Time Domain Complex-valued ViscoElastic Wave Equation in an Isotropic, Homogenous Medium

June 3, 2025

The idea presented here is based on paper:

Yang, Jidong, and Hejun Zhu. "A Time-Domain Complex Valued Wave Equation for Modeling Visco-Acoustic Wave Propagation." ***Geophysical journal international*** **215.2 (2018): 1064-1079.**

The tensor of elastic moduli for an isotropic medium is given as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}$$

The 3D equations of motion for an isotropic linear-elastic medium can then be written as

$$\begin{aligned} \rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \\ \sigma_{xx} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx} \\ \sigma_{yy} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy} \\ \sigma_{zz} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\ \sigma_{xy} &= 2\mu\epsilon_{xy} \\ \sigma_{yz} &= 2\mu\epsilon_{yz} \\ \sigma_{zx} &= 2\mu\epsilon_{zx} \end{aligned} \tag{1}$$

$$\begin{aligned} \mu &= v_s^2\rho = \frac{v_p^2\rho}{3} \\ \lambda &= v_p^2\rho - 2\mu \end{aligned} \tag{2}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \tag{3}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[\lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})+2\mu\epsilon_{xx}]}{\partial x} + \frac{\partial[2\mu\epsilon_{xy}]}{\partial y} + \frac{\partial[2\mu\epsilon_{xz}]}{\partial z}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[\lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})]}{\partial x} + \frac{\partial[2\mu\epsilon_{xx}]}{\partial x} + \frac{\partial[2\mu\epsilon_{xy}]}{\partial y} + \frac{\partial[2\mu\epsilon_{xz}]}{\partial z}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[(v_p^2\rho-2\mu)(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})]}{\partial x} + \frac{\partial[2\mu\epsilon_{xx}]}{\partial x} + \frac{\partial[2\mu\epsilon_{xy}]}{\partial y} + \frac{\partial[2\mu\epsilon_{xz}]}{\partial z}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[v_p^2\rho\epsilon_{xx}+v_p^2\rho\epsilon_{yy}+v_p^2\rho\epsilon_{zz}-2\mu\epsilon_{xx}-2\mu\epsilon_{yy}-2\mu\epsilon_{zz}]}{\partial x} + \frac{\partial[2\mu\epsilon_{xx}]}{\partial x} + \frac{\partial[2\mu\epsilon_{xy}]}{\partial y} + \frac{\partial[2\mu\epsilon_{xz}]}{\partial z}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[v_p^2\rho(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})]}{\partial x} + \frac{\partial(-2\mu\epsilon_{yy}-2\mu\epsilon_{zz})}{\partial x} + \frac{\partial[2\mu\epsilon_{xy}]}{\partial y} + \frac{\partial[2\mu\epsilon_{xz}]}{\partial z}$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[v_p^2\rho(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})]}{\partial x} + 2\mu\left(-\frac{\partial(\epsilon_{yy}+\epsilon_{zz})}{\partial x} + \frac{\partial\epsilon_{xy}}{\partial y} + \frac{\partial\epsilon_{xz}}{\partial z}\right)$$

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial[v_p^2\rho(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})]}{\partial x} + 2\frac{v_p^2\rho}{3}\left(-\frac{\partial(\epsilon_{yy}+\epsilon_{zz})}{\partial x} + \frac{\partial\epsilon_{xy}}{\partial y} + \frac{\partial\epsilon_{xz}}{\partial z}\right)$$

$$\frac{1}{v_p^2}\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})}{\partial x} + \frac{2}{3}\left(-\frac{\partial(\epsilon_{yy}+\epsilon_{zz})}{\partial x} + \frac{\partial\epsilon_{xy}}{\partial y} + \frac{\partial\epsilon_{xz}}{\partial z}\right)$$

$$\frac{1}{v_p^2}\frac{\partial^2 u_x}{\partial t^2} = \frac{1}{3}\left(\frac{\partial(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})}{\partial x} + 2\left(\frac{\partial\epsilon_{xx}}{\partial x} + \frac{\partial\epsilon_{xy}}{\partial y} + \frac{\partial\epsilon_{xz}}{\partial z}\right)\right)$$

$$\boxed{\frac{1}{v_p^2}\frac{\partial^2 u_x}{\partial t^2} = \left[\frac{\partial}{\partial x}\left(\epsilon_{xx}+\frac{1}{3}\epsilon_{yy}+\frac{1}{3}\epsilon_{zz}\right)\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{xy}}{\partial y}\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{xz}}{\partial z}\right]} \tag{4}$$

$$\boxed{\frac{1}{v_p^2}\frac{\partial^2 u_y}{\partial t^2} = \left[\frac{\partial}{\partial y}\left(\frac{1}{3}\epsilon_{xx}+\epsilon_{yy}+\frac{1}{3}\epsilon_{zz}\right)\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{yx}}{\partial x}\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{yz}}{\partial z}\right]} \tag{5}$$

$$\boxed{\frac{1}{v_p^2}\frac{\partial^2 u_z}{\partial t^2} = \left[\frac{\partial}{\partial z}\left(\frac{1}{3}\epsilon_{xx}+\frac{1}{3}\epsilon_{yy}+\epsilon_{zz}\right)\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{zy}}{\partial y}\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{zx}}{\partial x}\right]} \tag{6}$$

In frequency domain, the elastic equation in the x - direction (equation 4) can be written as:

$$\frac{-\omega^2 u_x}{c(\omega)^2} = \left[\frac{\partial}{\partial x}\left(\epsilon_{xx}+\frac{1}{3}\epsilon_{yy}+\frac{1}{3}\epsilon_{zz}\right)\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{xy}}{\partial y}\right] + \left[\frac{2}{3}\frac{\partial\epsilon_{xz}}{\partial z}\right]$$

$c(\omega)$ is a frequency-dependent phase velocity that can be modeled using the Kolsky-Futterman model [1]:

$$c(\omega) = c(\omega_0)\left[1 + \frac{1}{\pi Q}\ln\frac{\omega}{\omega_0} - \frac{i\,\mathrm{sgn}(\omega)}{2Q}\right]$$

$$\frac{1}{c(\omega)^2} = \frac{1}{c(\omega_0)^2}\left[1 - \frac{2}{\pi Q}\ln\frac{\omega}{\omega_0} + \frac{i\,\mathrm{sgn}(\omega)}{Q}\right] \tag{7}$$

This equation describes how the velocity $c(\omega)$ depends on frequency due to attenuation, $\left[\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]$, and dispersion, $\left[-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}\right]$.

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]\hat{u}_x=\left[\frac{\partial}{\partial x}\left(\epsilon_{xx}+\frac{1}{3}\epsilon_{yy}+\frac{1}{3}\epsilon_{zz}\right)\right]+\left[\frac{2}{3}\frac{\partial\epsilon_{xy}}{\partial y}\right]+\left[\frac{2}{3}\frac{\partial\epsilon_{xz}}{\partial z}\right]$$

$$\begin{aligned}\epsilon_{xx}&=\frac{1}{2}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_x}{\partial x}\right),\epsilon_{yy}=\frac{1}{2}\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_y}{\partial y}\right),\epsilon_{zz}=\frac{1}{2}\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_z}{\partial z}\right)\\ \epsilon_{xy}&=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),\epsilon_{xz}=\frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\end{aligned}$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]\hat{u}_x=\frac{1}{3}\left(2\frac{\partial^2 u_x}{\partial x^2}+2\frac{\partial^2 u_y}{\partial x\partial y}+2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]\hat{u}_x=\frac{1}{3}\left(3\frac{\partial^2 u_x}{\partial x^2}+2\frac{\partial^2 u_y}{\partial x\partial y}+2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

Let U_x represent the analytical function of u_x which is the complex wavefield, encapsulating both the amplitude and phase information. So, function is $U_x = u + iv$

we can define u and v as $-isgn(\omega)v$ and $isgn(\omega)u$, respectively. $sgn(\omega)$ is the signum function.

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]\hat{u}_x=\frac{1}{3}\left(3\frac{\partial^2 u_x}{\partial x^2}+2\frac{\partial^2 u_y}{\partial x\partial y}+2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1-\frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{i\,\mathrm{sgn}(\omega)}{Q}\right]\hat{v}_x=\frac{1}{3}\left(3\frac{\partial^2 v_x}{\partial x^2}+2\frac{\partial^2 v_y}{\partial x\partial y}+2\frac{\partial^2 v_z}{\partial x\partial z}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

this becomes;

$$\frac{-\omega^2}{c(\omega_0)^2}\left[\hat{u}_x-\frac{2\hat{u}_x}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{\hat{v}_x}{Q}\right]=\frac{1}{3}\left(3\frac{\partial^2 u_x}{\partial x^2}+2\frac{\partial^2 u_y}{\partial x\partial y}+2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right) \tag{8}$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[\hat{v}_x-\frac{2\hat{v}_x}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{-\hat{u}_x}{Q}\right]=\frac{1}{3}\left(3\frac{\partial^2 v_x}{\partial x^2}+2\frac{\partial^2 v_y}{\partial x\partial y}+2\frac{\partial^2 v_z}{\partial x\partial z}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right) \tag{9}$$

multiply equation 9 by i, then:

$$\frac{-\omega^2}{c(\omega_0)^2}\left[i\hat{v}_x-\frac{2i\hat{v}_x}{\pi Q}\ln\frac{\omega}{\omega_0}+\frac{-i\hat{u}_x}{Q}\right]=\frac{1}{3}\left(3i\frac{\partial^2 v_x}{\partial x^2}+2i\frac{\partial^2 v_y}{\partial x\partial y}+2i\frac{\partial^2 v_z}{\partial x\partial z}+i\frac{\partial^2 v_x}{\partial y^2}+i\frac{\partial^2 v_x}{\partial z^2}\right) \tag{10}$$

Equation 8 + Equation 10; $U = u + iv$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[\hat{U}_x - \frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}\hat{U}_x + \frac{\hat{v}_x - i\hat{u}_x}{Q}\right] = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[\hat{U}_x - \frac{2}{\pi Q}\ln\frac{\omega}{\omega_0}\hat{U}_x - \frac{i\hat{U}_x}{Q}\right] = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\frac{-\omega^2}{c(\omega_0)^2}\left[1 - \frac{2}{\pi Q}\ln\frac{\omega}{\omega_0} - \frac{i}{Q}\right]\hat{U}_x = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

Use approximation: [2]

$$\omega^2 \ln\frac{\omega}{\omega_0} \approx a\omega^2 + b\omega + d$$

where

$$a = 5.7356, \quad b = -762.1606, \quad d = 4.6054\times 10^4$$

$$-\frac{1}{c(\omega_0)^2}\left[\left(1 - \frac{2a}{\pi Q} - \frac{i}{Q}\right)\omega^2 - \frac{2b}{\pi Q}\omega - \frac{2d}{\pi Q}\right]U(\omega) = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\left(1 - \frac{2a}{\pi Q} - \frac{i}{Q}\right)\ddot{U} - \frac{2bi}{\pi Q}\dot{U} + \frac{2d}{\pi Q}U = \frac{c(\omega_0)^2}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

Discretization: $U(z,x,t) \to U_{ij}^n$, where $z = i\Delta z = ih$, $x = jh$, $t = n\Delta t$

Approximations:

$$\dot{U} \approx \frac{U^{n+1} - U^{n-1}}{2\Delta t}, \quad \ddot{U} \approx \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}$$

let:

$$\ddot{U}_x = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_y}{\partial x\partial y} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\ddot{U}_x = \frac{1}{3}\left(3\frac{\partial^2 U_x}{\partial x^2} + 2\frac{\partial^2 U_z}{\partial x\partial z} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$\left(1 - \frac{2a}{\pi Q} - \frac{i}{Q}\right)\frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} - \frac{2bi}{\pi Q}\frac{U^{n+1} - U^{n-1}}{2\Delta t} + \frac{2d}{\pi Q}U^n = c_0^2\ddot{U}_x^n,$$

$$\left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} - \frac{bi}{\pi Q}\Delta t\right)U_x^{n+1} - \left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} - \frac{d}{\pi Q}\Delta t^2\right)2U_x^n + \left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} + \frac{bi}{\pi Q}\Delta t\right)U_x^{n-1} = c_0^2\Delta t^2\ddot{U}_x^n$$

$$\boxed{\left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} - \frac{bi}{\pi Q}\Delta t\right)U_x^{n+1} = \left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} - \frac{d}{\pi Q}\Delta t^2\right)2U_x^n - \left(1 - \frac{2a}{\pi Q} - \frac{i}{Q} + \frac{bi}{\pi Q}\Delta t\right)U_x^{n-1} + c_0^2\Delta t^2\ddot{U}_x^n}$$

$$\ddot{U}_x(i,k) \approx \frac{1}{3}\Bigg(\frac{3\big(U_{x(i+1,k)} - 2U_{x(i,k)} + U_{x(i-1,k)}\big)}{h_x^2} + \frac{2\big(U_{z(i+1,k+1)} - U_{z(i+1,k-1)} - U_{z(i-1,k+1)} + U_{z(i-1,k-1)}\big)}{4h_xh_z} + \frac{\big(U_{x(i,k+1)} - 2U_{x(i,k)} + U_{x(i,k-1)}\big)}{h_z^2}\Bigg)$$

References

[1] Walter I Futterman. “Dispersive body waves”. In: *Journal of Geophysical research* 67.13 (1962), pp. 5279–5291.

[2] Jidong Yang and Hejun Zhu. “A time-domain complex-valued wave equation for modelling visco-acoustic wave propagation”. In: *Geophysical journal international* 215.2 (2018), pp. 1064–1079.