PMATH 351 - Real Analysis

Fantastic Theorems and How to Prove Them

Jose Luis Avilez
Faculty of Mathematics
University of Waterloo

Contents

Chapter 1

Cardinality and the Real Numbers

This is a course in the three Cs: completeness, compactness, and continuity. Office hours are on Tuesdays from 3-4pm and Wednesdays from 3:30 to 4:30 in MC5306.

Definition 1.1. Two sets, A, B have the same **cardinality** if there is a bijection between them. In this case we write $|A| = |B|$. We say that $|A| \leq |B|$ if there is an injection from A to B.

Theorem 1.2. *Cardinality is an equivalence relation.*

Proof. Reflexivity is achieved through the identity map; the symmetric property is achieved by f and f^{-1}; transitivity is achieved by the fact that if f and g are bijections then gf is a bijection. ■

Example 1.3. Some infinite sets. We have that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$; thus, the embedding mapping is an injection and we get, for free, that $|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}|$. Enumerations exist to show that the natural, integer, and rational numbers have the same cardinality.

Definition 1.4. We say that A is **countably infinite** if $|A| = |\mathbb{N}|$. We say that A is **countable** if it is either finite or countably infinite. A set that is not countable is said to be **uncountable**.

Example 1.5. The set $\mathbb{N} \times \mathbb{N}$ is countably infinite. We achieve such an enumeration by walking across the lattice diagonally, thus $|\mathbb{N} \times \mathbb{N}| = \mathbb{N}$.

Example 1.6. A similar method as above (walking on $\mathbb{Z} \times \mathbb{N}$) leads to the fact that $|\mathbb{N}| = |\mathbb{Q}|$

Theorem 1.7. *Any countable union of countable sets is countable. That is,*

$$\bigcup_{i=1}^{\infty} A_i$$

is countable if each set A_i is countable.

Proof. We can list each set $A_i = \{a_{i1}, a_{i2}, \ldots\}$. We use a similar argument as above. We have

$$U = \bigcup_{i=1}^{\infty} A_i = \{a_{ij} | i, j \in \mathbb{N}\}$$

Thus, there is a bijection from the union U to $\mathbb{N} \times \mathbb{N}$ and the union is countable. ■

Theorem 1.8. *Every infinite subset of $\mathbb{N}$ is countably infinite.*

Proof. Let B be an infinite subset of $\mathbb{N}$. By the well-ordering principle, B has a least element, say b_1. Naturally, $B \setminus \{b_1\}$ is infinite and has a least element, say b_2. By repeating this procedure, we find a bijection between B and $\mathbb{N}$. More formally, suppose $b \in B$. Consider $\{n \in B : n \leq b\}$, which is a finite set with cardinality, say k. Then we would have arrived at b after k steps; that is, $b = b_k$. ■

Theorem 1.9. *If A is infinite and $|A| \leq |\mathbb{N}|$ then $|A| = |\mathbb{N}|$.*

Proof. Let $j : A \to \mathbb{N}$ be an injection. Let $B = j(A) \subseteq \mathbb{N}$. Notice that $j : A \to B$ is an bijection and $|A| = |B|$ and B is infinite. By Theorem 1.8, it follows that B is countably infinite and, thus, so is A. ∎

Theorem 1.10. *The set of real numbers $A = \{x : 0 \leq x < 1\}$ is uncountable.*

Proof. **Cantor's argument.** Arguing by contradiction, suppose A is countable, say $[0, 1) = \{r_i : i \in \mathbb{N}\}]$. Denote the decimal expansion of $r_i = 0.r_{i1}r_{i2}\ldots$ with $r_{ij} \in \{0, 1, \ldots, 9\}$. Define $a = 0.a_1a_2a_3\ldots$ where $a_k = 1$ if $r_{kk} \geq 5$ and $a_k = 8$ if $r_{kk} < 5$. Note that a has a unique representation because it does not end in a tail of zeroes or nines. Not that a cannot equal any of the r_is, since $a_k \neq r_{kk}$ for any k. Since $[0, 1) \neq \{r_i\}$, a contradiction. Thus A is uncountable. ∎

Theorem 1.11. $\mathbb{R}$ *is uncountable.*

Proof. Follows trivially from Theorem 1.10 using the embedding map $f : [0, 1) \to \mathbb{R}$ and reaching a contradiction.

Definition 1.12. We say that $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = \aleph_1$.

Theorem 1.13. *The Continuum Hypothesis is undecidable. That is, the question, is there a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$, is unanswerable given the standard axioms of set theory.*

Definition 1.14. Given a set A, the power set of A, $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 1.15. *For a finite set A with $|A| = n$, $|\mathcal{P}(A)| = 2^n$.*

Proof. Trivial.

Theorem 1.16. ***Cantor's Theorem.*** *For any set A, $|A| \leq |\mathcal{P}(A)|$ and $|A| \neq |\mathcal{P}(A)|$.*

Proof. Define an injection by sending an element to its singleton set. Now, arguing by contradiction, suppose we have a bijection $g : A \to \mathcal{P}(A)$. Let $B = \{a \in A : a \in g(a)\} \subseteq A$. Since $B \subseteq A$, $B \in \mathcal{P}(A)$. Hence $\exists x \in A$ such that $g(x) = B$. Now we ask whether $x \in B$. Suppose yes; that is $x \in B$. Then $x \notin g(x)$ and $x \notin B$, a contradiction. Now, suppose $x \notin B$, then by the definition of B, x must be in B, a contradiction.

Either way, we have just shown that there cannot exist such a bijection g. Thus, $|A| \neq |\mathcal{P}(A)|$. ∎

Definition 1.17. We use the notation $2^A = \{f : A \to \{0, 1\}\}$.

Theorem 1.18. $|2^A| = |\mathcal{P}(A)|$

Proof. Define $g : \mathcal{P}(A) \to 2^A$ as $B \mapsto 1_B$ where $1_B(x) = 0$ if $x \notin B$ and $1_B(x) = 1$ if $x \in B$. It remains to prove that g is a bijection. ∎

Theorem 1.19. $|A| < |2^A|$

Proof. It follows from Theorems 1.16 and 1.18. ∎

Theorem 1.20. ***Schroeder-Bernstein Theorem.*** *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof. The strategy is to determine whether we can arrange a $D^c = g(f(D)^c)$. Let $f : A \to B$ and $g : B \to A$ be bijections. Define $Q : \mathcal{P}(A) \to \mathcal{P}(A)$ as $E \mapsto [g(f(E)^c)]^c$. We would like to argue that Q has a fixed point. We claim that if $E \subseteq F \subseteq A$ then $Q(E) \subseteq Q(F)$. Clearly, $E \subseteq F$ implies that $f(E) \subseteq f(F)$ and $f(F)^c \subseteq f(E)^c$. From here, we note that $Q(F) = g(f(F)^c) \subseteq g(f(E)^c) = Q(E)$.

Now let $\mathcal{D} = \{E \subseteq A : E \subseteq Q(E)\}$ and put

$$D = \bigcup_{E \in \mathcal{D}} E \subseteq A$$

If $E \in \mathcal{D}$ them $E \subseteq D$. By the claim, $Q(E) \subseteq Q(D)$. If $E \in \mathcal{D}$ then $E \subseteq Q(E) \subseteq (D)$, since $E \subseteq D$. So

$$\bigcup_{E\mathcal{D}} E \subseteq Q(D)$$

which means that $Q(D) \subseteq Q(Q(D))$ and $Q(D) \in \mathcal{D}$, which implies that $Q(D) \subseteq D$ and $D = Q(D)$ (that is, D is our desired fixed point). This argument is simply some symbolic manipulation using an order-preserving argument.

Now, we have that $D = Q(D) = (g(f(D)^c))^c$; i.e. $D^c = g(f(D)^c)$. Define $h : A \to B$ as

$$h(x) = \begin{cases} f(x) & x \in D \\ g^{-1}(x) & x \in D^c \end{cases}$$

where g^{-1} is the restriction of $g : f(D)^c \to g(f(D)^c)$. By construction, h is both injective and surjective, and thus a bijection from A to B. Hence $|A| = |B|$. ■

Example 1.21. If $A_1 \subseteq A_2 \subseteq A_3$ and $|A_1| = |A_3|$, then $|A_1| = |A_2| = |A_3|$. This follows trivially since the embedding maps provide injections in the forward direction and the equal cardinality of A_1 and A_3 provides the existence of a bijection from A_3 to A_1. Then the composition of the embedding of A_2 to A_3 with the bijection is an injection from A_2 to A_1.

Example 1.22. $|(0, 1)| = |[0, 1)| = |\mathbb{R}|$. Since these are nested sets, the embeddings serve as forward injections. We can construct an explicit bijection $h : \mathbb{R} \to (0, 1)$ as

$$h(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$$

and we are done.

Example 1.23. $|\mathbb{R}| = |2^{\mathbb{N}}|$. Our strategy is to show that $|[0, 1)| = |2^{\mathbb{N}}|$. Take $r \in [0, 1)$ and write $r = 0.r_1r_2r_3\dots$ where $r_j \in \{0, 1\}$, that is its binary representation. Note that the only non-unique representations are tails of 1s and tails of 0s. Define $f_r : \mathbb{N} \to \{0, 1\}$ as $f_r(n) = r_n$ (the n-th digit in its binary representation. Define $i : [0, 1) \to 2^{\mathbb{N}}$ as $r \mapsto f_r$. This is an injection since two distinct numbers will have distinct binary representations. Thus $|[0, 1)| \leq |2^{\mathbb{N}}|$.

Now, for the other direction, define the map $f \in 2^{\mathbb{N}} \mapsto 0.0f(1)0f(2)0f(3)\dots \in [0, 1)$. This is injective because non-unique representations end with a tail of 1s or 0s, but this function will never map to the representation with the tail of ones. Thus, by the Schroeder-Bernstein theorem, the result follows.

Definition 1.24. We say that a sequence $\{x_k\}_{k\in\mathbb{N}}$ **converges** to L if and only if, for all $\epsilon > 0$ there exists a natural number $n \in \mathbb{N}$ such that $|x_k - L| < \epsilon$ for all $k > n$.

Definition 1.25. Completeness axiom. We say that a set is **complete** if every bounded increasing sequence converges.

Theorem 1.26. *The completeness axiom is equivalent to:*

1. *Every Cauchy sequence converges.*
2. *The least upper bound property.*

Proof. Exercise. ■

Definition 1.27. We say that the **real numbers**, $\mathbb{R}$ is an ordered field that contains $\mathbb{Q}$ and satisfies the completeness axiom.

Theorem 1.28. *Archimedean Principle.* *Given any $r \in \mathbb{R}$, there exists a natural number N such that $N \geq r$.*

Proof. Arguing by contradiction, suppose this is not true; that is, there exists $r \in \mathbb{R}$ such that $N \leq r$ for all $N \in \mathbb{N}$. We put $x_n = n$. Since it is bounded (by r), it must converge to, say, $L \in \mathbb{R}$.

Take $\epsilon = \frac{1}{3}$. Then, there exists an N such that $|x_n - L| < \frac{1}{3}$ for all $n \geq N$. Then,

$$1 = |x_N - x_{N+1}| \leq |x_N - L| + |L - x_{N+1}| < \frac{2}{3}$$

This is a contradiction, hence, the Archimedean Principle holds. ■

Example 1.29. For every $x < y$ with $x, y \in \mathbb{R}$, there exists a rational number $\frac{p}{q} \in \mathbb{Q}$ such that $x < \frac{p}{q} < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Definition 1.30. Let $S \subseteq \mathbb{R}$. We say that $r \in \mathbb{R}$ is an **upper bound** for S if for every $x \in S$, $x \leq r$. If a set has an upper bound then we say it is **bounded above**.

Definition 1.31. We say that $r \in \mathbb{R}$ is the **least upper bound** of a set S if, for all $y < r$, y is not an upper bound. We say that $r = \sup S$ or $r = \text{lub}S$. Note that r is unique. In a similar manner we can define a **greatest upper bound**.

Theorem 1.32. *Some true facts about least upper bounds:*

1. *If $b \in S$ and b is an upper bound then $b = \sup S$.*
2. *If $\{x_k\}$ is an increasing and bounded sequence and $S = \{x_1, x_2, \ldots\}$ then $\sup S = \lim_{k\to\infty} x_k$.*
3. *$B = \sup S$ if and only if B is an upper bound for S and $\forall \epsilon > 0$, there exists an $x \in S$ such that $x > B - \epsilon$.*

Proof. Exercise. ■

Theorem 1.33. ***Completeness Theorem.*** *If S is a non-empty subset of $\mathbb{R}$ that is bounded above, then $\sup S$ exists.*

Proof. The strategy is to construct a bounded increasing sequence that will converge and argue that the limit of that sequence will in fact be the supremum.

We relax notation and say that $x \geq S$ if $z \geq x$ for all $x \in S$. Pick $y \in S$ (we can do this since $S \neq \emptyset$). Let $x_0 = y - 1$. Now, pick N_0 to be the least integer such that $x_0 + N_0 \geq S$. Note that $N_0 > 0$ and it must exist, because S is bounded above.

Now, put $x_1 = x_0 + N_0 - 1 \geq x_0$. Thus, $x_0 + N_0 - 1 \not\geq S$. Thus, there exists $s_1 \in S$ such that $s_1 > x_1$. Note that $x_1 + 1 = x_0 + N_0 \geq S$.

Choose the least integer N_1 such that $x_1 + \frac{N_1}{2} \geq S$. Clearly $N_1 \neq 0$, so $N_1 \geq 1$. Since $x_1 + 1 \geq S$, then $N_1 \leq 2$. That is, N_1 is either one or two!

Put $x_2 = x_1 + \frac{N_1 - 1}{2} \geq x_1$. By the definition of N_1, there exists an $s_2 \in S$ with $s_2 > x_2$. Simultaneously, $x_2 + frac12 = x_1 + \frac{N_1}{2} \geq S$.

Inductively, define $x_n = x_{n-1} + \frac{N_{n-1} - 1}{n}$ where N_{n-1} is the least integer such that $x_{n-1} + \frac{N_{n-1}}{n} \geq S$. Then, there exists $s_n \in S$ such that $x_n > s_n$ and $x_n + \frac{1}{n} \geq S$. At the next step, $N_n \geq 1$ implies that $x_n \leq x_{n+1}$.

Thus, the sequence $\{x_n\}$ is increasing and bounded above by any upper bound for S. Hence, by the completeness axiom $x_n \to L \in \mathbb{R}$. We claim that $L = \sup S$. First, we note that L is an upper bound for S. Next, suppose there exists $s \in S$ such that $s > L$; then, there exists an N such that $s > L + \frac{1}{N} \geq x_n + \frac{1}{N}$. The remainder of the proof is an exercise. ■

Definition 1.34. We say that a sequence is **Cauchy** if for all $\epsilon > 0$, there exists an $N \in N$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \epsilon$.

Theorem 1.35. *The following are true facts about Cauchy sequences:*

1. *A Cauchy sequence is bounded*
2. *Any convergent sequence is Cauchy*
3. ***Completeness property.*** *Cauchy sequences converge.*

Proof. The first two properties follow from the definition. The third is quite profound and requires some work. We refer to the MATH 147 notes for a proof. The strategy is to prove that there are convergent subsequences with limit L and that the limit of the original Cauchy sequence must be L.

Alternatively, we may argue by the upcoming Theorem 1.40. Since Cauchy sequences are bounded, their limit superior exists (by the completeness property) and there is a subsequence convergent to it, say L. Since the sequence is Cauchy, it follows that it converges to L. ∎

Definition 1.36. We define the terms **limit superior** and **limit inferior** of a bounded sequence $\{x_n\}$. Let $A_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$; note that A_n is increasing and bounded, and thus converges to L. We say that

$$L = \lim A_n = \liminf x_n = \sup_n A_n = \sup_n \left(\inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}\right)$$

is the limit inferior. We can define the limit superior similarly as

$$\limsup x_n = \lim \left(\sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}\right)$$

Note that $\liminf x_n \leq \limsup x_n$ since each $A_n \leq B_n$.

Example 1.37. Define the following sequence,

$$x_n = \begin{cases} 1 + \frac{1}{n} & \text{for } n \text{ even} \\ -\frac{1}{n} & \text{for } n \text{ odd} \end{cases}$$

Then $\limsup x_n = 1$ and $\liminf x_n = 0$.

Theorem 1.38. *$L = \limsup x_n$ if and only if for all $\epsilon > 0$ then $x_n < L + \epsilon$ for all but finitely many n and $x_n > L - \epsilon$ for infinitely many n. Likewise, $L = \liminf x_n$ if and only if for all $\epsilon > 0$ then $x_n < L + \epsilon$ for infinitely many n and $x_n > L - \epsilon$ for all but finitely many n.*

Theorem 1.39. *Every bounded sequence $\{x_n\}$ has a subsequence that converges to $\limsup x_n$ and a subsequence that converges to $\liminf x_n$.*

Proof. For all k we can find $\{n_k\}$ such that $L - \frac{1}{k} < x_{n_k} < L + \frac{1}{k}$ where $L = \limsup x_n$. The terms $\{x_{n_k}\}$ converge to L and the result follows from the squeeze theorem. ∎

Theorem 1.40. *A sequence $\{x_n\}$ converges if and only if $\limsup x_n = \liminf x_n$.*

Proof. ($\Longrightarrow$) Duh.

($\Longleftarrow$) Follows from the characterisation of Theorem 1.38. ∎

Theorem 1.41. ***Bolzano-Weierstrass Theorem.*** *Every bounded sequence has a convergent subsequence.*

Proof. Follows by the existence of the limit superior and limit inferior. ∎

Chapter 2

Metric Spaces

Definition 2.1. A set X is called a **metric space** if there is a function $d : X \times X \to [0, \infty)$ (called a **distance function** or **metric**) which satisfies the following properties:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (the triangle inequality)

We write (X, d) to denote the metric space X with metric d.

Example 2.2. In $\mathbb{R}^n$ we have the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Example 2.3. $(\mathbb{R}^2, d_1)$ is a metric space where $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

Example 2.4. $(\mathbb{R}^2, d_\infty)$ is a metric space where $d_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$.

Example 2.5. If we look at the sets $\{x \in X | d(x, 0) < 1\}$, then we have a circle with the standard norm, a diamond with the d_1 metric, and a square with the d_∞ metric.

Example 2.6. Let X be any set. We define the discrete metric as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Then (X, d) is a metric space.

Example 2.7. Let $X = \{(x_n)_{n=1}^{\infty} | \text{bounded sequences}\} = l^\infty$ and $d_\infty(x, y) = \sup_n |x_n - y_n|$. Then (X, d_∞) is a metric space. It is also a vector space. An interesting vector subspace is c_0, the set of all sequences which converge to zero.

Example 2.8. Let $X = \{(x_n)_{n=1}^{\infty} : \sum |x_n| < \infty\} = l^1$ and $d_1(x, y) = \sum |x_n - y_n|$. This is a metric space.

Example 2.9. Let $X = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\} = l^2$ and $d_2(x, y) = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2}$. This is a metric space which generalises the Euclidean metric. l^2 is an inner product space with the inner product $\langle x, y\rangle = \sum x_n y_n$

Example 2.10. For any real inner product space X with inner product $\langle \cdot, \cdot \rangle$, we can define a metric $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$. The condition which requires some work is the triangle inequality. We can prove this using the Cauchy-Schwartz inequality $\|\langle x, y\rangle\| \leq \|x\| \, \|y\|$.

Definition 2.11. Convergence. Let (X, d) be a metric space and let $x_n, x_0 \in X$. We say that $x_n \to X_0$ if $d(x_n, x_0) \to 0$.

Now we discuss some topology. Cool!

Definition 2.12. We say that the **ball** in (X, d) is the set $B(x_o, r) = \{x \in \mathbb{R}^n : d(x, x_0) < r\}$. Some balls were observed in Example 2.5.

Example 2.13. The balls in the discrete metric are either the whole space X for $r > 1$ or the singleton $\{x_0\}$ for $r \leq 1$. Remark: analysis on this space is either great, because there is only one thing to do, or tragic, because there are very few things to do.

Definition 2.14. Let $U \subseteq X$. We say that $x_0 \in U$ is an **interior point** if there exists an $r > 0$ such that $B(x_0, r) \subseteq U$. We denote the set of interior points of U as $\mathrm{Int}(U)$.

Definition 2.15. We say that a set is **open** if every point in U is an interior point.

Example 2.16. In $\mathbb{R}^2$ the set $\{(x, y) : x \in (0, 1)\}$ is open with respect to d_1, d_2, d_∞. In fact (exercise), in $\mathbb{R}^2$ a set which is open in one metric is open in the other two.

Example 2.17. X and the empty set are always open on any metric space.

Example 2.18. Balls are open sets. The proof of this fact follows from the intuition of what happens in $\mathbb{R}^2$.

Theorem 2.19. *Any finite intersection of open sets is open.*

Proof. It suffices to prove for two open sets U_1, U_2. Let $x \in U_1 \cap U_2$. Since U_j is open and $x \in U_j$ then $\exists r_j > 0$ such that $B(x, r_j) \subseteq U_j$. Take $r = \min(r_1, r_2) > 0$. Then

$$B(x, r) \subseteq B(x, r_1) \cap B(c, r_2) \subseteq U_1 \cap U_2$$

which completes the proof. ■

Theorem 2.20. *Any arbitrary union of open sets is open.*

Proof. Let $x \in \cup_{\alpha \in I} U_\alpha$. Pick α such that $x \in U_\alpha$. Since U_α is open, there exists an r such that $B(x, r) \subseteq U_\alpha \subseteq \cup_{\alpha \in I} U_\alpha$, and we are done. ■

Theorem 2.21. *U is an open set if and only if U is the union of balls.*

($\Longleftarrow$) Balls are open so their union is open.

($\Longrightarrow$) For every $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$. Then $\cup_{x \in U} B(x, r_x) = U$. Easy-piecey! ■

Theorem 2.22. $\mathrm{Int}(U)$ *is the union of all open subsets of U.*

Proof. Let $x \in \mathrm{Int}(U)$. Then, there exists an $r > 0$ such that $B(x, r) \subseteq U$ and that ball is open so, x is contained in the right hand sand expression.

Conversely, if y is in the union of all open sets, then y is in an open set V contained in U. Then $y \in B(y, r) \subseteq V \subseteq U$. Thus $y \in \mathrm{Int}(U)$. ■

Definition 2.23. A set U is said to be **closed** if $U^c = X \setminus U$ is open.

Example 2.24. In X with the discrete metric, every set is closed since every set is open.

Example 2.25. In any metric space, X and $\emptyset$ are closed.

Example 2.26. The singleton set in every metric space is closed.

Theorem 2.27. *Any finite union of closed sets is closed.*

Proof. Let A_1, A_2. Then, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$, where A_i^c is open and their intersection is open by Theorem 2.19. Thus $(A_1 \cup A_2)^c$ is closed. ■

Theorem 2.28. *Any intersection of closed sets is closed.*

Proof. If $A_i \in I$ are closed then, by De Morgan's Laws,

$$\left(\bigcap_{A_i \in I} A_i\right)^c = \bigcup_{A_i \in I} A_i^c$$

of which every set in the right-hand side is open. Thus, by Theorem 2.20, their union is open so the arbitrary intersection is closed. ∎

Definition 2.29. Let $E \subseteq X$. A point $x \in X$ is called an **accumulation point of** E (a.k.a. **cluster or limit point**) if for all $r > 0$, $B(x, r)$ contains a point of E other than x. That is

$$B(x, r) \cap (E \setminus \{x\}) \neq \emptyset \qquad \forall r > 0$$

The points in E that are not accumulation points of E are called **isolated point**.

Theorem 2.30. *The following are awesome facts:*

1. *If x is an accumulation point of E then every ball around x contains infinitely many points of E.*
2. *If x is an accumulation point then any open set containing x has infinitely many points in E.*
3. *A finite set has no accumulation points.*

Proof. From MATH 247. ∎

Theorem 2.31. *A set E is closed if and only if it contains all its accumulation points*[1].

Proof. ($\Longrightarrow$) Assume E is closed. Let $x \notin E$; we claim that x is not an accumulation point of E. Then $x \in E^c$, which is open since E was closed. Thus, we can obtain a ball $B(x, r) \subseteq E^c$ and $B(x, r)$ does not contain any points in E. Hence x is not an accumulation point.

($\Longleftarrow$) Suppose E contains all its accumulation point; we want to show that E^c. The strategy is to take $x \in E^c$ and show it is an interior point of E^c. Since we know that x is not an accumulation point of E, there exists an $r > 0$ such that $B(x, r) \cap E = \emptyset$ and thus $B(x, r) \subset E^c$ and x is an interior point of E^c. Since x was arbitrary, every point in E^c is an interior point so E^c is open. ∎

Definition 2.32. We denote the **closure** of E as $\text{cl}E = \overline{E} = E \cup \{\text{accumulation points of } E\}$. Note that E is closed if and only if $E = \overline{E}$.

Definition 2.33. We say that $E \subseteq X$ is **dense in** X if $\overline{E} = X$.

Example 2.34. Consider the set of all absolutely summable sequences l_1, the set of all sequences which tend to 0, c_0, and the set of all bounded sequences l_∞. Clearly $l_1 \subseteq c_0 \subseteq l_\infty$. Take the metric $d_\infty(x, y) = \sup_n |x_n - y_n|$. Then l_1 is dense in c_0. To prove this we show that $\overline{l_1} = c_0$ by taking $x \in c_0 \setminus l_0$ and showing it is an accumulation point of l_1.

Theorem 2.35. *The following are closed facts of life:*

1. *$\overline{E}$ is closed*
2. *$\overline{E} = \cap_{B \supseteq E} B$ for all closed B.*

Proof.

1. We start at the beginning. Let $x \in (\overline{E})^c$. Then $x \notin E$ and it is not an accumulation point of E. Hence there exists $r > 0$ such that $B(x, r) \cap E = \emptyset$. Furthermore, we claim that $B(x, r) \cap \overline{E} = \emptyset$. Suppose, for contradiction, $z \in B(x, r) \cap \overline{E}$, then z is an accumulation point. Then, every open set containing z contains points in E. So $B(x, r)$ must contain points of E, as it is open and contains z, which is a contradiction. Thus $B(x, r) \cap \overline{E} = \emptyset$. Thus $B(x, r) \subseteq (\overline{E})^c$, making any arbitrary x an interior point of $(\overline{E})^c$, which is open.

[1] Recall that MATH 247 Assignment which had a billion multi-part questions; lest we forget.

2. It should be clear that $\overline{E} \supseteq \cap_{B \supseteq E} B$ since $\overline{E}$ is closed. For the opposite inclusion, we show that each $B \supseteq E$ contains all the accumulation points of E. Let x be an accumulation point of E. Then every $B(x,r) \cap E \neq \emptyset$, thus $B(x,r) \cap B \neq \emptyset$ for all $r > 0$. Thus $x \in B$ or x is an accumulation point of B. But B is closed, so either way, $x \in B$. Hence $E \cup \{\text{accumulation points of } E\} \subseteq B$; namely, $\overline{E} \subseteq B$ for all B closed.

Definition 2.36. The **boundary** of E is the set $\text{bdy}(E) = \overline{E} \cap \overline{E^c}$.

Theorem 2.37. *x is a boundary point of E if and only if every open set that contains x contains points both in E and E^c.*

Definition 2.38. We say that A is **bounded** if $A \subseteq (X,d)$ if there exists x_0 and M such that $A \subseteq B(x_0, M)$.

Definition 2.39. In a metric space (X,d), we say that a sequence (x_n) converges to x if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Theorem 2.40. *$(x_n) \to x$ if and only if every open set containing x contains all but finitely many x_n.*

Proof. ($\Longrightarrow$) For the forward direction, get $B(x,\epsilon) \subseteq U$ and get N such that $x_n \in B(x,\epsilon)$ for all $n \geq N$.

($\Longleftarrow$) $B(x,\epsilon)$ is an open set containing x, so we get N such that $x_n \in B(x,\epsilon)$ for all $n \geq N$. ■

Theorem 2.41. *Limits are unique.*

Proof. Arguing by contradiction, suppose $x_n \to x$ and $x_n \to y$ where $x \neq y$. Then, there exist, $r_1, r_2 > 0$ such that $B(x,r_1) \cap B(y,r_2) = \emptyset$. But we cannot have all but finitely many points falling around both points. Hence the result follows. ■

Theorem 2.42. *Any convergent sequence (x_n) is bounded.*

Proof. Say $d(x_n, x) < 1$ for all $n \geq N$. Then $(x_n) \subseteq B(x, \max(1, 1 + d(x, x_j) : j = 1, \ldots, N-1))$. ■

Definition 2.43. We say that (x_n) is **Cauchy** if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Theorem 2.44. *The following are true about Cauchy sequences:*

1. *Any Cauchy sequence is bounded.*
2. *Every convergent sequence is Cauchy.*
3. *If a Cauchy sequence has a convergent subsequence with limit x, then the Cauchy sequence converges to x.*
4. *Not all Cauchy sequences in a metric space (X,d) converge.*

Proof. The proofs for the first three facts are the same as those for Cauchy sequences in $\mathbb{R}^n$ provided in Chapter 1. For the fourth point, take $\mathbb{Q}$ with the usual metric and take a sequence of rational numbers with converge to $\sqrt{2}$. ■

Theorem 2.45. *$x \in \overline{E}$ if and only if there is a sequence (x_n) in E such that $x_n \to x$.*

Proof. ($\Longrightarrow$) Suppose $x \in \overline{E}$. Then for all n, $B(x, \frac{1}{n}) \cap E \neq \emptyset$. Let $x_n \in B(x, \frac{1}{n}) \cap E$. Since $d(x_n, x) < \frac{1}{n}$ we have that $x_n \to x$.

($\Longleftarrow$) Conversely, let $x = \lim x_n$ with $x_n \in E$. Then, for all $\epsilon > 0$, $B(x,\epsilon) \cap E \neq \emptyset$. In fact, there are infinitely many points in this set for all ϵ Thus $x \in \overline{E}$. ■

Theorem 2.46. *E is closed if and only if whenever $x_n \in E$ and $x_n \to x$ then $x \in E$.*

Proof. E is closed if and only if $E = \overline{E}$. The remainder follows from Theorem 2.45. ■

Definition 2.47. We say that a metric space (X,d) if **complete** if every Cauchy sequence in (X,d) converges to an element in X.

Example 2.48. $\mathbb{R}^n$ is complete. $\mathbb{Q}$ is not complete. Discrete metric spaces are complete since all Cauchy sequences have constant tails. By the same reasoning, $\mathbb{Z}$ is complete.

Theorem 2.49. *If (X,d) is complete and E is closed then E is complete.*

Proof. Let (x_n) be a Cauchy sequence in E. Then it is also a Cauchy sequence in X. Since X is complete, (x_n) has a limit in X, say x_0. Since E is closed, $x_0 \in E$. Thus, any Cauchy sequence in E converges in E and the result follows. ∎

Definition 2.50. An **open cover** of A is a family of open sets $\{G_\alpha\}$ such that $\bigcup_\alpha G_\alpha \supseteq A$. A **subcover of an open cover** is a subset of $\{G_\alpha\}$ which still covers A.

Definition 2.51. A subset of a metric space $A \subseteq X$ is **compact** if every open cover of A has a finite subcover.

Example 2.52. $\mathbb{R}$ is not compact because there are open covers which do not permit finite subcovers; an example is the family of intervals $(-n,n)$ with $n \in \mathbb{N}$. $(0,1)$ is not compact; take the open cover $\left(\frac{1}{n}, 1-\frac{1}{n}\right)$.

Example 2.53. In any metric space, the singleton set is compact. Furthermore, any finite set in X is compact. The weirder fact: in the discrete metric space X, $A \subseteq X$ is compact if and only if A is finite.

Theorem 2.54. *For $A \subseteq \mathbb{R}^n$, the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded*
3. *Every sequence in A has a convergent subsequence with its limit in A.*

*The equivalence of the first two is called the **Heine-Borel Theorem**. The equivalence of* 1 *and* 3 *is called the **Bolzano-Weierstrass Theorem**. Note that Heine-Borel is not true in general metric spaces (see Example 2.53 for a counterexample).*

Proof. We first prove that compactness implies boundedness. Let K be a compact set. We look at the collection of balls $B(x,1)$ for $x \in K$. This collection is an open cover of K and, by compactness, it has a finite subcover, say $\{B(x_1,1),\dots,B(x_N,1)\}$. That is $K \subseteq \bigcup_{j=1}^N B(x_j,1)$. Now we construct a ball that contains all of these balls. Start with x_1 and calculate $d(x_1,x_j)$ for $j=2,\dots,N$. Let $d = \max(d(x_1,x_j) : j=2,\dots,N)$. It can be seen that $K \subseteq B(x_1, 1+d)$, so K is bounded.

(To the reader: need to fill in notes from Friday, May 25.)

Theorem 2.55. ***Cantor's Intersection Theorem.*** *If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are non-empty closed sets in a complete metric space X and $diam A_n \to 0$, where $diam A = \sup\{d(x,y) : x,y \in A\}$, then*

$$\bigcap_{n=1}^{\infty} A_n$$

is exactly one point.

Proof. For every n, choose $x_n \in A_n$. If $n \geq N$, then $x_n \in A_n \subseteq A_N$. So $\{x_n : n \geq N\} \subseteq A_N$. If $m,n \geq N$, $d(x_m,x_n) \leq \text{diam} A_N \to 0$. So, $(x_n)_{n=1}^\infty$ is Cauchy. Since X is complete, $\exists x_0$ such that $x_n \to x_0$. Since $x_n \in A_N, \forall n \geq N$, it must be that $x \in \overline{A_N} = A_n$ (since A_n is closed for every n). Hence, $x_0 \in \cap_{N=1}^\infty A_N$.

Arguing by contradiction, suppose there exists $y \neq x$ such that $y \in \cap_{N=1}^\infty A_N$. Then $d(x,y) \leq \text{diam} A_n$ for all n. Then $d(x,y) = 0$, a contradiction, hence $x = y$. ∎

Definition 2.56. Finite intersection property. We say that a collection of sets has the finite intersection property if every finite intersection is non-empty.

Example 2.57. Nested non-empty sets have the finite intersection property.

Theorem 2.58. *This is a big theorem. The following are equivalent for a metric space X:*

1. *X is compact.*
2. *Every collection of closed subsets of X with the finite intersection property, has non-empty intersection.*
3. *Every sequence in X has convergent subsequence (in X).*
4. *X is complete and totally bounded.*

Remark. The equivalence (1) $\iff$ (3) is the Bolzano-Weierstrass Theorem. The equivalence (1) $\iff$ (4) is the generalised Heine-Borel theorem. In fact, we get the Heine-Borel theorem for free from this, since, in $\mathbb{R}^n$ E is closed if and only if it is complete and E is bounded if and only if E is totally bounded.

Proof. ($1 \implies 2$) Suppose X is compact. Arguing by contradiction, suppose $A_\alpha \subset X$ are closed and $\cap_\alpha A_\alpha = \emptyset$ (the contrapositive of (2)). We will show that this collection does not have the finite intersection property.

We look at A_α^c, which are open and $\cup_\alpha A^c = (\cap_\alpha A_\alpha)^c = \emptyset^c = X$. Thus the collection $\{A_\alpha^c\}$ is an open cover of X. Since X was assumed to be compact, there are finitely many sets $A_{\alpha_1}^c, \ldots, A_{\alpha_k}^c$ whose union covers X. Thus

$$\bigcap_{i=1}^k A_{\alpha_i} = \left(\bigcup_{i=1}^k A_{\alpha_i}^c\right)^c = X^c = \emptyset$$

Hence $\{A_\alpha\}$ does not have the finite intersection property, our desired contradiction. Hence ($1 \implies 2$).

($2 \implies 3$) Let (x_n) be a sequence in X. Define $S_n = \{x_k : k \geq n\}$. Take $\overline{S_n}$ which is closed and non-empty. Since $S_n \supseteq S_{n+1}$, we have that $\overline{S_n} \supseteq \overline{S_{n+1}}$. Being nested, $\{\overline{S_n}\}$ has the finite intersection property. By (2), there exists $x \in \bigcap_{n=1}^\infty \overline{S_n}$. We have that $x \in \overline{S_n}$ for all n. Since $x \in \overline{S_n}$, for all $\epsilon > 0$, there exists $y \in S_n$ such that $d(x, y) < \epsilon$ where $y = x_k$ for some $k \geq n$.

Start at $\epsilon = 1$ and $n = 1$ and get $y \in S$ such that $d(x, y) < 1$. Say y_{k_1}. Apply again with $n = k + 1$ and $\epsilon = \frac{1}{2}$. We get $y = x_{k_2} \in S_n$ so $k_2 > k_1$ and $d(y, x) < \frac{1}{2}$. Repeat this to get $k_1 < k_2 < k_3 < \ldots$ and $y = x_{k_j}$ where $d(x, y) < \frac{1}{2^{j-1}}$. Thus $(x_{k_j})_{j=1}^\infty$ is a subsequence of (x_n) where $d(x, x_{k_j}) < \frac{1}{2^{j-1}} \to 0$ as $j \to \infty$. Hence $(x_{k_j}) \to x$.

($3 \implies 4$) Assume every sequence has a convergent subsequence; we will show that X is complete. Take a Cauchy sequence (x_n); by assumption, it has a convergent subsequence. But if a Cauchy sequence has a convergent subsequence, then the sequence actually converges. So X is complete.

To prove that X is totally bounded, we argue by contradiction; namely, assume X is not totally bounded. Then, $\exists \epsilon > 0$ such that X has no ϵ-net. In particular, $\{x_1\}$ is not an ϵ-net. So $\exists x_2 \in X \setminus B(x, \epsilon)$. That is $d(x_1, x_2) \geq \epsilon$. We now focus on $\{x_1, x_2\}$. So $B(x_1, \epsilon) \cup B(x_2, \epsilon) \neq X$. Hence $\exists x_3$ such that $d(x_j, x_3) \geq \epsilon$ for $j = 1, 2$. We can repeat and get $x_1, x_2, x_3, \ldots$ with the property that $d(x_j, x_i) \geq \epsilon$ for all $i = 1, 2, 3, \ldots, j - 1$. Namely, $d(x_j, x_k) \geq \epsilon$ for all $j \neq k$. Thus, the sequence $(x_n)_{n=1}^\infty$ has no convergent subsequence; a contradiction to (3). Hence, X is totally bounded.

($4 \implies 1$) Assume that X is complete and totally bounded; we need to check that X is compact. Arguing by contradiction, suppose X is not compact; that is, suppose X has an open cover that does not have a finite subcover; call it $\{U_\alpha\}$. Since X is totally bounded, it has a $\frac{1}{2}$-net, say $\{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_{N_1}^{(1)}\}$. Write $D(x, r) = \{y \in X : d(x, y) \leq r\}$, which are closed. We have that

$$\bigcup_{j=1}^N D(x_j^{(1)}, \frac{1}{2}) = X$$

Since X cannot be covered by finitely many $\{U_\alpha\}$, the same is true for at least one of $D(x_j^{(1)}, \frac{1}{2})$, for $j = 1, \ldots, N$; say this occurs for $j = 1$. Hence, we cannot cover $D(x_1^{(1)}, \frac{1}{2}) = X_0$ with finitely many U_α's. Notice that

$diam X_0 \leq 1 = \frac{1}{2^0}$.

Since subsets of totally bounded sets are totally bounded, we get a $\frac{1}{4}$-net for $X_0 \subseteq X_0$. Say the $\frac{1}{4}$-net is $\{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_{N_2}^{(2)}\}$. We have

$$X_0 \subseteq \bigcup_{j=1}^{N_2} D(x_j^{(2)}, \frac{1}{4}) \cap X_0$$

Again, one of the sets $D(x_1^{(2)}, \frac{1}{4}) \cap X_0 = X_1$ cannot be covered by finitely many U_α's. Notice that $X_1 \subseteq X_0$ and $diam X_1 \leq \frac{1}{2}$ with X_1.

We can repeat this process to get the collection $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ where each X_j is closed, non-empty, and cannot be covered by finitely many U_α's, and $diam X_j \leq \frac{1}{2^j} \to 0$. Hence, by the Cantor Intersection Theorem (Theorem 2.55), the intersection of these sets is a singleton:

$$\bigcap_{n=0}^{\infty} X_n = \{x_0\}$$

Since the family $\{U_\alpha\}$ covers X, there exists some index α_0 such that $x_0 \in U_{\alpha_0}$, where U_{α_0} is an open set. Hence $\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U_{\alpha_0}$. Choose N such that $\frac{1}{2^N} < \epsilon$. We know that $x_0 \in X_N$. If $y \in X_N$, then $d(x, y) \leq diam X_N \leq \frac{1}{2^N} < \epsilon$, which imples that $y \in B(x_0, \epsilon) \subseteq U_{\alpha_0}$. This is the same as saying $X_N \subseteq U_{\alpha_0}$, which is a finite subcover. This is a contradiction to the assumption that no X_j had a finite subcover had a finite subcover from the family $\{U_\alpha\}$.

Hence, X is compact. ■

Example 2.59. The Cantor Set. There exists a set $C \subseteq [0, 1]$ which is compact, has an empty interior, is uncountable, and is perfect (that is, it is closed and every point is an accumulation point). We can actually construct such a set.

Let $C_0 = [0, 1]$. Now, we remove the open interval $(\frac{1}{3}, \frac{2}{3})$ to obtain $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, which is the union of intervals, each of length $\frac{1}{3}$. We keep removing the middle thirds of each interval for each step. For instance, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, which is the union of four closed intervals, each of length $\frac{1}{3^2}$.

We can repeat the construction to get C_n, which is the union of 2^n closed intervals, each of length $\frac{1}{3^n}$. Since it is the finite union of closed sets, C_n is closed. Define the Cantor set as

$$C = \bigcap_{n=0}^{\infty} C_n$$

Note that C is closed because it is the intersection of closed sets. Furthermore, note that the endpoint of each closed interval is contained in the construction of C; this collection of endpoints is countable since it is contained in $\mathbb{Q}$. Moreover, C is compact, since it is closed and bounded in $\mathbb{R}$ (using Heine-Borel).

We note that at each C_n we have that the intervals are separated by gaps of length at least $\frac{1}{3^n}$. This implies that the interior is empty; assume by contradiction there is an interior point x which has a ball $(a, b) \subseteq C$ with $b - a \geq \frac{1}{3^n}$. Then $(a, b) \subseteq C_n$ for all C_n, which is impossible, since C_n is the union of intervals of length 3^{-n} separated by gaps.

We show every point is an accumulation point. Let $x \in C$ and we show that for all $\epsilon > 0$, there exists $y \in C$, $y \neq x$ such that $d(x, y) < \epsilon$. Pick N such that $3^{-N} < \epsilon$. Then $x \in C_N$ and x is in one of the levels in the construction of C_N (of length 3^{-N}). Say, without loss of generality, that x is at the endpoint of one of these levels and pick y the other endpoint (so $y \neq x$) of the interval. Then, by construction $d(x, y) = \frac{1}{3^N} < \epsilon$. If we pick a point in the interior of the level, then we can pick y to be either endpoint, and their distance is even less than 3^{-N}.

Now we prove C is uncountable. Write the ternary representation of $x \in C$ as

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} \qquad \text{where } a_j \in \{0, 1, 2\}$$

We can do this for every $x \in [0, 1]$, with $x = .a_1 a_2 a_3 \ldots$. Some arithmetic gymnastics shows that

$$C = \left\{ x = \sum_{j=1}^{\infty} : a_j \in \{0, 2\} \right\}$$

that is, the number 1 will not show up in the ternary representation of any element in C. This construction defines a bijection with the set of binary sequences, which has uncountable cardinality; in fact $|C| = |\mathbb{R}|$. ■