dt} \right|_{0^-} = \frac{A}{\rho}$$

Assuming zero initial velocity before impulse: $\frac{d\tilde{G}}{dt}(0^-) = 0$, so:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} = \frac{A}{\rho}$$

Final Solution

We apply the initial conditions:

- $\tilde{G}(0) = 0$
- $\frac{d\tilde{G}}{dt}(0^+) = \frac{A}{\rho}$

Recall the general solution:

$$\tilde{G}(t) = e^{-\gamma t}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

- Apply $\tilde{G}(0) = 0 \Rightarrow C_1 = 0$
- Compute the derivative:

$$\frac{d\tilde{G}}{dt} = e^{-\gamma t} C_2 (-\gamma \sin(\omega_d t) + \omega_d \cos(\omega_d t))$$

- Evaluate at $t = 0$:

$$\frac{d\tilde{G}}{dt}(0) = C_2 \omega_d = \frac{A}{\rho} \Rightarrow C_2 = \frac{A}{\rho \omega_d}$$

Final Solution in Wavenumber Domain

$$\tilde{G}(k,t) = \frac{A}{\rho\omega_d} e^{-\gamma t} \sin(\omega_d t), \quad t \geq 0,$$

where:

- $\gamma = \alpha k^2$ (damping coefficient)
- $\omega_d = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$ (damped frequency)

The solution can be written compactly using the Heaviside step function $H(t)$:

$$\boxed{\tilde{G}(k_x, k_z, t) = \frac{A}{\rho\omega_d} e^{-\gamma t} \sin(\omega_d t)\, H(t).}$$

$$\boxed{\tilde{G}(t) = \frac{A}{\rho\sqrt{\beta^2 k^2 - \alpha^2 k^4}}\, e^{-\alpha k^2 t} \sin\left(\sqrt{\beta^2 k^2 - \alpha^2 k^4}\, t\right) H(t)}$$

Key Observations

1. **Damping Effect**: The term $e^{-\alpha k^2 t}$ shows **wavenumber-dependent damping** (higher k modes decay faster).
2. **Dispersion**: Frequency ω_d depends on k, indicating dispersive waves.
3. **Singularity at $t = 0$**: Matches the delta source behavior.

Inverse Transform and Bessel Integral

Green's Function Solution

The wavenumber-domain solution is:

$$\tilde{G}(k_x, k_z, t) = \frac{A}{\rho\sqrt{\beta^2 k^2 - \alpha^2 k^4}} e^{-\alpha k^2 t} \sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right) H(t)$$

where $k^2 = k_x^2 + k_z^2$.

Inverse Fourier Transform

To obtain the space-time solution, we compute the 2D inverse Fourier transform:

$$G(x,z,t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}(k_x,k_z,t) e^{i(k_x x + k_z z)}\, dk_x\, dk_z$$

Step 1: Use Radial Symmetry

Let $r = \sqrt{x^2+z^2}$. Since $\tilde{G}(k_x,k_z,t)$ depends on $k = \sqrt{k_x^2+k_z^2}$, we switch to polar coordinates in k-space:

Let:

- $k_x = k\cos\theta$
- $k_z = k\sin\theta$
- $dk_x dk_z = k\, dk\, d\theta$

Also:

$$k_x x + k_z z = kr\cos(\theta - \phi)$$

where $\phi = \tan^{-1}(z/x)$.

So:

$$G(x,z,t) = \frac{1}{(2\pi)^2} \int_0^{\infty} \int_0^{2\pi} \tilde{G}(k_x,k_z,t) e^{ikr\cos(\theta-\phi)} k\, d\theta\, dk$$

Simplify the angular integral using the Bessel function. Use the identity:

$$\int_0^{2\pi} e^{ikr\cos(\theta-\phi)} d\theta = 2\pi J_0(kr)$$

Thus:

$$G(x,z,t) = \frac{1}{2\pi} \int_0^{\infty} \tilde{G}(k_x,k_z,t) k J_0(kr)\, dk$$

Step 2: Plug in Expression for $\tilde{G}(k,t)$

Substitute the expression for $G(k,t)$:

$$G(x,z,t) = \frac{A}{2\pi\rho} \int_0^{\infty} H(t) \frac{kJ_0(kr)}{\sqrt{\beta^2k^2 - \alpha^2k^4}} e^{-\alpha k^2 t} \sin\left(t\sqrt{\beta^2k^2-\alpha^2k^4}\right)\, dk \tag{8}$$

$$\boxed{G(x,z,t) = \frac{AH(t)}{2\pi\rho} \int_0^{\infty} \frac{e^{-\alpha k^2 t} \sin\left(kt\sqrt{\beta^2 - \alpha^2 k^2}\right)}{\sqrt{\beta^2 - \alpha^2 k^2}} J_0(kr)\, dk.} \tag{9}$$

where:

$$\alpha = \frac{\eta}{2\rho}$$

This is the exact solution for the Green's function in the time-space domain for a damped 2D SH wave equation.

Explanation of Equation 9

- Bessel Function $J_0(kr)$: Represents radial symmetry in cylindrical coordinates.
- Exponential Decay $e^{-\alpha k^2 t}$: Damping term, common in dissipative media or systems.
- $\sin\left(kt\sqrt{\beta^2 - \alpha^2 k^2}\right)$: This is the oscillatory term. It represents dispersive wave propagation.
- Denominator $\sqrt{\beta^2 - \alpha^2 k^2}$: Modifies the amplitude due to dispersion.

Equation 9 is the final solution, but can we simplify it even further? Let's try!

Step 2.1: Express $\sin(\cdot)$ Using Complex Exponentials

Express $\sin(\cdot)$ Using Complex Exponentials

Remember Euler's formula is given as:

$$e^{ix} = cosx + isinx$$

Thus, the sine function can be rewritten using Euler's formula:

$$\sin(x) = \text{Im}[e^{ix}], \quad \text{where Im denotes the imaginary part.}$$

Applying this to the sine term in the integral:

$$\sin\left(kt\sqrt{\beta^2 - \alpha^2 k^2}\right) = \text{Im}\left[e^{ikt\sqrt{\beta^2 - \alpha^2 k^2}}\right].$$

Thus, the integral becomes:

$$I = \mathrm{Im}\left[\int_0^\infty \frac{e^{-\alpha k^2 t + ikt\sqrt{\beta^2-\alpha^2k^2}}}{\sqrt{\beta^2-\alpha^2k^2}} J_0(kr)\,dk\right].$$

$$\boxed{G(x,z,t) = \frac{AH(t)}{2\pi\rho}\mathrm{Im}\left[\int_0^\infty \frac{e^{-\alpha k^2 t + ikt\sqrt{\beta^2-\alpha^2k^2}}}{\sqrt{\beta^2-\alpha^2k^2}} J_0(kr)\,dk\right]} \tag{10}$$

<u>**Simplify the Exponent**</u>

Let $\omega(k) = k\sqrt{\beta^2-\alpha^2k^2}$. $\omega(k)$ is what is called the **Dispersion Relation**. For $\alpha^2k^2 \ll \beta^2$ (weak damping), we can expand $\omega(k)$ as:

$$\omega(k) = k\sqrt{\beta^2-\alpha^2k^2} = \beta k\sqrt{1-\frac{\alpha^2k^2}{\beta^2}} = \beta k\left(1-\frac{\alpha^2k^2}{\beta^2}\right)^{\frac{1}{2}}.$$

Taylor Expansion of $\sqrt{1-x}$

For $|x| \ll 1$, the square root function can be expanded as:

$$\sqrt{1-x} \approx 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \cdots.$$

Here, $x = \frac{\alpha^2k^2}{\beta^2}$, so:

$$\sqrt{1-\frac{\alpha^2k^2}{\beta^2}} \approx 1 - \frac{\alpha^2k^2}{2\beta^2} - \frac{\alpha^4k^4}{8\beta^4} - \frac{\alpha^6k^6}{16\beta^6} - \cdots.$$

Multiply by βk

Substitute the expansion back into $\omega(k)$:

$$\omega(k) \approx \beta k\left(1 - \frac{\alpha^2k^2}{2\beta^2} - \frac{\alpha^4k^4}{8\beta^4} - \frac{\alpha^6k^6}{16\beta^6} - \cdots\right).$$

Distribute βk:

$$\boxed{\omega(k) \approx \beta k - \frac{\alpha^2k^3}{2\beta} - \frac{\alpha^4k^5}{8\beta^3} - \frac{\alpha^6k^7}{16\beta^5} - \cdots.}$$

$$
\omega(k) \approx \beta k - \frac{\alpha^2 k^3}{2\beta} + \mathcal{O}(\alpha^4).
$$

Truncate for Small α

For weak damping ($\alpha^2 k^2 \ll \beta^2$) ($\alpha \ll \beta$), higher-order terms ($\alpha^4, \alpha^6, \ldots$) are negligible. Keeping only the first two terms:

$$
\omega(k) \approx \beta k - \frac{\alpha^2 k^3}{2\beta}.
$$

Interpretation of Terms

1. **Leading Term (βk):** - Represents linear dispersion (waves propagate at constant phase velocity β). - Matches the undamped case ($\alpha = 0$).
2. **First Correction ($-\frac{\alpha^2 k^3}{2\beta}$):** - Introduces nonlinearity (wave speed depends on k). - Causes dispersion (different frequencies travel at different speeds). - Arises due to damping ($\alpha \neq 0$).

Validity Condition

The expansion is valid when:

$$
\left| \frac{\alpha^2 k^2}{\beta^2} \right| \ll 1 \quad \Rightarrow \quad k \ll \frac{\beta}{\alpha}.
$$

For larger k, higher-order terms become significant, and the full expression must be used.

Final Expanded Form

$$
\boxed{\omega(k) \approx \beta k - \frac{\alpha^2 k^3}{2\beta} \quad \text{(for small } \alpha\text{)}}
$$

Now, substitute the dispersion relation for small α into the exponent in equation 10:

$$
-\alpha k^2 t + i\omega(k)t \approx -\alpha k^2 t + i\beta k t - i\frac{\alpha^2 k^3 t}{2\beta}.
$$

The integral now takes the form:

$$I \approx \mathrm{Im}\left[\int_0^\infty \frac{e^{-\alpha k^2 t + i\beta k t - i\frac{\alpha^2 k^3 t}{2\beta}}}{\beta - \frac{\alpha^2 k^2}{2\beta}} J_0(kr)\, dk\right].$$

The solution is:

$$\boxed{G(x,z,t) = \frac{AH(t)}{2\pi\rho}\mathrm{Im}\left[\int_0^\infty \frac{e^{-\alpha k^2 t + i\beta k t - i\frac{\alpha^2 k^3 t}{2\beta}}}{\beta - \frac{\alpha^2 k^2}{2\beta}} J_0(kr)\, dk\right]} \tag{11}$$

For a very small damping term, α, or as it tends to zero, the solution to the 2D SH viscoelastic wave equation reduces to a purely elastic case (see Box 1). However, for a more general case (i.e an arbitrary α, but still $\alpha^2\beta^2 < k^2$), the integral is more challenging.

Box 1.0 - Small Damping Approximation: $\alpha \to 0$ (Weakly Damped $\alpha \ll \beta$)

When damping is very small, the solution to the underdamped system (equation 9) becomes nearly that of a pure 2D SH wave equation. This happens when the viscosity, η, approaches zero, such that α also approaches zero. For small α, then:

$$\sqrt{\beta^2 - \alpha^2 k^2} \approx \beta$$

$$\sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right) \approx \sin(\beta k t)$$

The exponential becomes:

$$e^{-\alpha k^2 t} \approx 1$$

This leads to:

$$\boxed{G(x,z,t) \approx \frac{AH(t)}{2\pi\rho}\int_0^\infty \frac{\sin(\beta k t)}{\beta k} \cdot k J_0(kR)\, dk = \frac{AH(t)}{2\pi\rho\beta}\int_0^\infty \sin(\beta k t) J_0(kr)\, dk}$$

This is the pure 2D SH wave solution.

Box 2.0 - Critically Damped Solution of Second-Order ODE with Delta Source

We start from the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t)$$

Let:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Taking the Laplace transform with zero initial conditions:

$$s^2 \tilde{G}(s) + 2\gamma s \tilde{G}(s) + \omega_0^2 \tilde{G}(s) = \frac{A}{\rho}$$

$$\tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

—

Critically damped case: $\gamma^2 = \omega_0^2 \Rightarrow \alpha^2 k^4 = \beta^2 k^2$
This implies the characteristic equation has a repeated real root:

$$s^2 + 2\gamma s + \gamma^2 = (s + \gamma)^2$$

So:

$$\tilde{G}(s) = \frac{A/\rho}{(s+\gamma)^2}$$

Using the Laplace transform identity:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}$$

We get:

$$\tilde{G}(t) = \frac{A}{\rho} \cdot te^{-\gamma t} \cdot H(t)$$

Substituting back $\gamma = \alpha k^2$:

$$\boxed{\tilde{G}(t) = \frac{A}{\rho} \cdot te^{-\alpha k^2 t} \cdot H(t)}$$

This solution exhibits the behaviour of a critically damped system: no oscillation and the fastest return to equilibrium without overshooting.

Box 3.0 - Overdamped Solution of Second-Order ODE with Delta Source

We are given the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t)$$

Let:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Take the Laplace transform of both sides (assuming zero initial conditions):

$$s^2\tilde{G}(s) + 2\gamma s\tilde{G}(s) + \omega_0^2\tilde{G}(s) = \frac{A}{\rho}$$

$$\tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

—

Overdamped case: $\gamma^2 > \omega_0^2 \Rightarrow \alpha^2 k^4 > \beta^2 k^2$

Define:

$$\lambda = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\alpha^2 k^4 - \beta^2 k^2}$$

Then the characteristic roots are:

$$s_1 = -\gamma + \lambda, \quad s_2 = -\gamma - \lambda$$

So the Laplace domain solution becomes:

$$\tilde{G}(s) = \frac{A/\rho}{(s - s_1)(s - s_2)} = \frac{A}{2\lambda\rho}\left(\frac{1}{s - s_1} - \frac{1}{s - s_2}\right)$$

Take the inverse Laplace transform:

$$\tilde{G}(t) = \frac{A}{2\lambda\rho}\left(e^{s_1 t} - e^{s_2 t}\right) H(t)$$

Substitute s_1 and s_2 explicitly:

$$\boxed{\tilde{G}(t) = \frac{A}{2\rho\sqrt{\alpha^2 k^4 - \beta^2 k^2}}\left[e^{(-\alpha k^2 + \sqrt{\alpha^2 k^4 - \beta^2 k^2})t} - e^{(-\alpha k^2 - \sqrt{\alpha^2 k^4 - \beta^2 k^2})t}\right] H(t)}$$

This solution is real and decaying (no oscillations), characteristic of overdamped systems.

Box 4.0 – Underdamped Solution of Second-Order ODE with Delta Source

Consider the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t)$$

Define:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Step-by-step Solution Using Laplace Transform

We apply the Laplace transform $\mathcal{L}\{\cdot\}$ with respect to time t, using the standard transforms:

- $\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 \tilde{f}(s) - s f(0) - f'(0)$,
- $\mathcal{L}\left\{\frac{df}{dt}\right\} = s\tilde{f}(s) - f(0)$,
- $\mathcal{L}\{\delta(t)\} = 1$.

Assume zero initial conditions (i.e., $\tilde{G}(0) = 0, \tilde{G}'(0) = 0$). Taking the Laplace transform of both sides:

$$\mathcal{L}\left\{\frac{\partial^2 \tilde{G}}{\partial t^2}\right\} + 2\alpha k^2 \mathcal{L}\left\{\frac{\partial \tilde{G}}{\partial t}\right\} + \beta^2 k^2 \mathcal{L}\left\{\tilde{G}\right\} = \frac{A}{\rho}$$

—

Laplace transforming both sides (assuming zero initial conditions) gives:

$$s^2 \tilde{G}(s) + 2\gamma s \tilde{G}(s) + \omega_0^2 \tilde{G}(s) = \frac{A}{\rho} \quad \Rightarrow \quad \tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

—

Underdamped case: $\gamma^2 < \omega_0^2 \Rightarrow \alpha^2 k^4 < \beta^2 k^2$

Let:

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$$

Then the denominator becomes:

$$s^2 + 2\gamma s + \omega_0^2 = (s + \gamma)^2 + \omega_d^2$$

Box 4.0 – Underdamped Solution of Second-Order ODE with Delta Source – Contd

So:

$$\tilde{G}(s) = \frac{A/\rho}{(s+\gamma)^2 + \omega_d^2}$$

Using the Laplace inverse identity:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2 + \omega^2}\right\} = \frac{1}{\omega} e^{-at} \sin(\omega t)$$

We obtain:

$$\tilde{G}(t) = \frac{A}{\rho\omega_d} e^{-\gamma t} \sin(\omega_d t) \cdot H(t)$$

Substitute $\gamma = \alpha k^2$, $\omega_d = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$:

$$\boxed{\tilde{G}(t) = \frac{A}{\rho\sqrt{\beta^2 k^2 - \alpha^2 k^4}} \cdot e^{-\alpha k^2 t} \sin\left(\sqrt{\beta^2 k^2 - \alpha^2 k^4}\, t\right) H(t)}$$

This solution captures the oscillatory decay of the underdamped system — a sine wave modulated by an exponential decay.