Analytical Solution to 2D SH (Shear Horizontal) Wave Equation Using Cagniard–De Hoop Method

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Governing Wave Equation

2D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \tag{1}$$

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} \tag{2}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z}$$

1D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + f_y,$$

- ρ: Mass density (kg/m^3)
- $u_y(x, t)$: Transverse displacement field (m)
- σ_{yx}: Shear stress component (Pa)
- f_y: External force density (N/m^3)
- x: Spatial coordinate along propagation direction
- t: Time coordinate

The Cagniard–De Hoop Method

Given the 2D SH Wave Equation:

$$\rho\frac{\partial^2 u_y}{\partial t^2} = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + f_y.$$

$f_y = (0,\ A\delta(x)\delta(z)\delta(t),\ 0)$ is the line source. Only the $y-$ component is excited by this source. A is a constant having the dimension of impulse per unit length

$$\rho\frac{\partial^2 u_y}{\partial t^2} = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + A\delta(x)\delta(z)\delta(t).$$

Since $\beta = \sqrt{\frac{\mu}{\rho}}$. The equation becomes:

$$\boxed{\frac{\partial^2 u_y}{\partial t^2} = \beta^2\nabla^2 u_y + \frac{A}{\rho}\delta(x)\delta(z)\delta(t),} \tag{3}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

we seek to find the causal Green's function $u_y(x, z, t)$ for the wave equation in (3) i.e the response to the source $A\delta(x)\delta(z)\delta(t)$.

Step 1: Take the Laplace Transform with Respect to Time

The Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = \tilde{f}(s) = \int_0^\infty f(t)e^{-st}dt.$$

Apply the Laplace transform to both sides of the equation. We will use the following properties of the Laplace transform:

1. $\mathcal{L}\left\{\frac{\partial^2 u_y}{\partial t^2}\right\} = s^2\tilde{u}_y - su_y(x, z, 0) - \frac{\partial u_y}{\partial t}(x, z, 0).$
2. $\mathcal{L}\{\delta(t)\} = 1.$

Assuming initial conditions are zero:

$$u_y(x, z, 0) = 0, \quad \frac{\partial u_y}{\partial t}(x, z, 0) = 0$$

The Laplace transform simplifies the equation to:

$$\rho\left(s^2\tilde{u}_y\right) = \mu\left(\frac{\partial^2\tilde{u}_y}{\partial x^2} + \frac{\partial^2\tilde{u}_y}{\partial z^2}\right) + A\delta(x)\delta(z).$$

$$\rho s^2\tilde{u}_y = \mu\left(\frac{\partial^2\tilde{u}_y}{\partial x^2} + \frac{\partial^2\tilde{u}_y}{\partial z^2}\right) + A\delta(x)\delta(z).$$

Rearrange the Equation

Divide both sides by ρ to isolate $\tilde{u}_y$:

$$s^2\tilde{u}_y(x,z,s) = \frac{\mu}{\rho}\left(\frac{\partial^2\tilde{u}_y}{\partial x^2} + \frac{\partial^2\tilde{u}_y}{\partial z^2}\right) + \frac{A}{\rho}\delta(x)\delta(z).$$

$$s^2\tilde{u}_y(x,z,s) = \beta^2\left(\frac{\partial^2\tilde{u}_y}{\partial x^2} + \frac{\partial^2\tilde{u}_y}{\partial z^2}\right) + \frac{A}{\rho}\delta(x)\delta(z).$$

Final Form of the Laplace-Transformed Equation

$$\boxed{\frac{\partial^2\tilde{u}_y}{\partial x^2} + \frac{\partial^2\tilde{u}_y}{\partial z^2} - \frac{s^2}{\beta^2}\tilde{u}_y = -\frac{A}{\rho\beta^2}\delta(x)\delta(z).}$$

Step 2: Take the Fourier Transform in the x-Direction

The Fourier transform of a function $f(x)$ is defined as:

$$\mathcal{F}\{f(x)\} = \hat{f}(k_x) = \int_{-\infty}^{\infty} f(x)e^{-ik_x x}dx.$$

Apply the Fourier transform to both sides of the equation. We will use the following properties of the Fourier transform:

1. $\mathcal{F}\left\{\frac{\partial^2\tilde{u}_y}{\partial x^2}\right\} = -k_x^2\hat{\tilde{u}}_y$.
2. $\mathcal{F}\left\{\frac{\partial^2\tilde{u}_y}{\partial z^2}\right\} = \frac{\partial^2\hat{\tilde{u}}_y}{\partial z^2}$ (since the FT is taken only in x).
3. $\mathcal{F}\{\tilde{u}_y\} = \hat{\tilde{u}}_y$.
4. $\mathcal{F}\{\delta(x)\} = 1$.

Applying the Fourier transform to the equation:

$$\mathcal{F}\left\{\frac{\partial^2\tilde{u}_y}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2\tilde{u}_y}{\partial z^2}\right\} - \frac{s^2}{\beta^2}\mathcal{F}\{\tilde{u}_y\} = -\frac{A}{\rho\beta^2}\mathcal{F}\{\delta(x)\}\delta(z).$$

Substituting the FT properties:

$$-k_x^2\hat{\tilde{u}}_y + \frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \frac{s^2}{\beta^2}\hat{\tilde{u}}_y = -\frac{A}{\rho\beta^2} \cdot 1 \cdot \delta(z).$$

Combine Like Terms

Combine the terms involving $\hat{\tilde{u}}_y$:

$$\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \left(k_x^2 + \frac{s^2}{\beta^2}\right)\hat{\tilde{u}}_y = -\frac{A}{\rho\beta^2}\delta(z).$$

Let $\kappa^2 = k_x^2 + \frac{s^2}{\beta^2}$, then the equation simplifies to:

$$\boxed{\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \kappa^2(k,s)\hat{\tilde{u}}_y(k,z,s) = -\frac{A}{\rho\beta^2}\delta(z).}$$

$$\boxed{\kappa^2(k,s) = k_x^2 + \frac{s^2}{\beta^2}}$$

with $\mathrm{Re}(\kappa) > 0$ for **boundedness**.

Step 3: Solve the ODE in the z-Direction

The equation above is a second-order ordinary differential equation (ODE) in z with a delta function source.

The general solution for $z \neq 0$ is:

Homogeneous Equation Solution (without the source term on the RHS)

The homogeneous equation is:

$$\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \kappa^2\hat{\tilde{u}}_y = 0.$$

This is a linear second-order ODE with constant coefficients. The general solution is:

$$\hat{\tilde{u}}_y^{\mathrm{hom}}(z) = \begin{cases} C_1 e^{\kappa z} + C_2 e^{-\kappa z}, & z < 0 \\ C_3 e^{\kappa z} + C_4 e^{-\kappa z}, & z > 0 \end{cases}$$

But we require **boundedness at infinity as $|z| \to \infty$**:

- As $z \to -\infty$, $e^{-\kappa z} \to \infty \implies C_2 = 0$
- As $z \to +\infty$, $e^{\kappa z} \to \infty \implies C_3 = 0$

So the physically admissible (bounded) solution becomes:

$$\hat{\tilde{u}}_y^{\text{hom}}(z) = \begin{cases} C_1 e^{\kappa z}, & z < 0 \\ C_4 e^{-\kappa z}, & z > 0 \end{cases}$$

Apply Discontinuity (Jump) Condition from δ-function

To account for the delta function at $z = 0$, we impose continuity and a jump condition in the derivative:

1. **Continuity at $z = 0$:**

$$\hat{\tilde{u}}_y(0^-) = \hat{\tilde{u}}_y(0^+).$$

$$\lim_{z \to 0^-} \hat{\tilde{u}}_y = \lim_{z \to 0^+} \hat{\tilde{u}}_y$$

$$C_1 e^{\kappa(0)} = C_4 e^{-\kappa(0)}$$

$$C_1 = C_4 \equiv C$$

2. **Jump condition in the derivative:** Integrate the ODE across an infinitesimally small interval around $z = 0$:

$$\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \kappa^2 \hat{\tilde{u}}_y = -\frac{A}{\rho \beta^2} \delta(z).$$

$$\int_{0^-}^{0^+} \frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} dz - \kappa^2 \int_{0^-}^{0^+} \hat{\tilde{u}}_y dz = -\frac{A}{\rho \beta^2} \int_{0^-}^{0^+} \delta(z) dz.$$

$$\left[\frac{\partial \hat{\tilde{u}}_y}{\partial z} \right]_{z=0^-}^{z=0^+} - (\kappa^2.0) = -\frac{A}{\rho \beta^2}.(1)$$

The first integral gives the jump in the derivative:

$$\left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^+} - \left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^-} = -\frac{A}{\rho\beta^2}.$$

The second integral vanishes because $\hat{\tilde{u}}_y$ is continuous.

For a symmetric solution (assuming decay as $|z| \to \infty$):

- For $z > 0$: $\hat{\tilde{u}}_y(z) = Ce^{-\kappa z}$
- For $z < 0$: $\hat{\tilde{u}}_y(z) = Ce^{\kappa z}$

Applying the jump condition:

$$\left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^+} = -\kappa C e^{-\kappa(0)}, \quad \left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^-} = \kappa C e^{\kappa(0)}.$$

$$\left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^+} = -\kappa C, \quad \left.\frac{\partial \hat{\tilde{u}}_y}{\partial z}\right|_{0^-} = \kappa C.$$

Thus:

$$-\kappa C - \kappa C = -\frac{A}{\rho\beta^2} \implies -2\kappa C = -\frac{A}{\rho\beta^2}.$$

Solving for C:

$$\boxed{C = \frac{A}{2\kappa\rho\beta^2}}$$

Final Solution in Fourier-Laplace Space

$$\boxed{\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\kappa\rho\beta^2} e^{-\kappa|z|}}$$

This is the Fourier–Laplace transformed displacement field $\hat{\tilde{u}}_y(k_x, z, s)$ for the 2D SH wave equation under a point force source.

where:

$$\kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2}}$$

.

κ encodes the wavenumber k_x and laplace parameter (s), ensuring the solution decays exponentially away from the source, $e^{-\kappa|z|}$. The choice $\text{Re}(\kappa) > 0$ ensures boundedness and causality (no energy from $z = \pm\infty$)

Step 4: Take the Inverse Spatial Fourier Transform of the Fourier-Laplace Transformed Solution

Given Fourier-Laplace Space Solution

$$\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\kappa\rho\beta^2} e^{-\kappa|z|},$$

where

$$\kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2}}.$$

The solution becomes:

$$\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\rho\beta^2} \cdot \frac{e^{-\kappa|z|}}{\kappa}.$$

Inverse Fourier Transform in x ($k_x \to x$)

The inverse Fourier transform is defined as:

$$\tilde{u}_y(x, z, s) = \mathcal{F}^{-1}\{\hat{\tilde{u}}_y(k_x, z, s)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{u}}_y(k_x, z, s) e^{ik_x x}\, dk_x.$$

Substitute $\hat{\tilde{u}}_y$:

$$\tilde{u}_y(x, z, s) = \frac{A}{4\pi\rho\beta^2} \int_{-\infty}^{\infty} \frac{e^{-\kappa|z|}}{\kappa} e^{ik_x x}\, dk_x. \tag{4}$$

Simplify the Integral

The integral to solve is:

$$I = \int_{-\infty}^{\infty} \frac{e^{-\kappa|z|}}{\kappa} e^{ik_x x}\, dk_x.$$

Let $\kappa = \sqrt{k_x^2 + \alpha^2}$, where $\alpha^2 = \frac{s^2}{\beta^2}$. The integral becomes:

$$I = \int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\alpha^2}|z|}}{\sqrt{k_x^2 + \alpha^2}} e^{ik_x x}\, dk_x.$$

To simplify the integral, I, above, we can use a known integral representation, or we could manipulate the integral path.

Using Known Integral Results

The integral above is a known integral representation of the modified Bessel function of the second kind K_0. This integral is a standard form whose result is known from tables of Fourier transforms or Green's functions. The result is:

$$\int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\alpha^2}|z|}}{\sqrt{k_x^2+\alpha^2}} e^{ik_x x} dk_x = 2K_0(\alpha\sqrt{x^2+z^2}),$$

where K_0 is the modified Bessel function of the second kind of order zero. However, another common representation is:

$$\int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\alpha^2}|z|}}{\sqrt{k_x^2+\alpha^2}} e^{ik_x x} dk_x = \pi H_0^{(1)}(i\alpha\sqrt{x^2+z^2}),$$

where $H_0^{(1)}$ is the Hankel function of the first kind. But for decaying exponentials, the correct form is:

$$I = 2K_0\left(\alpha\sqrt{x^2+z^2}\right).$$

Thus, the inverse Fourier transform yields:

$$\tilde{u}_y(x,z,s) = \frac{A}{4\pi\rho\beta^2} \cdot 2K_0\left(\alpha\sqrt{x^2+z^2}\right). \tag{5}$$

where:

$$\alpha = \sqrt{\frac{s^2}{\beta^2}}.$$

The final Laplace-transformed solution is:

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho\beta^2} K_0\left(\alpha\sqrt{x^2+z^2}\right)} \tag{6}$$

Integral Manipulation

Substitute $k_x = isp$ into the inverse FT in equation 4 before explicitly solving it. This substitution will also make p complex since k_x is real. By so doing, we intend to parameterise the solution for further analysis (for an inverse Laplace transform).

$$\tilde{u}_y(x,z,s) = \mathcal{F}^{-1}\{\hat{\tilde{u}}_y(k_x,z,s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\tilde{u}}_y(k_x,z,s)e^{ik_x x}\, dk_x.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\tilde{u}}_y(isp,z,s)e^{i(isp)x}\,(isdp).$$

The Fourier-Laplace solution before the inverse Fourier transform was:

$$\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\rho\beta^2} \cdot \frac{e^{-\kappa|z|}}{\kappa}, \quad \kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2}}.$$

$$\kappa = \sqrt{(isp)^2 + \frac{s^2}{\beta^2}} = \sqrt{-s^2p^2 + \frac{s^2}{\beta^2}} = s\sqrt{\frac{1}{\beta^2} - p^2}.$$

Let $\eta = \sqrt{\frac{1}{\beta^2} - p^2}$, so $\kappa = s\eta$.

$$\hat{\tilde{u}}_y(isp, z, s) = \frac{A}{2\rho\beta^2} \cdot \frac{e^{-s\eta|z|}}{s\eta}$$

The inverse Fourier transform of $\hat{\tilde{u}}_y$ would involve an integral over p and it is complex:

$$\tilde{u}_y(x, z, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{u}}_y(isp, z, s) e^{-spx}\, d(isp).$$

The integral over $k_x \in [-\infty, \infty]$ will maps to $p \in [i\infty, -i\infty]$ because when:

$$k_x = +\infty, \quad p = \frac{k_x}{is} = \frac{+\infty}{is} = -i\infty$$

$$k_x = -\infty, \quad p = \frac{k_x}{is} = \frac{-\infty}{is} = i\infty$$

So,

$$\tilde{u}_y(x, z, s) = \frac{is}{2\pi} \int_{+i\infty}^{-i\infty} \frac{Ae^{-s\eta|z|}e^{-spx}}{2\rho\beta^2 s\eta}\, dp.$$

$$\boxed{\tilde{u}_y(x, z, s) = \frac{A}{4\pi\rho\beta^2} \int_{+i\infty}^{-i\infty} \frac{ie^{-s(px+\eta|z|)}}{\eta}\, dp = \frac{-A}{4\pi\rho\beta^2} \int_{-i\infty}^{+i\infty} \frac{ie^{-s(px+\eta|z|)}}{\eta}\, dp.} \tag{7}$$

$$\boxed{\eta = \sqrt{\frac{1}{\beta^2} - p^2} \quad \text{and} \quad \beta^2 = \frac{\mu}{\rho}}$$

Let's try to simplify equation 7 even further. Since:

$$\tilde{u}_y(x, z, s) = \frac{-A}{4\pi\rho\beta^2} \int_{-i\infty}^{+i\infty} \frac{ie^{-s(px+\eta|z|)}}{\eta}\, dp.$$

It happens that if we decompose the integrand in equation 7 into even (E) and odd (O) parts with respect to p, we have: (**see Box 3.0 below for derivation**.)

$$\int_{-i\infty}^{+i\infty} \frac{-ie^{-s(px+\eta|z|)}}{\eta} dp = 2\text{Im}\left\{\int_{0}^{+i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta} dp\right\}$$

So,

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{4\pi\rho\beta^2}\int_{-i\infty}^{+i\infty} \frac{-ie^{-s(px+\eta|z|)}}{\eta} dp = \frac{A}{2\pi\rho\beta^2}\,\text{Im}\left\{\int_{0}^{+i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta} dp\right\}} \tag{8}$$

Step 5: The Cagniard Path and Deformation

Now, we will try to force equation 8 into the form of a laplace transform. To achieve this, we must investigate the path C in the complex $p-$ plane for which $px + \eta|z|$ is real

let:

$$t = px + \eta|z| = px + |z|\sqrt{\frac{1}{\beta^2} - p^2} \tag{9}$$

Note: C is the cagniard path given by $p = p(t)$, where t is real and positive.

Solving for p, we get: (**see Box 1.0 below for derivation**)

$$\boxed{p = \frac{xt \pm |z|\sqrt{\frac{R^2}{\beta^2} - t^2}}{R^2}} \tag{10}$$

$$p = \begin{cases} \frac{xt + i|z|\sqrt{t^2 - \frac{R^2}{\beta^2}}}{R^2}, & t > \frac{R}{\beta} \\ \frac{xt - |z|\sqrt{\frac{R^2}{\beta^2} - t^2}}{R^2}, & t < \frac{R}{\beta} \end{cases}$$

Thus, for $t < \frac{R}{\beta}$ the integrand in equation 8 is purely **real**, contributing nothing to the imaginary part. However, for $t > \frac{R}{\beta}$, the path C ensures t increase **monotonically**.

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho\beta^2}\,\text{Im}\left\{\int_{C} \frac{e^{-s(px+\eta|z|)}}{\eta} dp\right\}} \tag{11}$$

Differentiate t in equation 9 above with respect to t and make $\frac{dp}{dt}$ (**see Box 2.0 below for derivation**)

$$\boxed{\frac{dp}{dt} = \frac{\sqrt{\frac{1}{\beta^2} - p}}{x\sqrt{\frac{1}{\beta^2} - p} - p|z|} = \frac{\eta}{x\eta - p|z|}} \tag{12}$$

Given that:

$$t = px + \eta|z| = px + |z|\sqrt{\frac{1}{\beta^2} - p^2}$$

Take the square of equation 9

$$t^2 = p^2x^2 + \eta^2 z^2 + 2px\eta|z|$$

$$t^2 = p^2x^2 + \left(\frac{1}{\beta^2} - p^2\right)z^2 + 2px\eta|z|$$

$$\boxed{p^2x^2 - p^2z^2 = t^2 - \left(\frac{1}{\beta^2}\right)z^2 - 2px\eta|z|}$$

From equation 12 above, take the square of the denominator and substitute $p^2x^2 - p^2z^2$, we have:

$$\left(x\eta - p|z|\right)^2 = x^2\eta^2 - 2xp\eta|z| + p^2z^2$$

$$= x^2\left(\frac{1}{\beta^2} - p^2\right) - 2xp\eta|z| + p^2z^2$$

$$= \frac{x^2}{\beta^2} - x^2p^2 - 2xp\eta|z| + p^2z^2$$

$$= -x^2p^2 + p^2z^2 - 2xp\eta|z| + \frac{x^2}{\beta^2}$$

$$= -(x^2p^2 - p^2z^2) - 2xp\eta|z| + \frac{x^2}{\beta^2}$$

$$= -\left(t^2 - \frac{z^2}{\beta^2} - 2px\eta|z|\right) - 2xp\eta|z| + \frac{x^2}{\beta^2}$$

$$= -t^2 + \frac{z^2}{\beta^2} + \cancel{2px\eta|z|} - \cancel{2xp\eta|z|} + \frac{x^2}{\beta^2}$$

$$= -t^2 + \frac{z^2}{\beta^2} + \frac{x^2}{\beta^2}$$

$$= \frac{x^2}{\beta^2} + \frac{z^2}{\beta^2} - t^2$$

$$= \frac{x^2 + z^2}{\beta^2} - t^2 = \frac{R^2}{\beta^2} - t^2$$

$$(x\eta - p|z|)^2 = \frac{R^2}{\beta^2} - t^2$$

$$\boxed{x\eta - p|z| = \sqrt{\frac{R^2}{\beta^2} - t^2}}$$

$$x\eta - p|z| = \sqrt{\frac{R^2}{\beta^2} - t^2} \quad \text{for} \quad t < \sqrt{\frac{R^2}{\beta^2}}$$

Or:

$$x\eta - p|z| = i\sqrt{t^2 - \frac{R^2}{\beta^2}} \quad \text{for} \quad t > \sqrt{\frac{R^2}{\beta^2}}$$

where:

$$R = \sqrt{x^2 + z^2}$$

So, equation 11 becomes:

$$\boxed{\frac{dp}{dt} = \frac{\sqrt{\frac{1}{\beta^2} - p}}{x\sqrt{\frac{1}{\beta^2} - p - p|z|}} = \frac{\eta}{x\eta - p|z|} = \frac{\eta}{i\sqrt{t^2 - \frac{R^2}{\beta^2}}}} \quad \text{on} \quad C \quad \text{for} \quad t > \frac{R}{\beta} \tag{13}$$

$$\frac{dp}{dt} = \frac{\eta}{i\sqrt{t^2 - \frac{R^2}{\beta^2}}} = \frac{-i\eta}{\sqrt{t^2 - \frac{R^2}{\beta^2}}}$$

Interpretation

- The singularity at $t = \frac{R}{\beta}$ marks the wavefront arrival time.
- The factor $\frac{1}{\sqrt{t^2 - \frac{R^2}{\beta^2}}}$ represent the 2D geometric spreading of the wave.
- The imaginary unit, i, arises beacuse the Cagniard path, C, lies in the complex $p-$plane where $x\eta - p|z|$ is purely imaginary for $t > \frac{R}{\beta}$.
- Equation 13 is central to the Cagniard-De Hoop method as it enables the converison of the integral (equation 11) into a laplace-transform-like expression, from which time-domain solution can be directly known.

Now, let's substitute equation (13) into equation(11)

FRom equation 13, we have:

$$dp = \frac{-i\eta}{\sqrt{t^2 - \frac{R^2}{\beta^2}}} dt$$

$$\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho\beta^2} \operatorname{Im}\left\{ \int_C \frac{e^{-s(px+\eta|z|)}}{\eta} \, dp \right\}$$

$$\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho\beta^2} \operatorname{Im}\left\{ \int_{\frac{R}{\beta}}^{\infty} \frac{e^{-s(px+\eta|z|)}}{\cancel{\eta}} \cdot \frac{-i\cancel{\eta}}{\sqrt{t^2 - \frac{R^2}{\beta^2}}} dt \right\} \tag{14}$$

η is real on C for $t > \frac{R}{\beta}$. The term inside the imaginary part is purely imaginary ($-i\times$ real function). Therefore:

$$\operatorname{Im}\{-i \times (\text{real integral})\} = \operatorname{Re}\{(\text{real integral})\}.$$

$$\operatorname{Im}\left\{ -i \int \cdots \right\} = \operatorname{Re}\left\{ \int \cdots \right\}$$

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho\beta^2} \int_{\frac{R}{\beta}}^{\infty} \frac{1}{\sqrt{t^2 - \frac{R^2}{\beta^2}}} \, e^{-s\,t} \, dt} \tag{15}$$

Equation 15 is a Laplace transform equation.

Step 6: Inverse Laplace Transformation

Now, if we find the inverse laplace transform of equation 15. we get:

$$\boxed{u_y(x,z,t) = \frac{A}{2\pi\rho\beta^2} \, \frac{1}{\sqrt{t^2 - \frac{R^2}{\beta^2}}} \, \mathcal{H}\left(t - \frac{R}{\beta}\right)} \tag{16}$$

Equation 16 is the time-domain solution $u_y(x,z,t)$

Box 1.0 - The Cagniard path

$$t = px + |z|\sqrt{\frac{1}{\beta^2} - p^2}$$

Isolate the square root term:

$$t - px = |z|\sqrt{\frac{1}{\beta^2} - p^2}$$

Divide both sides by $|z|$:

$$\frac{t - px}{|z|} = \sqrt{\frac{1}{\beta^2} - p^2}$$

Now square both sides:

$$\left(\frac{t - px}{|z|}\right)^2 = \frac{1}{\beta^2} - p^2$$

Multiply both sides by $|z|^2$:

$$(t - px)^2 = |z|^2 \left(\frac{1}{\beta^2} - p^2\right)$$

Expand the left-hand side:

$$t^2 - 2tpx + p^2x^2 = \frac{|z|^2}{\beta^2} - |z|^2p^2$$

Bring all terms to one side:

$$t^2 - 2tpx + p^2x^2 + |z|^2p^2 - \frac{|z|^2}{\beta^2} = 0$$

Group like terms:

$$p^2(x^2 + |z|^2) - 2tx \cdot p + \left(t^2 - \frac{|z|^2}{\beta^2}\right) = 0$$

This is a quadratic equation in p. Solving using the quadratic formula:

$$p = \frac{2tx \pm \sqrt{(2tx)^2 - 4(x^2 + |z|^2)\left(t^2 - \frac{|z|^2}{\beta^2}\right)}}{2(x^2 + |z|^2)}$$

Box 1.0 - The cagniard path - Contd

Simplify:

$$p = \frac{tx \pm \sqrt{t^2x^2 - (x^2+|z|^2)\left(t^2 - \frac{|z|^2}{\beta^2}\right)}}{x^2+|z|^2}$$

$$p = \frac{xt \pm \sqrt{\cancel{x^2t^2} - \cancel{x^2t^2} + \frac{x^2|z|^2}{\beta^2} - z^2t^2 + \frac{|z|^2|z|^2}{\beta^2}}}{x^2+|z|^2}$$

$$p = \frac{xt \pm \sqrt{\frac{x^2|z|^2}{\beta^2} + \frac{|z|^2|z|^2}{\beta^2} - |z|^2t^2}}{x^2+|z|^2}$$

$$p = \frac{xt \pm |z|\sqrt{\frac{x^2}{\beta^2} + \frac{|z|^2}{\beta^2} - t^2}}{x^2+|z|^2}$$

$$\boxed{p = \frac{xt \pm |z|\sqrt{\frac{x^2+|z|^2}{\beta^2} - t^2}}{x^2+|z|^2} = \frac{xt \pm |z|\sqrt{\frac{R^2}{\beta^2} - t^2}}{R^2}}$$

$$p = \begin{cases} \frac{xt+i|z|\sqrt{t^2-\frac{R^2}{\beta^2}}}{R^2}, & t > \frac{R}{\beta} \\ \frac{xt-|z|\sqrt{\frac{R^2}{\beta^2}-t^2}}{R^2}, & t < \frac{R}{\beta} \end{cases}$$

Notes: We can see that p is purely **real**, contributing **nothing** to the imaginary part when $t < t_0$ i.e when $t < \frac{R}{\beta} = \frac{x^2+|z|^2}{\beta}$. p is only imaginary for any $t > t_0$ which is actually the part we are interested in. We are not interested in anything that arrives before t_0 (causality). So, the objective would be to **deform** the path such that p is real and positive for with $t > t_0$. See above.

Box 2.0 - The time derivative along the cagniard path

Given the equation:

$$t = px + \eta|z| \quad \text{where} \quad \eta = \sqrt{\frac{1}{\beta^2} - p^2}$$

Differentiate both sides with respect to t

Left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t} t = 1$$

Right-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(px + |z|\sqrt{\frac{1}{\beta^2} - p^2}\right) = x\frac{\mathrm{d}p}{\mathrm{d}t} + |z| \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\frac{1}{\beta^2} - p^2}\right)$$

Differentiate the square root term using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\frac{1}{\beta^2} - p^2}\right) = \frac{1}{2}\left(\frac{1}{\beta^2} - p^2\right)^{-1/2} \cdot (-2p)\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{p}{\sqrt{\frac{1}{\beta^2} - p^2}}\frac{\mathrm{d}p}{\mathrm{d}t}$$

So the right-hand side becomes:

$$x\frac{\mathrm{d}p}{\mathrm{d}t} - |z| \cdot \frac{p}{\sqrt{\frac{1}{\beta^2} - p^2}}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Factor out $\frac{\mathrm{d}p}{\mathrm{d}t}$

$$1 = \left(x - |z| \cdot \frac{p}{\sqrt{\frac{1}{\beta^2} - p^2}}\right)\frac{\mathrm{d}p}{\mathrm{d}t}$$

Solve for $\frac{\mathrm{d}p}{\mathrm{d}t}$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{x - |z| \cdot \frac{p}{\sqrt{\frac{1}{\beta^2} - p^2}}} = \frac{1}{x - \frac{|z|p}{\sqrt{\frac{1}{\beta^2} - p^2}}}$$

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{x - \frac{|z|p}{\eta}} = \frac{\eta}{x\eta - |z|p}}$$

Box 3.0 – Detailed Derivation of the Integral Simplification in Equation 7

We aim to show that:

$$\boxed{\int_{-i\infty}^{+i\infty} \frac{-ie^{-s(px+\eta|z|)}}{\eta}\,dp = 2\,\mathrm{Im}\left\{\int_{0}^{+i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta}\,dp\right\},}$$

where $\eta = \sqrt{\frac{1}{\beta^2} - p^2}$ (η is complex since p is complex)

Parametrize the Contour

Let $p = iy$, where y is real, y $\in \mathbf{R}$. Then:

- $dp = i\,dy$,
- $\eta = \sqrt{\frac{1}{\beta^2} + y^2}$ (η is now real and positive for real y).

The integral becomes:

$$\int_{-i\infty}^{+i\infty} f(p)\,dp = \int_{-\infty}^{+\infty} \frac{-ie^{-s(iyx+\eta|z|)}}{\eta} \cdot i\,dy = \int_{-\infty}^{+\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\,dy.$$

Odd/Even Decomposition of the integrand

Following Euler's formula, the integrand can be split into **real (even; cosine function)** and **imaginary (odd; sine function)** parts:

$$e^{-isyx} = \cos(syx) - i\sin(syx).$$

Thus:

$$\frac{e^{-s\eta|z|}}{\eta}\Big(\cos(syx) - i\sin(syx)\Big) = \underbrace{\frac{e^{-s\eta|z|}\cos(syx)}{\eta}}_{E(y)} \quad - \quad i\,\underbrace{\frac{e^{s\eta|z|}\sin(syx)}{\eta}}_{O(y)}.$$

$$\frac{e^{-s\eta|z|}}{\eta}\Big(\cos(syx) - i\sin(syx)\Big) = E(y) - i\,O(y).$$

- $E(y)$: Even in y (since cos is even, η is even).
- $O(y)$: Odd in y (since sin is odd, and $-i$ flips sign with y).

The even part $E(y)$ is:

$$\boxed{E(y) = \frac{e^{-s\eta|z|}\cos(syx)}{\eta}.}$$

Box 3.0 – Detailed Derivation of the Integral Simplification in Equation 7 - Contd.

The imaginary part of the integrand is:

$$\boxed{\underbrace{\mathrm{Im}\left\{\frac{e^{-s(px+\eta|z|)}}{\eta}\right\}}_{E(p)} = \underbrace{\mathrm{Im}\left\{\frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\right\}}_{E(y)} = -\frac{e^{-s\eta|z|}\sin(syx)}{\eta} = -O(y),}$$

where $O(y)$ is the odd part.

However, the original integrand is **purely imaginary**. Thus, we focus on the **imaginary contribution**:

$$\int_{-i\infty}^{+i\infty} \frac{-ie^{-s(px+\eta|z|)}}{\eta}\,dp = \int_{-\infty}^{+\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\,dy.$$

$$\int_{-\infty}^{+\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\,dy = \int_{-\infty}^{+\infty} E(y) - iO(y)\,dy = \int_{-\infty}^{0} \cdots\,dy + \int_{0}^{\infty} \cdots\,dy.$$

$$= \int_{-\infty}^{0} E(y) - i\,O(y)\,dy + \int_{0}^{\infty} E(y) - i\,O(y)\,dy.$$

Use symmetry

Since $E(y)$ is even: $E(-y) = E(y)$
Since $O(y)$ is odd: $O(-y) = -O(y)$
So:

$$\int_{-\infty}^{0} E(y)\,dy = \int_{0}^{\infty} E(y)\,dy, \quad \int_{-\infty}^{0} O(y)\,dy = -\int_{0}^{\infty} O(y)\,dy$$

Then:

$$\int_{-\infty}^{0} (E(y) - iO(y))\,dy = \int_{0}^{\infty} E(y)\,dy + i\int_{0}^{\infty} O(y)\,dy$$

$$\int_{0}^{\infty} (E(y) - iO(y))\,dy = \int_{0}^{\infty} E(y)\,dy - i\int_{0}^{\infty} O(y)\,dy$$

Add both:

$$\left[\int_{0}^{\infty} E(y)\,dy + i\int_{0}^{\infty} O(y)\,dy\right] + \left[\int_{0}^{\infty} E(y)\,dy - i\int_{0}^{\infty} O(y)\,dy\right]$$

$$= \int_{0}^{\infty} E(y)\,dy + i\int_{0}^{\infty} O(y)\,dy + \int_{0}^{\infty} E(y)\,dy - i\int_{0}^{\infty} O(y)\,dy.$$

Box 3.0 – Detailed Derivation of the Integral Simplification in Equation 7 - Contd.

$$= \int_0^\infty E(y)\,dy + i\cancel{\int_0^\infty O(y)\,dy} + \int_0^\infty E(y)\,dy - i\cancel{\int_0^\infty O(y)\,dy}.$$

$$= 2\int_0^\infty E(y)\,dy.$$

$$\boxed{\int_{-\infty}^{+\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\,dy = 2\int_0^\infty E(y)\,dy = 2\int_0^\infty \underbrace{\frac{e^{-s\eta|z|}\cos(syx)}{\eta}}_{E(y)}\,dy}$$

Remember $p = i\,y$, so $y = -i\,p$. Thus:

$$= -2i\int_0^{i\infty} E(y)\,dp.$$

$$= -2i\int_0^{i\infty} \underbrace{\frac{e^{-s\eta|z|}\cos(syx)}{\eta}}_{E(y)}\,dp = -2i\int_0^{i\infty} \underbrace{\frac{e^{-s\eta|z|}\cos(-ispx)}{\eta}}_{E(p)}\,dp$$

$$\int_{-\infty}^{+\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta}\,dy = \int_{-\infty}^{+\infty} E(y) - iO(y)\,dy = -2i\int_0^{i\infty} E(y)\,dp.$$

—

From the symmetry of the integral:

$$\int_0^{i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta}\,dp = \int_0^{\infty} \frac{e^{-s\eta|z|}e^{-isyx}}{\eta} i\,dy = \int_0^\infty (E(y) - iO(y))\,i\,dy.$$

$$= \int_0^\infty i\,E(y) + O(y)dy.$$

- The real part here is $\int_0^\infty O(y)dy$
- The imaginary part is $\int_0^\infty E(y)dy$

Taking the imaginary part:

$$\operatorname{Im}\left\{\int_0^{i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta}\,dp\right\} = \int_0^\infty E(y)\,dy.$$

Now, recall that:

$$-2i\int_0^{i\infty} E(y)\,dp = -2i\cdot i\int_0^\infty E(y)\,dy = 2\int_0^\infty E(y)\,dy.$$

Thus:

$$\boxed{-2i\int_0^{i\infty} E(y)\,dp = 2\operatorname{Im}\left\{\int_0^{+i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta}\,dp\right\}.}$$