Image Restoration

March 15, 2019

Restoration

Principal goal of Image restoration is to improve the quality of an image in some predefined sense. Although the aim is same as in Image enhancement, there is a subtle difference between the two - in image enhancement, subjectivity plays a crucial part, while image restoration is mostly an objective process. Restoration tries to recover an image that has been degraded by using a priori information about the degradation process. If the information is not available a priori, first the degradation model is developed from the image itself and then an inverse process is applied to recover the original image.

Restoration vs Enhancement

We will have to formulate a criterion of goodness to yield an optimal estimate of the image. In contrast, image enhancement is heuristic which involves manipulation of the image to take advantage of the psychophysical aspects of the human visual system. In view of the last three words, enhanced images might not pass muster in computer aided design or robotics. Thus, contrast stretching is an enhancement procedure, whereas, image deblurring is considered a restortion technique.

Approaches for Image restoration

Both spatial domain and frequency domain approaches are used.
Spatial domain technique is applicable when the only degradation is additive noise.
However, image blur is difficult to be tackled in spatial domain. In such cases, frequency domain filters are used.
Model of the Image Degradation/Restoration Process:
Degradation process is modelled as a degradation function together with an additive noise term, which operates on an input image $f(x, y)$ to produce a degraded image $g(x, y)$. Thus given $g(x, y)$ and with some knowledge of the degradation function H and some knowledge of the additive noise $\eta(x, y)$, the objective of restoration is to obtain an estimate $\hat{f}(x, y)$ of the original image. We want this estimate to be as close as possible to the original image. The more we know about H and η, closer will be the estimation.

Degradation/Restoration Models

If H is a linear position invariant process, then the degraded image is given in the spatial domain by

$$g(x, y) = h(x, y) \star f(x, y) + \eta(x, y)$$

In frequency domain, this translates to

$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$

We start our discussions by assuming H to be identity operator - that is the image is degraded only by additive noise.

Noise Models

Assumption: Noise is independent of spatial coordinates and is uncorrelated with respect to the image.
Some Noise models:

- Gaussian Noise: The PDF of a Gaussian random variable is given by

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}}$$

  $\bar{z}$ is the mean value of z and σ the std.

Some Noise PDFs

- Rayleigh Noise:

$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-\frac{(z-a)^2}{b}} & \text{if } z \geqslant a \\ 0 & \text{if } z < a \end{cases}$$

  Mean is given by $\bar{z} = a + \sqrt{\frac{\pi b}{4}}$; variance is $\sigma^2 = \frac{b(4-\pi)}{4}$

- Gamma noise:

$$p(z) = \begin{cases} \frac{a^b z^{b-1}}{(b-1)!}e^{-az} & \text{if } z \geqslant 0 \\ 0 & \text{if } z < 0 \end{cases}$$

  where $a > 0$ and b is a positive integer. The mean and variance are given by $\bar{z} = \frac{b}{a}$ and $\sigma^2 = \frac{b}{a^2}$

- Exponential noise:

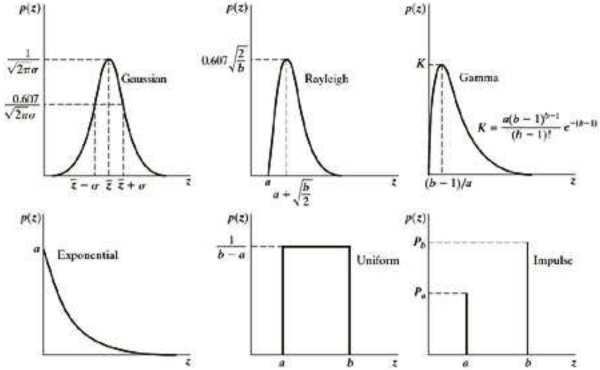
$$p(z) = \begin{cases} ae^{-az} & \text{if } z \geqslant 0 \\ 0 & \text{if } z < 0 \end{cases}$$

  where $a > 0$. The mean and variance are $\bar{z} = \frac{1}{a}$, $\sigma^2 = \frac{1}{a^2}$

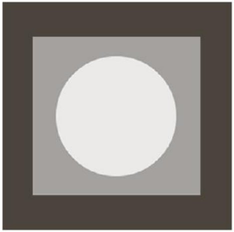
Some Noise PDFs

- Uniform noise: $p(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leqslant z \leqslant b \\ 0 & \text{if otherwise} \end{cases}$. The mean and variance are given by $\bar{z} = \frac{1+b}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$
- Salt and Pepper Noise (Impuse noise):
  $$p(z) = \begin{cases} P_a & \text{if for } z = a \\ P_b & \text{if for } z = b \\ 0 & \text{if otherwise} \end{cases}$$
  If $b > a$, the intensity b will appear as a light dot in the image and a will appear as a dark dot. If neighther of the probabilities is zero and approximately equal, the noise values will resemble granules of salt and pepper randomly distributed over the image.

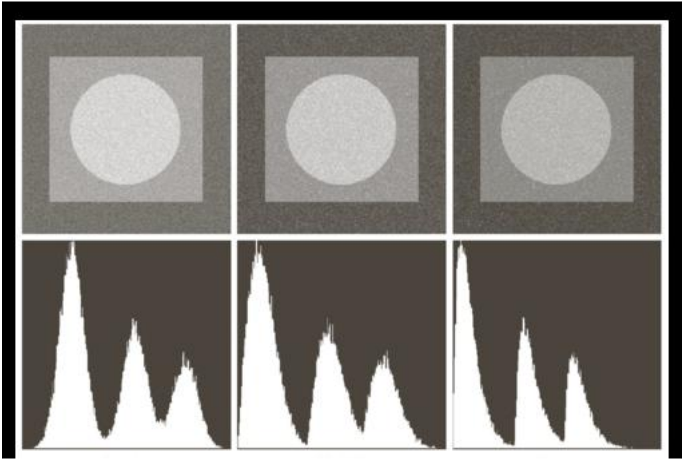
Some Random Noise



Noise on Standard Image

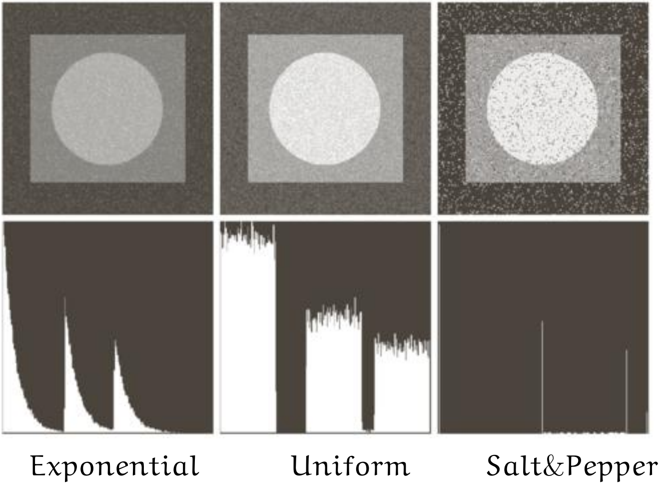


Noise on Standard Image



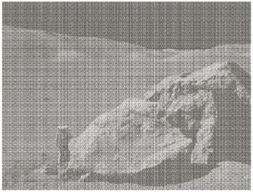
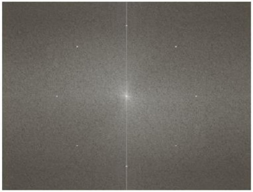
Gaussian Rayleigh Gamma

Noise on Standard Image



Explanation

A standard image with three different intensity values is taken and is corrupted with the noise PDF's mentioned earlier. The resulting images are displayed. If we take the histograms of the corrupted images, you see a clear similarity to the PDFs of the noise function added.
Below is the figure of an image corrupted by sinusoidal noise. The image and the frequency spectrum are displayed here.

Estimation of Noise Parameters

The parameters of periodic noise estimated by inspection of the Fourier spectrum of the image. Periodic noise tends to produce frequency spikes which can be detected by the eye. Of course this is possible only in simple cases.
The parameters of the noise PDF can be estimated if the imaging system is accessible. In that case, a board which is uniformly illuminated is imaged and the Fourier spectrum of the image gives an estimate of the noise.

Estimation of Noise Parameters

If the imaging system is not available, then the parameters have to be estimated from the image itself. Towards this, small patches of reasonably constant intensity is extracted. The histrograms of these patches are generated and visually inspected. Comparing the shapes with the histograms of standard noise PDFs give the shape of the noise. The parameters of the noise is now estimated as

$$\bar{z} = \sum_{i=0}^{L-1} z_i P_p(z_i)$$
$$\sigma^2 = \sum_{i=0}^{L-1} (z_i - \bar{z})^2 P_p(z_i)$$

Model of the Imaging system

Let $f(x, y)$ be the image and $h(x, y)$ a degradation filter. Then the observed image is given by

$$g(x, y) = f(x, y) \star h(x, y) + n(x, y)$$

Here $n(x, y)$ represents the noise.
In the frequency domain, this reduces to

$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$

Restoration in the Presence of Noise Only-Spatial Filtering

In this case, $h(x, y) = 0$. If the noise is periodic, then $N(u, v)$ can be estimated from the image and subtracted and then an inverse Fourier Transform gives the reconstructed image. When only additive random noise is present, spatial filtering is used.

Spatial Filters for removing additive random noise-Mean Filter

- Arithmetic mean filter: Simplest filter. S_{xy} is set of coordinates of a rectangular subimage window of size $m \times n$ entered at (x, y). The restored gray count at (x, y) is given by

  $$\hat{f} = \frac{1}{mn} \sum_{(x,t) \in s_{xy}} g(s, t)$$

  This filter can be implemented by using a spatial filter of size $m \times n$ in which all coefficients are $\frac{1}{mn}$. This filter smooths the local variations and thus noise is reduced, but the image gets blurred.

Mean Filters

- Geometric mean filter: The image is restored by the formula
$$\hat{f}(x,y) = [\Pi_{(s,t)\in S_{xy}} g(s,t)]^{\frac{1}{mn}}$$
This achieves smoothing comparable to arithmetic mean filter, but it tends to lose less image detail.

Mean Filters

- Harmonic Mean filter: Formula is
$$\hat{f}(x, y) = \frac{mn}{\sum_{(x,y) \in S_{xy}} \frac{1}{g(s,t)}}$$
Works well for salt noise, but fails for pepper noise. It does well with other types of noise as well like Gaussian noise.

Mean filters

- Contraharmonic mean filter:
$$\hat{f}(x, y) = \frac{\sum_{(x,y)\in S_{xy}} g(s,t)^{Q+1}}{\sum_{(x,y)\in S_{xy}} g(s,t)^{Q}}$$
Q is called the order of the filter. It can eliminate salt and pepper noise individually. For positive Q, salt noise is removed, while for negative Q pepper noise is removed. But both cannot be removed simultaneously. Observe that for $Q = 1$ it reduces to arithmetic mean filter, while for $Q = -1$, it reduces to harmonic mean filter.

Order statistic filters

Order statistic filters are spatial filtrs whose response is based on ordering the values of the pixels contained in the image area encompassed by the filter - window. The ranking result determines the response of the filter.

- Median filter: $\hat{f}(x, y) = median\{g(s, t), (s, t) \in S_{xy}\}$
- Max filter: $\hat{f}(x, y) = max\{g(s, t), (s, t) \in S_{xy}\}$
  This filter is useful for finding the brightest points in an image- Astronomical studies. It reduces the pepper noise in the image.
- Min filter:$\hat{f}(x, y) = min\{g(s, t), (s, t) \in S_{xy}\}$
  This reduces the salt noise.

Order statistic filters

- Midpoint filter: $\hat{f}(x,y) = \frac{1}{2}[max\{g(s,t),(s,t) \in S_{xy}\} + min\{g(s,t),(s,t) \in S_{xy}\}]$
  This works best for randomly distributed se like Gaussian or uniform noise.
- Alpha-trimmed mean filter. When we omit the least $d/2$ internsities and the highest $d/2$ intensities and find the mean of the remaining pixels, we get Alpha-trimmed mean filter. Here d is a parameter can vary from to $mn-1$. When $d=0$ this reduces to arithmetic mean filter and when $d = mn-1$, it becomes geometric filter.

Order statistic filters use the percentile concept in a frequency distribution. 100^th percentile gives the max filter and 0^th percentile gives the min filter.

Inverse Filtering

The inverse filter is a straightforward image restoration method. If we know the exact point spread function model in the image degradation system and ignore the nose effet, the degraded image can be restored using the inverse filter approach. In practice, the PSF models of the blurred images are usually unknown and the degraded process is also affected by noise. Hence the restoration result with inverse filter is not usually perfect. But the major adavantage of inverse filter based image restoration is that is is simple. From the image degradation model mentioned earlier, we have

$$g(x, y) = f(x, y) \star h(x, y) + n(x, y)$$
$$\text{i.e., } g = Hf + n$$

Inverse filtering

We will ignore $n(x, y)$ and try to estimate f. If $\hat{f}$ is the recovered function, then the error is given by

$$J(\hat{f}) = \|g - H\hat{f}\|^2$$

We wish to minimise J. There is no constraints on $\hat{f}$, and hene it can be termed uncontstrained restoration. The above equation can be written as

$$J(\hat{f}) = \|g - H\hat{f}\|^2 = g^2 + H^2\hat{f}^2 - 2hH\hat{f}$$

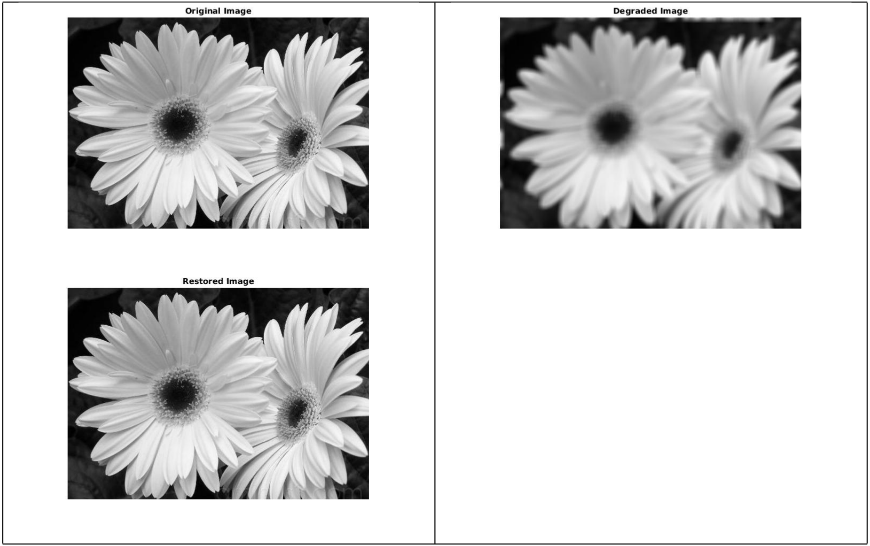
Inverse Filtering

To minimise $J(\hat{f})$, the above equation is differentiated wrt $\hat{f}$, and equated to zero, which gives

$$\hat{f} = H^{-1}g$$

Taking the Fourier Transform of the above equation, we get $\hat{F} = \frac{G}{H}$. The advantage of this method is that we need only the blur point spread function as a priori knowledge. The inverse filter produces perfect reconstruction in the absence of noise.

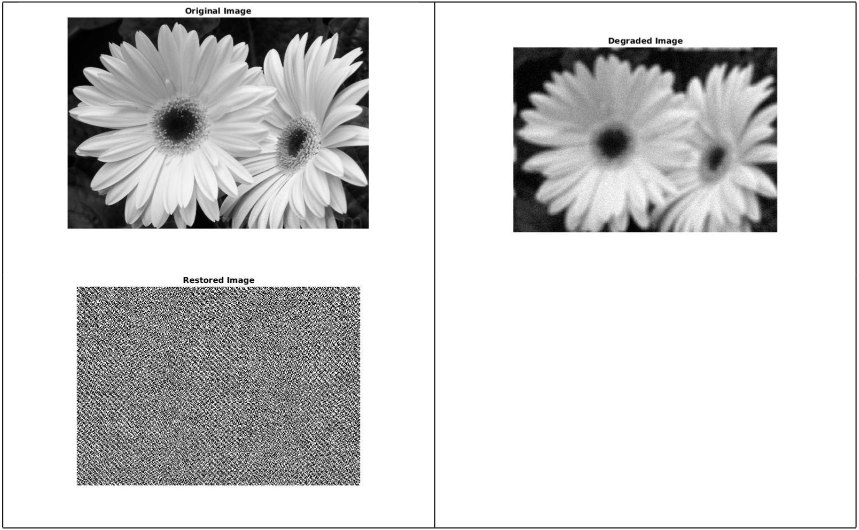
Inverse Filtering



Inverse Filtering

Drawbacks of Inverse filtering: The main drawback is that it is not always possible to obtain an inverse. For an inverse to exist, the matrix should be non-singular. In case it is difficult to obtain inverse filtering, we use a pseudo-inverse filter. The inverse filter will not perform well in the present of noise. If noise is present in the image, the inverse filter tend to amplify the noise.

Inverse Filtering



Pseudo Inverse Filter

Inverse filter equation if $\hat{F}(k,l) = \frac{G(k,l)}{H(k,l)}$. Here $H(k,l)$ represents the PSF. Most of the time, the PSF is a low pass filter, that means $H(k,l) \approx 0$ at high frequencies. The division by $H(k,l)$ will thus lead to large amplification at high frequencies, where the noise dominates over the image. This leads to significant errors in the resoted image as seen above. To avoid these problems, a pseudo-inverse filter is defined as

$$\frac{1}{H} = \begin{cases} \frac{1}{H} & \text{if } H > \epsilon \\ \epsilon & \text{if } H \leqslant \epsilon \end{cases}$$

The result of restoration depends on the value of ϵ

Pseudo Inverse Filtering

Estimation of Degradation Function

If we know the degradation function $H(u, v)$ in frequency domain or in spatial domain, we can use an inverse filter of a pseudo inverse filter to restore the image. In some cases, we will have to take recourse to Wiener Filtering to be discussed later.

Some standard degradation function:

Atmospheric turbulunce, which plays a crucial role in Astronomial imaging has been modelled by the function

$$H(u, v) = e^{-k(u^2+v^2)^{5/6}}$$

Here k is a parameter which determines the level of atmospheric disturbance.

Motion Blur Degradation Function

Suppose the imaging camera has velocity $(x_0(t), y_0(t))$ at the time of imaging and T is the exposure time. Then if $f(x, y)$ is the object, the image will be given by

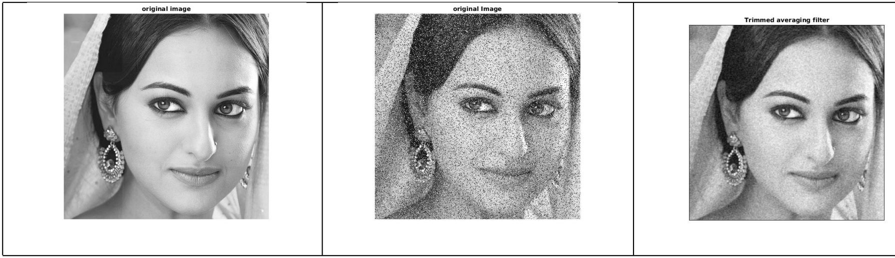
$$g(x, y) = \int_0^T f(x - x_0(t), y - y_0(t))dt$$

Taking the Fourier Transform of this equation yields

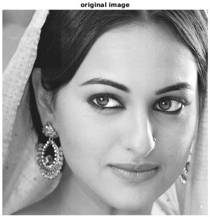
$$G(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-2i\pi(ux+vy)} dxdy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\int_0^T f(x - x_0(t), y - y_0(t))dt] e^{-2i\pi(ux+vy)} dxdy$$
$$= F(u, v) \int_0^T e^{-2i\pi(ux_0(t)+vy_0(t))} dt$$

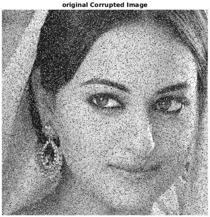
Hence, $H(u, v) = \int_0^T e^{-2i\pi(ux_0(t)+vy_0(t))} dt$
Hence if the motion parameters are known in terms of $(x_0(t), y_0(t))$, the degradation function is easily obtained. Once this is known, the blur can be removed by using inverse filtering or pseudo inverse filtering or in a more general way Wiener Filtering.

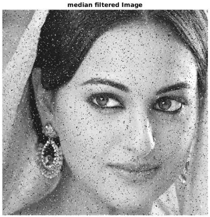
Trimmed Average Filter Performance

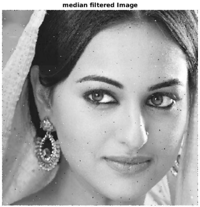


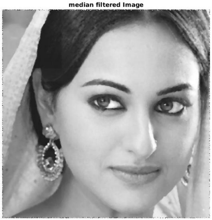
Effect of window size on Median filter





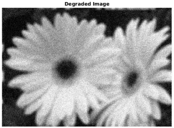
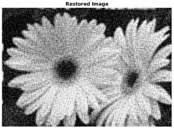
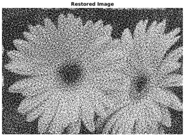






Two By Two	Three By Three	Five By Five

Performance of Pseudo Inverse Filter

		
Degraded Image	PseudoInverse_0.02	PseudoInverse_0.002

Wiener Filter

The inverse filtering approach presented earlier was "ad hoc" in the choise of ϵ. Wiener filtering is more objective and incorporates the characteristics of noise and the degradation function in the restoration process.
Assumption: Image and noise are considered as random variables.
Aim is to find an estimate $\hat{f}(x, y)$ of the uncorrupted image $f(x, y)$ such that the mean square error between the two is minimum.

Wiener Filtering

MSE is defined as $e^2 = E\{(f - \hat{f})^2\}$.
Model is $g(x, y) = h(x, y) \star f(x, y) + n(x, y)$
In Fourier Domain this is $G(u, v) = H(u, v)F(u, v) + N(u, v)$
MSE=$e^2 = E\{(f(x, y) - \hat{f}(x, y))^2\}$
$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y) - \hat{f}(x, y)|^2 dxdy$
$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v) - \hat{F}(u, v)|^2 dudv$ by Parseval's Theorem
Now, $\hat{F}$ is the restored signal in Fourier Domain =
$WG = WHF + WN$
Hence, $F - \hat{F} = (1 - WH)F - WN$

Wiener Filtering

Hence, $e^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(F-\hat{F})^2 du dv$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(1-WH)F - WN\}^2 du dv$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{|(1-WH)F|^2 + |WN|^2\} du dv$
since F and N are uncorrelated by assumption.
We have to find W such that this integral is minimum.
Hence, $\frac{\partial e^2}{\partial W} = 0$
$\Rightarrow (-(1-W^*H^*)H|F|^2 + W^*|N|^2 = 0$
We have used the concept of Writtinger derivative, which is defined as $\frac{\partial}{\partial z} = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$
This gives, $W^* = \frac{H|F|^2}{|H|^2|F|^2 + |N|^2}$
$\Rightarrow W = \frac{H^*|F|^2}{|H|^2|F|^2 + |N|^2} = \frac{H^*}{|H|^2 + |N|^2/|F|^2}$
The filter is derived in the Fourier Domain!!! Observation: The Wiener filter does not have the problem of inverse filtering, with zeroes of the degradation function, unless the entire denominator in the expression is zero for the same values of u and v.
If the noise is zero, Wiener filter reduces to inverse filter.

Wiener Filtering

Hence, the restored image is given by $\hat{F} = WG$

$= \frac{H^*(u,v)}{|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}}$

where, $S_f(u,v) = |F(u,v)|^2$= Power spectrum of undegraded image

$S_\eta(u,v) = |N(u,v)|^2$=Power Spectrum of noise

The quantity

$$\frac{\sum_{u=0}^{M-1}\sum_{n=0}^{N-1}|F(u,v)|^2}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}|N(u,v)|^2}$$

is defined as Signal to Noise ratio SNR

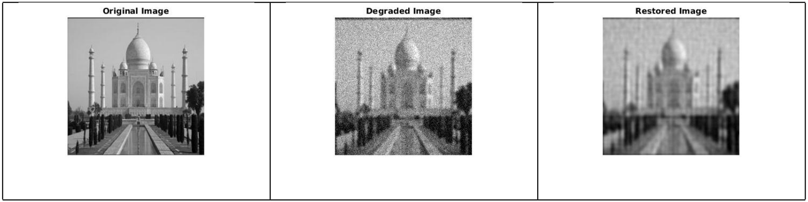
Wiener Filtering

Since the power spectrum of the undegrded image is seldom known, for practical implementation, the restoration is done by using the formula

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2+K}\right] G(u,v)$$

K is a constant and by trial and error only the best value can be found.

Wiener Filter



Adaptive local noise reduction filter - Spatial domain

Mean and variance are the parameters on which we derive an adaptive filter. Mean gives an average intensity while variance gives a measure of the noise.
The filter is to operate on a local region S_{xy}. The response of the filter at any point (x, y) on which the region is centered depends on four parameters - (a) $g(x, y)$ value of the noisy image at (x, y); (b) σ_η^2, the variance of the noise corrupting $f(x, y)$ to form $g(x, y)$;(c) m_L, the ocal mean of the pixels in S_{xy}; and (d)σ_L^2, the local vaiance of the pixels in S_{xy}. We want the behaviour of the filter as follows:

- If $\sigma_\eta^2 = 0$, teh filter should return the value of $g(x, y)$. This is the zero noise case, when $g(x, y) = f(x, y)$.
- If σ_η^2 is high, the filter should return $g(x, y)$ - for a high local variance suggests existence of edges.
- If the two variances are equal, we want the filter to return the arithmetic mean value of the pixels in S_{xy}. This condition means that the local noise has to be reduced

Adaptive local noise reduction filter-spatial domain

With these assumptions, a simple expression for obtaining $\hat{f}(x, y)$ is

$$\hat{f}(x, y) = g(x, y) - \frac{\sigma_\eta^2}{\sigma_L^2}[g(x, y) - m_L]$$

We will estimate σ_η^2 from the entire image. If $\sigma_\eta^2 > \sigma_L^2$, then $\hat{f}(x, y)$ will turn out to be negative. Allowing the negative values and then rescaling at the end is one approach that can be adopted. Another way to avoid the negative intensity values, is to assume that $\sigma_\eta^2 \leqslant \sigma_L^2$, which is in general a reasonable assumption.