Discrete Fourier Transforms (DFT)

February 9, 2019

DFT- Definition

Suppose a function $f(t)$ is defined at discrete set of points $k = 0, 1, 2, ..., N - 1$. The $1 - D$ DFT is defined as

$$F[m] = \frac{1}{N} \sum_{k=0}^{N-1} f[k] e^{-i\frac{2\pi mk}{N}}, m = 0, 1, 2, ..., N - 1$$

Since $e^{i\frac{2\pi mk}{N}}$ is N^{th} root of unity, it can be denoted by ω_N. Hence, the above definition can be simplified as

$$F[m] = \frac{1}{N} \sum_{k=0}^{N-1} f[k] \omega_N^{-mk}, m = 0, 1, 2, ..., N - 1$$

Inverse Discrete Fourier Transform

To solve for $f[n]$ from the above equation, we multiply both sides of the equation by ω^{mj} and add and get

$$\sum_{m=0}^{N-1} F[m]\omega_N^{mj} = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} f[k]\omega_N^{m(j-k)} = Nf[j]$$

$$\text{Hence, } f[n] = \sum_{m=0}^{N-1} F[m]\omega_N^{mn}, n = 0, 1, 2, ..., N-1$$

Matrix Representation of DFT

The matrix representation of DFT is obtained by writing the elements in terms of ω_N^{-mn}, both m and n taking values $0, 1, 2, ..., N-1$.

Hence, the DFT matrix representation is given by

$$\mathcal{F} = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & ... & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & ... & \omega_N^{-2(N-1)} \\ ... & ... & ... & ... & ... \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & ... & \omega_N^{-(N-1)(N-1)} \end{pmatrix}$$

Considering the columns of this matrix as vectors, $\mathbf{q_i}$, it can be seen that $\mathbf{q_i}.\mathbf{q_j} = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{N} & \text{if } i = j \end{cases}$.
Hence the matrix $\mathcal{F}$ is not unitary but symmetric. (Since, $\mathcal{F}.\mathcal{F}^* = \frac{1}{N}I$)

DFT- Matrix representation

Hence the $N-$pt DFT of the vector $(f[0], f[1], f[2], ..., f[N-1])^T$ is given by

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \dots \\ F[N-1] \end{pmatrix} = \mathcal{F} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \dots \\ f[N-1] \end{pmatrix}$$

Clearly, for each component of F, there are N multiplications and hence to evaluate the DFT of an n component vector, the number of multiplications is N^2. Of course multiplications are complex multiplications.

Some arithmetic of ω

1. $\sum_{m=0}^{S-1} e^{2i\pi t\frac{m}{S}} = S\delta(t)$
This is a GP with ratio $= q = e^{2i\pi\frac{t}{S}}$
The Sum of this GP $= \frac{1-q^S}{1-q} = 0$, if $q \neq 1$, and $= S$ if $q = 1$
Hence the result.

Example of a DFT of a 4×4 image

Clearly the 4×4 DFT matrix is

$$U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega^{-1\times 0} & \omega^{-1\times 1} & \omega^{-1\times 2} & \omega^{-1\times 3} \\ \omega^{-2\times 0} & \omega^{-2\times 1} & \omega^{-2\times 2} & \omega^{-2\times 3} \\ \omega^{-3\times 0} & \omega^{-3\times 1} & \omega^{-3\times 2} & \omega^{-3\times 3} \end{pmatrix}$$

Here $\omega = e^{2i\pi/4} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2} = i$

Hence,

$\omega^{-1} = -i^{-1} = 1/i = -i; \omega^{-2} = -1; \omega^{-3} = i^{-3} = \frac{1}{-i} = i;$

$\omega^{-4} = 1; \omega^{-6} = -1; \omega^{-9} = -i$

$$\text{Hence } U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Example of DFT

Calculate the DFT of the 4×4 image given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

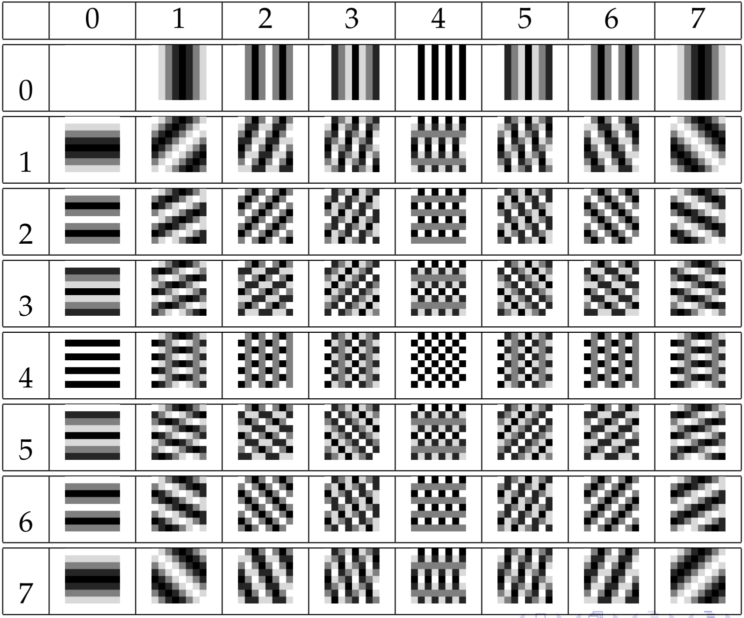
Clearly the DFT of this image is given by $B = UAU$ - since U is symmetric, $U^T = U$

Hence, $B = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

DFT matrix

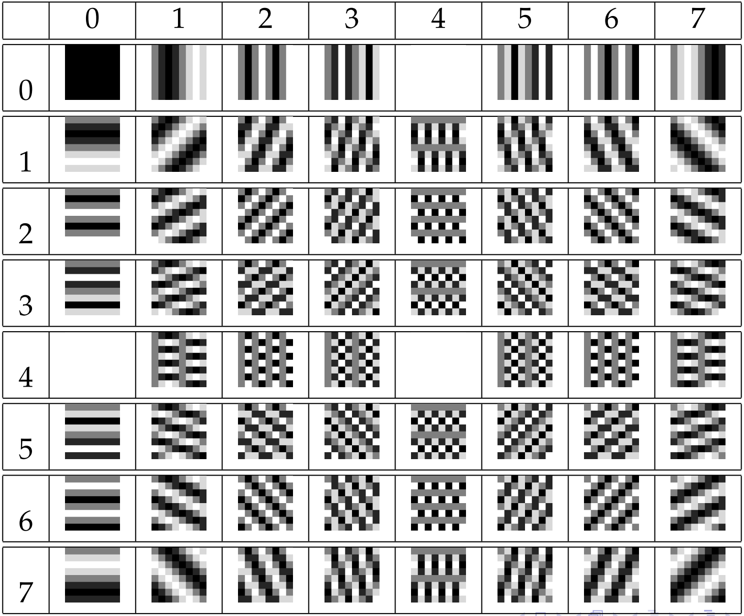
Observation: DFT matrix has complex entries. Hence, we cannot talk of orthogonality. When the entries of a matrix are complex and $A^{-1} = A^{T*}$, then the matrix is said to be unitary. This is equivalent of orthogonal in matrices with real entries. DFT matrix is not unitary, since $UU^{T*} \neq I$. The product is in fact $\frac{1}{N}I$. Hence, we can make the matrix unitary by defining $\hat{U} = \frac{1}{\sqrt{N}}U$.

Basis images of Real Fourier Transform



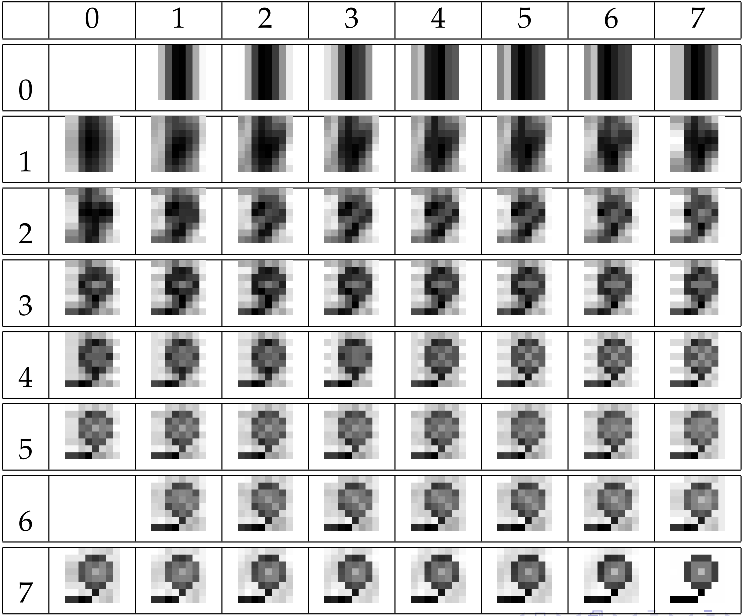
8 × 8 Real Fourier basis images

Basis images of Imag Fourier Transform



8 × 8 Imaginary Part Fourier basis images

FT Real components of Flower



8 × 8 Real Part Fourier Reconstruction of Flower images

Convolution in continuous domain

If $f(x)$ and $g(x)$ are two functions, their convolution is defined as

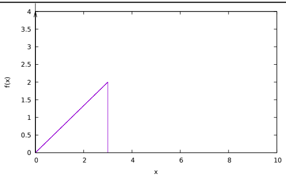
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - \alpha)g(\alpha)$$

It can easily be proved that $f * g = g * f$, i.e., convolution is commutative. It distributes over addition, i.e.,
$f * (g + h) = f * g + f * h$
$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(x - \alpha)\delta(\alpha)d\alpha = \int_{-\infty}^{\infty} f(\alpha)\delta(x - \alpha)d\alpha = f(x)$
$Lf(x)$=$L(f(x) * \delta(x))$=$L\int_{-\infty}^{\infty} f(\alpha)\delta(x - \alpha)d\alpha =$
$\int_{-\infty}^{\infty} f(\alpha)L\delta(x - \alpha)d\alpha = \int_{-\infty}^{\infty} f(\alpha)h(\delta(x - \alpha))d\alpha = f * h$
The effect of a LSI on any function is obtained by convolving the function with the PSF of the LSI

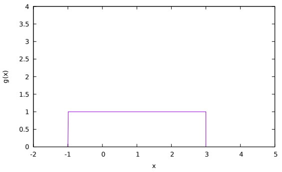
Graphical Procedure for Convolution

1. First graph the function $f(\alpha)$ using the dummy integraton variable α for the horizontal coordinate.
2. Choose a convenient value for x, say $x = 0$ and graph the function $h(x-\alpha) = h(-\alpha)$ below that of $f(\alpha)$. Observe that $h(-\alpha)$ is simply a mirror image of the functon $h(\alpha)$ about the $y-axis$.
3. The product $f(\alpha)h(x-\alpha) = f(\alpha)h(-\alpha)$ is calculated and graphed.
4. The area of this product is calculated- this value is equal to the convolution for the particular value of x chosen
5. Go to step 2 and change x to 1 and repeat the process.
6. The computed areas are graphed to give the convoluted function.

Convolution-Graphical Explanation

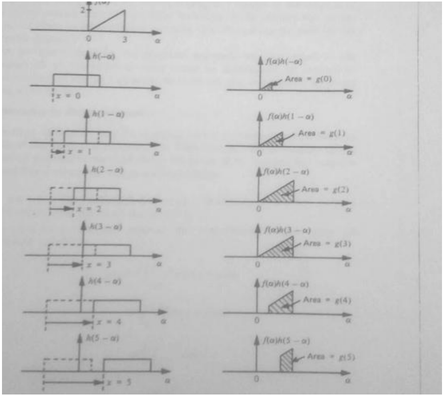


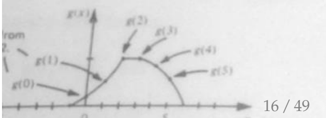
Input function f(x)



Input function h(x)

Convolution-Graphical Explanation





Properties of Convolution

- Commutative- the functions can be exchanged - $f(x) * g(x) = g(x) * f(x)$
- Associative - $f(x) * (g(x) * h(x)) = (f(x) * g(x)) * h(x)$
- Distributive over addition - $f(x) * (g(x) + h(x)) = f(x) * g(x) + f(x) * h(x)$
- Scaling- $\alpha(f(x) * g(x)) = (\alpha f(x)) * g(x) = f(x) * (\alpha g(x))$
- Identity- The dirac delta (impuse) is to convolution as 1 is to multiplication- $f(x) * \delta(x) = f(x)$
- Complex conjugation - The conjugate of the convolution is same as the convolution of the conjugates- $\overline{f(x) * g(x)} = \overline{f(x)} * \overline{g(x)}$

Convolution Properties

The integral of the convolution is the product of the integrals
$\int (f * g)(x)dx = (\int f(x)dx)(\int g(x)dx)$

Differentiation - The derivative of the convolution is the same as the convolution of the derivative of one function with the other function
$\frac{df(x)*g(x)}{dx} = \frac{df(x)}{dx} * g(x) = f(x) * \frac{dg(x)}{dx}$

Convolution Theorem: Most important property - The Fourier Transform of the convolution of two functions equals the product of the Fourier Transforms of the two functions
$\mathcal{F}(f(x) * g(x) = \mathcal{F}(f(x))(F)(g(x))$
This can be obtained from the integration property

Convolution of FT

Convolution is a smoothing operator.
Let $f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(t)h(x-t)dt$

Hence $\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)dx$

$= \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \int_{-\infty}^{\infty} g(t)h(x-t)dt$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i\omega x} g(t)h(x-t)dt$

Let $u = x - t$
Then the integral becomes

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i\omega u} e^{-2\pi i\omega t} g(t)h(u)dtdu$

$= \int_{-\infty}^{\infty} e^{-2\pi i\omega t} g(t)dt \int_{-\infty}^{\infty} e^{-2i\pi\omega u} h(u)du$

$= \mathcal{F}(g(x)).\mathcal{F}(h(x))$

Convolution Theorem in Discrete Domain

Suppose f and g are two discrete funtions we convolve.Then
$u(n) = \sum_{n'=0}^{N-1} f(n-n')g(n')$
To take DFT of the above equation, multiply by $\frac{1}{N}e^{-2i\pi pn/N}$ both sides and sum over n
Thus, the LHS will be $\frac{1}{N}\sum_{n=0}^{N-1} u(n)e^{2i\pi pn/N}$ and this is nothing but the DFT of $u = \hat{u}(p)$
The RHS becomes $\frac{1}{N}\sum_{n=0}^{N-1} e^{-2i\pi pn/N} \sum_{n'=0}^{N-1} f(n-n')g(n')$
$= \frac{1}{N}\sum_{n'=0}^{N-1}\sum_{n=0}^{N-1} e^{-2i\pi pn/N} f(n-n')g(n')$
Let $n - n' = n''$
Then the limits for n'' are:
When $n = 0, n'' = -n'$;
When $n = N-1, n'' = N-1-n'$
Then the above summation becomes
$\frac{1}{N}\sum_{n'=0}^{N-1} f(n')e^{-2i\pi pn'/N} \sum_{n''=-n'}^{N-1-n'} g(n'')e^{-2i\pi pn''/N} g(n'')$

Convolution in DFT

Let $T = \sum_{n''=-n'}^{N-1-n'} e^{-2i\pi p n''/N} g(n'')$
$= (\sum_{n''=-n'}^{-1} + \sum_{n''=0}^{N-1-n'}) e^{-2i\pi p n''/N} g(n'')$
Obviously the functions are not defined for negative indices.
Consider the first sum, $= \sum_{n''=-n'}^{-1} e^{-2i\pi p n''/N} g(n'')$
Define $n''' = N + n'' \Rightarrow n'' = n''' - N$
The above expression becomes
$= \sum_{n'''=N-n'}^{-1} e^{-2i\pi p n'''/N} e^{2i\pi p} g(n''' - N)$
$= \sum_{n'''=N-n'}^{-1} e^{-2i\pi p n'''/N} g(n''' - N)$
We define $g(n''' - N) = g(n''')$
Then the above sum becomes
$= \sum_{n'''=N-n'}^{-1} e^{-2i\pi p n'''/N} g(n''')$
This term is added to the term $\sum_{n''=0}^{N-1-n'} e^{-2i\pi p n''/N} g(n'')$

Convolution and DFT

The two terms together can be written as
$\sum_{n''=0}^{N-1} e^{-2i\pi pn''/N} g(n'') = N\}$
The summation involving $f(n')$ can be recognised as $N\mathcal{F}f$.
Hence, $\mathcal{F}(f * g) = N\mathcal{F}f\mathcal{F}g$
We periodically extend the function g to obtain this Convolution Result.

Properties of DFT of images

Given an image $f(k, l)$, its DFT is given by

$$\hat{g}(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{-j2\pi \frac{km+ln}{N}}$$

What happens to the DFT of an image when it is shifted?
Suppose the image is shifted to $(k_0, l-0)$, so that the image becomes $g(k-k_0, l-l_0)$
Then using the above formula, the DFT of the shifted image will be

$$\hat{\hat{g}}(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k-k_0, l-l_0) e^{-j2\pi \frac{km+ln}{N}}$$

Substitute $k' = k - k_0, l' = l - l_0$
Then the above equation becomes

$$\hat{\hat{g}}(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k', l') e^{-j2\pi \frac{k'm+l'n}{N}} e^{-j2\pi \frac{k_0m+l_0n}{N}}$$

Properties of DFT

What happens to the DFT of an image when the image is rotated?
The formula for DFT is

$$\hat{g}(m, n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{-j2\pi \frac{km+ln}{N}}$$

Introduce polar coordinates in the (k, l) plane and in (m, n) plane as $k = r\cos\theta, l = r\sin\theta$ and $m = \omega\cos\phi, n = \omega\sin\phi$. Observe that $km + ln = r\omega\cos(\theta - \phi)$. Hence, the above equation becomes

$$\hat{g}(\omega, \phi) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(r, \theta) e^{-j2\pi \frac{r\omega\cos(\theta-\phi)}{N}}$$

Fast Fourier Transform FFT

Recall we have defined DFT of a vector $f[0], f[1], ..., f[N-1]$ using matrix notation as

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ ... \\ F[N-1] \end{pmatrix} = \mathcal{F} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ ... \\ f[N-1] \end{pmatrix}$$

where $\mathcal{F}$ is the big $N \times N$ matrix involving ω's.
Clearly for every entry $F(i)$, there are N multiplications.
Hence, the total number of operations to calculate the DFT of f is N^2 multiplications.
Can we reduce this number by some trick?
Yes.

DFT

$F[m] = \sum_{k=0}^{N-1} f[k]\omega_N^{-km}, \quad m = 0, 1, 2, ..., N-1$

We split this sum into sum of odd indices and even indices. Thus,

$$F[m] = \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_N^{-2km} + \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_N^{-(2k+1)m}$$

$$m = 0, 1, 2, ..., N$$

Now, $\omega_N^{-2km} = e^{-(2i\frac{\pi}{N})2km} = e^{-(2i\pi/\frac{N}{2})km} = \omega_{\frac{N}{2}}^{-km}$

Similarly, $\omega_N^{-(2k+1)m} = \omega_N^{-m}\omega_{\frac{N}{2}}^{-km}$

Hence,

$F[m] = \mathcal{F}_{\frac{N}{2}} f_{even}[m] + \omega_N^{-m}\mathcal{F}_{\frac{N}{2}} f_{odd}[m], m = 0, 1, 2, ..., \frac{N}{2} - 1$

FFT

Thus, if we take m to go from 0 to $\frac{N}{2} - 1$, then we get the first $\frac{N}{2}$ outputs of $F[m]$ in terms of DFT of $\frac{N}{2}$ length vectors of odd and even elements.

Now for m going from $\frac{N}{2}$ to $N - 1$, we write $m = m' + \frac{N}{2}$

Then, m' will vary from 0 to $\frac{N}{2} - 1$

Hence, for $m \in (\frac{N}{2}, N - 1)$

$F[m] = F[m' + \frac{N}{2}] = \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_N^{-2k(m'+\frac{N}{2})}$

$+ \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_N^{-(2k+1)(m'+\frac{N}{2})}$

$= \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_N^{-2km}.\omega_N^{-kN}$

$+ \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_N^{-2k(m'+\frac{N}{2})}\omega_N^{-(m'+\frac{N}{2})}$

FFT

$$= \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_{\frac{N}{2}}^{-km'} + \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_{N}^{-2km'}\omega_{N}^{-m'}\omega_{N}^{-\frac{N}{2}}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_{\frac{N}{2}}^{-km'} + \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_{\frac{N}{2}}^{-km'}\omega_{N}^{-m'}(-1)$$

$$= \sum_{k=0}^{\frac{N}{2}-1} f[2k]\omega_{\frac{N}{2}}^{-km'} - \sum_{k=0}^{\frac{N}{2}-1} f[2k+1]\omega_{\frac{N}{2}}^{-km'}\omega_{N}^{-m'}$$

Hence,

$$F[m + \tfrac{N}{2}] = \mathcal{F}_{\frac{N}{2}} f_{even}[m] - \omega_N^{-m}\mathcal{F}_{\frac{N}{2}} f_{odd}[m],$$

$m = 0, 1, 2, \cdots, \frac{N}{2} - 1$

From earlier result,

$$F[m] = \mathcal{F}_{\frac{N}{2}} f_{even}[m] + \omega_N^{-m}\mathcal{F}_{\frac{N}{2}} f_{odd}[m],$$

$m = 0, 1, 2, ..., \frac{N}{2} - 1$

Thus, by doing the $\frac{N}{2}$ DFT twice, and some simple operations, we are able to get the DFT of N length vector.

FFT

Number of operations:
The number of operations of an $N-pt$ FFT is $T(N)$. Then this is equal to 2 times the number of operations of an $N/2-pt$ FFT added to N multiplications from the second terms.
Suppose $N = 2^M$
Then,
$T(N) = T(2^M) = 2T(2^{M-1}) + N$
$T(\frac{N}{2}) = T(2^{M-1}) = 2T(2^{M-2}) + \frac{N}{2}$
$\cdots\cdots\cdots\cdots\cdots\cdots$
$T(2) = 2T(1) + \frac{N}{2^{M-1}}$
Multiplying these equations by $1, 2, \cdots, 2^{M-1}$, we get $T(N) = MN = N\log_2 N$, since $T(1) = 0$, number of operations for a one point DFT.
Hence the number of operations of an $N-pt$ DFT which was N^2 has been reduced to $N\log_2 N$, which is a significant reduction.

Discrete Cosine Transform DCT

This is a real valued Unitary Transform.
If $x[n]$ is the signal of length N, the Fourier Transform of the signal $x[n]$ is given by
$X[k] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi kn}{N}}, k = 0, 1, 2, ..., N-1$
Now consider the extension of the signal $x[n]$, we denote it by $x_e[n]$, so that the length of the extended sequence is 2N. The sequence can be extended in two ways-
1. by simply copying the original sequence
2. by copying the original sequence in a folded manner.
The advantage of the second method is it avoids "ringing".

DCT

The DFT of the extended sequence is given by
$X_e[k] = \sum_{n=0}^{2N-1} x_e[n] e^{\frac{-2i\pi kn}{2N}}$

We split the interval into two halves

$X_e[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{2i\pi kn}{2N}} + \sum_{n=N}^{2N-1} x[2N-1-n] e^{\frac{-2i\pi kn}{2N}}$

This is by using the definition of x_e as folded sequence of x.

Let $m = 2N - 1 - n$.
Substituting in the above equations, we get

$X_e[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-2i\pi kn}{2N}}$

$+ \sum_{m=0}^{N-1} x[m] e^{-\frac{2i\pi k}{2N} 2N} e^{\frac{i2\pi(m+1)k}{2N}}$

$= \sum_{n=0}^{N-1} x[n] e^{\frac{-2i\pi kn}{2N}} + \sum_{m=0}^{N-1} x[m] e^{\frac{i2\pi(m+1)k}{2N}}$

$= \sum_{n=0}^{N-1} x[n] \{ e^{-\frac{2i\pi kn}{2N}} + e^{\frac{i2\pi(n+1)k}{2N}} \}$

DCT

Multiplying both sides of the above equation by $e^{-\frac{i\pi k}{2N}}$, we get

$X_e[k]e^{-\frac{i\pi k}{2N}} = \sum_{n=0}^{N-1} x[n]\{e^{-\frac{i2\pi kn}{2N}} + e^{\frac{i2\pi(n+1)k}{2N}}\}e^{-\frac{i\pi k}{2N}}$

Taking $e^{-\frac{i\pi k}{2N}}$ inside, we get

$X_e[k]e^{-\frac{i\pi k}{2N}} = \sum_{n=0}^{N-1} x[n]\{e^{-\frac{i2\pi kn}{2N}} e^{-\frac{i\pi k}{2N}} + e^{\frac{i2\pi(n+1)k}{2N}} e^{-\frac{i\pi k}{2N}}\}$

i.e., $X_e[k]e^{-\frac{i\pi k}{2N}} = \sum_{n=0}^{N-1} x[n]\{e^{-\frac{i\pi k(2n+1)}{2N}} + e^{\frac{i\pi(2n+1)k}{2N}}\}$

Hence, $X_e[k]e^{-\frac{i\pi k}{2N}} = 2\sum_{n=0}^{N-1} x[n]\cos\{\frac{2n+1)\pi k}{2N}\}$

But the LHS is the DFT of a sample shifted by $1/2$.

Thus DCT is a DFT of an even number of samples symmetric about the origin.

DCT and Inverse DCT

To make it orthogonal we add some factor and the final result is
$X[k] = \alpha(k) \sum_{n=0}^{N-1} x[n] \cos(\frac{(2n+1)\pi k}{2N})$,

$$\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } k = 0 \\ \sqrt{\frac{2}{N}} & \text{if } k \neq 0 \end{cases}$$

The corresponding inverse DCT formula is
$x[k] = \alpha(k) \sum_{n=0}^{N-1} X[n] \cos(\frac{(2n+1)\pi k}{2N})$,

$$\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } k = 0 \\ \sqrt{\frac{2}{N}} & \text{if } k \neq 0 \end{cases}$$

DCT matrix

The DCT matrix is defined as

$U_{kn} = \alpha(k)\cos\frac{(2n+1)\pi k}{2N}, k, n = 0, 1, 2, ..., N-1$

Here $\alpha(0) = \sqrt{\frac{1}{N}}, \alpha(i) = \sqrt{\frac{2}{N}}, i \neq 0$

This matrix is orthogonal.

Proof: Consider any column of this matrix, say v_n

$v_{kn} = \sqrt{2/N}\cos(\frac{(2n+1)k\pi}{2N}), k = 1, 2, ..., N-1$ and
$v_{0n} = 1/\sqrt{N}$.
Hence, $v_n.v_n = \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1}\cos^2(\frac{(2n+1)k\pi}{2N})$

$= \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}(1 + \cos(\frac{(2n+1)k\pi)}{N}$

$= \frac{1}{N} + \frac{N-1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}\cos(\frac{(2n+1)k\pi}{N})$

DCT matrix

$$= \frac{1}{N} + 1 - \frac{1}{N} + \frac{\sin\frac{(N-1)(2n+1)\pi}{2N}}{\sin\frac{(2n+1)\pi}{2N}} \cos\left(\frac{(2n+1)\pi}{2N}(1+N-1)\right)$$

$$= 1 + \frac{\sin\frac{(N-1)(2n+1)\pi}{2N}}{\sin\frac{(2n+1)\pi}{2N}} \cos\left(\frac{(2n+1)\pi}{2}\right) = 1$$

Now, $v_n.v_m$, for $n \neq m$

$$= \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1} \cos\left(\frac{(2n+1)\pi k}{2N}\right)\cos\left(\frac{(2m+1)\pi k}{2N}\right)$$

$$= \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}\left(\cos\frac{(n+m+1)\pi k}{N} + \cos\frac{(n-m)\pi k}{N}\right)$$

DCT Matrix

Let $\frac{((n+m+1)\pi}{2N} = \alpha$, $\frac{(n-m)\pi}{2N} = \beta$

Then $v_n.v_m = \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}(\cos 2\alpha k + \cos 2\beta k)$

$= \frac{1}{N} + \frac{1}{N}\frac{\sin(N-1)\alpha}{\sin\alpha}\cos(N\alpha) + \frac{1}{N}\frac{\sin(N-1)\beta}{\sin\beta}\cos(N\beta)$

$= \frac{1}{N} + \frac{1}{N}(\frac{\sin((N-1)\alpha)\cos(N\alpha)\sin(\beta)+\sin((N-1)\beta)\cos(N\beta)\sin(\alpha)}{\sin(\alpha)\sin(\beta)})$

Numerator of the big expression on the right

$= \sin(N\alpha)\cos(\alpha)\cos(N\alpha)\sin(\beta) - \sin(\alpha)\cos^2(N\alpha)\sin(\beta) +$
$\sin(N\beta)\cos(\beta)\cos(N\beta)\sin(\alpha) - \sin(\beta)\cos^2(N\beta)\sin(\alpha)$

$= 1/2\cos\alpha\sin\beta\sin 2N\alpha + 1/2\cos\beta\sin\alpha\sin 2N\beta -$
$\sin\alpha\sin\beta(\cos^2 N\alpha + \cos^2 N\beta)$

But $\sin 2N\alpha = \sin(m+n+1)\pi = 0$
Similarly, $\sin 2N\beta = \sin(n-m)\pi = 0$

DCT Matrix

Hence numerator of big expression

$= -\sin\alpha \sin\beta(\cos^2\frac{(m+n+1)\pi}{2} + \cos^2\frac{(n-m)\pi}{2})$

$= -\sin\alpha \sin\beta(\sin^2\frac{(m+n)\pi}{2} + \cos^2\frac{(n-m)\pi}{2})$

$= -\sin\alpha \sin\beta(\sin^2\frac{(n-m+2m)\pi}{2} + \cos^2\frac{(n-m)\pi}{2})$

$= -\sin\alpha \sin\beta(\sin^2\frac{(n-m)\pi}{2} + \cos^2\frac{(n-m)\pi}{2})$

$= -\sin\alpha \sin\beta$

Hence, $v_n.v_m = 0$

Hence, the DCT matrix is orthogonal

DCT Matrix for $N = 4$

Let us compute the DCT matrix for $N = 4$
The formula is $X[k] = \alpha[k] \sum_{n=0}^{3} \cos[\frac{(2n+1)\pi k}{8}], k = 0, 1, 2, 3$
Substituting the different values of k and summing up, we get the DCT matrix of size 4 as

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 05 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6533 & 0.6533 & -0.2706 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Properties of DCT Matrix

- The DCT is real and orthogonal
- The discrete DCT has excellent energy ompaction for highly correlated data
- 2D DCT can be computed as two separate one dimensional transforms and hence the DCT of an image can be computed first in terms of rows and then in terms of columns.

Greatest application of DCT is in jpg compression

DCT for Image Processing

As before, if $f(x, y)$ is the image, the DCT transform of this image is given by
$g(u, v) = f(x, y)UU^T$

Karhunen-Loeve Transform

This is also called Principal Component Analysis.
Given an image in the form of a matrix
$A = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$ Aim is to transform this set of column vectors into another domain so that the correlation between the columns are eliminated.To do this we find the covariance matrix of A.
$Cov(A) = (A - \bar{A})^T (A - \bar{A})$
where $\bar{A}$ is the matrix in which each column is defined as
$q_i - \bar{a_i})$
Essentially this gives
$Cov(A)_{ij} = \sum (q_{ik} - \bar{q_i})(q_{jk} - \bar{q_j})$
Thus the $(ij)^{th}$ element of $Cov(A)$ will give the covariance between the i^{th} column and j^{th} column. Hence, the diagonal elements will give the variance of each of the columns themselves.

Karhunen-Loeve Transform

This matrix is clearly symmetric and hence will have real eigenvalues. Find the eigenvalues and the corresponding eigenvectors. Express the matrix in terms of the eigenvectors as new coordinate systems and hence the Covariance matrix will be diagonalized.
In this new system of coordinates, the correlation between the different columns will be zero. Thus, the redundancy in data is eliminated. In fact you might even ignore some of the smaller eigenvalues and thus get the PRINCIPAL COMPONENTS of the data. This analysis is very useful in Remote Sensing.

Karhunen-Loeve Transform

Typically a Remote Sensing data will have multispectral bands and sometimes hyperspectral bands. However for some applications, not all bands are relevant but at the same time you do not want to miss out on any information from any of the bands. In such situations, we do PCA, and get only the few important bands and do the analysis. This can also be used for Image Compression.
Compare this method with SVD.

Definitions

An operator H is called *time invariant* (if x represents time), *spatially invariant* (if x is a spatial variable), or simply *fixed parameter*, for some class of inputs $\{f(x)\}$ if

$$g_i(x) = H(f_i(x)) \Rightarrow g_i(x - x_0) = H[f_i(x - x_0)], \quad \forall f_i(x) \in \{f(x)\} \text{ and } \forall x_0$$

A system described by a fixed parameter operator is said to be a **fixed-parameter system**.
NUTSHELL: Offsetting the independent variable of the input by x_0 causes the same offset in the independent variable of the output. Hence, the input-output relationship remains the same.

Definitions

An operator H is said to be *causal*, and hence the system described by H is a *causal system*, if there is no output before there is an input. In other words,
$f(x) = 0, x < x_0 \Rightarrow g(x) = H[f(x)] = 0$ for $x < x_0$
Finally, a linear system H is said to be *stable* if its response to any bounded input is bounded. That is, if

$$|f(x)| < K \Rightarrow |g(x)| < cK,$$

where c, K are constants.

Definitions

1. Suppose the operator H is the integral operator between the limits $-\infty$ and x. Then the output in terms of the input is given by

$$g(x) = \int_{-\infty}^{x} f(w)dw$$

This system is linear because

$$\begin{aligned}\int_{-\infty}^{x} [\alpha_i f_i(w) + \alpha_j f_j(w)]dw &= \alpha_i \int_{-\infty}^{x} f_i(w)dw + \alpha_j \int_{-\infty}^{x} f_j(w)dw \\ &= \alpha_i g_i(x) + \alpha_j g_j(x)\end{aligned}$$

Impulse

A unit impulse at a denoted by $\delta(x-a)$ is defined by its effect on a function by the expression

$$\int_{-\infty}^{\infty} f(\alpha)\delta(x-\alpha)d\alpha = f(x)$$

If $g(x)$ is the output of a system H for an input $f(x)$, then

$$\begin{aligned}
g(x) &= H[f(x)] \\
&= H[\int_{-\infty}^{\infty} f(\alpha)\delta(x-\alpha)d\alpha] \\
&= \int_{-\infty}^{\infty} H[f(\alpha)\delta(x-\alpha)]d\alpha \\
&= \int_{-\infty}^{\infty} f(\alpha)H[\delta(x-\alpha)]d\alpha \\
&= \int_{-\infty}^{\infty} f(\alpha)h(x,\alpha)d\alpha
\end{aligned}$$

Impulse response

The term $h(x, \alpha) = H[\delta(x - \alpha)]$ is called **the impulse response of** H.
The expression
$g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x, \alpha)d\alpha$
is called the *superposition(or Fredholm) integral of the first kind*. It states that if the response of H to a unit impulse ($h(x, \alpha)$) is known, then the response to any input f can be computed by the above integral. **Thus, the response of a linear system is completely characterized by its impulse response**

System Characterization

If H is a fixed-parameter operator, then

$$H[\delta(x-\alpha)] = h(x-\alpha)$$

and the superposition integral becomes

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha$$

This expression is the Convolution Integral introduced in Mathematical Prelimaries. **It states that the response of a linear, fixed-parameter system is completely characterized by the convolution of the input with the system impulse response.**
Convolution as stated earlier is represented by
$g(x) = f(x) * h(x)$.
We have also seen in the topic of Fourier Transforms, proved that $F(f * g) = F(f).F(g)$. This result will be used very often in Digital Image Processing.

$\vdots$