Frequency Domain Filtering

March 4, 2019

Special Functions

Dirac Delta Function δ: Dirac Delta function or an Impulse located at $x = 0$ is defined as

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and is also constrained by the condition $\int_{-\infty}^{\infty} \delta(x)dx = 1$. The impulse has the so called sifting property

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

for every continuous function $f(x)$.
Similarly, $\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x - x_0)$

Discrete Form of Delta Function - Impulse

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

This definition satisfies the condition $\sum_{x=-\infty}^{\infty} \delta(x) = 1$.
The sifting property of discrete variables has the form

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Shah function or Impulse Train

Consider an impulse train,$s_{\Delta X}(x)$, defined as the sum of **infinitey many periodic impulses** ΔX **units apart**:

$$s_{\Delta X(x)} = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta X)$$

Fourier Transform of one variable - continuous functions

Definition

The Fourier Transform of a continuous function $f(x)$ of a continuous variable, x is denoted by

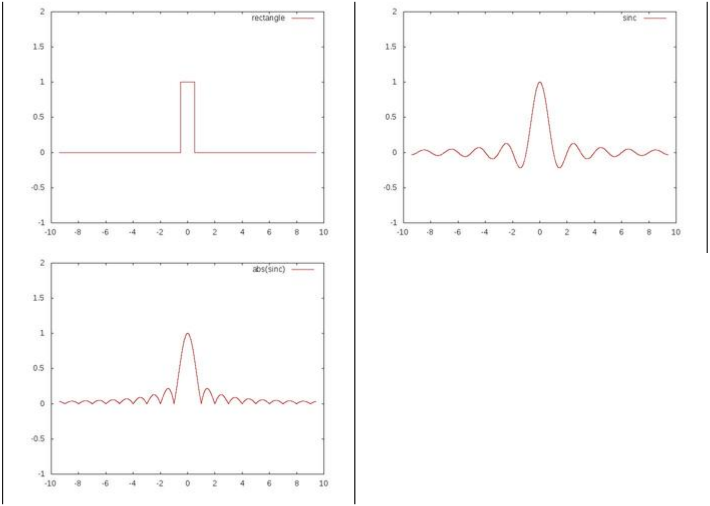
$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-2i\pi\mu x}dx$$

where μ is a u variable.
Hence, $F(\mu) = \int_{-\infty}^{\infty} f(x)e^{-2i\pi\mu x}dx$.
The inverse Fourier transform is given by $f(x) = (F)^{-1}\{F(\mu)\}$, written as

$$f(x) = \int_{-\infty}^{\infty} F(\mu)e^{2i\pi\mu x}d\mu$$

The Fourier Transform is in general complex. It is customary to display only the magnitude of the transform, which is a real quantity, and it is called the Fourier Spectrum or the frequency spectrum.

Frequency spectrum example



Fourier Transform of a unit impulse

Fourier Transform of a unit impulse at the origin=$F(\mu) = \int_{-\infty}^{\infty} \delta(x)e^{-2i\pi\mu x}dx = 1.$
Thus the Fourier Transform of an impulse at origin equals 1.
Similarly the Fourier Transform of an impulse located at $x = x_0$ is $e^{-2i\pi\mu x_0} = \cos(2\pi\mu x_0) - i\sin(2\pi\mu x_0)$
Observe that the only difference in the form of the equations of the Fourier Transform and its inverse is the sign of the exponential. Hence, if $\mathcal{F}(f(x)) = F(\mu)$, then, $\mathcal{F}f(x) = f(-\mu)$. This is the symmetric property of Fourier Transform and its inverse.Thus,

$$F(\mu) = \int_{-\infty}^{\infty} f(x)e^{-2i\pi\mu x}dx.$$
$$\mathcal{F}(f)(t) = f(-\mu)$$

Fourier Transform of standard functions

Using this symmetry property and from the fact that $\mathcal{F}(\delta(x - x_0)) = e^{-2i\pi\mu x_0}$, it follows that $\mathcal{F}(e^{-2i\pi x_0 x}) = \delta(-\mu - x_0)$. Substituting $-x_0 = a$, it follows that $\mathcal{F}(e^{2i\pi a x}) = \delta(-\mu + a) = \delta(\mu - a)$.
The impulse train $s_{\Delta X}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta X)$ is periodic with period ΔX. Hence it has a Fourier Series expansion as

$$s_{\Delta X}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2i\frac{\pi n}{\Delta X}x}, \text{ where}$$
$$c_n = \frac{1}{\Delta X}\int_{-\frac{\Delta X}{2}}^{\frac{\Delta X}{2}} s_{\Delta X}(x) e^{-2i\frac{\pi n}{\Delta X}x}$$

We see that the integral in the interval $[-\frac{\Delta X}{2}, \frac{\Delta X}{2}]$ contains only the impulse of $s_{\Delta X}(x)$ located at the origin, and hence

$$c_n = \frac{1}{\Delta X}\int_{-\frac{\Delta X}{2}}^{\frac{\Delta X}{2}} \delta(x) e^{-\frac{2i\pi n}{\Delta X}x} dx = \frac{1}{\Delta X}e^0 = \frac{1}{\Delta X}$$

Fourier Transform of standard functions

The Fourier series expansion of the impulse becomes

$$s_{\Delta X}(x) = \frac{1}{\Delta X} \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{\Delta X} x}$$

Hence, the Fourier Transform of the impulse equals the FT of the infinite sum on the RHS and since FT is a linear operator, we get

$$\mathcal{F}(s_{\Delta X}(x)) = \frac{1}{\Delta X} \sum_{n=-\infty}^{\infty} \mathcal{F}(e^{\frac{2\pi n}{\Delta X} x}) = \sum_{n=-\infty}^{\infty} \frac{1}{\Delta X} \delta(\mu - \frac{n}{\Delta X})$$

Thus the Fourier Transform an impulse train with period is also an impulse train, whose period is $\frac{1}{\Delta X}$

Convolution

Convolution of two functions $f(x)$ and $h(x)$ is defined as

$$f(x) \star h(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau$$

Minus sign accounts for the flipping of the function $h(x)$ and x is the displacement needed to slide one function past the other. The Fourier Transform of the convolution is given by

$$\begin{aligned} \mathcal{F}\{f(x) \star h(x)\} &= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau] e^{-i2\pi\mu x} dx \\ &= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} h(x-\tau)e^{-i2\pi\mu x} dx] d\tau \end{aligned}$$

The term inside the square bracket is the Fourier Transform of $h(x-\tau) = \int_{-\infty}^{\infty} h(x-\tau)e^{-2i\pi x\mu}dx$
Substitute $x - \tau = y$ and this expression becomes $e^{-2i\pi\tau}\mathcal{F}(h(x)) = e^{-2i\pi\tau}H(\mu)$.

Convolution

Using this result, we obtain

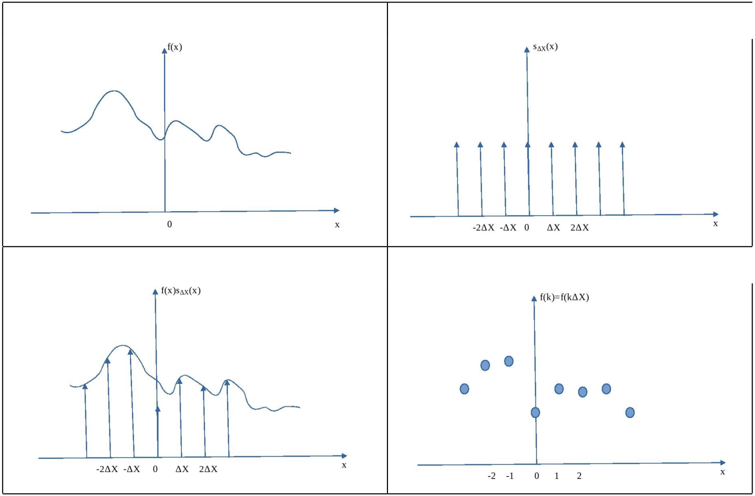
$$\mathcal{F}\{f(x) \star h(x)\} = \int_{-\infty}^{\infty} f(\tau)[H(\mu)e^{-i2\pi\mu\tau}d\tau$$
$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-2i\pi\mu\tau}d\tau = H(\mu)F(\mu)$$

Thus,
Convolution Theorem:Fourier Transform of Convolution of two functions in spatial domain is the product of the respective Fourier Transforms. Conversely, if we have the product of the Fourier Transforms of two functions, we can recover the convolution of the original functions in the spatial domain by finding the inverse Fourier Transform
$f(x) \star h(x) \iff F(\mu)H(\mu)$
This theorem is the foundation of Frequency domain filtering

Sampling and the Fourier Transform of Sampled Functions

Sampling is needed when we want to discrete a continuous signal. In image processing, the process of discretising the spatial coordinate is called Sampling, while the process of discretising the intensity values is called Quantisation.

Sampling and Fourier Transforms



Sampling and Fourier Transforms

Suppose we wish to sample a continuous function $f(x)$ at uniform interals ΔX apart. We assume that the function extends over the entire real line. A simple way to model this sampling is to multiply $f(x)$ by a sampling function equal to a train of impulses ΔX apart. That is,

$$\tilde{f}(x) = f(x)s_{\Delta X}(x) = \textstyle\sum_{n=-\infty}^{\infty} f(x)\delta(x - n\Delta x)$$

Each component of the summation is an impulse weighted by the value of $f(x)$ at the location of the impulse. The value of each sample is then given by the strength of the weighted impulse, which is obtained by integration. Thus,

$$f_k = \textstyle\int_{-\infty}^{\infty} f(x)\delta(x - k\Delta X)dx = f(k\Delta X)$$
$$k = \cdots, -2, -1, 0, 1, 2, \cdots$$

Fourier Transform of Sampled Functions

Let $F(\mu)$ denote the Fourier Transform of the continuous function $f(x)$. The corresponding sampled function $\tilde{f}(x)$ is obtained by multiplying $f(x)$ by the impulse train $s_{\Delta}(x)$. But the Fourier transform of the product of two functions is the convolution of the respective Fourier Transforms (Already proved). Thus, $\tilde{F}(\mu)$, the Fourier Transform of the sampled function is given by

$$\begin{aligned}\tilde{F}(\mu) &= \mathcal{F}\{\tilde{f}(t)\} \\ &= \mathcal{F}\{f(x)s_{\Delta X}(x)\} \\ &= F(\mu) \star S(\mu)\end{aligned}$$

where $S(\mu)$ is the Fourier transform of the impulse train which is given by

$$S(\mu) = \frac{1}{\Delta X} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta X})$$

Fourier Transform of Sampled Functions

Now we obtain the convolution of $F(\mu)$ and $S(\mu)$ directly from the defintion as

$$\tilde{F(\mu)} = F(\mu) \star S(\mu)$$

$$= \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)d\tau$$

$$= \frac{1}{\Delta X} \int_{-\infty}^{\infty} F(\tau) \sum_{-\infty}^{\infty} \delta(\mu - \tau - \frac{n}{\Delta X})d\tau$$

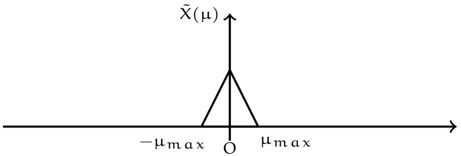
$$= \frac{1}{\Delta X} \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau - \frac{n}{\Delta X})d\tau$$

$$= \frac{1}{\Delta X} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta X})$$

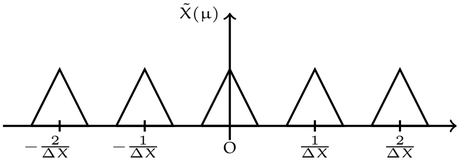
Fourier Transform of sampled functions

The last summation shows that the Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$ is an infinite, periodic sequence of copies of $F(\mu)$, the transform of the original continuous function. The separation between the copies is determined by the value of $\frac{1}{\Delta X}$. Note that although $\tilde{f}(t)$ is a sample function, its transform $\tilde{F}(\mu)$ is continuous because it consists of copies of $F(\mu)$ which is continuous.

xx

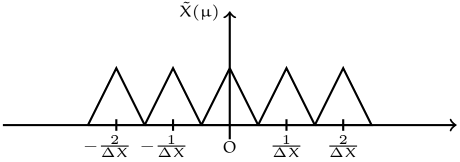


Fourier Transform of Continuous Function

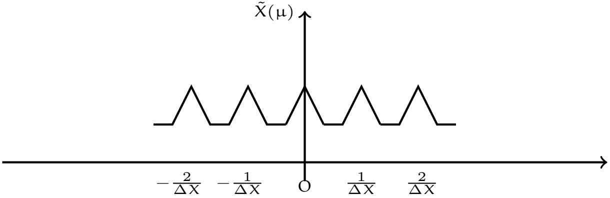


Fourier Transform of Oversampled Function

xy

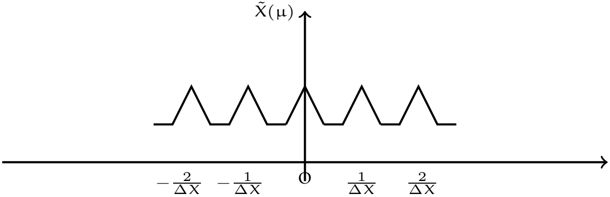


Fourier Transform of critically sampled function



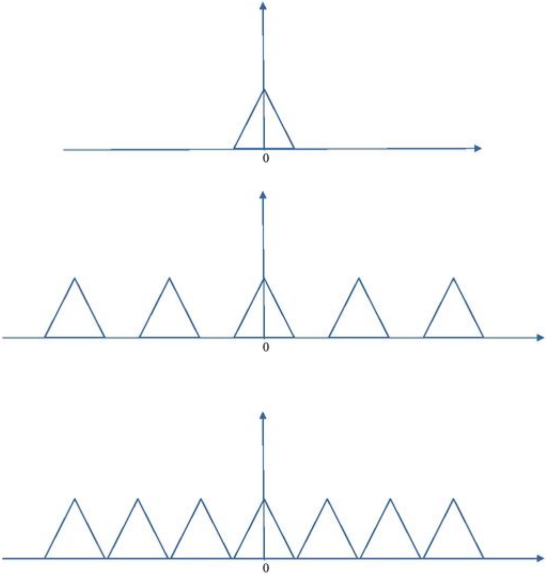
Fourier Transform of under sampled function

XZ



$\tilde{X}(\mu)$
$-\frac{2}{\Delta X}$
$-\frac{1}{\Delta X}$
0
$\frac{1}{\Delta X}$
$\frac{2}{\Delta X}$

Fourier Transform of Sampled Functions



Fourier Transform of Sampled functions

The figures show a summary of the results. The first figure is the FT of a continuous function, while the second is the FT of a the same function sampled. The distance between the copies of the FTs is the reciprocal of the sampling interval. In the second figure, the sampling was sufficient to provide enough gap between the copies. However, in the third figure, the sampling was just critical, enough to preserve $F(\mu)$. The second one is when the sampling is oversampled and the third is when the sampling is done critically.

Fourier Transform of Sampled functions

A function $f(x)$ whose Fourier Transform is zero for values of frequencies outsie a finite interval (band) [-μ_{max}, μ_{max}] about the origin is called a band limited function. The first figure is the Fourier Transform of such a function, and for such a function, the FT of the sampled functions are all copies but separated by inverse of sampling interval. Since there are gaps between the copies, the frequencies are well separate Second Figure.

The third figure is the FT of the sampled function, when the samples are located just at sufficient distance. In this case, the the cut off frequency is $\frac{1}{2\Delta X}$.

We can recover $f(x)$ from its sampled version if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contined in $\tilde{F}(\mu)$, the transform of the sampled function $\tilde{f}(x)$.

Fourier Transform of Sampled Functions

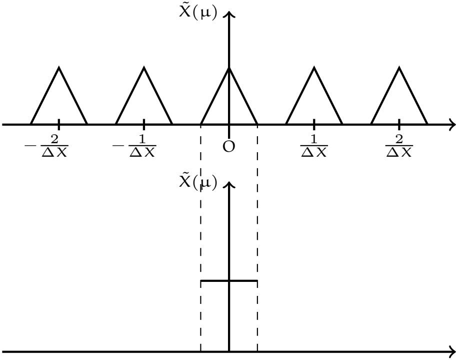
Extracting $\tilde{F}(\mu)$, a single period that is equal to $F(\mu)$ is possible if the separation between the copies is sufficient. This is guaranteed if $\frac{1}{2\Delta X} > \mu_{max}$, or $\frac{1}{\Delta X} > 2\mu_{max}$.
This shows that a continuous, band limited function can be recovered completely from a set of its samples if the samplesa re acquired at rate exceeding twice the highest frequencty content of the function. This result is known as Nyquist Sampling Theorem. That is no information is lost if a continuous, band limited function is represented by samples acquired at a rate greater than twice the highest frequency content of the function.
Conversely, the maximum frequency that can be captured by sampling a signal at the rate $\frac{1}{\Delta X}$ is $\mu_{max} = \frac{1}{2\Delta x}$. This critical sampling rate is called Nyquist rate.

Recovery of sampled functions

If it sampled exactly at the Nyquist rate, then also the function can be recovered. However, if the sampling is less than the Nyquist rate, would cause the period in $\tilde{F}(\mu)$ to merge. This is called aliasing. Lower frequencies from one copy will act as if it is higher frequency of the previou copy.
Thus, extracting $\tilde{F}(\mu)$ a single period that equals $F(\mu)$ is possible if the separation between copies is sufficient. Under such condition, how to recover the original signal?

Recovery of sampled signal

Consider the following diagram, which shows the Fourier transform of a function sampled at a rate higher than the Nyquist rate.

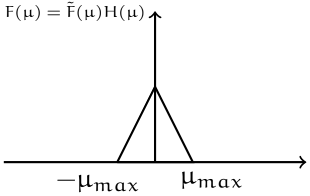


Recovery of Sampled Signal

The function

$$H(\mu) = \begin{cases} \Delta X & \text{if } -\mu_{max} \leqslant \mu_{max} \\ 0 & \text{if otherwise} \end{cases}$$

is depicted below in the above figure.
When this function multiplies the periodic sequence of copies of Fourier Transforms of $f(x)$, it isolates the period centred at origin.
Thus as shown in the figure below, we get $F(\mu) = H(\mu)\tilde{F}(\mu)$



Recovery of Sampled signals

Once we get $F(\mu)$, we can recover $f(x)$ by inverse Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(\mu) e^{2i\pi\mu x} d\mu$$

Thus, theoretically it is possible to recover a band-limited function from samples of the function obtained at a rate exceeding twice the highest frequency content of the function.
However, if a function is to be band limited, then it has to extend from $-\infty$ to ∞.
For, $\tilde{f}(s) = \Pi(s)\tilde{f}(s)$
Hence, by inverse Fourier Transform, $f(x) = sinc(x) * f(x)$
Since sinc is a function which extends to whole real line, this convolution will also be unlimited in both directions. Hence the proof.

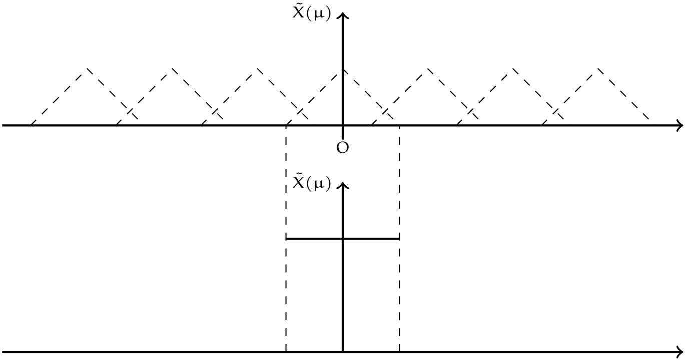
Recovery of signals from Samples

The function H_μ defined earlier is called a *low pass filter*, because it allows frequences at the lower end of the spectrum, but eliminates all higher frequencies. These low pass filters will be used for different image processing operations.

Aliasing: What happens if a band limited function is sampled at a rate less than twice the highest frequency? Check the figure given earlier. The effect of this sampling is that the frequencies from adjacent periods overlap, and hence it is impossible to isolate one period. Hence, inverse transform will not deliver the pure orginal signal, but a corrupted one. This effect is called frequency aliasing.
Aliasing is a process in which high frequeny components of a continuous function "masquerade" as ower frequencies in the sampled function.

Recovery of signals from its samples

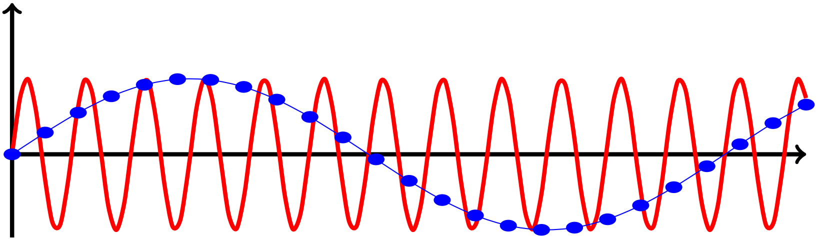
Aliasing is always present, for even if the original signal is band limited, the moment we limit the extent of the function by a function of the form of a rectangle function, will introduce infinite frequencies, as proved earlier.



Recovering signal from sampled data

To overcome this problem of aliaing, whenver we work with digital data, we shall smooth the input function to attenuate its higher frequencies - this amounts to defocussing the image. This process is called anti-aliasing, has to be dne before the function is sampled, because aliasing is a sampling issue that cannot be removed later.

Undersampling Reconstruction-Aliasing



Red curve represents $sin(12x)$. Blue dots represent the sample points. Blue curve represents the reconstructed curve. So much different from the original!

Function recovery from sampled data

Suppose we have the function as a set of its samples. In image processing even displaying images means reconstructing the signal from its sampled data. Image reconstruction is a fundamental problem of Image Processing.
Suppose the sampling is done at better than twice the highest frequency, then, we have seen that

$$\begin{aligned} f(x) &= \mathcal{F}^{-1}\{F(\mu)\} \\ &= \mathcal{F}^{-1}\{H(\mu)\tilde{F}(\mu)\} \\ &= h(x) \star \tilde{f}(x) \end{aligned}$$

But we have seen that $\tilde{f}(x) = \sum_{n=-\infty}^{\infty} f(x)\delta(x - n\Delta X)$
Substituting this in the above equation, we get

$$f(x) = h(x) \star \sum_{n=-\infty}^{\infty} f(x)\delta(x - n\Delta X)$$

$$=\sum_{n=-\infty}^{\infty} f(n\Delta X)\mathrm{sinc}[(x - n\Delta X)/n\Delta X]$$

The Discrete Fourier Transform DFT

We have seen that the Fourier Transform of a sampled, band limited function extending from $-\infty$ to ∞ is a continuous periodic function, that also extends from $-\infty$ to ∞. However, we always deal with finite number of samples in Image Processing.
Fourier Transform of a sampled function $\tilde{f}$ is given by

$$\tilde{F} = \int_{-\infty}^{\infty} \tilde{f}(x) e^{-2i\pi\mu x} dx$$

Writing the sampled function in terms of impulse train, this reduces to

$$\begin{aligned}\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(x)\delta(x - n\Delta X) e^{-2i\pi\mu x} dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\delta(x - n\Delta X) dx \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-2i\pi\mu n\Delta X}\end{aligned}$$

DFT

Thus, though f_n is a discrete function, its Fourier Transform $\tilde{F}(\mu)$ is continuous and infinitely period, with period $1/\Delta X$. Thus to characterize $\tilde{F}(\mu)$, we need only one period of $\tilde{F}(\mu)$, and this is the basis of DFT.
Suppose we want to obtain M equally spaced samples of $\tilde{F}(\mu)$ taken over a period $\mu = 0$ to $\mu = 1/\Delta X$. We do this by taking samples at the following frequencies

$$\mu = \tfrac{m}{M\Delta X}, m = 0, 1, 2, \ldots, M-1$$

Thus we get the DFT of $f(x)$ as

$$F_m = \textstyle\sum_{n=0}^{M-1} f_n e^{-2i\pi mn/M}, \; m = 0, 1, 2, \ldots, M-1$$

IDFT is given by

$$f_n = \tfrac{1}{M} \textstyle\sum_{m=0}^{M-1} F_m e^{2i\pi mn/M}, \; n = 0, 1, 2, \ldots, M-1$$

DFT

Both forward and inverse discrete transforms are infinitely periodic with period M - $F(u) = F(u + kM), f(x) = f(x + kM)$, k is an integer.
Discrete equivalent of convolution is

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m), \text{ for } m = 0, 1, 2, \ldots, M-1$$

The convolution is also periodic.
Relation between sampling and frequency intervals: $f(x)$ consists of M samples of a function separated by ΔX, then the total distance covered by the function is $X = M\Delta X$. The total frequency content of the DFT is $\mu = \frac{1}{\Delta X}$. Hence the separation in frequency of the DFT is $\mu/M = \frac{1}{M\Delta X} = \frac{1}{X}$.

Extension to Function of Two variables

Two dimensional impulse $\delta(t, z)$ is defined as

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{if otherwise} \end{cases}$$

and

$$\int_{\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dt dz = 1$$

The 2D impulse exhibits sifting property under integration, that is,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t, z) dt dz = f(0, 0)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t - t_0, z - z_0) dt dz = f(t_0, z_0)$$

2D Discrete variables

The 2D discrete impulse is defined as

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{if otherwise} \end{cases}$$

The sifting property is

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)$$

2D Continuous Fourer Transform Pair

$f(t, z)$ is a continuous function of two variables t, z. The 2D Fourier Transform pair is given by

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-2i\pi(\mu t + \nu z)} dt dz$$

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{-2i\pi(\mu t + \nu z)} d\mu d\nu$$

Here μ, ν are the frequency variables.

2D Sampling and the 2D sampling theorem

The sampling in 2D can be modelled using the 2D sampling function (impulse train) as

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

where ΔT and ΔZ are the separations between samples along the t and z axes of the continuous function $f(t, z)$.
Function $f(t, z)$ is said to be band-limited if its Fourier Transform is zero outside a rectangle established by the intervals $[-\mu_{max}, \mu_{max}]$ and $[-\nu_{max}, \nu_{max}]$, that is if

$$f(\mu, \nu) = 0, \text{ for } |\mu| \geqslant \mu_{max}, \text{ and } |\nu| \geqslant \nu_{max}$$

2D Sampling theorem

A continuous band limited function $f(t, z)$ can be recovered with no error from a set of its samples if the samplng intervals are

$$\Delta T < \frac{1}{2\mu_{max}} \text{ and } \Delta Z < \frac{1}{2\nu_{max}}$$

Aliasing in 2D follows the same pattern as in 1D. Check the images generated for this purpose. How to avoid?

Aliasing explained through images

The left image is obtained by skipping every alternate pixel/scan in the original image. Thus it is a compressed image. The right is the same but after an average filter has been applied. Notice the ringing effect in the left, which is removed in the right. Explanation - high frequencies are cut and hence frequency merging does not happen.

2D Discrete Fourier Transform

Discrete Fourier transform of $f(x, y)$ is defined as

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-2i\pi(ux/M + vy/N)}$$

where $f(x, y)$ is the digital image of size $M \times N$. The equation must be evaluated for values of the discrete variables u and v in the range $u = 0, 1, \ldots, M-1$ and $v = 0, 1, \ldots, N-1$. $f(x, y)$ is obtained from $F(u, v)$ by inverse digital transform

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{2i\pi(ux/M + vy/N)}$$

where $x = 0, 1, \ldots, M-1$ and $y = 0, 1, \ldots, N-1$.

Properties of 2D DFT

- Relationship between Spatial and Frequency Intervals
  Similar to 1D case, these are related as
  $\Delta u = \frac{1}{M\Delta X}, \Delta v = \frac{1}{N\Delta Z}$
- Translation and Rotation:
  $f(x,y)e^{2i\pi(u_0x/M+v_0y/N)} \iff F(u-u_0, v-v_0)$ and
  $f(x-x_0, y-y_0) \iff F(u,v)e^{-2i\pi(x_0u/M+y_0v/N)}$.
  Multiplying by exponential shift the origin of the DFT to $u_0, v_0)$, and multiplying $F(uv)$ by the negative of that exponential shifts the origin of $f(x,y)$ to (x_0, y_0). The translation does not have any effect on the magnitude of the $F(u,v)$.
  $f(r, \theta+\theta_0) \iff F(\omega, \phi+\theta_0)$. Thus rotating by an angle θ_0 in the $x-y$ plane, rotates the Fourier Transform by the same angle.

Properties of 2D DFT

- Periodicity:
  $F(u, v) = F(u + k_1 M, v + k_2 N,$
  $k_1, k_2; f(x, y) = f(x + k_1 M, y + k_2 M), k_1, k_2$ are integers.

Periodicity of the transform and its inverse are very important in the implementation of DFT based algorithms.
Consider the 1D spectrum in the figure. The transform data in the interval from 0 to $M - 1$ consists of two back to back half periods meeting at the point $M/2$. For display and filtering purposes, it is more convenient to have in this interval a complete period of the transform in which the data are contiguous as in the second figure. From the properties of FT, we know that

$$f(x)e^{2i\pi(u_0 x/M)} \iff F(u - M/2)$$

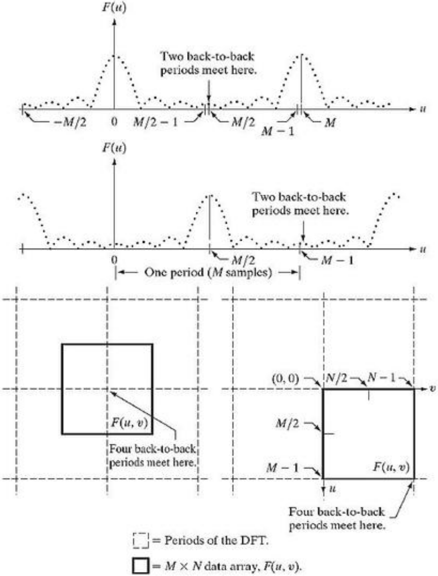
If we take $u_0 = M/2$, then this equation reduces to

$$f(x)(-1)^x \iff F(u - M/2)$$

Practical DFT

Thus $F(0)$ is shifted to the center of the interval $[0, M-1]$, which corresponds to the second figure.
A similar argument applies to 2D FFT, wherein the spectrum shifts from one corner of the image to the centre of the image as shown in the subsequent figure.

Folding of DFT



Fourier Spectrum and Phase Angle

2D DFT is complex and hence can be expressed as Amplitude and phase in polar form as

$$F(u, v) = |F(u, v)|e^{-i\phi(u,v)}$$

The magnitude $|F(u, v)|$ is called the Fourier Spectrum and ϕ is called the phase angle.
The Fourier transform of a real function is conjugate symmetric that is $|F(u, v)| = |F(-u, -v)|$.
We also know that
$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = MN.\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$.
Since MN is large, $F(0, 0)$, the zero frequency component is several orders larger than the other terms. In view of this, for display purposes, we display the log of the amplitude, $\log(1 + F(u, v))$.

Summary of 2D DFT properties

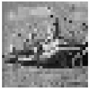
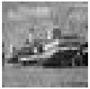
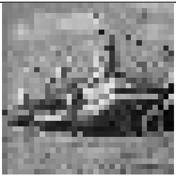
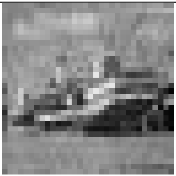
Name	Expression
1. DFT of $f(x,y)$	$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2i\pi(ux/M+vy/N)}$
2. IDFT of $F(u,v)$	$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{2i\pi(ux/M+vy/N)}$
3. Polar Representation	$F(u,v) = \lvert F(u,v)\rvert e^{i\phi(u,v)}$
4. Spectrum	$\lvert F(u,v)\rvert$
5. Phase angle	$\phi(u,v) = atan[\frac{Imag(F(u,v))}{Real(F(u,v))}]$
6. Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
7. Average value	$f(\tilde{x},y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN} F(0,0)$
8. Periodicity	$F(u+k_1M, v+k_2N) = F(u,v)$
9. Convolution	$f \star g = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$
10. Separability	2D DFT calculated in terms of two 1D DFT

(uv)

Frequency domain operations-zooming

	
ZOOMED BY 2; 1:0.15	Original 1:0.15

Frequency Domain operations - compression

			
SKIP by 16	FFT	SKIP 16 1:2	FFT 1:2
			
SKIP by 4	FFT method		

Basics of Filtering in Frequency Domain

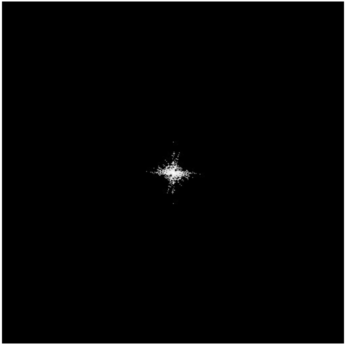
By definition, $F(u, v)$ contains a finite sum of terms, each of which contains values of $f(x, y)$ multiplied by exponential terms. Hence, $F(u, v)$ is a linear combination of the $f(x, y)$ values weighted by the exponential terms. Hence, it is impossible to make any direct association between $F(u, v)$ values and $f(x, y)$ values.

Since the frequency is related to the spatial rate of change, we can associate the frequencies in the Fourier transform with patterns of intensity variations in the image. We have seen that $F(0, 0)$ is average intensity of the image and hence corresponds to the dc term. As we move away from origin of the frequencies, we include more and more spatial variations. However, the location of these variations are not available from Fourier Transforms.

Frequency Domain filtering

The general philosophy of Frequency Domain filtering is to modify the Fourier Transform of the image to achieve a given objective and then find the inverse Fourier Transform of this modified transform.

A typical Fourier spectrum is given below (of the boat image)



Frequency Domain Filtering

Filtering in the frequency domain consists of modifying the Fourier Transfom of the image and then computing the inverse transform. Thus, given a digital image, $f(x, y)$ of size $M \times N$, the basic filtering equation is

$$g(x, y) = \mathcal{F}^{-1}[H(u, v)F(u, v)]$$

Here $\mathcal{F}^{-1}$ is the Inverse DFT of $F(u, v)$, which in turn is the Fourier transform of $f(x, y)$. $H(u, v)$ is the filter function. The filter function is specified for all values of u, v in the range. This is made easy by using functions that are symmetric about their centre.

Frequency Domain Filtering

We had seen that convolution in spatial domain is same as multiplying in the Frequency domain. This multiplication is element by element. That is we must multiply the components of $F(u, v)$ and $H(u, v)$ which correspond to the same frequencies. To facilitate this operation, we generally take both the image and the filter function of the same size. Since specification of the filter is made easier by considering symmetry about the origin, we shift the origin of the Fourier Transform of the image also to the centre. This is possible by multiplyiing the image by $(-1)^{x+y}$ before computing the transform.

Frequency domain filtering - pitfalls

One of the major problems in Frequency domain filtering is the wraparound error. The 2D convolution is given by

$$f(x,y) \star h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$
$$\text{for } x = 0, 1, 2, \ldots, M-1 \text{ and } y = 0, 1, 2, \ldots, N-1$$

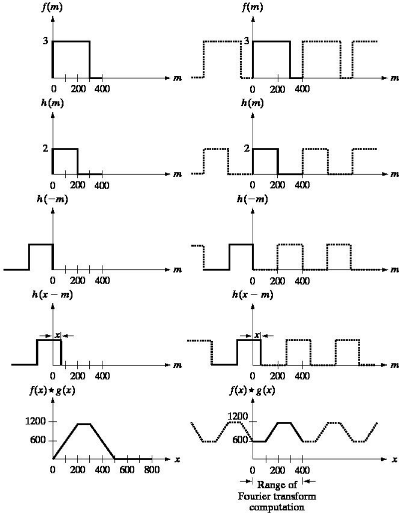
This gives one period of a 2D periodic sequence.
The 2D convolution theorem is given by

$$f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

and conversely.
Since we are dealing with discrete samples, the Fourier transform is carried out with DFT algorithm. IDFT being the product of two transforms will give rise to periodic issues.

Pitfalls of Frequency Domain filtering

Consider 1D signal $f(x)$ and filter $h(x)$.



Frequency domain filtering pitfalls - how to overcome

Let them be of same size. For convolution to be completely successful, the sliding function h should completely slide across f. The last diagram in the right column gives the wraparound error when this sliding is not possible. To overcome this, we have to pad the two functions in the following way: If $f(x)$ and $h(x)$ are two functions composed of A and B samples, append zeroes to both functions so that thy have the same length, say P, where $P \geqslant A + B - 1$.

For 2D case, we have two functions $f(x, y), g(x, y)$ of sizes $A \times B$ and $C \times D$. Then, we form two new functions

$$f_p(x, y) = \begin{cases} f(x, y) & \text{if } 0 \leqslant x \leqslant A - 1, 0 \leqslant y \leqslant B - 1 \\ 0 & \text{if } A \leqslant x \leqslant P \text{ or } B \leqslant y \leqslant Q \end{cases}$$

and

$$h_p(x, y) = \begin{cases} h(x, y) & \text{if } 0 \leqslant x \leqslant C - 1, 0 \leqslant y \leqslant D - 1 \\ 0 & \text{if } C \leqslant x \leqslant P \text{ or } D \leqslant y \leqslant Q \end{cases}$$

with $P \geqslant A + C - 1$ and $Q \geqslant B + D - 1$

Frequency domain filtering - specific cases

- Simplest case - consider $H(u, v)$ such that
$$H(u,v) = \begin{cases} 0 & \text{if } 0 \\ 1 & \text{if } otherwise \end{cases}$$
This filter rejects the dc term and passes all other terms of $F(u, v)$ unchanged. We now that the dc term is responsible for the average intensity of the image, and so making it zero will reduce the average internsity of the image to zero. The resultant image becomes darker than the original.

No DC component filtering

Since average intensity is zero, obviously there are negative intensities, which get wrapped around.

Frequency domain Filtering

- Low Pass Filtering:In this case, we define the filter as
$$H(u, v) = \begin{cases} 1 & \text{if } \sqrt{u^2 + v^2} < D_0 \\ 0 & \text{if otherwise} \end{cases}$$
D_0 is called the cut-off frequency

Low Pass Filter Effect

	
Original	Cut Off frequency 40

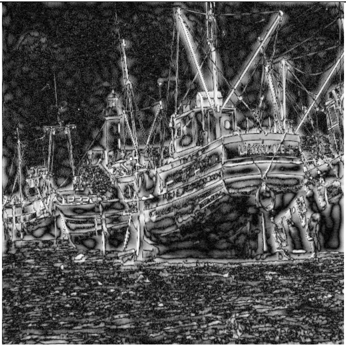
Low Pass Filter Effect

	
Cut Off frequency 150	Cut Off frequency 240

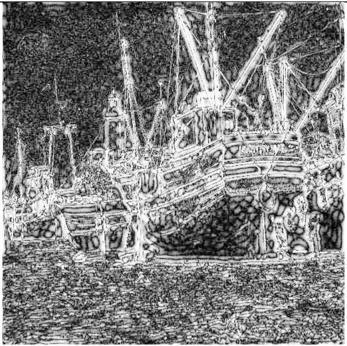
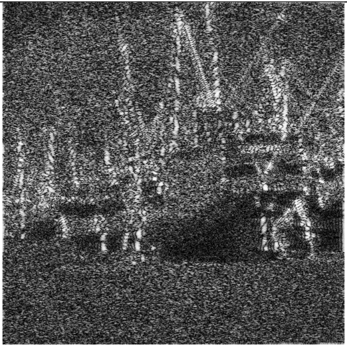
Frequency Domain Filtering

- High Pass Filtering: In this case we define the filter as
$$H(u,v) = \begin{cases} 1 & \text{if } \sqrt{u^2+v^2} > D_0 \\ 0 & \text{if } otherwise \end{cases}$$
D_0 is called the cut-off frequency

High Pass Filter Effect

	
Original	Cut Off 20

High Pass Filter Effect

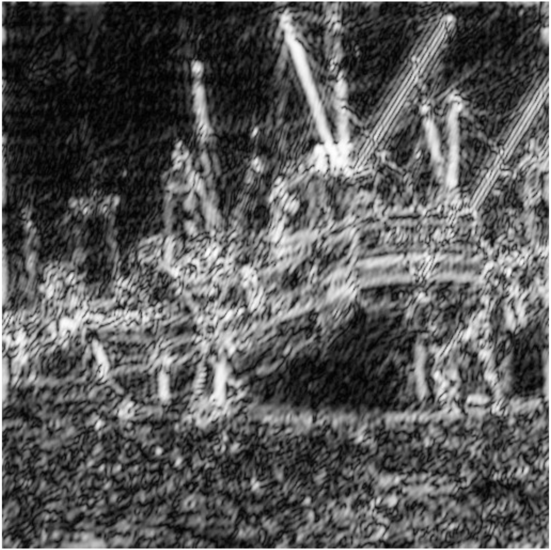
	
Cut Off 30	Cut Off 200

Band Pass Filter

- Band Pass Filter: In this case, we define the filter as

$$H(u,v) = \begin{cases} 1 & \text{if } D_0 < \sqrt{u^2+v^2} < D_1 \\ 0 & \text{if otherwise} \end{cases}$$

Effect of Band Pass Filter

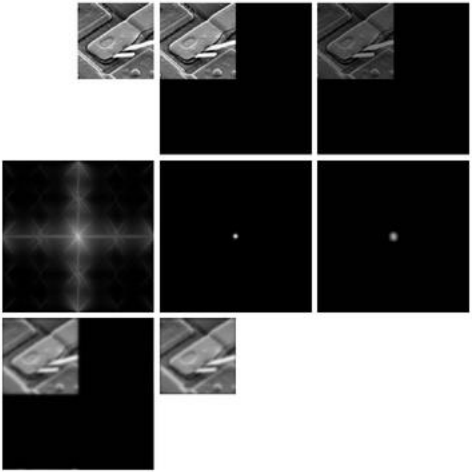


Band Pass locations are 100 and 110

Summary of Steps in Frequency Domain Filtering

- Given an input image $f(x, y)$ of size $M \times N$, for a padded image $f_p(x, y)$ of size $P \times Q$ by appending the necessary zeroes to $f(x, y)$
- Multiply $f_p(x, y)$ to centre the transform
- Compute DFT $F(u, v)$ of the image from previous step
- Generate a real, symmetric filter function, $H(u, v)$ of size $P \times Q$ with center at coordiantes $(P/2, Q/2)$. Form the product $G(u, v) = H(u, v)F(u, v)$ using elementwise multiplication $G(i, k) = H(i, k)F(i, k)$
- Obtain the processed image $g_p(x, y) = \{\mathrm{real}[\mathcal{F}^{-1}[G(u, v)]]\}(-1)^{x+y}$. The real part is selected to ignore the small complex components due to computational inaccuracies
- Obtain the final processed results, $g(x, y)$ by extracting the $M \times N$ from the top, left quadrant of $g_p(x, y)$.

Frequency domain filtering example



Frequency filters-Low pass filters

- Gaussian filter: $H(u) = Ae^{-\frac{u^2}{2\sigma^2}}$
- Difference of Gaussian - DOG: $H(u) = Ae^{-\frac{u^2}{2\sigma_1^2}} - Be^{-\frac{u^2}{2\sigma_2^2}}$. This filter captures the edges very well.
- Low pass filters. $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leqslant D_0 \\ 0 & \text{if otherwise} \end{cases}$
- Butterworth Low Pass Filter: $H(u,v) = \frac{1}{[1+\frac{D(u,v)}{D_0}]^{2n}}$, where $D(u,v)$ is as in the previous case
- Gaussian Low pass filter: $H(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}}$, where $D(u,v)$ is same as above. σ is the spread of the filter. If we set $\sigma = D_0$, then $H(u,v) = e^{-\frac{D^2(u,v)}{2D_0^2}}$, D_0 is the cutoff frequency.

Frequency filters-High Pass filters

- A high pass filter is obtained from a given low pass filter by using the equation
$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$
- Butterworth High Pass filter - $H(u,v) = \frac{1}{[1+\frac{D_0}{D(u,v)}]^{2n}}$, where $D(u,v)$ is as in the previous case

Frequency domain filtering - Edges Laplacian

We had seen that Laplacian in spatial domain gives good edges. The same in frequency domain can be done. the Fourier transform of the Laplacian is $H(u, v) = -4\pi^2(u^2 + v^2)$. With respect to (uthe centre of the frequency rectangle, it will $H(u, v) = -4\pi^2[(u - P/2)^2 + (v - Q/2)^2] = -4\pi^2 D^2(u, v)$. Hence the laplacian of the image is obtained as $\Delta^2 f(x, y) = \mathcal{F}^{-1} H(u, v) F(u, v)$ Enhancement of edges is achieved by adding a multiple of the Laplacian to the original iage. Thus, $g(x, y) = f(x, y) + c^2 \Delta^2 f(x, y)$ accentuates the edges. The same expression in Fourier domain will give $g(x, y) = \mathcal{F}^{-1}\{F(u, v) - H(u, v)F(u, v)\}$ (negative sign because $H(u, v)$ has an inherent negative sign)=$\mathcal{F}^{-1}\{[1 - H(u, v)]F(u, v)\}$=$\mathcal{F}^{-1}\{[1 + 4\pi^2 D^2(u, v)]F(u, v)\}$

Homomorphic filtering

A simple image formation model- Let $f(x,y)$ be an image. Clearly it is obtained by the amount of radiation reflected by some objects and the amount of light incident on those objects. Thus $f(x,y)$ must lie between 0 and ∞. This function $f(x,y)$ can thus be modelled as $f(x,y) = i(x,y)r(x,y)$, where $0 < i(x,y) < \infty$ and $0 < r(x,y) < 1$, where i and r are the illumination and reflectance components respectively.
The nature of $i(x,y)$ is determined by the illumination source characteristics while $f(x,y)$ is determined by the characteristics of the imaged objects.

Homomorphic Filtering

This equation cannot be used directly to operate on the frequency domain, because $\mathcal{F}[f(x,y)] \neq \mathcal{F}[i(x,y)]\mathcal{F}[r(x,y)]$. To overcome this problem, we define

$$z(x,y) = \ln(f(x,y) = \ln i(x,y) + \ln r(x,y)$$

Then, taking Fourier transforms, we get
$Z(u,v) = F_i(u,v) + F_r(u,v)$, where F_i and F_r are the Fourier transforms of $\ln i(x,y)$ and $\ln r(x,y)$.
We can filter $Z(u,v)$ using a filter $H(u,v)$ so that
$S(u,v) = H(u,v)Z(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$

Homomorphic Filtering

The filtered image in the spatial domain is
$s(x, y) = \mathcal{F}^{-1}\{S(u, v)\} =$
$\mathcal{F}^{-1}\{H(u, v)F_i(u, v)\} + \mathcal{F}^{-1}H(u, v)F_r(u, v)\}$
Define $i'(x, y) = \mathcal{F}^{-1}\{H(u, v)F_i(u, v)$, and
$r'(x, y) = \mathcal{F}^{-1}\{H(u, v)F_r(u, v)$, and we obtain
$s(x, y) = i'(x, y) + r'(x, y)$.

Since we had taken logarithms in the beginning, the final filtered image is obtained by the formula $g(x, y) = e^{s(x,y)}$. Low frequency variations correspond to illumination and high frequency variations correspond to the object. This type of filtering is very useful in Medical imaging.

Homomorphic filtering Example