Diffraction in Homogeneous Elastic Media.

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In an isotropic, homogeneous elastic solid with small deformations, the equation of motion that describes how waves propagate in this medium is given by the Navier-Cauchy equation (or elastodynamic wave equation) [1] as:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{f} \tag{1}$$

Where:

- $\mathbf{u}(\mathbf{x})$ = displacement vector field
- λ, μ = Lamé parameters (elastic constants)
- $\mathbf{F}$ = body force per unit volume (e.g., gravity)
- ρ = mass density of the material
- ∇^2 = Vector Laplacian (applied component-wise).
- $\nabla \cdot \mathbf{u}$ = divergence of displacement (volumetric strain)

You may have also come across a different form of equation 1 in textbook.

Alternative Form

Sometimes, the equation is also written using the **stress tensor $\boldsymbol{\sigma}$**:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$$

with

$$\sigma = \lambda (\nabla \cdot \mathbf{u}) I + 2\mu\varepsilon \quad \text{with} \quad \varepsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

$$\boldsymbol{\sigma} = \lambda (\nabla \cdot \mathbf{u}) \mathbf{I} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right)$$

being the **Cauchy stress tensor** for isotropic linear elasticity.

$$\rho\ddot{\mathbf{\vec{u}}} = \nabla(\lambda\nabla\cdot\mathbf{\vec{u}}) + \nabla\cdot(\mu(\nabla\mathbf{\vec{u}} + (\nabla\mathbf{\vec{u}})^T)) + \vec{f} \tag{2}$$

So, fear not; the two equations are essentially equivalent for isotropic linear elasticity. Equation 1 is a simplified isotropic form and a more compact vector identity-based form, whereas Equation 2 expands the stress-divergence term using tensor notation, explicitly representing how stress relates to displacement gradients. It can also be extended to anisotropic materials. The choice between them will depend on the context (e.g., theoretical derivation vs. numerical implementation).

Derivation of the scattered wavefield due to model perturbation in Homogeneous Elastic Media

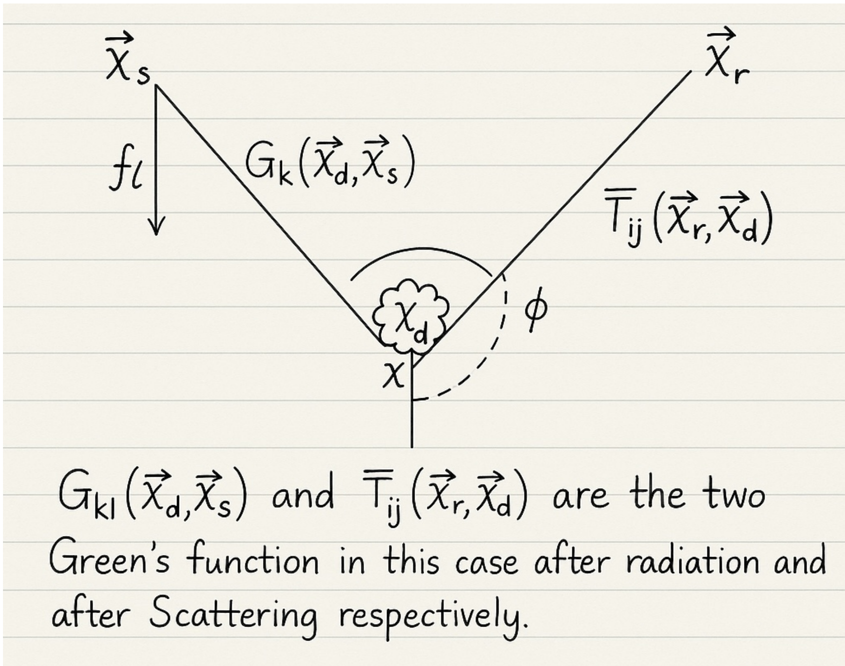

Figure 1: Diffraction Pattern

Where:

- $\vec{X}_s$: Source position
- $\vec{X}_r$: Receiver position
- $\vec{X}_d$: Diffraction/scatter point
- $G_{kl}(\vec{X}_d, \vec{X}_s)$: Green's function from source to scatterer (after radiation)

- $\Gamma_{ij}(\vec{X}_r, \vec{X}_d)$: Green's function from scatterer to receiver (after scattering)
- g_k: Direction cosine from source to scatterer
- γ_i: Direction cosine from scatterer to receiver
- θ: Incidence angle
- ϕ: Scattering angle

—

In the derivation, we shall examine how perturbations in elastic parameters (λ, μ) scatter incident wavefields in a homogeneous elastic medium. The analysis uses Green's functions and the Born approximation to develop expressions for diffraction patterns that are fundamental to elastic full waveform inversion (FWI).

Unperturbed System

For a material in its reference or unperturbed state, the unperturbed elastic wave equation with background parameters $(\rho_0, \lambda_0, \mu_0)$ and displacement field $\vec{U}_0$ can be written as:

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{f}$$

$$\rho_0\frac{\partial^2 \vec{U}_0}{\partial t^2} = \nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \mu_0\nabla^2\vec{U}_0 + \vec{F} \tag{3}$$

$$\rho_0\ddot{\vec{U}}_0 = \nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \vec{F} \tag{4}$$

where:

- ρ_0: background density
- λ_0, μ_0: background Lamé parameters
- $\vec{U}_0$: displacement field

Based on equation 2, we can also write that:

$$\rho_0\frac{\partial^2 \vec{U}_0}{\partial t^2} = \nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla\cdot[\mu_0(\nabla\vec{U}_0 + (\nabla\vec{U}_0)^T)] + \vec{F}$$

Perturbed System

Add small perturbations to the parameters and wavefield. The perturbed parameters are:

$$\rho = \rho_0 + \delta\rho$$
$$\lambda = \lambda_0 + \delta\lambda$$
$$\mu = \mu_0 + \delta\mu$$
$$\vec{U} = \vec{U}_0 + \delta\vec{U}$$

Substituting these perturbations into equation 1, the perturbed wave equation becomes:

$$\begin{aligned}(\rho_0 + \delta\rho)\frac{\partial^2(\vec{U}_0 + \delta\vec{U})}{\partial t^2} &= \nabla\left[(\lambda_0 + \delta\lambda)\nabla\cdot(\vec{U}_0 + \delta\vec{U})\right] \\ &\quad + \nabla\left[(\mu_0 + \delta\mu)\nabla\cdot(\vec{U}_0 + \delta\vec{U})\right] + (\mu_0 + \delta\mu)\nabla^2(\vec{U}_0 + \delta\vec{U}) + \vec{F}\end{aligned} \tag{5}$$

Key Components:

- $\delta\rho$, $\delta\lambda$, $\delta\mu$ are parameter perturbations
- $\delta\vec{U}$ is the scattered wavefield
- The equation now contains both background and perturbation terms
- First-order approximation will lead to the Born scattering formulation

Expand the Perturbed Equation

LHS Expansion of Equation 5

$$\begin{aligned}\text{LHS} &= (\rho_0 + \delta\rho)\partial_t^2(\vec{U}_0 + \delta\vec{U}) \\ &= (\rho_0 + \delta\rho)\frac{\partial^2(\vec{U}_0 + \delta\vec{U})}{\partial t^2} \\ &= \underbrace{\rho_0\partial_t^2\vec{U}_0}_{\text{Order 0}} + \underbrace{\rho_0\partial_t^2\delta\vec{U} + \delta\rho\partial_t^2\vec{U}_0}_{\text{Order 1}} + \underbrace{\delta\rho\partial_t^2\delta\vec{U}}_{\text{Order 2}}\end{aligned} \tag{6}$$

- Order 0: Original equation
- Order 1: Linear perturbations (Born approximation)
- Order 2: Nonlinear term (neglected)

RHS Expansion of Equation 5

$$\begin{aligned}\text{RHS} = &\nabla[(\lambda_0 + \delta\lambda)\nabla \cdot (\vec{U}_0 + \delta\vec{U})] \\ &+ \nabla[(\mu_0 + \delta\mu)\nabla \cdot (\vec{U}_0 + \delta\vec{U})] \\ &+ (\mu_0 + \delta\mu)\nabla^2(\vec{U}_0 + \delta\vec{U})\end{aligned} \tag{7}$$

1. λ Term

$$\nabla(\lambda_0\nabla \cdot \vec{U}_0) + \nabla(\lambda_0\nabla \cdot \delta\vec{U}) + \nabla(\delta\lambda\nabla \cdot \vec{U}_0) + \nabla(\delta\lambda\nabla \cdot \delta\vec{U}) \tag{8}$$

2. μ Divergence Term

$$\nabla(\mu_0\nabla \cdot \vec{U}_0) + \nabla(\mu_0\nabla \cdot \delta\vec{U}) + \nabla(\delta\mu\nabla \cdot \vec{U}_0) + \nabla(\delta\mu\nabla \cdot \delta\vec{U}) \tag{9}$$

3. Shear Term

$$\mu_0\nabla^2\vec{U}_0 + \mu_0\nabla^2\delta\vec{U} + \delta\mu\nabla^2\vec{U}_0 + \delta\mu\nabla^2\delta\vec{U} \tag{10}$$

Combine all terms

Combine equations 6, 8, 9, and 10 and simplify

$$\begin{aligned}&\rho_0\ddot{\vec{U}}_0 + \rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0 \\ &= \nabla(\lambda_0\nabla \cdot \vec{U}_0) + \nabla(\mu_0\nabla \cdot \vec{U}_0) + \mu_0\nabla^2\vec{U}_0 \\ &\quad + \nabla(\lambda_0\nabla \cdot \delta\vec{U}) + \nabla(\mu_0\nabla \cdot \delta\vec{U}) + \mu_0\nabla^2\delta\vec{U} \\ &\quad + \nabla(\delta\lambda\nabla \cdot \vec{U}_0) + \nabla(\delta\mu\nabla \cdot \vec{U}_0) + \delta\mu\nabla^2\vec{U}_0 \\ &\quad + \boxed{\nabla(\delta\lambda\nabla \cdot \delta\vec{U}) + \nabla(\delta\mu\nabla \cdot \delta\vec{U}) + \delta\mu\nabla^2\delta\vec{U} - \delta\rho\delta\ddot{\vec{U}}} + \vec{F}\end{aligned}$$

The expression in the box all represent the second order terms (Higher order)

By the Born approximation, it follows that wavefield perturbations due to the medium heterogeneities are **small**, allowing for a first-order approximation of the scattered wavefield. Born approximation is a linearised scattering theory. So, if we neglect the higher-order terms (second-order terms) in the perturbed wave equation, the linearised perturbed equation becomes:

$$
\begin{aligned}
&\underbrace{\rho_0\ddot{\vec{U}}_0 + \rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0}_{\text{1st order}} + \cancel{\delta\rho\delta\ddot{\vec{U}}}^{\text{2nd order}} \\
&= \underbrace{\nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \mu_0\nabla^2\vec{U}_0}_{\text{Background (order 0)}} \\
&\quad + \underbrace{\nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \nabla(\mu_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U}}_{\text{Wave propagation}} \\
&\quad + \underbrace{\nabla(\delta\lambda\nabla\cdot\vec{U}_0) + \nabla(\delta\mu\nabla\cdot\vec{U}_0) + \delta\mu\nabla^2\vec{U}_0}_{\text{Scattering sources}} \\
&\quad + \cancel{\nabla(\delta\lambda\nabla\cdot\delta\vec{U}) + \nabla(\delta\mu\nabla\cdot\delta\vec{U}) + \delta\mu\nabla^2\delta\vec{U}}^{\text{2nd order}} + \vec{F}
\end{aligned}
$$

- Black terms: Background solution (order 0)
- Blue terms: Wave propagation in background medium
- Red terms: Scattering from parameter perturbations
- Crossed terms: Neglected 2^{nd}-order terms (Born approximation)

So,

$$
\begin{aligned}
&\rho_0\ddot{\vec{U}}_0 + \rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0 \\
&= \nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \mu_0\nabla^2\vec{U}_0 \\
&\quad + \nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \nabla(\mu_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U} \\
&\quad + \nabla(\delta\lambda\nabla\cdot\vec{U}_0) + \nabla(\delta\mu\nabla\cdot\vec{U}_0) + \delta\mu\nabla^2\vec{U}_0 + \vec{F}
\end{aligned}
\tag{11}
$$

Why Neglect Second-Order Terms?

The Born approximation assumes:

1. Perturbations $(\delta\rho, \delta\lambda, \delta\mu)$ are small
2. The scattered field $\delta\vec{U}$ is much weaker than $\vec{U}_0$ $(\|\delta\vec{U}\| \ll \|\vec{U}_0\|)$
3. Products of perturbations (e.g., $\delta\rho\,\delta\vec{U}$) are negligible compared to linear terms

This linearization will enable solving for $\delta\vec{U}$ using Green's functions as we will see below.

Linearized Perturbed Equation

Subtract the background solution from equation 12 and keep 1st-order terms:

equation 11 - equation 3

$$
\rho_0\ddot{\vec{U}}_0 = \nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \mu_0\nabla^2\vec{U}_0 + \vec{F} \tag{12}
$$

$$
\begin{aligned}
&\cancel{\rho_0\ddot{\vec{U}}_0} + \rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0 - \cancel{\rho_0\ddot{\vec{U}}_0} \\
&= \cancel{\nabla(\lambda_0\nabla\cdot\vec{U}_0) + \nabla(\mu_0\nabla\cdot\vec{U}_0) + \mu_0\nabla^2\vec{U}_0} \\
&\quad + \nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \nabla(\mu_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U} \\
&\quad + \nabla(\delta\lambda\nabla\cdot\vec{U}_0) + \nabla(\delta\mu\nabla\cdot\vec{U}_0) + \delta\mu\nabla^2\vec{U}_0 + \cancel{\vec{F}} \\
&\quad \cancel{- \nabla(\lambda_0\nabla\cdot\vec{U}_0) - \nabla(\mu_0\nabla\cdot\vec{U}_0) - \mu_0\nabla^2\vec{U}_0} - \cancel{\vec{F}}
\end{aligned}
\tag{13}
$$

$$
\boxed{
\begin{aligned}
&\rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0 \\
&= \quad \nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \nabla(\mu_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U} \\
&\quad + \nabla(\delta\lambda\nabla\cdot\vec{U}_0) + \nabla(\delta\mu\nabla\cdot\vec{U}_0) + \delta\mu\nabla^2\vec{U}_0
\end{aligned}
}
\tag{14}
$$

$$
\rho_0\delta\ddot{\vec{U}} + \delta\rho\ddot{\vec{U}}_0 = \underbrace{\nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \nabla(\mu_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U}}_{\text{Background propagation}} + \underbrace{\nabla(\delta\lambda\nabla\cdot\vec{U}_0) + \nabla(\delta\mu\nabla\cdot\vec{U}_0)}_{\text{Scattering terms}}
\tag{15}
$$

Key Notes and Interpretation:

- Left side: Wavefield perturbation + density scattering
- Right side:
  - Background wave propagation terms (order 0)
  - Parameter perturbation terms (order 1)
- Forms basis for Born approximation
- Terms like $\nabla(\lambda_0\nabla\cdot\delta\vec{U}) + \mu_0\nabla^2\delta\vec{U} + \mu_0\nabla^2\delta\vec{U}$ describe wave propagation in the background medium.
- $\delta\rho\,\ddot{\vec{U}}_0$: Inertial scattering due to density changes
- $\nabla(\delta\lambda\nabla\cdot\vec{U}_0)$: Scattering from bulk modulus perturbations (P-wave dominant)
- $\nabla(\delta\mu\nabla\cdot\vec{U}_0)$: Scattering from shear modulus perturbations (S-wave dominant)

Green's Function Representation

Remember the wavefield (solution to the wave equation) can be expressed using Green's tensors:

$$U_i(\vec{x},t) = \int_\Omega \Gamma_{ij}(\vec{x},\vec{x}_s,t) f_j(\vec{x}_s,t) dV(\vec{x}_s)$$

In the frequency domain, it can be expressed as:

$$U_i(\vec{x},\omega) = \int_\Omega \Gamma_{ij}(\vec{x},\vec{x}_s,\omega) f_j(\vec{x}_s,\omega) dV(\vec{x}_s)$$

where:

- $\Gamma_{ij}(\vec{x},\vec{x}_s,\omega)$ = Green's function (response at $\vec{x}$ due to a unit harmonic force at $\vec{x}_s$ in the direction$j$) in the frequency domain.
- $F_j(\vec{x}_s,\omega)$ is the source term at $\vec{x}_s$ in the frequency domain.
- $U_i(\vec{x},\omega)$ is the displacement field at $\vec{x}$,
- Ω represents the spatial region (or volume) where the perturbation occurs

Now, We shall try to solve the linearised perturbed elastic wave equation in equation 14

$$\boxed{\begin{aligned} &\rho_0 \delta \ddot{\vec{U}} + \delta\rho \ddot{\vec{U}}_0 \\ &= \quad \nabla(\lambda_0 \nabla \cdot \delta\vec{U}) + \nabla(\mu_0 \nabla \cdot \delta\vec{U}) + \mu_0 \nabla^2 \delta\vec{U} \\ &\quad + \nabla(\delta\lambda \nabla \cdot \vec{U}_0) + \nabla(\delta\mu \nabla \cdot \vec{U}_0) + \delta\mu \nabla^2 \vec{U}_0 \end{aligned}}$$

Rewrite the Linearized Perturbed Wave Equation in Index Notation and take the Fourier Transform in Time Domain

We start with the given linearized perturbed elastic wave equation:

$$\boxed{\begin{aligned} &\rho_0 \delta \ddot{\vec{U}} + \delta\rho \ddot{\vec{U}}_0 \\ &= \quad \nabla(\lambda_0 \nabla \cdot \delta\vec{U}) + \nabla(\mu_0 \nabla \cdot \delta\vec{U}) + \mu_0 \nabla^2 \delta\vec{U} \\ &\quad + \nabla(\delta\lambda \nabla \cdot \vec{U}_0) + \nabla(\delta\mu \nabla \cdot \vec{U}_0) + \delta\mu \nabla^2 \vec{U}_0 \end{aligned}}$$

$$\rho_0 \delta \ddot{\vec{U}} + \delta\rho \ddot{\vec{U}}_0 = \nabla(\lambda_0 \nabla \cdot \delta\vec{U}) + \nabla(\mu_0 \nabla \cdot \delta\vec{U}) + \mu_0 \nabla^2 \delta\vec{U} + \nabla(\delta\lambda \nabla \cdot \vec{U}_0) + \nabla(\delta\mu \nabla \cdot \vec{U}_0) + \delta\mu \nabla^2 \vec{U}_0.$$

1. Convert to Index Notation

We express all vector/tensor terms using Einstein summation convention (sum over repeated indices).

(a) Time Derivatives

- $\rho_0 \delta \ddot{U}_i = \rho_0 \partial_t^2 \delta U_i$,
- $\delta\rho \ddot{U}_{0,i} = \delta\rho \partial_t^2 U_{0,i}$.

(b) Stiffness Terms (Background Parameters λ_0, μ_0)

1. $\nabla(\lambda_0 \nabla \cdot \delta\vec{U}) \to \partial_i(\lambda_0 \partial_j \delta U_j)$,
2. $\nabla(\mu_0 \nabla \cdot \delta\vec{U}) \to \partial_i(\mu_0 \partial_j \delta U_j)$,
3. $\mu_0 \nabla^2 \delta\vec{U} \to \mu_0 \partial_j \partial_j \delta U_i$.

(c) Perturbation Terms ($\delta\lambda, \delta\mu$)

4. $\nabla(\delta\lambda \nabla \cdot \vec{U}_0) \to \partial_i(\delta\lambda \partial_j U_{0,j})$,
5. $\nabla(\delta\mu \nabla \cdot \vec{U}_0) \to \partial_i(\delta\mu \partial_j U_{0,j})$,
6. $\delta\mu \nabla^2 \vec{U}_0 \to \delta\mu \partial_j \partial_j U_{0,i}$.

(d) Combined Index Form

$$\begin{aligned}\rho_0\, \partial_t^2 \delta U_i + \delta\rho\, \partial_t^2 U_{0,i} = \partial_i(\lambda_0\, \partial_j \delta U_j) + \partial_i(\mu_0\, \partial_j \delta U_j) + \mu_0\, \partial_j^2 \delta U_i \\ + \partial_i(\delta\lambda\, \partial_j U_{0,j}) + \partial_i(\delta\mu\, \partial_j U_{0,j}) + \delta\mu\, \partial_j^2 U_{0,i}\end{aligned}$$

2. Simplify the Equation

Group terms involving δU_i (LHS) and perturbation terms (RHS):

Left-hand side:

$$\rho_0 \partial_t^2 \delta U_i - \partial_i \Bigg((\lambda_0 + \mu_0) \partial_j \delta U_j \Bigg) - \mu_0 \partial_j \partial_j \delta U_i.$$

Right-hand side (perturbation-induced sources):

$$-\delta\rho \partial_t^2 U_{0,i} + \partial_i(\delta\lambda \partial_j U_{0,j}) + \partial_i(\delta\mu \partial_j U_{0,j}) + \delta\mu \partial_j^2 U_{0,i}.$$

$$-\delta\rho \partial_t^2 U_{0,i} + \partial_i \left((\delta\lambda + \delta\mu) \partial_j U_{0j} \right) + \delta\mu \partial_j^2 U_{0,i}$$

These act as **sources** for the scattered field δU_i.

Final Simplified Index Form

$$\boxed{\begin{aligned}&\rho_0\partial_t^2\delta U_i - \partial_i\left[(\lambda_0+\mu_0)\partial_j\delta U_j\right] - \mu_0\partial_j^2\delta U_i\\&\quad = -\delta\rho\partial_t^2 U_{0,i} + \partial_i\left(\delta\lambda\partial_j U_{0,j}\right) + \partial_i\left(\delta\mu\partial_j U_{0,j}\right) + \delta\mu\partial_j^2 U_{0,i}.\end{aligned}}$$

3. Fourier Transform (Time Domain → Frequency Domain)

Apply the Fourier transform $\mathcal{F}[f(t)] = \int f(t)e^{-i\omega t}dt$ to both sides.

(a) Time Derivatives Become Multiplicative

- $\partial_t^2 \to -\omega^2$,
- $\partial_j^2 \to \partial_j^2$ (spatial derivatives unchanged).

(b) Transformed Equation

$$\begin{gathered}-\rho_0\omega^2\delta U_i - \partial_i\left[(\lambda_0+\mu_0)\,\partial_j\delta U_j\right] - \mu_0\,\partial_j^2\delta U_i =\\ \delta\rho\,\omega^2 U_{0,i} + \partial_i\left(\delta\lambda\,\partial_j U_{0,j}\right) + \partial_i\left(\delta\mu\,\partial_j U_{0,j}\right) + \delta\mu\,\partial_j^2 U_{0,i}.\end{gathered}$$

(c) Compact Frequency-Domain Form

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[(\lambda_0+\mu_0)\partial_j\delta U_j\right] + \mu_0\partial_j^2\delta U_i\\&+\delta\rho\omega^2 U_{0,i} + \partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_i(\delta\mu\partial_j U_{0,j}) + \delta\mu\partial_j^2 U_{0,i}.\end{aligned}$$

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[\lambda_0\partial_j\delta U_j\right] + \partial_i\left[\mu_0\partial_j\delta U_j\right] + \mu_0\partial_j^2\delta U_i\\&+\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_i(\delta\mu\partial_j U_{0,j}) + \delta\mu\partial_j^2 U_{0,i} + \delta\rho\omega^2 U_{0,i}.\end{aligned}$$

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[\lambda_0\partial_j\delta U_j\right] + \partial_j\left[\mu_0\partial_i\delta U_j\right] + \mu_0\partial_j^2\delta U_i\\&+\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_j(\delta\mu\partial_i U_{0,j}) + \delta\mu\partial_j^2 U_{0,i} + \delta\rho\omega^2 U_{0,i}.\end{aligned}$$

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[\lambda_0\partial_j\delta U_j\right] + \mu_0\partial_j^2\delta U_i + \partial_j\left[\mu_0\partial_i\delta U_j\right]\\&+\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_j(\delta\mu\partial_i U_{0,j}) + \delta\mu\partial_j^2 U_{0,i} + \delta\rho\omega^2 U_{0,i}.\end{aligned}$$

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[\lambda_0\partial_j\delta U_j\right] + \partial_j\big[\mu_0(\partial_j\delta U_i + \partial_i\delta U_j)\big]\\&+\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_j(\delta\mu\partial_i U_{0,j}) + \delta\mu\partial_j^2 U_{0,i} + \delta\rho\omega^2 U_{0,i}.\end{aligned}$$

$$\begin{aligned}-\omega^2\rho_0\delta U_i = {}&\partial_i\left[\lambda_0\partial_j\delta U_j\right] + \partial_j\big[\mu_0(\partial_j\delta U_i + \partial_i\delta U_j)\big]\\&+\partial_i(\delta\lambda\partial_j U_{0,j}) + \delta\mu\partial_j^2 U_{0,i} + \partial_j(\delta\mu\partial_i U_{0,j}) + \delta\rho\omega^2 U_{0,i}.\end{aligned}$$

$$\boxed{\begin{aligned}&-\omega^2\rho_0\delta U_i - \partial_i\left(\lambda_0\partial_j\delta U_j\right) - \partial_j\left[\mu_0\left(\partial_j\delta U_i + \partial_i\delta U_j\right)\right]\\ &\qquad\qquad = \partial_i\left(\delta\lambda\partial_j U_{0,j}\right) + \partial_j\left[\delta\mu\left(\partial_j U_{0,i} + \partial_i U_{0,j}\right)\right] + \delta\rho\omega^2 U_{0,i}\end{aligned}}$$

The equation governs the scattered field δu_i in the frequency domain.

Where:

- ρ_0: Background density.
- λ, μ: Lamé parameters (background + perturbations: $\lambda = \lambda_0 + \delta\lambda$, $\mu = \mu_0 + \delta\mu$).
- $U_{0,i}$: Background wavefield.
- δU_i: Scattered wavefield.

Key Observations

1. **LHS**: Describes wave propagation in the background medium ($-\omega^2\rho_0\delta U_i$ = inertia, remaining terms = elastic restoring forces).
2. **RHS**: Source terms due to perturbations:
   - $\delta\rho\omega^2 U_{0,i}$: Density contrast (monopole-like).
   - $\partial_i(\delta\lambda\partial_j U_{0,j})$: Bulk modulus contrast (dipole-like).
   - $\partial_j\left[\delta\mu\left(\partial_j U_{0,i} + \partial_i U_{0,j}\right)\right]$: Shear modulus contrast (quadrupole-like).

From the frequency-domain equation for the scattered wavefield δU_i, we can see that the **source terms** arise from perturbations in the medium's properties ($\delta\lambda, \delta\mu, \delta\rho$). These terms act as secondary sources that generate the scattered field. See the breakdown below.

Source Terms from Perturbations: Breakdown

The inhomogeneous wave equation for δu_i is:

$$-\omega^2\rho_0\delta U_i = \underbrace{\partial_i(\lambda_0\partial_j\delta U_j) + \partial_j\left[\mu_0(\partial_j\delta U_i + \partial_i\delta U_j)\right]}_{\text{Bacjground propagation}} + \underbrace{\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_j\left[\delta\mu(\partial_j U_{0,i} + \partial_i U_{0,j})\right] + \omega^2\delta\rho U_{0,i}}_{\text{Source terms}}.$$

Move the perturbation terms to the right-hand side (RHS) to treat them as sources for the scattered field:

$$\underbrace{-\omega^2\rho_0\delta U_i - \partial_i(\lambda_0\partial_j\delta U_j) - \partial_j[\mu_0(\partial_j\delta U_i + \partial_i\delta U_j)]}_{\text{Linear wave operator } \mathcal{L}[\delta U_i]} = \underbrace{\partial_i(\delta\lambda\partial_j U_{0,j}) + \partial_j[\delta\mu(\partial_j U_{0,i} + \partial_i U_{0,j})] + \omega^2\delta\rho U_{0,i}}_{\text{Source terms } S_i}$$

$$-\omega^2\rho_0\delta U_i = \partial_i(\lambda_0\partial_j\delta U_j) + \partial_j\left[\mu_0(\partial_j\delta U_i + \partial_i\delta U_j)\right] + \hat{f}_i^{\text{scatt}}(\vec{x},\omega).$$

$$\hat{f}_i^{\text{scatt}}(\vec{x},\omega) = \omega^2\delta\rho(\vec{x})\,U_{0,i}(\vec{x}) + \partial_i\left(\delta\lambda\partial_j U_{0,j}\right)(\vec{x}) + \partial_j\left[\delta\mu(\vec{x})\left(\partial_j U_{0,i} + \partial_i U_{0,j}(\vec{x})\right)\right]$$

This equation is the Navier - Cauchy equation for the displacement perturbation δU_i in a background medium in the frequency domain. In vector notation, it can be written as:

$$-\omega^2\rho_0\,\delta\mathbf{U} = \nabla\left(\lambda_0\nabla\cdot\delta\mathbf{U}\right) + \nabla\cdot\left[\mu_0\left(\nabla\delta\mathbf{U} + (\nabla\delta\mathbf{U})^T\right)\right] + \mathbf{f}$$

The right-hand side (RHS) of the frequency-domain equation acts as a **source** for δU_i. We identify:

1. **Bulk modulus perturbation** $\delta\lambda$ term:

   $$S_\lambda = \partial_i(\delta\lambda\partial_j U_{0,j}) = \nabla(\delta\lambda\,\nabla\cdot\boldsymbol{U}_0).$$

   Physically, this represents *compressional wave scattering* due to changes in λ (P-wave modulus).

2. **Shear modulus perturbation** $\delta\mu$ term:

   $$S_\mu = \partial_j\left[\delta\mu(\partial_j U_{0,i} + \partial_i U_{0,j})\right] = \nabla\cdot\left[\delta\mu(\nabla U_0 + (\nabla\boldsymbol{U_0})^T)\right].$$

   This generates *shear wave scattering* (S-waves) from changes in rigidity.

3. **Density perturbation** $\delta\rho$ term (if included):

   $$S_\rho = -\omega^2\delta\rho\,U_{0,i}.$$

   This accounts for *inertial effects* due to density variations.

Physical Interpretation of Source Terms

Term	Mathematical Form	Physical Meaning
$\delta\lambda$	$\nabla(\delta\lambda\nabla\cdot\boldsymbol{U}_0)$	Scattering from **bulk modulus changes** (e.g., fluid-filled cracks). Dominates P→P scattering.
$\delta\mu$	$\nabla\cdot\left[\delta\mu\left(\nabla\boldsymbol{U}_0+(\nabla\boldsymbol{U}_0)^T\right)\right]$	Scattering from **rigidity changes** (e.g., fractures). Generates P→S and S→S waves.
$\delta\rho$	$\omega^2\delta\rho\boldsymbol{U}_0$	Scattering from **density contrasts** (e.g., lithology boundaries). Affects all wave modes.

The **source terms** for the scattered wavefield are:

$$\boxed{S_{\text{total}}=\nabla(\delta\lambda\nabla\cdot\boldsymbol{U}_0)+\nabla\cdot\left[\delta\mu\left(\nabla\boldsymbol{U}_0+(\nabla\boldsymbol{U}_0)^T\right)\right]-\omega^2\delta\rho\boldsymbol{U}_0.}$$

These terms quantify how perturbations in λ, μ, and ρ generate measurable scattered waves, forming the basis for **elastic FWI** and **diffraction tomography**. The integral solution via Green's functions (as shown below) is the practical tool for modeling these effects.

Integral Solution via Green's Function

The scattered field δU_i can be expressed as an integral over the perturbation volume Ω using the **Green's function** $\Gamma_{ij}(\boldsymbol{x}_r,\boldsymbol{x}')$ (the response at $\boldsymbol{x}_r$ due to a unit force at $\boldsymbol{x}'$ in direction j):

$$\delta U_i(\boldsymbol{x}_r)=\int_\Omega\left[\Gamma_{ij}(\boldsymbol{x}_r,\boldsymbol{x}')\cdot(\text{source terms})\right]dV(\boldsymbol{x}').$$

$$\delta U_i(\boldsymbol{x}_r)=\int_\Omega\Gamma_{ij}(\boldsymbol{x}_r,\boldsymbol{x}')\cdot\left[S_\lambda+S_\mu+S_\rho\right]\,dV(\boldsymbol{x}').$$

The original **Born approximation** of the scattered wavefield $\delta U_i(\boldsymbol{x}_r)$ at the receiver location $\boldsymbol{x}_r$ is:

$$\delta U_i(\boldsymbol{x}_r)=\int_\Omega\Gamma_{ij}(\boldsymbol{x}_r,\boldsymbol{x}')\cdot\Big(\partial_i(\delta\lambda\partial_jU_{0,j})+\partial_j[\delta\mu(\partial_jU_{0,i}+\partial_iU_{0,j})]+\omega^2\delta\rho U_{0,i}\Big)\,dV(\boldsymbol{x}').$$

Where:

- $\Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}')$: Green's function tensor.
- $\delta\lambda, \delta\mu, \delta\rho$: perturbations in Lamé parameters and density.
- $U_{0,i}(\boldsymbol{x}')$: background displacement field.

This integral can be simplified by parts.

Integration by Parts

To simplify, we apply **integration by parts** to shift derivatives from the perturbation terms (the source term) to the Green's function. Why? Because we want the scattering term to appear as a product of Green's function derivatives and background field derivatives — a form that's easier to interpret physically and compute numerically.

First Term: $\partial_i(\delta\lambda\,\partial_j U_{0,j})$

$$\int \Gamma_{ij}\partial_i(\delta\lambda\partial_j U_{0,j})dV = -\int(\partial_i\Gamma_{ij})\delta\lambda\partial_j U_{0,j}dV + \text{(boundary terms, vanish)}$$

Note: Boundary terms are neglected, assuming a sufficiently large domain or absorbing boundaries.

Second Term: $\partial_j[\delta\mu(\partial_j U_{0,i} + \partial_i U_{0,j})]$

$$\int \Gamma_{ij}\partial_j(\delta\mu(\partial_j U_{0,i}+\partial_i U_{0,j}))dV = -\int(\partial_j\Gamma_{ij})\delta\mu(\partial_j U_{0,i}+\partial_i U_{0,j})dV + \text{(boundary terms, vanish)}$$

Third Term: $\omega^2\delta\rho U_{0,i}$

No derivative applied, so the term remains as unchanged:

$$\int \Gamma_{ij}\cdot\omega^2\delta\rho U_{0,i}\,dV = \int \omega^2\Gamma_{ij}\,\delta\rho U_{0,i}\,dV \tag{16}$$

By combining these, we get the final scattered equation:

$$\begin{aligned}\delta U_i(\vec{x}_r) = \int \Big[& -\partial_i\Gamma_{ij}(\vec{x}_r,\vec{x'})\,\delta\lambda(\vec{x'})\,\partial_j U_{0,j}(\vec{x'}) \\ & -\partial_j\Gamma_{ij}(\vec{x}_r,\vec{x'})\,\delta\mu(\vec{x'})\,\Big(\partial_j U_{0,i}(\vec{x'}) + \partial_i U_{0,j}(\vec{x'})\Big) \\ & +\omega^2\Gamma_{ij}(\vec{x}_r,\vec{x'})\,\delta\rho(\vec{x'})\,U_{0,i}(\vec{x'})\Big]\,dV(\vec{x'})\end{aligned}$$

More compact:

$$\delta U_i(\vec{x}_r) = \int \Big[-\partial_i \Gamma_{ij}\, \delta\lambda\, \partial_j U_{0,j} - \partial_j \Gamma_{ij}\, \delta\mu\, (\partial_j U_{0,i} + \partial_i U_{0,j}) + \omega^2 \Gamma_{ij}\, \delta\rho\, U_{0,i} \Big]\, dV$$

$$\boxed{\begin{aligned} \delta U_i(\boldsymbol{x}_r) = \int_\Omega \Big[&- \partial_i \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}')\, \partial_j U_{0,j}(\boldsymbol{x}')\, \delta\lambda(\boldsymbol{x}') \\ &- \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}') \big(\partial_j U_{0,i}(\boldsymbol{x}') + \partial_i U_{0,j}(\boldsymbol{x}') \big)\, \delta\mu(\boldsymbol{x}') \\ &+ \omega^2 \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}')\, U_{0,i}(\boldsymbol{x}')\, \delta\rho(\boldsymbol{x}') \Big]\, dV(\boldsymbol{x}'). \end{aligned}}$$

Key Notes and Interpretations

- The **scattered field** δU_i is a superposition of local scattering effects.
- **Green's function derivatives** transport the scattered wave from the perturbation point to the receiver.
- **Background wavefield derivatives** describe how incident waves interact with perturbations.
- Terms are separated into:
  - Material contrasts: $\delta\lambda, \delta\mu, \delta\rho$
  - Incident field: u_i
  - Green's function: Γ_{ij}

This representation is foundational for linearized inversion techniques such as Full Waveform Inversion (FWI) and Reverse Time Migration (RTM).

Many theoretical derivations in seismology assume that:

- Density perturbations $\delta\rho$ are **negligible** compared to $\delta\lambda$ and $\delta\mu$.
- The primary scattering is driven by **elastic moduli** (λ, μ), while ρ is treated as a secondary effect.

Assuming that $\delta\rho = 0$ (density is fixed), the $\delta\rho$ term may be dropped for simplicity, leaving only $\delta\lambda$ and $\delta\mu$ contributions:

$$\delta u_i(\vec{x}_r) = \int_\Omega [-\partial_i \Gamma_{ij} \partial_k u_k\, \delta\lambda - \partial_k \Gamma_{ij} (\partial_k u_j + \partial_j u_k)\, \delta\mu]\, dV.$$

$$\boxed{\begin{aligned} \delta U_i(\boldsymbol{x}_r) = \int_\Omega \Big[&- \partial_i \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}')\, \partial_j U_{0,j}(\boldsymbol{x}')\, \delta\lambda(\boldsymbol{x}') \\ &- \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}') \big(\partial_j U_{0,i}(\boldsymbol{x}') + \partial_i U_{0,j}(\boldsymbol{x}') \big)\, \delta\mu(\boldsymbol{x}'). \end{aligned}}$$

Simplify for Point Perturbations

For a **point perturbation** at $\boldsymbol{x}_d$:

$$\delta\lambda(\boldsymbol{x}') = \delta\lambda\, \delta(\boldsymbol{x}' - \boldsymbol{x}_d), \quad \delta\mu(\boldsymbol{x}') = \delta\mu\, \delta(\boldsymbol{x}' - \boldsymbol{x}_d), \quad \delta\rho(\boldsymbol{x}') = \delta\rho\, \delta(\boldsymbol{x}' - \boldsymbol{x}_d).$$

The integral collapses to:

$$\begin{aligned} \delta U_i(\boldsymbol{x}_r) = &- \partial_i \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\, \partial_j U_{0,j}(\boldsymbol{x}_d)\, \delta\lambda(\boldsymbol{x}_d) \\ &- \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d) \Big(\partial_j U_{0,i}(\boldsymbol{x}_d) + \partial_i U_{0,j}(\boldsymbol{x}_d) \Big)\, \delta\mu(\boldsymbol{x}_d) \end{aligned}$$

The key components are:

- Green's function derivatives $\partial_i \Gamma_{ij}, \partial_j \Gamma_{ij}$.
- Background field derivatives $\partial_j U_{0,j}, \partial_j U_{0,i} + \partial_i U_{0,j}$.
- Perturbation terms $\delta\lambda, \delta\mu, \delta\rho$.

Diffraction Pattern Components in Diffraction Analysis

Diffraction patterns describe how perturbations in elastic parameters (bulk modulus λ and shear modulus μ) scatter incident wave energy into different directions. These patterns are crucial for **elastic Full Waveform Inversion (FWI)** because they determine how well we can resolve subsurface parameters. In this section, we derive the diffraction patterns for P-P (compressional wave) and P-S (converted shear wave) scattering due to perturbations in elastic parameters ($\delta\lambda$ and $\delta\mu$). These patterns describe how the amplitude of diffracted waves varies with direction.

The perturbation in the displacement field $\delta U_i(\boldsymbol{x}_r)$ due to spatial variations in the Lamé parameters $\delta\lambda(\boldsymbol{x}')$ and $\delta\mu(\boldsymbol{x}')$ is given by:

$$\begin{aligned} \delta U_i(\boldsymbol{x}_r) = \int_\Omega \Big[&- \partial_i \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\, \partial_j U_{0,j}(\boldsymbol{x}_d)\, \delta\lambda(\boldsymbol{x}_d) \\ &- \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d) \Big(\partial_j U_{0,i}(\boldsymbol{x}_d) + \partial_i U_{0,j}(\boldsymbol{x}_d) \Big)\, \delta\mu(\boldsymbol{x}_d) \Big] dV(\boldsymbol{x}_d). \end{aligned} \tag{17}$$

This equation is fundamental in seismic inversion, where the goal is to infer spatial variations in λ and μ from observed perturbations in the wavefield.

Interpretation

- The first term $-\partial_i \Gamma_{ij}\, \partial_j U_{0,j}\, \delta\lambda$ represents the contribution from perturbations in λ, weighted by the divergence of the background field $\partial_j U_{0,j}$ and the gradient of the Green's function.
- The second term $-\partial_j \Gamma_{ij}(\partial_j U_{0,i} + \partial_i U_{0,j})\, \delta\mu$ represents the contribution from perturbations in μ, weighted by the strain components of the background field and the gradient of the Green's function.

Let's try to rewrite equation 17 in another form using the notation in figure 1

$$\delta U_i(\boldsymbol{x}_r) = \int_\Omega \Big[-\partial_i \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\, \partial_j U_{0,j}(\boldsymbol{x}_d, \boldsymbol{x}_s)\, \delta\lambda(\boldsymbol{x}_d) \\ - \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d) \Big(\partial_j U_{0,i}(\boldsymbol{x}_d, \boldsymbol{x}_s) + \partial_i U_{0,j}(\boldsymbol{x}_d, \boldsymbol{x}_s) \Big)\, \delta\mu(\boldsymbol{x}_d) \Big] dV(\boldsymbol{x}_d). \quad (18)$$

From the equation 17 above, $U_{0,j}$ is the background wavefield from the radiating source $\boldsymbol{x}_s$ that is incident at the scatterer $\boldsymbol{x}_d$. The wavefield can be represented in integral form using the convolution theorem as:

$$U_{0,j}(\boldsymbol{x}_d, \boldsymbol{x}_s) = \int G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s) f_l(\boldsymbol{x}_s) dV(\boldsymbol{x}_s)$$

For a point source,

$$U_{0,j}(\boldsymbol{x}_d, \boldsymbol{x}_s) = \sum_l G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s) f_l(\boldsymbol{x}_s)$$

Substitute the expression for the background wavefield into equation 18, change the index, and rewrite the final form of the integral solution for a point source. We have a different form of 17:

$$\delta U_i(\boldsymbol{x}_r) = \sum_{jkl} \Big[-\partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\, \partial_k G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s)\, \delta\lambda(\boldsymbol{x}_d) \\ - \partial_k \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d) \big(\partial_k G_{jl}(\boldsymbol{x}_d, \boldsymbol{x}_s) + \partial_j G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s) \big)\, \delta\mu(\boldsymbol{x}_d) \Big] f_l.$$

1. P-P (Compressional Wave) Scattering ($\delta\lambda$ Perturbation)

The scattered P-wave displacement due to a $\delta\lambda$ perturbation is:

From equation 17 above, we have:

$$\delta U_i^{PP}(\mathbf{x}_r) = -\sum_{jk} \partial_j \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\, \partial_k G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s)\, \delta\lambda(\boldsymbol{x}_d)$$

where:

- Γ_{ij} = Green's function (wave propagation from $\mathbf{x}_d$ (the scatterer) to $\mathbf{x}_r$).
- G_{kl} = Green's function (for the background model) from source to scatterer.
- $\delta\lambda$ = Perturbation in bulk modulus.

Far-Field Approximation

In the **far field**, the Green's function derivative scales as:

$$\partial_j \Gamma_{ij} \sim \gamma_j \gamma_i \gamma_j \frac{e^{ikr}}{r}$$

where:

- $\boldsymbol{\gamma} = (\cos\phi, \sin\phi)$ = Unit vector in **source direction**.
- $\gamma_1 = \cos\phi$ and $\gamma_2 = \sin\phi$
- k = Wavenumber, r = Distance.

The **incident P-wave divergence** is:

$$\partial_k G_{kl} \sim g_k g_k g_l \quad \text{(since P-waves are compressional)}$$

where:

- $\mathbf{g} = (\cos\theta, \sin\theta)$ = Unit vector in **receiver direction**.
- $g_1 = \cos\theta$ and $g_2 = \sin\theta$

Substitution and Simplification

For a **point perturbation at $\mathbf{x} = \mathbf{x}_0$**, $\delta\lambda(\mathbf{x}) = \delta\lambda\delta(\mathbf{x} - \mathbf{x}_0)$, so:

$$\delta U_i^{PP} = -\sum_{jk} \gamma_j \gamma_i \gamma_j g_k g_k g_l \delta\lambda$$

$$\delta U_i^{PP} = -\gamma_i g_l \left(\sum_j \gamma_j^2 \right) \left(\sum_k g_k^2 \right) \delta\lambda$$

Since $\boldsymbol{\gamma}$ and $\boldsymbol{g}$ are unit vectors:

Using $\sum_j \gamma_j^2 = 1$ and $\sum_k g_k^2 = 1$ (unit vectors):

$$\sum_j \gamma_j^2 = \cos^2\phi + \sin^2\phi = 1$$

$$\sum_k g_k^2 = \cos^2\theta + \sin^2\theta = 1$$

Thus, the expression reduces to:

$$PP\delta\lambda = \delta U_i^{PP} = -\gamma_i g_l \delta\lambda$$

Final PP Diffraction Pattern

$$\boxed{\delta U_i^{PP} = -\gamma_i g_l = \begin{cases} -\cos\phi\cos\theta & \text{(Horizontal)} \\ -\sin\phi\sin\theta & \text{(Vertical)} \end{cases}}$$

key Interpretation and Notes

- γ_i = direction cosines of the **scattered P-wave** (angle ϕ),
- g_l = direction cosines of the **incident P-wave** (angle θ).
- **Maximum (peak) scattering** occurs when:
  - $\theta = 0°$ (forward scattering, same direction as incident wave),
  - $\theta = 180°$ (backward scattering, opposite direction).
- **No scattering** at $\theta = 90°$ (sideways), meaning $\delta\lambda$ does not affect waves perpendicular to the source. i.e no sensitivity to $\delta\lambda$ sideways
- **Symmetry**: The pattern is **dipolar** (two lobes, positive and negative).

2. P-P (Converted Shear Wave) Scattering ($\delta\mu$ Perturbation)

The scattered S-wave displacement due to $\delta\mu$ is:

From equation 17, we have

$$\delta U_i^{PP}(\mathbf{x}_r) = -\sum_{jk} \partial_k \Gamma_{ij}(\boldsymbol{x}_r, \boldsymbol{x}_d)\bigg(\partial_k G_{jl}(\boldsymbol{x}_d, \boldsymbol{x}_s) + \partial_j G_{kl}(\boldsymbol{x}_d, \boldsymbol{x}_s)\bigg)\,\delta\mu(\boldsymbol{x}_d)$$

where:

- $(\partial_k G_{jl} + \partial_j G_{kl})$ = Strain tensor (shear deformation).

Far-Field Approximation

The Green's function derivative scales as:

$$\partial_k \Gamma_{ij} \sim \gamma_k \gamma_i \gamma_j \frac{e^{ikr}}{r}$$

The **strain term** for an incident P-wave is:

$$\partial_k G_{jl} + \partial_j G_{kl}) \sim g_k g_j g_l + g_j g_k g_l = 2 g_k g_j g_l$$

Substitution and Simplification

For a **point perturbation** ($\delta\mu(\mathbf{x}) = \delta\mu\delta(\mathbf{x} - \mathbf{x}_0)$):

$$\delta U_i^{PP} = \sum_{jk} -2\gamma_k \gamma_i \gamma_j g_k g_j g_l \delta\mu$$

$$\delta U_i^{PP} = -2\gamma_i \sum_{jk} \gamma_k \gamma_j g_k g_j g_l \delta\mu = -2\gamma_i \sum_{jk} \gamma_k g_k \gamma_j g_j g_l \delta\mu$$

Using trigonometric identities:

$$\sum_{jk} \gamma_k \gamma_j g_k g_j = \cos^2(\phi - \theta)$$

Thus:

$$PP\delta\mu = \delta U_i^{PP} = -2\gamma_i \cos^2(\phi - \theta) g_l \delta\mu$$

Final PP Diffraction Pattern

$$\boxed{\delta U_i^{PP} = -2\gamma_i \cos^2(\phi - \theta) g_l = \begin{cases} -2\cos\phi\cos^2(\phi - \theta)\cos\theta & \text{(Horizontal)} \\ 2\sin\phi\cos^2(\phi - \theta)\cos\theta & \text{(Vertical)} \end{cases}}$$

Key Notes and Interpretations

- $(\phi - \theta)$ = angle between incident and scattered waves.

- **Maximum scattering** occurs at $\theta \approx 45°$ (typical for shear wave conversion).
- **No scattering** when $\theta = \phi$ (forward direction), meaning $\delta\mu$ does not produce P-S conversion directly ahead.
- **Peaks at oblique angles**, showing strong **shear wave generation** when the wave changes direction.
- **Quadrupole-like pattern** (four lobes) See [2].

Summary of Key Steps Above

Step	**PP Scattering** ($\delta\lambda$)	**PP Scattering** ($\delta\mu$)
Governing equation	$\delta U_i = -\partial_j \Gamma_{ij} \partial_k G_{kl}\, \delta\lambda$	$\delta U_i = -\partial_k \Gamma_{ij} (\partial_k G_{jl} + \partial_j G_{kl})\, \delta\mu$
Far-field Green's function	$\partial_j \Gamma_{ij} \sim \gamma_j \gamma_i \gamma_j$	$\partial_k \Gamma_{ij} \sim \gamma_k \gamma_i \gamma_j$
Incident field term	$\partial_k G_{kl} \sim g_k g_k g_l$	$\partial_k G_{jl} + \partial_j G_{kl} \sim 2 g_k g_j g_l$
Simplification	$\delta U_i = -\gamma_i g_i$	$\delta U_i = -2\gamma_i \cos^2(\phi - \theta) g_l$

References

[1] Keiiti Aki and Paul G Richards. *Quantitative seismology*. 2002.

[2] W Zhou and X Liu. "Radiation and Diffraction Pattern Analyses for Elastic FWI Using Geophone and DAS Data". In: *86th EAGE Annual Conference & Exhibition*. Vol. 2025. 1. European Association of Geoscientists & Engineers. 2025, pp. 1–5.