SFU Macm 316
Final Exam: Final Exam, April 15, 2002

Name: ____________________________________

Instructions: Answer all questions. Closed book.
Mark very clearly on the question if you use the back of any page.

Time: 3 Hours.

1	/6
2	/6
3	/7
4	/5
5	/9
6	/8
7	/8
8	/3
9	/3
Total	/55

1. Some questions on iterative methods.

   (a) Is the following matrix positive definite? Explain.

   $$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

   (b) If

   $$b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

   name an iterative method that can be used to solve $Ax = b$?

   (c) Find the first two iterations of the method you proposed in (b) using the zero vector as an initial guess.

2. Some Chapter 1 stuff

   (a) The quadratic formula states that the roots of $ax^2+bx+c=0$ when $a \neq 0$ are

   $$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

   and

   $$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

   Suppose that you wish to calculate the roots of $x^2+100x+1=0$ in a way that minimizes roundoff error. How would you modify the expressions for x_1 and x_2 to accomplish this goal?

   (b) Find the rate of convergence as $h \to 0$:

   $$\lim_{h\to 0} \sin(h) - h\cos h = 0$$

   Show your steps.

3. Some questions on direct methods for solving linear systems follow.

   (a) Factor the following matrix into the LU decomposition using the LU factorization Algorithm with $l_{ii} = 1$ for all i.

$$A1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 3 & 14 & 28 \end{bmatrix}$$

   (b) Consider the following matrix $A2$. Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1s on its diagonal and U is upper triangular.

$$A2 = \begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 3 & 14 & 28 \end{bmatrix}$$

   (c) In general, can Choleski's Algorithm be applied to strictly diagonally dominant matrices? Explain why or why not.

4. Initial Value Problems.

   (a) Show that the following initial value problem is well-posed

   $$y' = t^3 + y, \quad 0 \le t \le 2, \quad y(0) = -1$$

   (b) Approximate $y(2)$ using one of the methods that was covered in this course. Use a step size of $h = 1$.

   (c) What is more significant here, roundoff error or truncation error? Explain.

5. Some questions on numerical differentiation and integration follow.

   (a) Derive an $O(h^2)$ three-point formula to approximate $f''(x_0)$ that uses $f(x_0 - h), f(x_0)$ and $f(x_0 + h)$.

   (b) The formula in part (a) should imply that the smaller h is, the smaller the error is. Is this always true in practice? Justify your answer assuming f is smooth.

   (c) Determine constants a, b, c and d that will produce a quadrature formula

   $$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

   that has degree of precision 3.

   (d) Suppose that $f(0) = 1, f(1/2) = \beta, f(1) = 2$ and $f(1/4) = f(3/4) = \alpha$. Find α, β if the midpoint rule applied applied to the integral $\int_0^1 f(x)dx$ gives 2.5 and the Composite Trapezoid Rule ($n = 4$) gives 1.75.

6. Some questions on solutions of nonlinear equations.

   (a) Does the sequence defined by $p_n = 1/n^2$ converge linearly or quadratically to $p = 0$? Explain.

   (b) Give a geometric interpretation of Newton's method.

   (c) Suppose Newton's method is used to solve the equation

$$0 = x(\cos(x) - 1).$$

      (c1) What is the value of p_1 using an initial guess of $p_0 = 0.1$?

      (c2) Why do we expect Newton's method to give less than quadratic convergence in this example?

      (c3) Give a modified algorithm that would give quadratic convergence.

7. Some questions on interpolation.

   (a) Determine the free cubic spline that interpolates the data $f(0) = 0, f(1) = 1$ and $f(2) = 2$.

   (b) We have seen a variety of interpolation techniques: These include Hermite interpolation, cubic spline interpolation and Bezier curves.
   Suppose that we have obtained an approximation at a single point p. If a new data point is added what is the number of operations required to update the approximation at p for each of the three cases?
   In each case, select one of the following as your answer: $O(1)$, $O(\log(n))$, $O(\sqrt{n})$, $O(n)$, $O(n \log(n))$ or $O(n^2)$ operations. Assume there are n data points and that we choose the optimal algorithm.

8. Every morning between 3am and 7am someone steals your friend's newspaper. Your friend rents a camera that will take a picture of his doorstep at a pre-programmed time. Suppose your friend programs the camera in the evening after looking at the previous photo. Explain how he should program the camera to determine when the newspaper disappears assuming it is stolen at exactly the same time each morning. If the thief is so bold as to spend 20 minutes reading your newspaper before walking off, what is the maximum number of nights your friend will have to wait before photographing the thief? NOTE: Efficiency counts since your friend is renting an expensive camera and he is on a strict budget.

9. Given

$$\begin{aligned} P_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) \\ &+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \end{aligned}$$

show that $a_2 = f[x_0, x_1, x_2]$.