LINEAR ALGEBRA to Digital Image Processing

January 14, 2018

Singular Value Decomposition

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is said to be orthogonal if the columns of A that is the vectors $q_1, q_2, ..., q_n$ are orthonormal vectors. This means they should satisfy the condition $q_i.q_j = \delta_{ij}$

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is said to be orthogonal if the columns of A that is the vectors $q_1, q_2, ..., q_n$ are orthonormal vectors. This means they should satisfy the condition $q_i.q_j = \delta_{ij}$
When the matrix $A_{m\times n}$ is orthogonal, it can be easily proved that $A^T A = I_{n\times n}$. However, $AA^T = I$, if and only if the matrix is square.

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Hence it will have n eigen values and n eigenvectors
- this follows from the fact that a real square symmetric matrix can always be diagonalized.

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Then, $||Av_i||^2 = (Av_i)^T(Av_i) = v_i^T A^T A v_i = v_i^T(A^T A v_i) = v_i^T \lambda_i v_i = \lambda_i ||v_i||^2 = \lambda_i$
Hence, eigenvalues of $A^T A$ are all non-negative.

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Hence, $\sigma_i = \sqrt{\lambda_i} = ||Av_i||, i = 1, 2, ..., n$ and they are the lengths of the vectors Av_i

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This implies that $N(A) \subset N(A^T A)$....(1)

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From (1) and (4), we get $N(A^T A) = N(A)$

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From (1) and (4), we get $N(A^T A) = N(A)$
Hence, dim($N(A^T A)$)=dim($N(A)$)

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Let now, $x \in N(A^TA) \Rightarrow A^TAx = 0$
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From (1) and (4), we get $N(A^T A) = N(A)$
Hence, dim($N(A^T A)$)=dim($N(A)$)
We also know that for any matrix, rank(A)+dim($N(A)$)= number of columns of A.
Since, both $A^T A$ and A have the same number of columns n, this means $r(A^T A) = r(A)$
Proved.

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This implies only r of these vectors $A^T Av_i$ can be linearly independent.

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We arrange the λ_i's such that
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$\lambda_1 \geq \lambda_2 \geq ...\lambda_r > 0$ and $\lambda_i = 0, i = r + 1, ..., n$
Consequently, $\sigma_i > 0, i = 1, 2, ..., r$ and $\sigma_i = 0, i = r + 1, ..., n$

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$$= [u_1 \;\; u_2 \;\; ...u_r] \begin{bmatrix} \sigma_1 & 0 & 0... & 0 \\ 0 & \sigma_2 & 0... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \sigma_r \end{bmatrix}$$

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Check: $A_{m \times n} V_{n \times r} = U_{m \times r} \Sigma_{r \times r}$

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However, V and U are not square!!

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Hence, we now define
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Now, $AV = A[v_1 \;\; v_2 \;\; ...v_r \;\; v_{r+1} \;\; ...v_n]$

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$V = [v_1 \;\; v_2 \;\; ...v_r \;\; v_{r+1} \;\; ...v_n]$
$U = [u_1 \;\; u_2 \;\; ...u_r \;\; u_{r+1} \;\; ...u_m]$
Now both the matrices V and U are square orthogonal matrices of orders $n \times n$ and $m \times m$ respectively
Now, $AV = A[v_1 \;\; v_2 \;\; ...v_r \;\; v_{r+1} \;\; ...v_n]$
$= [Av_1 \;\; Av_2 \;\; ..Av_r \;\; Av_{r+1} \;\; ...Av_n]$

Singular Value Decomposition-construction

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i.e., $A_{m\times n} V_{n\times n} = U_{m\times m} \Sigma_{m\times n}$

The matrix Σ has m rows and n columns, in the first r rows and r columns, the diagonal elements contain the singular values of A. The remaining rows and columns are zeroes.

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The last result is by virtue of V being square orthogonal.

Singular Value Decomposition

Let us expand the result $A = U\Sigma V^T$

$$U = \begin{bmatrix} u_{col_1} & u_{col_2} & \dots & u_{col_n} \end{bmatrix}, \ \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

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Then, $U\Sigma V^T = \sigma_1 u_{col_1} v_{row_1}^T + \sigma_2 u_{col_2} v_{row_2}^T + \dots + \sigma_r u_{col_r} v_{row_r}^T$

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The remaining terms in the summation are zeroes, by virtue of $\sigma_i = 0, i = r+1, \dots, n$

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However, Σ is unique.

SVD to Image Processing

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We shall talk only of Image Compression.

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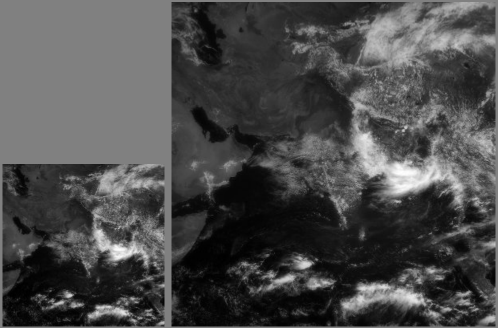


Figure: 1. Images at Resolutions 16Km and 8Km

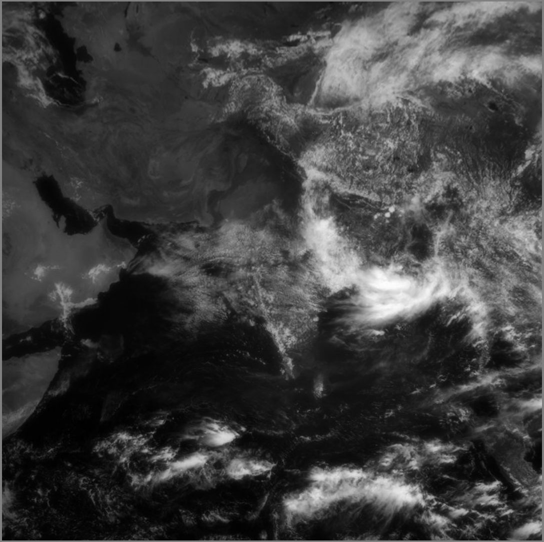
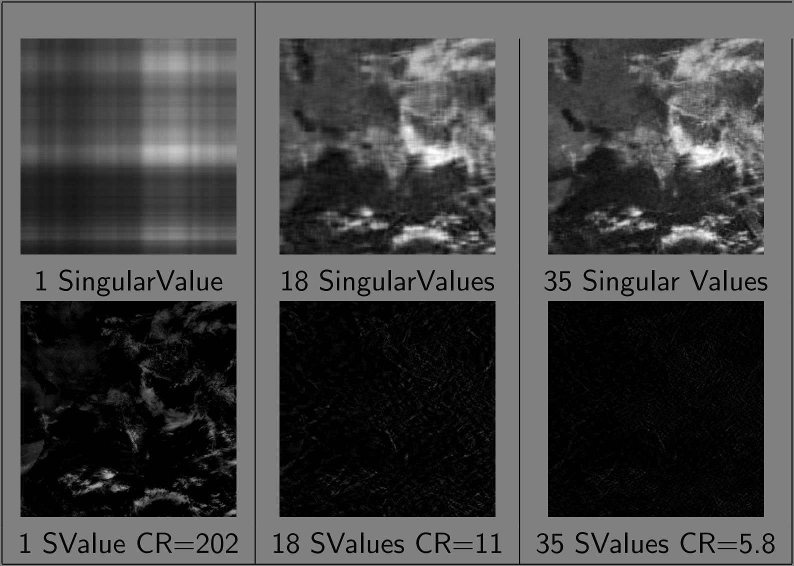


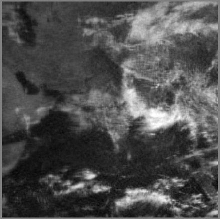
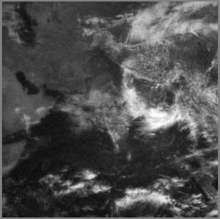
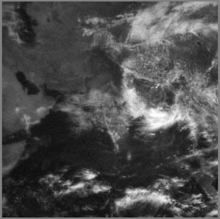
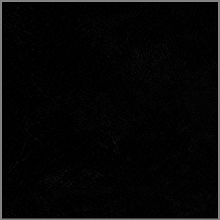
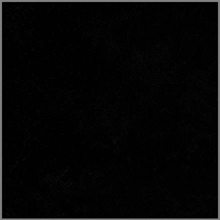
Figure: 2. Image at Resolution 4 Km

Result Satellite Image



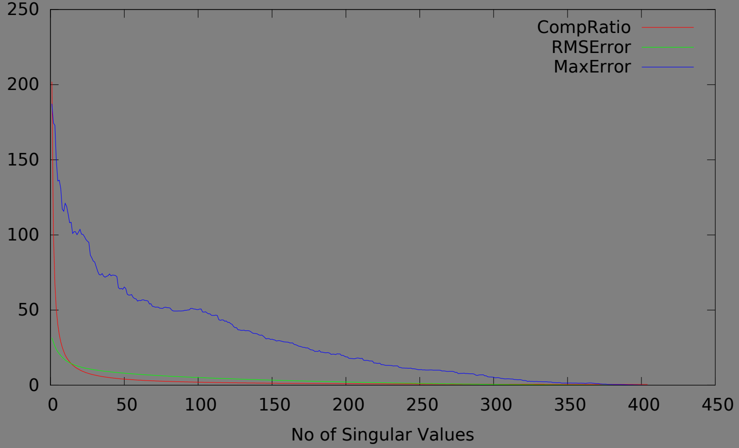
Reconstructed and Error Images

Result Satellite Image

		
52 SValue CR=3.9	69 SValues CR=2.9	86 SValues CR=2.3
		

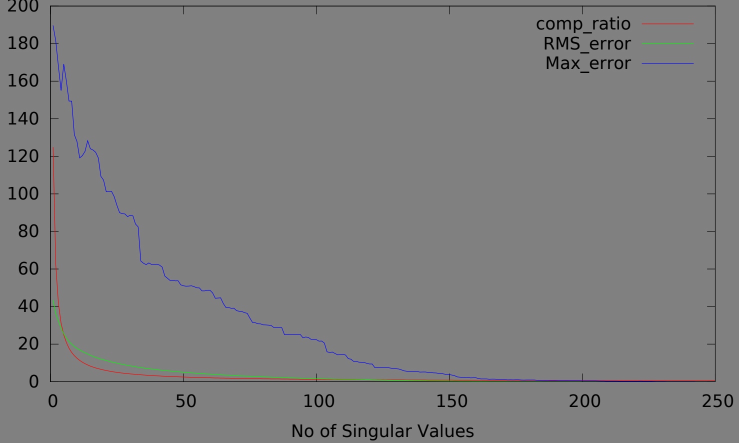
Reconstructed and Error Images

CompressionRatio/RMSError vs No of Singular Values



Results-Flower

Reconstructed and Error Images with Singular Values and Comp. Ratio


comp_ratio
RMS_error
Max_error
0
20
40
60
80
100
120
140
160
180
200
0
50
100
150
200
250
No of Singular Values