Marchenko Imaging: The Past, The Present, and The Future

1 Introduction

Green's function is crucial for reconstructing accurate reflectivity by linking recorded data to subsurface properties. Traditionally, these are often computed from forward modelling of the earth's subsurface, which requires medium properties such as the P-wave velocity or density model. However, the Marchenko method from the multidimensional Marchenko equation [2] has allowed direct retrieval of these functions from the recorded seismic response [1].

2 Marchenko Method – 1D

When an incident wave from a source propagating through a medium hits a heterogeneity, waves are scattered, and the scattered waves transport information about this heterogeneity to the receivers where they are recorded. Reconstruction of the properties (e.g., potential, density, or impedance) of the unknown medium from recordings of the scattered waves is known as ***the inverse scattering problem***. So, in this tutorial, we will try to mathematically "invert" the recorded reflections from a receiver to reveal the property of the medium. In 1D inverse scattering problems, information from a single receiver is sufficient to reconstruct the medium's properties. This is because waves in 1D propagate only in two directions (forward and backward), and the reflection data contains enough information.

To solve the inverse problem in 1D, it requires finding a solution to an integral equation called the "Marchenko Equation". It relates the reflection response (measured at the receiver) to the medium's properties. The Marchenko equation provides a **closed-form, exact solution** for 1D inverse problems (unlike iterative or approximate methods for 2D/3D cases). Derivation of the Marchenko equation in 1D is provided below.

The 1D wave equation is given as:

$$\rho(x)\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}\left[\rho(x)c^2(x)u(x,t)\right] = 0\,. \tag{1}$$

where:

- $u(x,t)$: Particle displacement.
- $\rho(x)$: Density of the medium.
- $c(x)$: Wave speed.

The objective is to recover $\rho(x)$ and $c(x)$ based on the recorded response, the observations of $u(x_0,t)$ at a fixed location x_0.

STEP 1: WAVE EQUATION TRANSFORMATION

In this section, we try to transform equation (1) into a different form of partial differential equation called the **Schrödinger equation** [3].

To simplify the problem, the spatial coordinate x is transformed to the traveltime domain using the one-way traveltime τ that the wave required to reach the position. This is defined by:

Transformation to Traveltime Domain
The one-way traveltime τ is defined as:

$$\tau(x) = \int_0^x \frac{dx'}{c(x')},$$

This represents the time it takes for a wave to travel from $x = 0$ to x. Its differential form is:

$$d\tau = \frac{dx}{c(x)}.$$

This substitution "re-parameterizes" space by the time it takes a wave to travel from $x = 0$ to x. By the chain rule, spatial derivatives transform as:

$$\frac{d}{dx} = \frac{d\tau}{dx}\frac{d}{d\tau} = \frac{1}{c(x)}\frac{d}{d\tau}.$$

Now, rewrite the original wave equation (1) in terms of τ instead of x.

Term 1: Time Derivative
This term remains unchanged as it doesn't involve $x-$ derivatives

$$\rho(x)\frac{\partial^2 u}{\partial t^2} \to \rho(\tau)\frac{\partial^2 u}{\partial t^2}.$$

Term 2: Spatial Derivative
Let $w(x,t) = \rho(x)c^2(x)u(x,t)$. The second derivative with respect to x is:

$$\frac{\partial^2}{\partial x^2}\left[\rho(x)c^2(x)u(x,t)\right] = \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\rho(x)c^2(x)\frac{\partial u(x,t)}{\partial x}\right).$$

Using the chain rule:

$$\frac{\partial}{\partial x}\left(\rho(x)c^2(x)\frac{d\tau}{dx}\frac{\partial u(x,t)}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(\rho(x)c^2(x)\frac{1}{c(x)}\frac{\partial u(x,t)}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right)$$

$$\frac{\partial}{\partial x}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right) = \frac{d\tau}{dx}\frac{\partial}{\partial \tau}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right) = \frac{1}{c(x)}\frac{\partial}{\partial \tau}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right)$$

Combining Terms

Substitute the transformed terms into the original equation in (1):

$$\rho(x)\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}\left[\rho(x)c^2(x)u(x,t)\right] = 0$$

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{1}{c(x)}\frac{\partial}{\partial \tau}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right) = 0$$

Multiply through by $c(x)$:

$$\rho(x)c(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial \tau}\left(\rho(x)c(x)\frac{\partial u(x,t)}{\partial \tau}\right) = 0$$

Here, the product ρc is the *impedance* of the medium.

Introduction of Scaled Wavefield and Schrödinger Equation

We can define a new impedance variable $\eta(\tau) = \sqrt{\rho c}$. Replacing $\rho c = \eta^2$, the above equation becomes:

$$\eta^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \tau}\left(\eta^2\frac{\partial u}{\partial \tau}\right) = 0 \tag{2}$$

Next we introduce the scaled wavefield $\psi(\tau,t) = \eta(\tau)u(\tau,t)$. This implies $u = \frac{\psi(\tau,t)}{\eta(\tau)}$

$$\boxed{\eta^2\frac{\partial^2 u}{\partial t^2} = \eta^2\frac{\partial^2}{\partial t^2}\left(\frac{\psi}{\eta}\right) = \eta\frac{\partial^2 \psi}{\partial t^2}}$$

Derivation of $\frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial u}{\partial \tau}\right)$

$$\frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial u}{\partial \tau}\right) = \frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial}{\partial \tau}\left(\frac{\psi}{\eta}\right)\right) = \frac{\partial}{\partial \tau}\left(\eta^2 \left(\frac{1}{\eta}\frac{\partial \psi}{\partial \tau} - \frac{\psi}{\eta^2}\frac{d\eta}{d\tau}\right)\right)$$

$$\frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial u}{\partial \tau}\right) = \frac{\partial}{\partial \tau}\left(\eta \frac{\partial \psi}{\partial \tau} - \psi \frac{d\eta}{d\tau}\right)$$

$$\begin{aligned}
\frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial u}{\partial \tau}\right) &= \frac{\partial}{\partial \tau}\left(\eta \frac{\partial \psi}{\partial \tau} - \psi \frac{d\eta}{d\tau}\right) \\
&= \frac{\partial}{\partial \tau}\left(\eta \frac{\partial \psi}{\partial \tau}\right) - \frac{\partial}{\partial \tau}\left(\psi \frac{d\eta}{d\tau}\right) \\
&= \frac{d\eta}{d\tau}\frac{\partial \psi}{\partial \tau} + \eta \frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial \psi}{\partial \tau}\frac{d\eta}{d\tau} - \psi \frac{d^2 \eta}{d\tau^2} \\
&= \eta \frac{\partial^2 \psi}{\partial \tau^2} - \psi \frac{d^2 \eta}{d\tau^2}
\end{aligned}$$

By substituting the above, equation 2 becomes:

$$\eta^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \tau}\left(\eta^2 \frac{\partial u}{\partial \tau}\right) = 0$$

$$\eta \frac{\partial^2 \psi}{\partial t^2} - \left(\eta \frac{\partial^2 \psi}{\partial \tau^2} - \psi \frac{d^2 \eta}{d\tau^2}\right) = 0$$

$$\Rightarrow \eta \left(\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \tau^2}\right) + \psi \frac{d^2 \eta}{d\tau^2} = 0$$

Divide through by η:

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \tau^2} + \frac{1}{\eta}\frac{d^2 \eta}{d\tau^2}\psi = 0 \tag{3}$$

The terms involving derivatives of η can be grouped into the scattering potential q:

$$q = \frac{1}{\eta}\frac{d^2 \eta}{d\tau^2}$$

Here $q(\tau)$ depends on the material distribution (ρ, c), and it is nonzero only where the medium is inhomogeneous. This led to the famous Schrödinger equation:

$$\boxed{\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial\tau^2} + q\psi = 0} \tag{4}$$

This is a wave equation with a potential term, often called a scattering wave equation.

STEP 2: ANALYTICAL SOLUTION TO THE SCHRÖDINGER EQUATION

Next, we shall try to find an analytical solution to the Schrödinger equation in equation (4).

Propagating Pulses and the Ray Expansion (ansatz)

we know that solutions, $\psi(\tau, t)$, to the Schrödinger equation in equation 4 are wave-like. So, we assume solution of the form:

$$\boxed{\psi(\tau, t) = \sum_{n=0}^{\infty} a_n(\tau) f_n[t - \phi(\tau)]} \tag{5}$$

The solution is assumed to be a ray expansion (or propagating pulse series). From equation 5, there are 3 unknowns: a_n, f_n, and ϕ.

where:

- $\phi(\tau)$ is the **phase function** (traveltime).
- $a_n(\tau)$ are **amplitude coefficients** of the superposition of wave shapes or wavelets, f_n.
- f_n are **wavelets** each of which are shifted by a phase ϕ as t progresses.

To reduce the number of unknowns, we impose an additional relation between the wavelets with successive indices. So, the wavelet must satisfy the recursion:

$$\frac{df_n(z)}{dz} = f_{n-1}(z).$$

So, for a δ-pulse input ($f_0(z) = \delta(z)$), the higher-order wavelets are:

Wavelet Examples:

- $f_0(z) = \delta(z)$ (Dirac delta, impulse) ($\delta(z)$ is a good starting point).
- $f_1(z) = H(z)$ (Heaviside step function).
- $f_2(z) = zH(z)$
- $f_3(z) = \frac{z^2}{2}H(z)$
- $f_4(z) = \frac{z^3}{6}H(z)$
- $f_5(z) = \frac{z^4}{24}H(z)$, etc.

Generally,

$$f_n(z) = \frac{z^{n-1}}{(n-1)!}H(z) \quad \text{for } n > 0 \quad (\text{e.g., } f_1 = H(z),\ f_2 = zH(z), \text{ etc.}).$$

Substituting into the Schrödinger Equation

Now, substitute $\psi(\tau, t)$ in equation 5 (the assumed solution) into the Schrödinger wave equation (4) :

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \tau^2} + q\psi = 0$$

we get:

$$\sum_{n=0}^{\infty} \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \tau^2} + q(\tau) \right] (a_n(\tau) f_n[t - \phi(\tau)]) = 0.$$

Compute each term:

1. **Time derivative (∂_t^2):**

$$\frac{\partial \psi}{\partial t} = \sum_{n=0}^{\infty} a_n(\tau) \frac{\partial}{\partial t} f_n[t - \phi(\tau)] = \sum_{n=0}^{\infty} a_n(\tau) f_n'[t - \phi(\tau)].$$

$$\frac{\partial^2 \psi}{\partial t^2} = \sum_{n=0}^{\infty} a_n(\tau) f_n''[t - \phi(\tau)].$$

From the recursion $f_n' = f_{n-1}$, we have $f_n'' = f_{n-2}$. We can compute the other wavelets either by differentiation (to get lower indices) or integration (to get higher indices).

2. **Spatial derivative** (∂_τ^2):

$$\frac{\partial\psi}{\partial\tau} = \sum_{n=0}^{\infty} \left(\frac{da_n}{d\tau} f_n[t-\phi(\tau)] + a_n(\tau) \frac{\partial}{\partial\tau} f_n[t-\phi(\tau)] \right).$$

Since $\frac{\partial}{\partial\tau} f_n[t-\phi(\tau)] = -f_n'[t-\phi(\tau)]\frac{d\phi}{d\tau}$, this becomes:

$$\frac{\partial\psi}{\partial\tau} = \sum_{n=0}^{\infty} \left(\frac{da_n}{d\tau} f_n[t-\phi(\tau)] - a_n \frac{d\phi}{d\tau} f_n'[t-\phi(\tau)] \right).$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial\tau^2} = \sum_{n=0}^{\infty} \Bigg(&\frac{d^2a_n}{d\tau^2} f_n[t-\phi(\tau)] - 2\frac{da_n}{d\tau}\frac{d\phi}{d\tau} f_n'[t-\phi(\tau)] \\ &+ a_n \left(\frac{d\phi}{d\tau}\right)^2 f_n''[t-\phi(\tau)] - a_n \frac{d^2\phi}{d\tau^2} f_n'[t-\phi(\tau)] \Bigg).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial\tau^2} = \sum_{n=0}^{\infty} \big[&a_n''(\tau) f_n[t-\phi(\tau)] - 2a_n'(\tau)\phi'(\tau) f_n'[t-\phi(\tau)] \\ &+ a_n(\phi'(\tau))^2 f_n''[t-\phi(\tau)] - a_n \phi''(\tau) f_n'[t-\phi(\tau)] \big].\end{aligned}$$

3. **Scattering potential term**:

$$\sum_{n=0}^{\infty} \left[q(\tau) a_n(\tau) f_n[t-\phi(\tau)] \right].$$

Combine all terms into the Schrödinger equation:

$$\sum_{n=0}^{\infty} \left[a_n f'' - \left(\frac{d^2a_n}{d\tau^2} f_n - 2\frac{da_n}{d\tau}\frac{d\phi}{d\tau} f' + a_n \left(\frac{d\phi}{d\tau}\right)^2 f'' - a_n \frac{d^2\phi}{d\tau^2} f' \right) + q a_n f_n \right] = 0.$$

$$\sum_{n=0}^{\infty}\left[a_n f_{n-2} - \left(\frac{d^2a_n}{d\tau^2}f_n - 2\frac{da_n}{d\tau}\frac{d\phi}{d\tau}f_{n-1} + a_n\left(\frac{d\phi}{d\tau}\right)^2 f_{n-2} - a_n\frac{d^2\phi}{d\tau^2}f_{n-1}\right) + qa_nf_n\right] = 0.$$

$$\sum_{n=0}^{\infty}\left[a_nf_{n-2} - a_n''f_n + 2a_n'\phi'f_{n-1} - a_n(\phi')^2f_{n-2} + a_n\phi''f_{n-1} + qa_nf_n\right] = 0.$$

Simplify by grouping terms with the same f_{n-k}:

$$\sum_{n=0}^{\infty}\left[\left(a_n - a_n\left(\frac{d\phi}{d\tau}\right)^2\right)f_{n-2} + \left(2\frac{da_n}{d\tau}\frac{d\phi}{d\tau} + a_n\frac{d^2\phi}{d\tau^2}\right)f_{n-1} + \left(-\frac{d^2a_n}{d\tau^2} + qa_n\right)f_n\right] = 0.$$

Eikonal and Transport Equations

For the equation to hold for all t, the coefficients of f_{n-2}, f_{n-1}, f_n, etc., must vanish separately. This leads to 3 key equations:

(a) **For $n = 0$ (coefficient of f_{n-2})**:

$$a_0\left(1 - \left(\frac{d\phi}{d\tau}\right)^2\right) = 0.$$

Assuming $a_0 \neq 0$, this reduces to the **eikonal equation**:

$$\left(\frac{d\phi}{d\tau}\right)^2 = 1 \implies \frac{d\phi}{d\tau} = \pm 1.$$

Integrating yields the phase function solution:

$$\phi(\tau) = \pm\tau + C.$$

This describes waves propagating right to left $(-\tau)$ or from left to right $(+\tau)$ with unit speed.

(b) **For $n = 1$ (coefficient of f_{n-1})**:

The equation simplifies to:

$$2\frac{da_0}{d\tau}\frac{d\phi}{d\tau} + a_0\frac{d^2\phi}{d\tau^2} = 0.$$

From the eikonal equation, $\frac{d^2\phi}{d\tau^2} = 0$, so:

$$\frac{da_0}{d\tau} = 0 \implies a_0(\tau) = \text{constant} = 1 \quad \text{(normalization)}.$$

Typically, we set $a_0 = 1$ for a unit-amplitude δ-pulse.

(c) **For $n = 2$ (coefficient of f_n)**:

$$a_1\left(1 - (\phi')^2\right) + 2a_1'\phi' + a_1\phi'' + qa_0 = 0.$$

Using $\phi' = \pm 1$ and $\phi'' = 0$, this reduces to:

$$2a_1'\phi' + qa_0 = 0.$$

Since $a_0 = 1$ and $\phi' = \pm 1$, we obtain a special form of **transport equation** which describes how amplitude is transported through the medium:

$$2\frac{da_1}{d\tau} = -q(\tau)\frac{d\phi}{d\tau}.$$

For a right-going wave ($\phi' = +1$):

$$\boxed{\frac{da_1}{d\tau} = -\frac{q(\tau)}{2}.}$$

This relates the amplitude $a_1(\tau)$ to the scattering potential $q(\tau)$.

Solution for the Wavefield

For a right-going δ-pulse, the wavefield is constructed as:

$$\psi(\tau, t) = \delta(t - \tau) + a_1(\tau)H(t - \tau) + a_2(\tau)(t - \tau)H(t - \tau) + \cdots .$$

$$\psi(\tau, t) = \delta(t-\tau) + a_1(\tau)H(t-\tau) + \sum_{n=2}^{\infty} a_n(\tau)\frac{(t-\tau)^{n-1}}{(n-1)!}H(t-\tau),$$

where:

- $\delta(t-\tau)$ is the incident pulse propagating with constant amplitude ($a_0 = 1$).
- $a_1(\tau)$ is determined by integrating $2\frac{da_1}{d\tau} = \mp q(\tau)$.
- The Heaviside term $H(t-\tau)$ has amplitude $a_1(\tau)$ determined by:

$$a_1(\tau) = -\frac{1}{2}\int_0^{\tau} q(\tau')d\tau'.$$

- Higher-order terms (e.g., $a_2(\tau)$) are derived similarly from the recursion.

Key Observations:

1. The δ-pulse $\delta(t-\tau)$ propagates undistorted.
2. The Heaviside term $H(t-\tau)$ has amplitude $a_1(\tau)$, governed by $q(\tau)$.
3. Higher-order terms (e.g., $(t-\tau)H(t-\tau)$) are corrections for more complex media.

Example Solution

For a **delta-function scattering potential** $q(\tau) = \delta(\tau)$:

- The amplitude $a_1(\tau)$ is:

$$a_1(\tau) = -\frac{1}{2}\int_0^{\tau} \delta(\tau')d\tau' = -\frac{1}{2}H(\tau).$$

- The wavefield becomes:

$$\psi(\tau, t) = \delta(t-\tau) - \frac{1}{2}H(\tau)H(t-\tau) + \cdots.$$

Summary

1. **Ray Expansion**: The wavefield is expressed as a sum of propagating pulses f_n.
2. **Eikonal Equation**: Determines the phase $\phi(\tau) = \pm\tau$.

3. **Transport Equation**: Relates $a_1(\tau)$ to $q(\tau)$.

This method systematically constructs the wavefield solution for arbitrary $q(\tau)$, enabling the solution of the inverse scattering problem via the Marchenko equation.

STEP 3: The Annihilator Wavefield and Marchenko Equation

For an incoming δ-pulse, the fundamental solution ψ_f is:

Assume that the wavefield has the form

$$\psi_f(\tau, t) = \delta(\tau - t) + R(\tau, t), \tag{6}$$

where $R(\tau, t)$ is the scattered wavefield. For $\tau < 0$, this simplifies to:

$$\psi_f(\tau, t) = \delta(\tau - t) + R(t + \tau).$$

The reflection data $R(t)$ is recorded at $\tau_0 = x_0$. In the τ-coordinate, the source is at $\tau = 0$ and recordings are at some $\tau = \tau_0$.

Remember that the goal is to recover $q(\tau)$ from the measured reflection response $R(t)$ (the recorded wave for $\tau < 0$). So, let's construct a special wavefield called the "**annihilator wavefield**" that cancels the scatterer's response, leading to an integral equation whose solution yields $q(\tau)$.

To eliminate the scattered wave, an **annihilator wavefield** $A(\tau, t)$ is introduced. It follows that:

$$\psi_a(\tau, t) = \delta(t - \tau) + A(\tau, t).$$

The annihilator is constructed such that it cancels the scattered wave, leaving only the rightward-propagating pulse. Its support is restricted to $\tau \geq |t|$.

It follows that the non delta component $A(\tau, t)$ is given as:

$$A(\tau, t) = a_1 H(t-\tau) + \sum_{n=2}^{\infty} a_n(\tau) f_n(t-\tau).$$

Observing the wavefield ψ_a at position τ at time $t = \tau$, gives

$$A(\tau, \tau) = a_1,$$

Extension to Higher Dimensions (2-D or 3-D)

As we show above, the transformation to the travel-time domain (τ) was quite straightforward. This transformation enables the formation of a new PDE called the **Schrödinger Equation**. Unfortunately, extending this approach to 2-D or 3-D is **not straightforward** for some reasons:

1. **Non-Uniqueness of Traveltime Paths**: In 1-D, there's only one path between two points, but in higher dimensions, multiple paths exist (e.g., rays bending due to refraction).

2. **Directional Dependence**: The traveltime τ becomes a function of both position and direction, complicating the transformation.

3. **Wavefront Curvature**: In 2-D/3-D, wavefronts curve, making it hard to define a single τ that simplifies the wave equation globally.

Possible Approaches for 2-D Extension

1. **Ray-Based Methods (High-Frequency Approximation)**

   - Use **ray theory** to define $\tau(x, y)$ along specific paths.
   - The **eikonal equation** $|\nabla\tau| = 1/c(x, y)$ governs traveltimes.
   - Works well for smoothly varying media but fails at caustics (where rays cross).

2. **Wave-Equation Redatuming (Data-Driven Approaches)**

   - Instead of a strict traveltime transform, use **wavefield extrapolation** to "redatum" data to a virtual receiver level.
   - The **2-D Marchenko method** [6] uses focusing functions to estimate internal reflections.

3. **Coordinate Transformations (Numerical Approaches)**

   - Define $\tau(x, y)$ numerically via **fast marching methods** or **level-set techniques**.
   - The wave equation can be rewritten in (τ, ξ) coordinates, where ξ is orthogonal

to τ.

- However, the resulting equations are more complex and may not yield a simple Schrödinger-like form.

Challenges in 2-D/3-D

- **No Exact Marchenko Solution**: Unlike 1-D, the integral equation does not decouple neatly.
- **Multi-Pathing**: Reflections and refractions create overlapping wavefields, making inversion harder.
- **Numerical Cost**: Solving the generalized Marchenko equation in 2-D/3-D requires advanced computational methods.

Summary

Dimensional Limitations of the Traveltime Approach

In 1D, the wave can only go left or right, so traveltime $\tau(x)$ is uniquely defined. In higher dimensions, wavefronts spread in many directions. Traveltime to a point depends on **direction**—this is **not uniquely invertible** like in 1D.

In 2D (or 3D), you can't always find a single scalar coordinate like $\tau(x, z)$ that makes all waves propagate with speed 1 and collapses the medium into a Schrödinger-like equation. This is due to:

- Ray bending,
- Multi-pathing,
- Mode conversions,
- Anisotropy (if present).

So, since exact transformation to a traveltime coordinate like in 1D is not possible due to the geometry of wave propagation in higher dimension. Researchers [4, 5, 6] have developed powerful approximate methods that extend the ideas behind the 1D Marchenko method based on the **reciprocity theorems** and **focusing functions** using coupled systems of equations and multidimensional convolutions. These approximate methods don't require the exact traveltime transformation but rely on wavefield reciprocity and data-driven focusing. These methods also require more input data (e.g., multi-offset reflection data).

The idea is to construct Green's functions inside the medium from surface measurements without knowing the internal structure in detail.

Marchenko in 2D: Reciprocity-Based Extension

Coupled Marchenko System (Single-Sided Illumination)

Two equations are solved simultaneously:

$$
\begin{aligned}
f_1^-(x_0, t) + \int_0^t R(x_0, x', t - t') f_1^+(x', t')\, dt' &= 0, \\
f_1^+(x_0, t) + \int_0^t R(x', x_0, t - t') f_1^-(x', t')\, dt' &= \delta(t),
\end{aligned}
$$

where:

- $R(x, x', t)$ is the reflection response at surface location x due to a source at x',
- f_1^+ is the downgoing focusing function,
- f_1^- is the upgoing focusing function,
- x_0 is the subsurface focus point.

These equations resemble a Fredholm system of the second kind, and are solved iteratively.

Physical Interpretation

The focusing function f_1 is designed to "focus" energy at a target depth (subsurface point), analogous to time-reversal.

When injected from the surface, it cancels all multiple reflections from above the focus point and retrieves the **Green's function** from the surface to that point.

From this, we can reconstruct:

- Primary reflections,

- Transmission responses,
- Internal multiples (for imaging or inversion).

Summary Table

Aspect	**1D Marchenko**	**2D/3D Marchenko**
Coordinate transform	Exact via $\tau(x) = \int dx/c(x)$	Not Exact. Approximate via ray-based coordinates
Focusing	Directly at τ (depth/time)	Via space- and time-dependent focusing functions
Data requirement	Single receiver trace	Complete surface reflection response
Solution type	Single integral equation for $A(\tau, t)$	Coupled integral equations for focusing functions
Green's function	Exact from one reflection trace	Approximated from single-sided reflection data
Computational complexity	Low	High due to multidimensional coupling
Internal multiple elimination	Implicit via inversion	Explicit through focusing function construction
Velocity model dependency	Minimal (only smooth background)	Requires more accurate background velocity
Marchenko method	Fully analytical	Numerical, based on integral equations

Table 1: Comparison of 1D and 2D/3D Marchenko methods

References

[1] Angus Lomas and Andrew Curtis. "An introduction to Marchenko methods for imaging". In: *Geophysics* 84.2 (2019), F35–F45.

[2] Vladimir Aleksandrovich Marchenko. "On reconstruction of the potential energy from phases of the scattered waves". In: vol. 104. 5. Society of Exploration Geophysicists, 1955, pp. 695–698.

[3] Erwin Schrödinger. "An undulatory theory of the mechanics of atoms and molecules". In: *Physical review* 28.6 (1926), p. 1049.

[4] Evert Slob. "Green's function retrieval and Marchenko imaging in a dissipative acoustic medium". In: *Physical review letters* 116.16 (2016), p. 164301.

[5] Evert Slob et al. "Seismic reflector imaging using internal multiples with Marchenko-type equations". In: *Geophysics* 79.2 (2014), S63–S76.

[6] Kees Wapenaar et al. "Marchenko imaging". In: *Geophysics* 79.3 (2014), WA39–WA57.