Derivation of the Integral Representation of the Scattered Wavefield

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1 Governing Acoustic Wave Equation

The pressure wavefield $p(x, \omega)$ in an acoustic medium in frequency domain is given as:

$$-\omega^2 \kappa^{-1}(x) p - \nabla \cdot (\rho^{-1}(x) \nabla p) = s(x, \omega), \tag{1}$$

where:

- $\kappa(x)$ is the **bulk modulus**,
- $\rho(x)$ is the **density**,
- $s(x, \omega)$ is the **source term**.

For a point source at x_s, we assume:

$$s(x, \omega) = \delta(x - x_s). \tag{2}$$

2 Decomposing the Wavefield Using Born Approximation

We assume the background model $m_0(x)$ and a small perturbation $\delta m(x)$:

$$m(x) = m_0(x) + \delta m(x). \tag{3}$$

The pressure wavefield is decomposed into:

$$p(x_r, \omega; x_s, m) = p_0(x_r, \omega; x_s, m_0) + \delta p(x_r, \omega; x_s, m), \tag{4}$$

where:

- $p_0(x_r, \omega; x_s, m_0)$ is the **background (unperturbed) wavefield**,
- $\delta p(x_r, \omega; x_s, m)$ is the **scattered wavefield**.

3 Substituting into the Frequency-Domain Acoustic Wave Equation

For the background wavefield $p_0(x_r, \omega; x_s, m_0)$, we substitute $m_0(x)$ into the wave equation:

$$-\omega^2 \kappa_0^{-1}(x) p_0 - \nabla \cdot (\rho_0^{-1}(x) \nabla p_0) = \delta(x - x_s). \tag{5}$$

For the total wavefield $p(x, \omega)$, we substitute $m(x) = m_0(x) + \delta m(x)$:

$$-\omega^2 (\kappa_0^{-1} + \delta\kappa^{-1})(p_0 + \delta p) - \nabla \cdot [(\rho_0^{-1} + \delta\rho^{-1}) \nabla (p_0 + \delta p)] = \delta(x - x_s). \tag{6}$$

Expanding this equation:

$$\begin{aligned}
&-\omega^2 \kappa_0^{-1} p_0 - \omega^2 \kappa_0^{-1} \delta p - \omega^2 \delta\kappa^{-1} p_0 - \omega^2 \delta\kappa^{-1} \delta p \\
&-\nabla \cdot (\rho_0^{-1} \nabla p_0) - \nabla \cdot (\rho_0^{-1} \nabla \delta p) - \nabla \cdot (\delta\rho^{-1} \nabla p_0) - \nabla \cdot (\delta\rho^{-1} \nabla \delta p) = \delta(x - x_s).
\end{aligned} \tag{7}$$

Since $p_0(x, \omega)$ satisfies the background equation, the terms involving only p_0 cancel. This is same as doing equation 5 - equation 7

$$-\omega^2 \kappa_0^{-1} \delta p - \nabla \cdot (\rho_0^{-1} \nabla \delta p) = \omega^2 \delta\kappa^{-1} p_0 + \nabla \cdot (\delta\rho^{-1} \nabla p_0) + O(\delta m^2). \tag{8}$$

Neglecting higher-order perturbation terms (Born approximation), we obtain:

$$-\omega^2 \kappa_0^{-1} \delta p - \nabla \cdot (\rho_0^{-1} \nabla \delta p) = \omega^2 \delta\kappa^{-1} p_0 + \nabla \cdot (\delta\rho^{-1} \nabla p_0). \tag{9}$$

4 Using Green's Function for Integral Representation

The Green's function $G_0(x_r, \omega, y)$ satisfies:

$$-\omega^2 \kappa_0^{-1}(y) G_0 - \nabla \cdot (\rho_0^{-1}(y) \nabla G_0) = \delta(x - y). \tag{10}$$

From equation 9, we have

$$-\omega^2 \kappa_0^{-1} \delta p - \nabla \cdot (\rho_0^{-1} \nabla \delta p) = \omega^2 \delta\kappa^{-1} p_0 + \nabla \cdot (\delta\rho^{-1} \nabla p_0).$$

This equation is consistent with Equation 5 above and represents the wave equation for the perturbed (or scattered) wavefield.

$$\boxed{-\omega^2 \kappa_0^{-1} \delta p - \nabla \cdot (\rho_0^{-1} \nabla \delta p) = \omega^2 \delta\kappa^{-1} p_0 + \nabla \cdot (\delta\rho^{-1} \nabla p_0).}$$

$$\boxed{\left(\frac{1}{\kappa_0} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left(\frac{1}{\rho_0} \nabla \right) \right) \delta p = -\delta \left(\frac{1}{\kappa} \right) \frac{\partial^2 p_0}{\partial t^2} + \nabla \cdot \left(\delta \left(\frac{1}{\rho} \right) \nabla p_0 \right).}$$

From this equation, $\delta(x - y)$ is represented as:

$$\delta(x - y) = \omega^2 \delta\kappa^{-1}(y) p_0(y, \omega) + \nabla \cdot (\delta\rho^{-1}(y) \nabla p_0(y, \omega)) \tag{11}$$

Using the convolution property:

$$\delta p(x_r, \omega; x_s) = \int_\Omega G_0(x_r, \omega, y) \delta(x - y) dy. \tag{12}$$

$$\delta p(x_r, \omega; x_s) = \int_\Omega G_0(x_r, \omega, y) \left[\omega^2 \delta\kappa^{-1}(y) p_0(y, \omega) + \nabla \cdot (\delta\rho^{-1}(y) \nabla p_0(y, \omega)) \right] dy. \tag{13}$$

$p_0(y, \omega)$ represents the background or unperturbed wavefield from the source point x_s to the scatterer at y which can be represented as $G_0(y, \omega, x_s)$

$$\boxed{p_0(y, \omega) = p_0(y, \omega, x_s) = G_0(y, \omega, x_s)}$$

so, equation 12 can also be written as

$$\delta p(x, \omega; x_s) = \int_\Omega G_0(x_r, \omega, y) \left[\omega^2 \delta\kappa^{-1}(y) G_0(y, \omega, x_s) + \nabla \cdot (\delta\rho^{-1}(y) \nabla G_0(y, \omega, x_s))\right] dy. \quad (14)$$

Applying integration by parts to the second term of equation 13:

$$\boxed{\int_\Omega G_0(x_r, \omega, y) \nabla \cdot (\delta\rho^{-1} \nabla G_0(y, \omega, x_s)) dy = -\int_\Omega \nabla G_0(x_r, \omega, y) \cdot (\delta\rho^{-1} \nabla G_0(y, \omega, x_s)) dy.} \quad (15)$$

Thus, we arrive at the final integral representation:

$$\boxed{\begin{aligned} \delta p(x_r, \omega; x_s) = \int_\Omega \Big[&\omega^2 \delta\kappa^{-1}(y) G_0(x_r, \omega, y) G_0(y, \omega, x_s) \\ &- \nabla G_0(x_r, \omega, y) \cdot \left(\delta\rho^{-1}(y) \nabla G_0(y, \omega, x_s)\right) \Big] dy. \end{aligned}}$$

This represents the scattered wavefield in terms of Green's function and the perturbation in medium properties.

5 Explanation of the Integration by part above

The original integral is:

$$\int_\Omega G_0(x_r, \omega, y) \nabla \cdot \left(\delta\rho^{-1} \nabla G_0(y, \omega, x_s)\right) \, dy$$

$$\int_\Omega G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right) \, dy$$

G_0^+ and G_0^- represent the upgoing and downgoing Green functions, respectively.

- $G_0(x_r, \omega, y)$: Green's function, which represents the response at receiver location x_r due to a source located at point y (from the scatterer) with frequency ω.
- ∇: The gradient operator.
- $\nabla \cdot$: The divergence operator.
- $\delta\rho^{-1}$: The inverse of a perturbation in density ρ.
- $G_0(y, \omega, x_s)$: Green's function, which represents the wavefield from the source location x_s to the scatterer at y.

- Ω: The integration domain.

Using the product rule for divergence:

$$\boxed{\nabla \cdot \left(G_0^+ \delta\rho^{-1} \nabla G_0^-\right) = G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right) + \nabla G_0^+ \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)}$$

$$\boxed{G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right) = \nabla \cdot \left(G_0^+ \delta\rho^{-1} \nabla G_0^-\right) - \nabla G_0^+ \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)}$$

Thus, the integral becomes:

$$\int_\Omega G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy = \int_\Omega \nabla \cdot \left(G_0^+ \delta\rho^{-1} \nabla G_0^-\right)\, dy - \int_\Omega \nabla G_0^+ \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy$$

By the divergence theorem:

$$\boxed{\int_\Omega \nabla \cdot \mathbf{F}\, dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS}$$

$$\int_\Omega \nabla \cdot \left(G_0^+ \delta\rho^{-1} \nabla G_0^-\right)\, dy = \int_{\partial\Omega} G_0^+ \delta\rho^{-1} \nabla G_0^- \cdot \mathbf{n}\, dS$$

Thus, we have:

$$\int_\Omega G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy = \int_{\partial\Omega} G_0^+ \delta\rho^{-1} \nabla G_0^- \cdot \mathbf{n}\, dS - \int_\Omega \nabla G_0^+ \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy$$

If we assume G_0^+ vanishes at the boundary $\partial\Omega$, then:

$$\boxed{\int_\Omega G_0^+ \nabla \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy = -\int_\Omega \nabla G_0^+ \cdot \left(\delta\rho^{-1} \nabla G_0^-\right)\, dy}$$

$$\boxed{\int_\Omega G_0(x_r, \omega, y) \nabla \cdot (\delta\rho^{-1} \nabla G_0(y, \omega, x_s)) dy = -\int_\Omega \nabla G_0(x_r, \omega, y) \cdot (\delta\rho^{-1} \nabla G_0(y, \omega, x_s)) dy.}$$

6 <u>Conclusion</u>

This derivation shows how the scattered wavefield arises due to small perturbations in bulk modulus and density. The result is crucial for seismic inversion, as it forms the Born modelling operator, which maps subsurface perturbations to observed data.

Multivariate Taylor Series Expansion of Nonlinear Forward Model

For a vector-valued function $G : \mathbb{R}^n \to \mathbb{R}^m$ (where m is an n-dimensional model parameter vector), the Taylor expansion around a reference model m_0 is:

$$G(m) = G(m_0 + \Delta m) = G(m_0) + J(m_0)\Delta m + \frac{1}{2}H(m_0)\Delta m^{\otimes 2} + \frac{1}{6}T(m_0)\Delta m^{\otimes 3} + \frac{1}{24}Q(m_0)\Delta m^{\otimes 4} + \frac{1}{120}P(m_0)\Delta m^{\otimes 5} + \mathcal{O}(\Delta m^6)$$

where:

- $\Delta m = m - m_0$ (perturbation vector)
- $J(m_0)$: Jacobian matrix (1st derivative, $m \times n$)
- $H(m_0)$: Hessian tensor (2nd derivative, $m \times n \times n$)
- $T(m_0)$: 3rd-order derivative tensor ($m \times n \times n \times n$)
- $Q(m_0)$: 4th-order derivative tensor
- $P(m_0)$: 5th-order derivative tensor
- $\Delta m^{\otimes k}$ denotes the k-th order Kronecker product of Δm with itself

2. Explicit Form of Each Term

(a) First-order term (Jacobian)

$$J(m_0)\Delta m = \sum_{j=1}^{n} \frac{\partial G}{\partial m_j}\bigg|_{m_0} \Delta m_j$$

(b) Second-order term (Hessian)

$$\frac{1}{2}H(m_0)\Delta m^{\otimes 2} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 G}{\partial m_j \partial m_k}\bigg|_{m_0} \Delta m_j \Delta m_k$$

(c) Third-order term

$$\frac{1}{6}T(m_0)\Delta m^{\otimes 3} = \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^3 G}{\partial m_j \partial m_k \partial m_l}\bigg|_{m_0} \Delta m_j \Delta m_k \Delta m_l$$

(d) Fourth-order term

$$\frac{1}{24}Q(m_0)\Delta m^{\otimes 4} = \frac{1}{24}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{p=1}^{n}\left.\frac{\partial^4 G}{\partial m_j \partial m_k \partial m_l \partial m_p}\right|_{m_0}\Delta m_j \Delta m_k \Delta m_l \Delta m_p$$

(e) Fifth-order term

$$\frac{1}{120}P(m_0)\Delta m^{\otimes 5} = \frac{1}{120}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\left.\frac{\partial^5 G}{\partial m_j \partial m_k \partial m_l \partial m_p \partial m_q}\right|_{m_0}\Delta m_j \Delta m_k \Delta m_l \Delta m_p \Delta m_q$$

$$\boxed{\begin{aligned} G(m) = G(m_0) & \\ &+ \sum_{i=1}^{n}\frac{\partial G}{\partial m_i}(m_0)\Delta m_i \\ &+ \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 G}{\partial m_i \partial m_j}(m_0)\Delta m_i \Delta m_j \\ &+ \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^3 G}{\partial m_i \partial m_j \partial m_k}(m_0)\Delta m_i \Delta m_j \Delta m_k \\ &+ \frac{1}{4!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^4 G}{\partial m_i \partial m_j \partial m_k \partial m_l}(m_0)\Delta m_i \Delta m_j \Delta m_k \Delta m_l \\ &+ \frac{1}{5!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{s=1}^{n}\frac{\partial^5 G}{\partial m_i \partial m_j \partial m_k \partial m_l \partial m_s}(m_0)\Delta m_i \Delta m_j \Delta m_k \Delta m_l \Delta m_s \end{aligned}}$$

Summary

The Taylor series is linear only if truncated at the 1st-order term. Any higher-order terms (2nd, 3rd, ..., 5th) introduce polynomial nonlinearities. In practice, inversion methods balance accuracy (more terms) versus computational cost (fewer terms).

Would you like a geophysical example (e.g., seismic FWI) to illustrate how nonlinear terms affect inversion?

- **1st order**: Gradient — linear approximation.
- **2nd order**: Hessian — curvature (quadratic shape).
- **3rd order**: (cubic nonlinearity).
- **Higher orders**: More complex variations of curvature.