SFU Macm 316

Final Exam: April 17, 2016

Last Name: _____

First Name: _____

Email: _____

ID: _____

Instructions: 3 hours. Answer all 12 questions. Closed book.
Two-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:
$$\lim_{h \to 0} \exp(h) - \sin(h) = 1$$

Show your steps.

2. (3 marks) Suppose A is an $n \times n$ matrix. Use the definition of matrix norm to show that $|| \cdot ||_*$, defined by

$$||A||_* = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

is a matrix norm.

3. (3 marks) Suppose $f(t, y) = \cos(yt) + t\sin(y)$.

(a) Does $f(t, y)$ satisfy a Lipschitz condition on
$D = \{(t, y) \mid 0 \leq t \leq 1, -\infty < y < \infty\}$? Explain.

(b) Is the initial value problem

$$y' = f(t, y), \ 0 \leq t \leq 1, y(0) = 1.$$

well-posed? Explain.

4. (3 marks) A clamped cubic spline S for a function f is defined by

$$S(x) = \begin{cases} S_0(x) & = & 1 + Bx + 2x^2 - 2x^3 & \text{on } [0, 1) \\ S_1(x) & = & 1 + b(x-1) + c(x-1)^2 + 7(x-1)^3 & \text{on } [1, 2] \end{cases}$$

Find B, b, c, $f'(0)$ and $f'(2)$.

5. (3 marks) Suppose a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$.

HINT: Recall the error formula for the Lagrange polynomial of degree n is

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

6. (3 marks) Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_{-1}^{0} f(x)dx = c_0 f(-1) + c_1 f(x_1)$$

has the highest possible degree of precision.

7. (3 marks) Consider the sequence

$$p_n = \exp(-3^n).$$

Clearly the sequence converges to zero as n tends to infinity. Does the sequence converge linearly ($\alpha = 1$), quadratically ($\alpha = 2$), or cubically ($\alpha = 3$)? Use the definition of order of convergence to justify your answer.

8. (3 marks) Derive the Trapezoid Rule with its error term.

9. (3 marks) Suppose that

$$2x_1 + x_2 + 3x_3 = 8$$
$$4x_1 + 6x_2 + 8x_3 = 5$$
$$6x_1 + \alpha x_2 + 10x_3 = 5$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

(a) $\alpha = 6$

(b) $\alpha = 9$

(c) $\alpha = -3$

10. (3 marks) Apply a theorem from class to prove that the sequence defined by
$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1$$
converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

11. (3 marks) Use Newton's method to solve

$$\sin(p/2) = 1$$

to two decimal places using an initial guess of $p_0 = 3$. Do you observe linear or quadratic convergence? Explain why this order of convergence is observed.

12. (3 marks) Using

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{1}{2}f''(x_0)h^2+\frac{1}{6}f'''(x_0)h^3+\frac{1}{24}f^{(4)}(\xi_1)h^4$$

and

$$f(x_0-h) = f(x_0)-f'(x_0)h+\frac{1}{2}f''(x_0)h^2-\frac{1}{6}f'''(x_0)h^3+\frac{1}{24}f^{(4)}(\xi_{-1})h^4$$

where $x_0 - h < \xi_{-1} < x_0 < \xi_1 < x_0 + h$, derive an approximation for $f''(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$ and $f(x_0 + h)$.

As part of your derivation, find an expression for the error of your approximation.