PMATH 351 - Real Analysis

Fantastic Theorems and How to Prove Them

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Contents

Part I

Cardinality and the Real Numbers

Chapter 1

Cardinality

This is a course in the three Cs: completeness, compactness, and continuity. Office hours are on Tuesdays from 3-4pm and Wednesdays from 3:30 to 4:30 in MC5306.

Definition 1.1. Two sets, A, B have the same **cardinality** if there is a bijection between them. In this case we write $|A| = |B|$. We say that $|A| \leq |B|$ if there is an injection from A to B.

Theorem 1.2. *Cardinality is an equivalence relation.*

Proof. Reflexivity is achieved through the identity map; the symmetric property is achieved by f and f^{-1}; transitivity is achieved by the fact that if f and g are bijections then gf is a bijection. ■

Example 1.3. Some infinite sets. We have that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$; thus, the embedding mapping is an injection and we get, for free, that $|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}|$. Enumerations exist to show that the natural, integer, and rational numbers have the same cardinality.

Definition 1.4. We say that A is **countably infinite** if $|A| = |\mathbb{N}|$. We say that A is **countable** if it is either finite or countably infinite. A set that is not countable is said to be **uncountable**.

Example 1.5. The set $\mathbb{N} \times \mathbb{N}$ is countably infinite. We achieve such an enumeration by walking across the lattice diagonally, thus $|\mathbb{N} \times \mathbb{N}| = \mathbb{N}$.

Example 1.6. A similar method as above (walking on $\mathbb{Z} \times \mathbb{N}$) leads to the fact that $|\mathbb{N}| = |\mathbb{Q}|$

Theorem 1.7. *Any countable union of countable sets is countable. That is,*

$$\bigcup_{i=1}^{\infty} A_i$$

is countable if each set A_i is countable.

Proof. We can list each set $A_i = \{a_{i1}, a_{i2}, \ldots\}$. We use a similar argument as above. We have

$$U = \bigcup_{i=1}^{\infty} A_i = \{a_{ij} | i, j \in \mathbb{N}\}$$

Thus, there is a bijection from the union U to $\mathbb{N} \times \mathbb{N}$ and the union is countable. ■

Theorem 1.8. *Every infinite subset of $\mathbb{N}$ is countably infinite.*

Proof. Let B be an infinite subset of $\mathbb{N}$. By the well-ordering principle, B has a least element, say b_1. Naturally, $B \setminus \{b_1\}$ is infinite and has a least element, say b_2. By repeating this procedure, we find a bijection between B and $\mathbb{N}$. More formally, suppose $b \in B$. Consider $\{n \in B : n \leq b\}$, which is a finite set with cardinality, say k. Then we would have arrived at b after k steps; that is, $b = b_k$. ■

Theorem 1.9. *If A is infinite and $|A| \leq |\mathbb{N}|$ then $|A| = |\mathbb{N}|$.*

Proof. Let $j : A \to \mathbb{N}$ be an injection. Let $B = j(A) \subseteq \mathbb{N}$. Notice that $j : A \to B$ is an bijection and $|A| = |B|$ and B is infinite. By Theorem 1.8, it follows that B is countably infinite and, thus, so is A. ■

Theorem 1.10. *The set of real numbers $A = \{x : 0 \leq x < 1\}$ is uncountable.*

Proof. **Cantor's argument.** Arguing by contradiction, suppose A is countable, say $[0, 1) = \{r_i : i \in \mathbb{N}\}]$. Denote the decimal expansion of $r_i = 0.r_{i1}r_{i2}\dots$ with $r_{ij} \in \{0, 1, \dots, 9\}$. Define $a = 0.a_1a_2a_3\dots$ where $a_k = 1$ if $r_{kk} \geq 5$ and $a_k = 8$ if $r_{kk} < 5$. Note that a has a unique representation because it does not end in a tail of zeroes or nines. Not that a cannot equal any of the r_is, since $a_k \neq r_{kk}$ for any k. Since $[0, 1) \neq \{r_i\}$, a contradiction. Thus A is uncountable. ■

Theorem 1.11. $\mathbb{R}$ *is uncountable.*

Proof. Follows trivially from Theorem 1.10 using the embedding map $f : [0, 1) \to \mathbb{R}$ and reaching a contradiction.

Definition 1.12. We say that $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = \aleph_1$.

Theorem 1.13. *The Continuum Hypothesis is undecidable. That is, the question, is there a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$, is unanswerable given the standard axioms of set theory.*

Definition 1.14. Given a set A, the power set of A, $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 1.15. *For a finite set A with $|A| = n$, $|\mathcal{P}(A)| = 2^n$.*

Proof. Trivial.

Theorem 1.16. ***Cantor's Theorem.*** *For any set A, $|A| \leq |\mathcal{P}(A)|$ and $|A| \neq |\mathcal{P}(A)|$.*

Proof. Define an injection by sending an element to its singleton set. Now, arguing by contradiction, suppose we have a bijection $g : A \to \mathcal{P}(A)$. Let $B = \{a \in A : a \notin g(a)\} \subseteq A$. Since $B \subseteq A$, $B \in \mathcal{P}(A)$. Hence $\exists x \in A$ such that $g(x) = B$. Now we ask whether $x \in B$. Suppose yes; that is $x \in B$. Then $x \notin g(x)$ and $x \notin B$, a contradiction. Now, suppose $x \notin B$, then by the definition of B, x must be in B, a contradiction.

Either way, we have just shown that there cannot exist such a bijection g. Thus, $|A| \neq |\mathcal{P}(A)|$. ■

Definition 1.17. We use the notation $2^A = \{f : A \to \{0, 1\}\}$.

Theorem 1.18. $|2^A| = |\mathcal{P}(A)|$

Proof. Define $g : \mathcal{P}(A) \to 2^A$ as $B \mapsto 1_B$ where $1_B(x) = 0$ if $x \notin B$ and $1_B(x) = 1$ if $x \in B$. It remains to prove that g is a bijection. ■

Theorem 1.19. $|A| < |2^A|$

Proof. It follows from Theorems 1.16 and 1.18. ■

Theorem 1.20. ***Schroeder-Bernstein Theorem.*** *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof. The strategy is to determine whether we can arrange a $D^c = g(f(D)^c)$. Let $f : A \to B$ and $g : B \to A$ be bijections. Define $Q : \mathcal{P}(A) \to \mathcal{P}(A)$ as $E \mapsto [g(f(E)^c)]^c$. We would like to argue that Q has a fixed point. We claim that if $E \subseteq F \subseteq A$ then $Q(E) \subseteq Q(F)$. Clearly, $E \subseteq F$ implies that $f(E) \subseteq f(F)$ and $f(F)^c \subseteq f(E)^c$. From here, we note that $Q(F) = g(f(F)^c) \subseteq g(f(E)^c) = Q(E)$.

Now let $\mathcal{D} = \{E \subseteq A : E \subseteq Q(E)\}$ and put

$$D = \bigcup_{E \in \mathcal{D}} E \subseteq A$$

If $E \in \mathcal{D}$ them $E \subseteq D$. By the claim, $Q(E) \subseteq Q(D)$. If $E \in \mathcal{D}$ then $E \subseteq Q(E) \subseteq (D)$, since $E \subseteq D$. So

$$\bigcup_{E\mathcal{D}} E \subseteq Q(D)$$

which means that $Q(D) \subseteq Q(Q(D))$ and $Q(D) \in \mathcal{D}$, which implies that $Q(D) \subseteq D$ and $D = Q(D)$ (that is, D is our desired fixed point). This argument is simply some symbolic manipulation using an order-preserving argument.

Now, we have that $D = Q(D) = (g(f(D)^c))^c$; i.e. $D^c = g(f(D)^c)$. Define $h : A \to B$ as

$$h(x) = \begin{cases} f(x) & x \in D \\ g^{-1}(x) & x \in D^c \end{cases}$$

where g^{-1} is the restriction of $g : f(D)^c \to g(f(D)^c)$. By construction, h is both injective and surjective, and thus a bijection from A to B. Hence $|A| = |B|$. ■

Example 1.21. If $A_1 \subseteq A_2 \subseteq A_3$ and $|A_1| = |A_3|$, then $|A_1| = |A_2| = |A_3|$. This follows trivially since the embedding maps provide injections in the forward direction and the equal cardinality of A_1 and A_3 provides the existence of a bijection from A_3 to A_1. Then the composition of the embedding of A_2 to A_3 with the bijection is an injection from A_2 to A_1.

Example 1.22. $|(0,1)| = |[0,1)| = |\mathbb{R}|$. Since these are nested sets, the embeddings serve as forward injections. We can construct an explicit bijection $h : \mathbb{R} \to (0,1)$ as

$$h(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$$

and we are done.

Example 1.23. $|\mathbb{R}| = |2^{\mathbb{N}}|$. Our strategy is to show that $|[0,1)| = |2^{\mathbb{N}}|$. Take $r \in [0,1)$ and write $r = 0.r_1r_2r_3\ldots$ where $r_j \in \{0,1\}$, that is its binary representation. Note that the only non-unique representations are tails of 1s and tails of 0s. Define $f_r : \mathbb{N} \to \{0,1\}$ as $f_r(n) = r_n$ (the n-th digit of its binary representation). Define $i : [0,1) \to 2^{\mathbb{N}}$ as $r \mapsto f_r$. This is an injection since two distinct numbers will have distinct binary representations. Thus $|[0,1)| \leq |2^{\mathbb{N}}|$.

Now, for the other direction, define the map $f \in 2^{\mathbb{N}} \mapsto 0.0f(1)0f(2)0f(3)\ldots \in [0,1)$. This is injective because non-unique representations end with a tail of 1s or 0s, but this function will never map to the representation with the tail of ones. Thus, by the Schroeder-Bernstein theorem, the result follows.

Definition 1.24. We say that a sequence $\{x_k\}_{k\in\mathbb{N}}$ **converges** to L if and only if, for all $\epsilon > 0$ there exists a natural number $n \in \mathbb{N}$ such that $|x_k - L| < \epsilon$ for all $k > n$.

Chapter 2

Sequences and Real Numbers

Definition 2.1. Completeness axiom. We say that a set in $\mathbb{R}$ is **complete** if every bounded increasing sequence converges.

Theorem 2.2. *The completeness axiom is equivalent to:*

1. *Every Cauchy sequence converges.*

2. *The least upper bound property.*

Proof. Exercise. ■

Definition 2.3. We say that the **real numbers**, $\mathbb{R}$ are an ordered field that contains $\mathbb{Q}$ and satisfies the completeness axiom.

Theorem 2.4. *Archimedean Principle.* *Given any $r \in \mathbb{R}$, there exists a natural number N such that $N \geq r$.*

Proof. Arguing by contradiction, suppose this is not true; that is, there exists $r \in \mathbb{R}$ such that $N \leq r$ for all $N \in \mathbb{N}$. We put $x_n = n$. Since it is bounded (by r), it must converge to, say, $L \in \mathbb{R}$.

Take $\epsilon = \frac{1}{3}$. Then, there exists an N such that $|x_n - L| < \frac{1}{3}$ for all $n \geq N$. Then,

$$1 = |x_N - x_{N+1}| \leq |x_N - L| + |L - x_{N+1}| < \frac{2}{3}$$

This is a contradiction, hence, the Archimedean Principle holds. ■

Example 2.5. For every $x < y$ with $x, y \in \mathbb{R}$, there exists a rational number $\frac{p}{q} \in \mathbb{Q}$ such that $x < \frac{p}{q} < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Definition 2.6. Let $S \subseteq \mathbb{R}$. We say that $r \in \mathbb{R}$ is an **upper bound** for S if for every $x \in S$, $x \leq r$. If a set has an upper bound then we say it is **bounded above**.

Definition 2.7. We say that $r \in \mathbb{R}$ is the **least upper bound** of a set S if, for all $y < r$, y is not an upper bound. We say that $r = \sup S$ or $r = \text{lub} S$. Note that r is unique. In a similar manner we can define a **greatest lower bound**.

Theorem 2.8. *Some true facts about least upper bounds:*

1. *If $b \in S$ and b is an upper bound then $b = \sup S$.*

2. *If $\{x_k\}$ is an increasing and bounded sequence and $S = \{x_1, x_2, \ldots\}$ then $\sup S = \lim_{k \to \infty} x_k$.*

3. *$B = \sup S$ if and only if B is an upper bound for S and $\forall \epsilon > 0$, there exists an $x \in S$ such that $x > B - \epsilon$.*

Proof. Exercise. ■

Theorem 2.9. ***Completeness Theorem.*** *If S is a non-empty subset of $\mathbb{R}$ that is bounded above, then* $\sup S$ *exists.*

Proof. The strategy is to construct a bounded increasing sequence that will converge and argue that the limit of that sequence will in fact be the supremum.

We relax notation and say that $z \geq S$ if $z \geq x$ for all $x \in S$. Pick $y \in S$ (we can do this since $S \neq \emptyset$). Let $x_0 = y - 1$. Now, pick N_0 to be the least integer such that $x_0 + N_0 \geq S$. Note that $N_0 > 0$ and it must exist, because S is bounded above.

Now, put $x_1 = x_0 + N_0 - 1 \geq x_0$. Thus, $x_0 + N_0 - 1 \ngeq S$. Thus, there exists $s_1 \in S$ such that $s_1 > x_1$. Note that $x_1 + 1 = x_0 + N_0 \geq S$.

Choose the least integer N_1 such that $x_1 + \frac{N_1}{2} \geq S$. Clearly $N_1 \neq 0$, so $N_1 \geq 1$. Since $x_1 + 1 \geq S$, then $N_1 \leq 2$. That is, N_1 is either one or two!

Put $x_2 = x_1 + \frac{N_1 - 1}{2} \geq x_1$. By the definition of N_1, there exists an $s_2 \in S$ with $s_2 > x_2$. Simultaneously, $x_2 + \frac{1}{2} = x_1 + \frac{N_1}{2} \geq S$.

Inductively, define $x_n = x_{n-1} + \frac{N_{n-1} - 1}{n}$ where N_{n-1} is the least integer such that $x_{n-1} + \frac{N_{n-1}}{n} \geq S$. Then, there exists $s_n \in S$ such that $x_n > s_n$ and $x_n + \frac{1}{n} \geq S$. At the next step, $N_n \geq 1$ implies that $x_n \leq x_{n+1}$.

Thus, the sequence $\{x_n\}$ is increasing and bounded above by any upper bound for S. Hence, by the completeness axiom $x_n \to L \in \mathbb{R}$. We claim that $L = \sup S$. First, we note that L is an upper bound for S. Next, suppose there exists $s \in S$ such that $s > L$; then, there exists an N such that $s > L + \frac{1}{N} \geq x_n + \frac{1}{N}$. The remainder of the proof is an exercise. ■

Definition 2.10. We say that a sequence is **Cauchy** if for all $\epsilon > 0$, there exists an $N \in N$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \epsilon$.

Theorem 2.11. *The following are true facts about Cauchy sequences:*

1. *A Cauchy sequence is bounded*
2. *Any convergent sequence is Cauchy*
3. ***Completeness property.*** *Cauchy sequences converge.*

Proof. The first two properties follow from the definition. The third is quite profound and requires some work. We refer to the MATH 147 notes for a proof. The strategy is to prove that there are convergent subsequences with limit L and that the limit of the original Cauchy sequence must be L.

Alternatively, we may argue by the upcoming Theorem 1.40. Since Cauchy sequences are bounded, their limit superior exists (by the completeness property) and there is a subsequence convergent to it, say L. Since the sequence is Cauchy, it follows that it converges to L. ■

Definition 2.12. We define the terms **limit superior** and **limit inferior** of a bounded sequence $\{x_n\}$. Let $A_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$; note that A_n is increasing and bounded, and thus converges to L. We say that

$$L = \lim A_n = \liminf x_n = \sup_n A_n = \sup_n \left(\inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}\right)$$

is the limit inferior. We can define the limit superior similarly as

$$\limsup x_n = \lim \left(\sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}\right)$$

Note that $\liminf x_n \leq \limsup x_n$ since each $A_n \leq B_n$.

Example 2.13. Define the following sequence,

$$x_n = \begin{cases} 1 + \frac{1}{n} & \text{for } n \text{ even} \\ -\frac{1}{n} & \text{for } n \text{ odd} \end{cases}$$

Then $\limsup x_n = 1$ and $\liminf x_n = 0$.

Theorem 2.14. *$L = \limsup x_n$ if and only if for all $\epsilon > 0$ then $x_n < L + \epsilon$ for all but finitely many n and $x_n > L - \epsilon$ for infinitely many n. Likewise, $L = \liminf x_n$ if and only if for all $\epsilon > 0$ then $x_n < L + \epsilon$ for infinitely many n and $x_n > L - \epsilon$ for all but finitely many n.*

Theorem 2.15. *Every bounded sequence $\{x_n\}$ has a subsequence that converges to $\limsup x_n$ and a subsequence that converges to $\liminf x_n$.*

Proof. For all k we can find $\{n_k\}$ such that $L - \frac{1}{k} < x_{n_k} < L + \frac{1}{k}$ where $L = \limsup x_n$. The terms $\{x_{n_k}\}$ converge to L and the result follows from the squeeze theorem. ■

Theorem 2.16. *A sequence $\{x_n\}$ converges if and only if $\limsup x_n = \liminf x_n$.*

Proof. ($\Longrightarrow$) Duh.

($\Longleftarrow$) Follows from the characterisation of Theorem 1.38. ■

Theorem 2.17. ***Bolzano-Weierstrass Theorem.*** *Every bounded sequence has a convergent subsequence.*

Proof. Follows by the existence of the limit superior and limit inferior. ■

Part II

Metric Spaces

Chapter 3

Topology of Metric Spaces

Definition 3.1. A set X is called a **metric space** if there is a function $d : X \times X \to [0, \infty)$ (called a **distance function** or **metric**) which satisfies the following properties:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (the triangle inequality)

We write (X, d) to denote the metric space X with metric d.

Example 3.2. In $\mathbb{R}^n$ we have the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Example 3.3. $(\mathbb{R}^2, d_1)$ is a metric space where $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

Example 3.4. $(\mathbb{R}^2, d_\infty)$ is a metric space where $d_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$.

Example 3.5. If we look at the sets $\{x \in X | d(x, 0) < 1\}$, then we have a circle with the standard norm, a diamond with the d_1 metric, and a square with the d_∞ metric.

Example 3.6. Let X be any set. We define the discrete metric as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Then (X, d) is a metric space.

Example 3.7. Let $X = \{(x_n)_{n=1}^{\infty} | \text{bounded sequences}\} = l^\infty$ and $d_\infty(x, y) = \sup_n |x_n - y_n|$. Then (X, d_∞) is a metric space. It is also a vector space. An interesting vector subspace is c_0, the set of all sequences which converge to zero.

Example 3.8. Let $X = \{(x_n)_{n=1}^{\infty} : \sum |x_n| < \infty\} = l^1$ and $d_1(x, y) = \sum |x_n - y_n|$. This is a metric space.

Example 3.9. Let $X = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\} = l^2$ and $d_2(x, y) = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2}$. This is a metric space which generalises the Euclidean metric. l^2 is an inner product space with the inner product $\langle x, y \rangle = \sum x_n y_n$

Example 3.10. For any real inner product space X with inner product $\langle \cdot, \cdot \rangle$, we can define a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$. The condition which requires some work is the triangle inequality. We can prove this using the Cauchy-Schwartz inequality $\|\langle x, y \rangle\| \leq \|x\| \, \|y\|$.

Definition 3.11. Convergence. Let (X, d) be a metric space and let $x_n, x_0 \in X$. We say that $x_n \to X_0$ if $d(x_n, x_0) \to 0$.

Now we discuss some topology. Cool!

Definition 3.12. We say that the **ball** in (X, d) is the set $B(x_0, r) = \{x \in \mathbb{R}^n : d(x, x_0) < r\}$. Some balls were observed in Example 2.5.

Example 3.13. The balls in the discrete metric are either the whole space X for $r > 1$ or the singleton $\{x_0\}$ for $r \leq 1$.

Remark. Analysis on the discrete metric space is either great, because there is only one thing to do, or tragic, because there are very few things to do.

Definition 3.14. Let $U \subseteq X$. We say that $x_0 \in U$ is an **interior point** if there exists an $r > 0$ such that $B(x_0, r) \subseteq U$. We denote the set of interior points of U as $\text{Int}(U)$ or U°.

Definition 3.15. We say that a set is **open** if every point in U is an interior point.

Example 3.16. In $\mathbb{R}^2$ the set $\{(x, y) : x \in (0, 1)\}$ is open with respect to d_1, d_2, d_∞. In fact (exercise), in $\mathbb{R}^2$ a set which is open in one metric is open in the other two.

Example 3.17. X and the empty set are always open on any metric space.

Example 3.18. Balls are open sets. The proof of this fact follows from the intuition of what happens in $\mathbb{R}^2$.

Theorem 3.19. *Any finite intersection of open sets is open.*

Proof. It suffices to prove for two open sets U_1, U_2. Let $x \in U_1 \cap U_2$. Since U_j is open and $x \in U_j$ then $\exists r_j > 0$ such that $B(x, r_j) \subseteq U_j$. Take $r = \min(r_1, r_2) > 0$. Then

$$B(x, r) \subseteq B(x, r_1) \cap B(c, r_2) \subseteq U_1 \cap U_2$$

which completes the proof. ∎

Theorem 3.20. *Any arbitrary union of open sets is open.*

Proof. Let $x \in \cup_{\alpha \in I} U_\alpha$. Pick α such that $x \in U_\alpha$. Since U_α is open, there exists an r such that $B(x, r) \subseteq U_\alpha \subseteq \cup_{\alpha \in I} U_\alpha$, and we are done. ∎

Theorem 3.21. *U is an open set if and only if U is the union of balls.*

($\Longleftarrow$) Balls are open so their union is open.

($\Longrightarrow$) For every $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$. Then $\cup_{x \in U} B(x, r_x) = U$. Easy-piecey! ∎

Theorem 3.22. $\text{Int}(U)$ *is the union of all open subsets of U.*

Proof. Let $x \in \text{Int}(U)$. Then, there exists an $r > 0$ such that $B(x, r) \subseteq U$ and that ball is open so, x is contained in the right hand sand expression.

Conversely, if y is in the union of all open sets, then y is in an open set V contained in U. Then $y \in B(y, r) \subseteq V \subseteq U$. Thus $y \in \text{Int}(U)$. ∎

Definition 3.23. A set U is said to be **closed** if $U^c = X \setminus U$ is open.

Example 3.24. In X with the discrete metric, every set is closed since every set is open.

Example 3.25. In any metric space, X and $\emptyset$ are closed.

Example 3.26. The singleton set in every metric space is closed.

Theorem 3.27. *Any finite union of closed sets is closed.*

Proof. Let A_1, A_2. Then, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$, where A_i^c is open and their intersection is open by Theorem 2.19. Thus $(A_1 \cup A_2)^c$ is closed. ∎

Theorem 3.28. *Any intersection of closed sets is closed.*

Proof. If $A_i \in I$ are closed then, by De Morgan's Laws,

$$\left(\bigcap_{A_i \in I} A_i\right)^c = \bigcup_{A_i \in I} A_i^c$$

of which every set in the right-hand side is open. Thus, by Theorem 2.20, their union is open so the arbitrary intersection is closed. ∎

Definition 3.29. Let $E \subseteq X$. A point $x \in X$ is called an **accumulation point of** E (a.k.a. **cluster or limit point**) if for all $r > 0$, $B(x,r)$ contains a point of E other than x. That is

$$B(x,r) \cap (E \setminus \{x\}) \neq \emptyset \qquad \forall r > 0$$

The points in E that are not accumulation points of E are called **isolated point**.

Theorem 3.30. *The following are awesome facts:*

1. *If x is an accumulation point of E then every ball around x contains infinitely many points of E.*
2. *If x is an accumulation point then any open set containing x has infinitely many points in E.*
3. *A finite set has no accumulation points.*

Proof. From MATH 247. ∎

Theorem 3.31. *A set E is closed if and only if it contains all its accumulation points*[1]*.*

Proof. ($\Longrightarrow$) Assume E is closed. Let $x \notin E$; we claim that x is not an accumulation point of E. Then $x \in E^c$, which is open since E was closed. Thus, we can obtain a ball $B(x,r) \subseteq E^c$ and $B(x,r)$ does not contain any points in E. Hence x is not an accumulation point.

($\Longleftarrow$) Suppose E contains all its accumulation point; we want to show that E^c is open. The strategy is to take $x \in E^c$ and show it is an interior point of E^c. Since we know that x is not an accumulation point of E, there exists an $r > 0$ such that $B(x,r) \cap E = \emptyset$ and thus $B(x,r) \subset E^c$ and x is an interior point of E^c. Since x was arbitrary, every point in E^c is an interior point so E^c is open. ∎

Definition 3.32. We denote the **closure** of E as $\mathrm{cl}E = \overline{E} = E \cup \{\text{accumulation points of } E\}$. Note that E is closed if and only if $E = \overline{E}$.

Definition 3.33. We say that $E \subseteq X$ is **dense in** X if $\overline{E} = X$.

Example 3.34. Consider the set of all absolutely summable sequences l_1, the set of all sequences which tend to 0, c_0, and the set of all bounded sequences l_∞. Clearly $l_1 \subseteq c_0 \subseteq l_\infty$. Take the metric $d_\infty(x,y) = \sup_n |x_n - y_n|$. Then l_1 is dense in c_0. To prove this we show that $\overline{l_1} = c_0$ by taking $x \in c_0 \setminus l_0$ and showing it is an accumulation point of l_1.

Theorem 3.35. *The following are closed facts of life:*

1. *$\overline{E}$ is closed*
2. *$\overline{E} = \cap_{B \supseteq E} B$ for all closed B.*

Proof.

1. We start at the beginning. Let $x \in (\overline{E})^c$. Then $x \notin E$ and it is not an accumulation point of E. Hence there exists $r > 0$ such that $B(x,r) \cap E = \emptyset$. Furthermore, we claim that $B(x,r) \cap \overline{E} = \emptyset$. Suppose, for contradiction, $z \in B(x,r) \cap \overline{E}$, then z is an accumulation point. Then, every open set containing z contains points in E. So $B(x,r)$ must contain points of E, as it is open and contains z, which is a contradiction. Thus $B(x,r) \cap \overline{E} = \emptyset$. Thus $B(x,r) \subseteq (\overline{E})^c$, making any arbitrary x an interior point of $(\overline{E})^c$, which is open.

[1] Recall that MATH 247 Assignment which had a billion multi-part questions; lest we forget.

2. It should be clear that $\overline{E} \supseteq \cap_{B \supseteq E} B$ since $\overline{E}$ is closed. For the opposite inclusion, we show that each $B \supseteq E$ contains all the accumulation points of E. Let x be an accumulation point of E. Then every $B(x,r) \cap E \neq \emptyset$, thus $B(x,r) \cap B \neq \emptyset$ for all $r > 0$. Thus $x \in B$ or x is an accumulation point of B. But B is closed, so either way, $x \in B$. Hence $E \cup \{\text{accumulation points of } E\} \subseteq B$; namely, $\overline{E} \subseteq B$ for all B closed.

Definition 3.36. The **boundary** of E is the set $\partial E = \text{bdy}(E) = \overline{E} \cap \overline{E^c}$.

Theorem 3.37. *x is a boundary point of E if and only if every open set that contains x contains points both in E and E^c.*

Definition 3.38. We say that A is **bounded** if $A \subseteq (X,d)$ if there exists x_0 and M such that $A \subseteq B(x_0, M)$.

Definition 3.39. In a metric space (X,d), we say that a sequence (x_n) converges to x if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Theorem 3.40. *$(x_n) \to x$ if and only if every open set containing x contains all but finitely many x_n.*

Proof. ($\Longrightarrow$) For the forward direction, get $B(x,\epsilon) \subseteq U$ and get N such that $x_n \in B(x,\epsilon)$ for all $n \geq N$.

($\Longleftarrow$) $B(x,\epsilon)$ is an open set containing x, so we get N such that $x_n \in B(x,\epsilon)$ for all $n \geq N$. ■

Theorem 3.41. *Limits are unique.*

Proof. Arguing by contradiction, suppose $x_n \to x$ and $x_n \to y$ where $x \neq y$. Then, there exist, $r_1, r_2 > 0$ such that $B(x, r_1) \cap B(y, r_2) = \emptyset$. But we cannot have all but finitely many points falling around both points. Hence the result follows. ■

Theorem 3.42. *Any convergent sequence (x_n) is bounded.*

Proof. Say $d(x_n, x) < 1$ for all $n \geq N$. Then $(x_n) \subseteq B(x, \max(1, 1 + d(x, x_j) : j = 1, \ldots, N-1))$. ■

Definition 3.43. We say that (x_n) is **Cauchy** if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Theorem 3.44. *The following are true about Cauchy sequences:*

1. *Any Cauchy sequence is bounded.*

2. *Every convergent sequence is Cauchy.*

3. *If a Cauchy sequence has a convergent subsequence with limit x, then the Cauchy sequence converges to x.*

4. *Not all Cauchy sequences in a metric space (X,d) converge.*

Proof. The proofs for the first three facts are the same as those for Cauchy sequences in $\mathbb{R}^n$ provided in Chapter 1. For the fourth point, take $\mathbb{Q}$ with the usual metric and take a sequence of rational numbers with converge to $\sqrt{2}$. ■

Theorem 3.45. *$x \in \overline{E}$ if and only if there is a sequence (x_n) in E such that $x_n \to x$.*

Proof. ($\Longrightarrow$) Suppose $x \in \overline{E}$. Then for all n, $B(x, \frac{1}{n}) \cap E \neq \emptyset$. Let $x_n \in B(x, \frac{1}{n}) \cap E$. Since $d(x_n, x) < \frac{1}{n}$ we have that $x_n \to x$.

($\Longleftarrow$) Conversely, let $x = \lim x_n$ with $x_n \in E$. Then, for all $\epsilon > 0$, $B(x,\epsilon) \cap E \neq \emptyset$. In fact, there are infinitely many points in this set for all ϵ Thus $x \in \overline{E}$. ■

Theorem 3.46. *E is closed if and only if whenever $x_n \in E$ and $x_n \to x$ then $x \in E$.*

Proof. E is closed if and only if $E = \overline{E}$. The remainder follows from Theorem 2.45. ■

Chapter 4

The Three Big C's: Completeness, Compactness, and Continuity

4.1 Completeness and Compactness

Definition 4.1. We say that a metric space (X, d) if **complete** if every Cauchy sequence in (X, d) converges to an element in X.

Example 4.2. $\mathbb{R}^n$ is complete. $\mathbb{Q}$ is not complete. Discrete metric spaces are complete since all Cauchy sequences have constant tails. By the same reasoning, $\mathbb{Z}$ is complete.

Theorem 4.3. *If (X, d) is complete and $E \subseteq X$ is closed then E is complete.*

Proof. Let (x_n) be a Cauchy sequence in E. Then it is also a Cauchy sequence in X. Since X is complete, (x_n) has a limit in X, say x_0. Since E is closed, $x_0 \in E$. Thus, any Cauchy sequence in E converges in E and the result follows. ■

Definition 4.4. An **open cover** of A is a family of open sets $\{G_\alpha\}$ such that $\bigcup_\alpha G_\alpha \supseteq A$. A **subcover of an open cover** is a subset of $\{G_\alpha\}$ which still covers A.

Definition 4.5. A subset of a metric space $A \subseteq X$ is **compact** if every open cover of A has a finite subcover.

Example 4.6. $\mathbb{R}$ is not compact because there are open covers which do not permit finite subcovers; an example is the family of intervals $(-n, n)$ with $n \in \mathbb{N}$. $(0, 1)$ is not compact; take the open cover $\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$.

Example 4.7. In any metric space, the singleton set is compact. Furthermore, any finite set in X is compact. The weirder fact: in the discrete metric space X, $A \subseteq X$ is compact if and only if A is finite.

Theorem 4.8. *For $A \subseteq \mathbb{R}^n$, the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded*
3. *Every sequence in A has a convergent subsequence with its limit in A.*

*The equivalence of the first two is called the **Heine-Borel Theorem**. The equivalence of* 1 *and* 3 *is called the **Bolzano-Weierstrass Theorem**. Note that Heine-Borel is not true in general metric spaces (see Example 2.53 for a counterexample). Thus, we provide the first part of a replacement theorem for a general metric space.*

Proof. We first prove that compactness implies boundedness. Let K be a compact set. We look at the collection of balls $B(x, 1)$ for $x \in K$. This collection is an open cover of K and, by compactness, it has a finite subcover, say $\{B(x_1, 1), \ldots, B(x_N, 1)\}$. That is $K \subseteq \bigcup_{j=1}^N B(x_j, 1)$. Now we construct a ball that contains all of these balls. Start with x_1 and calculate $d(x_1, x_j)$ for $j = 2, \ldots, N$. Let $d = \max(d(x_1, x_j) : j = 2, \ldots, N)$. It can be seen that $K \subseteq B(x_1, 1 + d)$, so K is bounded.

Now we show that compactness implies that the set is closed. Let $K \subseteq X$ be compact and pick $x \in K^c$. Consider $U_n = \{y \in X : d(x,y) > \frac{1}{n}\}$. Then each U_n is open. We observe that

$$\bigcup_{n=1}^{\infty} U_n = X \setminus \{x\}$$

thus making (U_n) an open cover of K. Since K is compact, we can product a finite subcover $U_{n_1}, U_{n_2}, \dots, U_{n_l}$ for K. Since these sets are nested, $K \subseteq U_{n_l}$. Thus $x \in B\left(x, \frac{1}{n_l}\right) \subseteq K^c$. Since x was arbitrarily chosen in K^c, K^c is open, thus K is closed. ■

Definition 4.9. A finite set $\{x_1, x_2, \dots, x_n\} \subseteq X$ is called an **ϵ-net** for $A \subseteq X$ if each point of A has distance less than ϵ for some point x_j for $j = 1, \dots, n$. Namely,

$$A \subseteq \bigcup_{k=1}^{n} B(x_k, \epsilon)$$

Definition 4.10. A said A is said to be **totally bounded** if it has an ϵ-net for all $\epsilon > 0$.

Example 4.11. Any infinite discrete metric space is not totally bounded because there can be no ϵ-net for $\epsilon \leq 1$.

Theorem 4.12. *If $A \subseteq X$ is totally bounded, it is bounded.*

Proof. Since A is totally bounded, take a 1-net of A conformed by $\{x_1, \dots, x_n\}$. Then

$$A \subseteq \bigcup_{j=1}^{n} B(x_j, 1)$$

Let $r = \max\{d(x_1, x_i) : 2 \leq i \leq n\}$. Then $A \subseteq B(x_1, r+1)$, which means A is bounded. ■

Remark. If A is totally bounded, any subset of A is bounded using the same ϵ-net.

Theorem 4.13. *Compact sets are totally bounded.*

Proof. Fix $\epsilon > 0$. Let K be a compact set with an open cover $\{B(x_i, \epsilon)\}$ where $x_i \in K$ which permits a finite subcover $B(x_{i_1}, \epsilon), \dots, B(x_{i_k}, \epsilon)$, which turns out to be an ϵ-net for free. ■

Theorem 4.14. *If $A \subseteq X$ is totally bounded then $\overline{A}$ is totally bounded.*

Proof. Fix $\epsilon > 0$. Let $\{x_1, \dots, x_n\}$ be an $\frac{\epsilon}{2}$-net for A. Let y be an accumulation point of A. Then, there exists some $x \in A$ satisfying $d(x,y) < \frac{\epsilon}{2}$. Thus, for some x_i in our net, we have

$$d(x_i, y) \leq d(x_i, x) + d(x, y) < \epsilon$$

thus forming an ϵ-net for $\overline{A}$. ■

Definition 4.15. For a non-empty set $E \subseteq X$, the diameter of E is defined as

$$\operatorname{diam}(E) = \sup\{d(x,y)\, x, y \in E\}$$

Theorem 4.16. ***Cantor's Intersection Theorem.*** *If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are non-empty closed sets in a complete metric space X and $\operatorname{diam} A_n \to 0$, where $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$, then*

$$\bigcap_{n=1}^{\infty} A_n$$

is exactly one point.

Proof. For every n, choose $x_n \in A_n$. If $n \geq N$, then $x_n \in A_n \subseteq A_N$. So $\{x_n : n \geq N\} \subseteq A_N$. If $m, n \geq N$, $d(x_m, x_n) \leq \text{diam} A_N \to 0$. So, $(x_n)_{n=1}^{\infty}$ is Cauchy. Since X is complete, $\exists x_0$ such that $x_n \to x_0$. Since $x_n \in A_N, \forall n \geq N$, it must be that $x_0 \in \overline{A_N} = A_N$ (since A_n is closed for every n). Hence, $x_0 \in \cap_{N=1}^{\infty} A_N$.

Arguing by contradiction, suppose there exists $y \neq x$ such that $y \in \cap_{N=1}^{\infty} A_N$. Then $d(x, y) \leq \text{diam} A_n$ for all n. Then $d(x, y) = 0$, a contradiction, hence $x = y$. ■

Definition 4.17. Finite intersection property. We say that a collection of sets has the finite intersection property if every finite intersection is non-empty.

Example 4.18. Nested non-empty sets have the finite intersection property.

Theorem 4.19. *This is a big theorem. The following are equivalent for a metric space X:*

1. *X is compact.*
2. *Every collection of closed subsets of X with the finite intersection property, has non-empty intersection.*
3. *Every sequence in X has convergent subsequence (in X).*
4. *X is complete and totally bounded.*

Remark. The equivalence (1) $\iff$ (3) is the Bolzano-Weierstrass Theorem. The equivalence (1) $\iff$ (4) is the generalised Heine-Borel theorem. In fact, we get the Heine-Borel theorem for free from this, since, in $\mathbb{R}^n$ E is closed if and only if it is complete and E is bounded if and only if E is totally bounded.

Proof. ($1 \implies 2$) Suppose X is compact; we prove the contrapositive of (2). Suppose $A_\alpha \subset X$ are closed and $\cap_\alpha A_\alpha = \emptyset$. We will show that this collection does not have the finite intersection property.

We look at A_α^c, which are open and $\cup_\alpha A^c = (\cap_\alpha A_\alpha)^c = \emptyset^c = X$. Thus the collection $\{A_\alpha^c\}$ is an open cover of X. Since X was assumed to be compact, there are finitely many sets $A_{\alpha_1}^c, \ldots, A_{\alpha_k}^c$ whose union covers X. Thus

$$\bigcap_{i=1}^{k} A_{\alpha_i} = \left(\bigcup_{i=1}^{k} A_{\alpha_i}^c \right)^c = X^c = \emptyset$$

Hence $\{A_\alpha\}$ does not have the finite intersection property, our desired contradiction. Hence ($1 \implies 2$).

($2 \implies 3$) Let (x_n) be a sequence in X. Define $S_n = \{x_k : k \geq n\}$. Take $\overline{S_n}$ which is closed and non-empty. Since $S_n \supseteq S_{n+1}$, we have that $\overline{S_n} \supseteq \overline{S_{n+1}}$. Being nested, $\{\overline{S_n}\}$ has the finite intersection property. By (2), there exists $x \in \bigcap_{n=1}^{\infty} \overline{S_n}$. We have that $x \in \overline{S_n}$ for all n. Since $x \in \overline{S_n}$, for all $\epsilon > 0$, there exists $y \in S_n$ such that $d(x, y) < \epsilon$ where $y = x_k$ for some $k \geq n$.

Start at $\epsilon = 1$ and $n = 1$ and get $y \in S$ such that $d(x, y) < 1$. Say y_{k_1}. Apply again with $n = k + 1$ and $\epsilon = \frac{1}{2}$. We get $y = x_{k_2} \in S_n$ so $k_2 > k_1$ and $d(y, x) < \frac{1}{2}$. Repeat this to get $k_1 < k_2 < k_3 < \ldots$ and $y = x_{k_j}$ where $d(x, y) < \frac{1}{2^{j-1}}$. Thus $(x_{k_j})_{j=1}^{\infty}$ is a subsequence of (x_n) where $d(x, x_{k_j}) < \frac{1}{2^{j-1}} \to 0$ as $j \to \infty$. Hence $(x_{k_j}) \to x$.

($3 \implies 4$) Assume every sequence has a convergent subsequence; we will show that X is complete. Take a Cauchy sequence (x_n); by assumption, it has a convergent subsequence. But if a Cauchy sequence has a convergent subsequence, then the sequence actually converges. So X is complete.

To prove that X is totally bounded, we argue by contradiction; namely, assume X is not totally bounded. Then, $\exists \epsilon > 0$ such that X has no ϵ-net. In particular, $\{x_1\}$ is not an ϵ-net. So $\exists x_2 \in X \setminus B(x_1, \epsilon)$. That is $d(x_1, x_2) \geq \epsilon$. We now focus on $\{x_1, x_2\}$. So $B(x_1, \epsilon) \cup B(x_2, \epsilon) \neq X$. Hence $\exists x_3$ such that $d(x_j, x_3) \geq \epsilon$ for $j = 1, 2$. We can repeat and get $x_1, x_2, x_3, \ldots$ with the property that $d(x_j, x_i) \geq \epsilon$ for all $i = 1, 2, 3, \ldots, j-1$. Namely, $d(x_j, x_k) \geq \epsilon$ for all $j \neq k$. Thus, the sequence $(x_n)_{n=1}^{\infty}$ has no convergent subsequence; a contradiction to (3). Hence, X is totally bounded.

($4 \implies 1$) Assume that X is complete and totally bounded; we need to check that X is compact. Arguing by contradiction, suppose X is not compact; that is, suppose X has an open cover that does not have a finite subcover; call it $\{U_\alpha\}$. Since X is totally bounded, it has a $\frac{1}{2}$-net, say $\{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_{N_1}^{(1)}\}$. Write $D(x,r) = \{y \in X : d(x,y) \leq r\}$, which are closed. We have that

$$\bigcup_{j=1}^{N} D(x_j^{(1)}, \frac{1}{2}) = X$$

Since X cannot be covered by finitely many $\{U_\alpha\}$, the same is true for at least one of $D(x_j^{(1)}, \frac{1}{2})$, for $j = 1, \ldots, N$; say this occurs for $j = 1$. Hence, we cannot cover $D(x_1^{(1)}, \frac{1}{2}) = X_0$ with finitely many U_α's. Notice that $diam X_0 \leq 1 = \frac{1}{2^0}$.

Since subsets of totally bounded sets are totally bounded, we get a $\frac{1}{4}$-net for $X_0 \subseteq X_0$. Say the $\frac{1}{4}$-net is $\{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_{N_2}^{(2)}\}$. We have

$$X_0 \subseteq \bigcup_{j=1}^{N_2} D(x_j^{(2)}, \frac{1}{4}) \cap X_0$$

Again, one of the sets $D(x_1^{(2)}, \frac{1}{4}) \cap X_0 = X_1$ cannot be covered by finitely many U_α's. Notice that $X_1 \subseteq X_0$ and $diam X_1 \leq \frac{1}{2}$ with X_1.

We can repeat this process to get the collection $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ where each X_j is closed, non-empty, and cannot be covered by finitely many U_α's, and $diam X_j \leq \frac{1}{2^j} \to 0$. Hence, by the Cantor Intersection Theorem (Theorem 2.55), the intersection of these sets is a singleton:

$$\bigcap_{n=0}^{\infty} X_n = \{x_0\}$$

Since the family $\{U_\alpha\}$ covers X, there exists some index α_0 such that $x_0 \in U_{\alpha_0}$, where U_{α_0} is an open set. Hence $\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U_{\alpha_0}$. Choose N such that $\frac{1}{2^N} < \epsilon$. We know that $x_0 \in X_N$. If $y \in X_N$, then $d(x,y) \leq diam X_N \leq \frac{1}{2^N} < \epsilon$, which imples that $y \in B(x_0, \epsilon) \subseteq U_{\alpha_0}$. This is the same as saying $X_N \subseteq U_{\alpha_0}$, which is a finite subcover. This is a contradiction to the assumption that no X_j had a finite subcover had a finite subcover from the family $\{U_\alpha\}$.

Hence, X is compact. ■

Example 4.20. The Cantor Set. There exists a set $C \subseteq [0,1]$ which is compact, has an empty interior, is uncountable, and is perfect (that is, it is closed and every point is an accumulation point). We can actually construct such a set.

Let $C_0 = [0,1]$. Now, we remove the open interval $(\frac{1}{3}, \frac{2}{3})$ to obtain $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, which is the union of intervals, each of length $\frac{1}{3}$. We keep removing the middle thirds of each interval for each step. For instance, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, which is the union of four closed intervals, each of length $\frac{1}{3^2}$.

We can repeat the construction to get C_n, which is the union of 2^n closed intervals, each of length $\frac{1}{3^n}$. Since it is the finite union of closed sets, C_n is closed. Define the Cantor set as

$$C = \bigcap_{n=0}^{\infty} C_n$$

Note that C is closed because it is the intersection of closed sets. Furthermore, note that the endpoint of each closed interval is contained in the construction of C; this collection of endpoints is countable since it is contained in $\mathbb{Q}$. Moreover, C is compact, since it is closed and bounded in $\mathbb{R}$ (using Heine-Borel).

We note that at each C_n we have that the intervals are separated by gaps of length at least $\frac{1}{3^n}$. This implies that the interior is empty; assume by contradiction there is an interior point x which has a ball $(a,b) \subseteq C$ with $b-a \geq \frac{1}{3^n}$. Then $(a,b) \subseteq C_n$ for all C_n, which is impossible, since C_n is the union of intervals of length 3^{-n} separated by gaps.

We show every point is an accumulation point. Let $x \in C$ and we show that for all $\epsilon > 0$, there exists $y \in C$, $y \neq x$ such that $d(x,y) < \epsilon$. Pick N such that $3^{-N} < \epsilon$. Then $x \in C_N$ and x is in one of the levels in the construction of C_N (of length 3^{-N}). Say, without loss of generality, that x is at the endpoint of one of these levels and pick y the other endpoint (so $y \neq x$) of the interval. Then, by construction $d(x,y) = \frac{1}{3^N} < \epsilon$. If we pick a point in the interior of the level, then we can pick y to be either endpoint, and their distance is even less than 3^{-N}.

Now we prove C is uncountable. Write the ternary representation of $x \in C$ as

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} \qquad \text{where } a_j \in \{0,1,2\}$$

We can do this for every $x \in [0,1]$, with $x = .a_1a_2a_3\ldots$. Some arithmetic gymnastics shows that

$$C = \left\{ x = \sum_{j=1}^{\infty} : a_j \in \{0,2\} \right\}$$

that is, the number 1 will not show up in the ternary representation of any element in C. This construction defines a bijection with the set of binary sequences, which has uncountable cardinality; in fact $|C| = |\mathbb{R}|$. ■

Theorem 4.21. *Any non-empty perfect set E in a complete metric space is uncountable.*

Remark. If we take the metric space to be $\mathbb{Q}$ then $\mathbb{Q}$ is a perfect subspace of $\mathbb{Q}$, but of course $\mathbb{Q}$ is not uncountable. This shows that completeness is a required property.

Proof. Since E is non-empty and perfect, it has at least one accumulation point, and if it has an accumulation point, it must be infinite. Suppose, arguing by contradiction, that E is not uncountable; that is, E is countably infinite. Write $E = \{x_n\}_{n=1}^{\infty}$

Look at x_1 and take $B(x_1, \frac{1}{2}) = V_1$ whose $\text{diam} V_1 \leq 1$. Since x_1 is an accumulation point of E (it is perfect), and V_1 is open, there exists $x_{k_2} \in V_1 \cap (E \setminus \{x_1\})$. That is $x_{k_2} \in V_1 \cap E$ and $x_{k_2} \neq x_1$ (put $k_1 = 1$). We pick k_2 to be the minimal natural number with this property, so $x_2, x_3, \ldots, x_{k_2-11} \notin V_1 \cap E$.

Consider $x_{k_2} (\neq x_{k_1})$. Get $B(x_{k_2}, r_2) \subseteq V_1, x_{k_1} \notin B(x_{k_2}, r_2)$ (we can pick $r_2 < d(x_{k_2}, x_{k_1})$). Put $V_2 = B(x_{k_2}, \frac{r_2}{2})$. Then $\overline{V_2} \subseteq B(x_{k_2}, r_2) \subseteq V_1$. Then $x_{k_1} \neq \overline{V_2}$ with $x_2, x_3, \ldots, x_{k_2-1} \notin \overline{V_2} \cap E$ either.

Without loss of generality, assume that $r_2 \leq \frac{1}{2}$ which implies that $\text{diam} V_2 \leq r_2 \leq \frac{1}{2}$. We repeat the argument with $x_{k_2} \in V_2 \cap E$. Get $x_{k_3} \in V_2 \cap (E \setminus \{x_2\})$ and we choose k_3 to be minimal. Namely, $x_{k_2+1}, \ldots, x_{k_3-1} \notin V_2 \cap E$. Since V_2 is open, there exists $B(x_{k_3}, r_3) \subseteq V_2$ such that $x_{k_2} \notin B(x_{k_3}, r_3)$. WLOG, $r_3 \leq \frac{1}{4}$. Put $V_3 = B(x_{k_3}, \frac{r_3}{2})$, so $\overline{V_3} \subseteq V_2$ with $\text{diam} V_3 \leq r_3 \leq \frac{1}{4}$, and $x_{k_3} \in V_3$, $x_{k_2+1}, \ldots x_{k_3-1} \notin \overline{V_3} \cap E$.

Repeat and get $k_1 < k_2 < k_3 < \ldots$ and a family of open sets $\{V_n\}$ with $\overline{V_n} \subseteq V_{n-1}$, with $\text{diam} V_n \leq \frac{1}{2^{n-1}}$, and $x_{k_n} \in V_{n-1} \cap E \setminus \{x_{k_{n-1}}\}$. Likewise, $x_{k_{n-1}} \notin \overline{V_n}$ and $x_{k_{n-1}+1}, \ldots x_{k_n-1} \notin V_{n-1} \cap E$ so they are also not in $\overline{V_n} \cap E$.

This process yields nested sets $\overline{V_n}$, none of which are empty since $x_{k_n} \in \overline{V_n}$ and $\text{diam}\overline{V_n} \to 0$, all of which live in a complete metric space. Thus, by applying Cantor's Intersection Theorem to the family $\{\overline{V_n} \cap E\}$ (each of which is non-empty since x_{k_n} is always contained in it), we find that

$$\bigcap_{n=1}^{\infty} \overline{V_n} \cap E \neq \emptyset \qquad (*)$$

We ask, what's in the intersection? Well, since $x_1 \notin \overline{V_2}$ then x_1 is not in the intersection and neither are $x_2, \ldots, x_{k_2-1} \notin \overline{V_2} \cap E$. Similarly, $x_{k_2} \notin \overline{V_3}$, so, $x_{k_2+1}, \ldots, x_{k_3-1}$ are not in the intersection. By construction, and repeating the above argument, we see that $x_{k_j} \notin \overline{V_{j+1}}$ and since each k_j was chosen to be minimal, nothing up to $x_{k_{j+1}-1}$ is contained in the intersection. Since $E = \{x_j\}_{j=1}^{\infty}$, we have that

$$\cap \overline{V_n} \cap E = \emptyset \quad (**)$$

We observe that $(*)$ and $(**)$ contradict one another, thus E must be uncountable. ■

4.2 Continuity

Definition 4.22. Let $f : X \to Y$ be a function where (X, d_X) and (Y, d_Y) are metric spaces. Then, we say that f is **continuous at** $a \in X$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$.

We say that f is **continuous** if it is continuous at every $a \in X$.

Definition 4.23. Let $f : X \to A \subseteq Y$. The **pre-image** of A under f is $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Theorem 4.24. *We have the following equivalences:*

$$\begin{aligned} f \text{ is continuous at } a &\iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_Y(f(x), f(a)) < \epsilon \text{ whenever } d_X(x, a) < \delta \\ &\iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in B(a, \delta) \text{ we have } f(x) \in B(f(a), \epsilon) \\ &\iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } f(B_X(a, \delta)) \subseteq B_Y(f(a), \epsilon) \\ &\iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon)) \end{aligned}$$

Proof. These are simply definitional equivalences. ■

Example 4.25. Constant functions are always continuous on any metric space.

Example 4.26. The identity map from X to X when we coerce the same metric on both sides is continuous.

Example 4.27. Let $f : (\mathbb{R}, d_2) \to (\mathbb{R}, d_D)$ where d_D is the discrete metric. Then f is not continuous. Pick $\epsilon = 1$, then $d_Y(f(x), f(a)) < 1$ implies that $x = a$.

Example 4.28. Let $f : (X, d_D) \to (Y, d_Y)$. Then this is always continuous. If we take $\delta = 1$ then, $B_X(a, \delta) = \{a\}$, so $f(B_X(a, \delta)) \subseteq B_Y(f(a), \epsilon)$ for all choices of $\epsilon > 0$.

Example 4.29. Let $f : X \to Y$. Take $a \in X$ such that a is not an accumulation point. Then f is continuous at a, because for some $\delta > 0$, the ball $B_X(a, \delta) = \{a\}$; the remainder of the argument follows as above.

Example 4.30. Let $f : (X, d_X) \to \mathbb{R}$, where $\mathbb{R}$ is endowed with its usual metric. Fix $x_0 \in X$. Define $f(x) = d_X(x, x_0)$. Then f is continuous.

Theorem 4.31. *The function f is continuous at x_0 if and only if, whenever $(x_n) \subseteq X$ converging to x_0, then $f(x_n)$ converges to $f(x_0)$.*

Proof. $(\Longrightarrow)$ Suppose f is continuous at x_0. Take a sequence $(x_n) \to x_0$. Now, we have to check that $\forall \epsilon > 0$, there exists an N such that $d(f(x_n), f(x_0)) < \epsilon$ if $n \geq N$.

Since f is continuous at x_0, for each $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon$ if $d(x, x_0) < \delta$. Pick N such that $d(x_n, x_0) < \delta$ for all $n \geq N$. Then $d(f(x_n), f(x_0)) < \epsilon$ for all $n \geq N$.

$(\Longleftarrow)$ Suppose that whenever $(x_n) \subseteq X$ converges to x_0, then $f(x_n)$ converges to $f(x_0)$. Arguing by contradiction, suppose that f is not continuous at x_0. So there exists $\epsilon > 0$, such that $\forall \delta > 0$ there exists $x \in B(x_0, \delta)$, but $f(x) \notin B(f(x_0), \epsilon)$.

Since we can do this for all δ, we do it in particular for $\delta = \frac{1}{n}$ $n \in \mathbb{N}$. So, for all n, get $x_n \in B(x_0, \frac{1}{n})$, but $f(x_n) \notin B(f(x_0), \epsilon)$. Then we have that $(x_n) \to x_0$, but $d(f(x_n), f(x_0)) \geq \epsilon$ for all $n \geq N$. So $(f(x_n))$ does not converge to $f(x_0)$, contradicting our assumption. Thus f is continuous. ■

Theorem 4.32. *Suppose $f, g : X \to \mathbb{R}$ are continuous. Then, so are $f \pm g, fg, \frac{f}{g}(g \neq 0)$.*

Proof. Exercise. ■

Theorem 4.33. *The following are equivalent for a function $f : X \to Y$:*

1. *f is continuous on X*
2. *For all open sets $V \subseteq Y$, $f^{-1}(V)$ is open in X*
3. *For all closed sets $W \subseteq Y$, $f^{-1}(W)$ is closed in X.*

Proof. ($1 \implies 2$) Take V open in Y. Take $x_0 \in f^{-1}(V)$ and we want to show it is an interior point of $f^{-1}(V)$. We know that $f(x_0) \in V$. Since V is open, $\exists \epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq V$. Since f is continuous on X, it is continuous at x_0, in particular. So, $\exists \delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq V$. This implies that[1]

$$B(x_0, \delta) \subseteq f^{-1}\left(B(f(x_0), \epsilon)\right) \subseteq f^{-1}(V)$$

Hence, $x_0 \in \text{Int}(f^{-1}(V))$, as required.

($2 \implies 1$) Exercise.

($2 \implies 3$) Let $W \subseteq Y$ be closed. Then W^c is open. Hence $f^{-1}(W^c)$ is open (by 2). Let's explore what this set is[2]

$$f^{-1}(W^c) = \{x : f(x) \in W^c\} = \{x : f(x) \notin W\} = (\{x : f(x) \in W\})^c = \left(f^{-1}(W)\right)^c$$

Since $\left(f^{-1}(W)\right)^c$ is open, $f^{-1}(W)$ is closed, as required.

($3 \implies 2$) Exercise. ■

Theorem 4.34. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.*

Proof. Let $V \subseteq Z$ be open. Then $(g \cdot f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right)$. By the Theorem above, $g^{-1}(V)$ is open and so is $f^{-1}\left(g^{-1}(V)\right)$ which implies that our composition is continuous. ■

Apology. I had to study for a midterm, so I have only posted the theorems studied on Friday June 8th's class (and not their proofs). I'll try to update this section soon.

Theorem 4.35. *Let K be a compact metric space and $f : K \to Y$ be a continuous function. Then $f(K)$ is compact.*

Theorem 4.36. ***Extreme Value Theorem.*** *Let K be compact and $f : K \to \mathbb{R}$ be continuous. Then, f has maximum and minimum values on K.*

Theorem 4.37. *If K is compact and $f : K \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in K$ then $\exists \epsilon > 0$ such that $f(x) \geq \epsilon$ for all $x \in K$.*

Theorem 4.38. *Suppose $f : X \to Y$ is continuous and bijective, and assume that X is compact. Then f^{-1} is continuous.*

Definition 4.39. We say that $f : X \to Y$ is **uniformly continuous** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$.

Example 4.40. In $\mathbb{R}$, $f(x) = x^2$ and $f : (0,1) \to \mathbb{R}$ and $f(x) = \frac{1}{x}$ are both continuous functions which are not uniformly continuous.

[1] Using that pre-image respects set-inclusion ordering.
[2] That is, pre-image respects complements.

Definition 4.41. A metric space X is **not connected** if $X = U \cup V$ where U and V are open non-empty sets and $U \cap V = \emptyset$.

Remark. Notice that $U^c = V$, thus V is closed and similarly for U. That is, U and V are both open and closed.

Definition 4.42. We say that $E \subseteq X$ is **connected** if $E \neq (E \cap U) \cup (E \cap V)$ where U, V are open in X, both $E \cap U$ and $E \cap V$ are non-empty, and $(E \cap U) \cap (E \cap V) = \emptyset$.

Example 4.43. The set $E = (0,1) \cup [2,3]$ is not connected.

Example 4.44. $\mathbb{Q}$ is not connected.

Example 4.45. A set in $\mathbb{R}$ is connected if and only if it is an interval.

Example 4.46. In the discrete metric space, singletons are the only non-empty connected sets.

Theorem 4.47. *If $f : X \to f(X)$ is continuous and X is connected then $f(X)$ is connected.*

Proof. Arguing by contradiction, suppose $f(X)$ is not connected. Then, we can write $f(X) = A \cup B$ where A and B are open in $f(X)$ and disjoint, with $A, B \neq \emptyset$.

We see that $X = f^{-1}(A) \cup f^{-1}(B)$, each of which is open since f is continuous, they are non-empty, and they are disjoint. Hence $f^{-1}(A), f^{-1}(B)$ form a disconnecting pair for X, a contradiction. Hence $f(X)$ is connected. ∎

Definition 4.48. Let $f : [a,b] \to X$ be a continuous function where $f(a) = x$ and $f(b) = y$ with $x, y \in X$. Then $I = f([a,b])$ is said to be a **path**.

Definition 4.49. The set E is said to be **path connected** if $\forall x, y \in E$, there exists a path $I \subseteq E$.

Theorem 4.50. *If E is path connected, it is connected.*

Proof. Arguing by contradiction, suppose E is disconnected but path connected. Say $E = A \cup B$, where A, B are open, disjoint, and non-empty. Let $x \in A$ and $y \in B$. Assume $f : [a,b] \to E$ is continuous and $f(a) = x$ and $f(b) = y$.

But, clearly $f([a,b])$ is connected, since f is continuous and $[a,b]$ is connected. But, $f([a,b]) = (f[a,b] \cap A) \cup (f[a,b] \cap B) = U \cup V \subseteq A \cup B$. Since A, B are disjoint, so are U and V; these sets are not empty, since $x \in U$ and $y \in V$. Thus U and V are a relatively open disconnecting pair disconnecting $f([a,b])$; a contradiction.

Hence path connectedness implies connectedness. ∎

Theorem 4.51. *If E is connected then any A, such that $E \subseteq A \subseteq \overline{E}$, is connected.*

Proof. Exercise. ∎

Example 4.52. Unfortunately, connectedness does not imply path connectedness. For instance the set $X = \{(x, \sin\frac{1}{x}) : x > 0\} \cup \{(0,0)\}$ be a set with the metric inherited from $\mathbb{R}^2$. This set is connected but not path connected. We can write $E = \{(x, \sin\frac{1}{x}) : x > 0\}$ and $X = \overline{E}$.

We show E is path connected. Let $(x_1, x_2), (y_1, y_2) \in E$. Then $(x_1, x_2) = \left(x, \sin\frac{1}{x_1}\right)$ and $(y_1, y_2) = \left(y_1, \sin\frac{1}{y_1}\right)$. Define $f : (0,1) \to E$ by $f(t) = \left(tx_1 + (1-t)y_1, \sin\frac{1}{tx_1 + (1-t)y_2}\right) \in E$. Clearly, f is continuous and $f(0) = (y_1, y_2)$ and $f(1) = (x_1, x_2)$. Hence E is path connected and thus connected. By the proposition above, this implies that $X = \overline{E}$ is connected.

Now we prove that X is not path connected. Suppose $f : [a,b] \to X$ is a "path" from $(0,0)$ to $(\frac{1}{\pi}, 0)$. We claim that all the points $P = \left\{\left(\frac{2}{5\pi + 4k}, 1\right)\right\}_{k=0}^{\infty}$ must be contained in the path. Suppose $(\frac{2}{5\pi}, 1) \notin f([a,b])$, for example. Then,

$$f[a,b] = \left(f[a,b] \cap \{(x,y) : x < \frac{2}{5\pi}\}\right) \bigcup \left(f([a,b]) \cap \{(x,y) : x > \frac{2}{5\pi}\}\right) = U \cup V$$

The only points possibly missing are the points $(x, y) \in X$ where $x = \frac{2}{5\pi}$ but then $y = \sin\frac{5\pi}{2} = 1$. But we assumed $(\frac{2}{5\pi}, 1) \notin f([a, b])$. Now we see that U, V form a disconnecting pair for $f([a, b])$, since they are open, disjoint, and non-empty. This is a contradiction, since $f([a, b])$ is the continuous image of a connected set. This contradiction proves the claim that all points in P must be in a path connecting $(0, 0)$ with $(\frac{1}{\pi}, 0)$.

Now we show that X is not path connected. Since $[a, b]$. $f([a, b])$ is compact, so it contains all its accumulation points. But the limit of the sequence P is not in X (it would converge to $(0, 1)$ in $\mathbb{R}^2$); more formally, note that P has no convergent subsequence in X, contradicting the Bolzano-Weierstrass characterisation of compactness). This contradicts the compactness of $f([a, b])$. This contradiction proves that X is not path connected. ■

Theorem 4.53. *Every open set which is connected is path connected.*

Proof. Exercise. ■

Chapter 5

Normed Vector Spaces

Definition 5.1. A **norm** is a map $\|\cdot\| : V \to \mathbb{R}$ satisfying

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$
2. $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$
3. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

Definition 5.2. A **normed vector space** is a pair $(V, \|\cdot\|)$ where V is a vector space and $\|\cdot\|$ is a norm.

Example 5.3. $\mathbb{R}^n$ with the Euclidean norm is a normed vector space.

Example 5.4. $\mathbb{R}^n$ with $\|x\|_\infty = \max_i |x_i|$ is a normed vector space. This induces the d_∞ metric.

Example 5.5. ℓ^∞ with $\|x\|_\infty$ is a normed vector space which induces the ℓ^∞ norm.

Example 5.6. ℓ^p space for $1 \leq p < \infty$ with $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$

Theorem 5.7. ***Minkowski's Inequality.*** *For $1 \leq p \leq \infty$ we have*

$$\|x + y\| \leq \|x\|_p + \|y\|_p$$

for all $x, y \in \ell^p$.

Proof. We shall only look at the finite case. The inequality is trivial if $\|x\|_p = 0$ and similarly for y. So we assume otherwise.

Let $\alpha = \|x\|_p$ and $\beta = \|y\|_p$; additionally, let

$$u = \frac{x}{\|x\|_p} \qquad v = \frac{y}{\|y\|_p} \qquad \Longrightarrow \qquad x = \alpha u, \quad y = \beta v$$

Put $\lambda = \frac{\alpha}{\alpha+\beta}$ and notice that

$$1 - \lambda = 1 - \frac{\alpha}{\alpha + \beta} = \frac{\beta}{\alpha + \beta}$$

Then, for each i, we have

$$\begin{aligned} |x_i + y_i|^p &\leq (|x_i| + |y_i|)^p \\ &= \left(\frac{\alpha|u_i| + \beta|v_i|}{\alpha + \beta}\right)^p (\alpha + \beta)^p \\ &= \left[\lambda|u_i| + (1 - \lambda)|v_i|\right]^p (\alpha + \beta)^p \end{aligned}$$

Note that the first term is a convex combination of $|u_i|, |v_i|$. Let us investigate the behaviour of $f(t) = t^p$ for $t \geq 0$[1]. By convexity we have

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$$

[1] I'll post a picture at a later date.

Hence,

$$(\lambda a+(1-\lambda)b)^p \leq \lambda a^p+(1-\lambda)b^p \qquad \forall a,b\geq 0$$

We can apply this convex inequality to $a=|u_i|, b=|v_i|$ to obtain

$$\begin{aligned}
\sum_{i=1}^{\infty}|x_i+y_i|^p &\leq \sum_{i=1}^{\infty}\left(\lambda|u_i|+(1-\lambda)|v_i|\right)^p(\alpha+\beta)^p \\
&\leq \sum_{i=1}^{\infty}\left(\lambda|u_i|^p+(1-\lambda)|v_i|^p\right)(\alpha+\beta)^p \\
&= (\alpha+\beta)^p\left(\lambda\sum_{i=1}^{\infty}|u_i|^p+(1-\lambda)\sum_{i=1}^{\infty}|v_i|^p\right) \\
&= (\alpha+\beta)^p\left(\lambda\|u\|_p^p+(1-\lambda)\|v\|_p^p\right) \qquad (*)
\end{aligned}$$

Now we start working our way up. We have

$$\|u\|_p=\left\|\frac{1}{\|x\|_p}x\right\|_p=\frac{1}{\|x\|_p}\times\|x\|_p=1$$

and similarly for $\|v\|_p$. We substitute this back to $(*)$ and obtain

$$\begin{aligned}
\sum_{i=1}^{\infty}|x_i+y_i|^p &\leq (\alpha+\beta)^p\left(\lambda\|u\|_p^p+(1-\lambda)\|v\|_p^p\right) \\
&= (\alpha+\beta)^p(\lambda\times 1+(1-\lambda)\times 1)=(\alpha+\beta)^p \\
&= (\|x\|_p+\|y\|_p)^p
\end{aligned}$$

Victory! ■

Remark. This shows that ℓ^p is a normed vector space.

Example 5.8. The space of polynomials is a normed vector space with the following norms:

1. $\|p\|=\max_{x\in[0,1]}|p(x)|$
2. $\|p\|_1=\int_0^1|p(x)|dx$

Neither of the spaces above is complete. Sad-face.

Example 5.9. The vector space $C[0,1]$ (that is the set of continuous function on $[0,1]$ with the two norms provided in the example above forms normed vector spaces. The first one of these is the one we like and we pick it as the standard norm (and denote it as d_∞), because it forms a complete metric space.

In fact, the space of polynomials over $[0,1]$ with the standard norm is dense in $C[0,1]$.

Part III

Function Spaces

Chapter 6

Convergence in Function Spaces

6.1 Pointwise and Uniform Convergence

Definition 6.1. Let $f_n : X \to Y$ be a function where X, Y are metric spaces. We say that (f_n) **converges pointwise** to $f : X \to Y$ if for all $\epsilon > 0$ and for all $x \in X$, there exists an $N = N(\epsilon, x)$ such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n \geq N$.

Definition 6.2. Using the above set-up, we say that (f_n) **converges uniformly** to f if $\forall \epsilon > 0$ there exists $N = N(\epsilon)$ such that $d(f_n(x), f(x)) < \epsilon$ for all $n \geq N$ for all $x \in X$.

Example 6.3. Let $f_n : [0,1] \to \mathbb{R}$. This function converges pointwise, but it does not converge uniformly (since the limit is not continuous)

Theorem 6.4. *A uniform limit of continuous functions is continuous.*

Proof. Let $\epsilon > 0$. R.T.P. that for all $x \in X$ there exists a δ such that $d(x, y) < \delta$ which implies that $d(f(x), f(y)) < \epsilon$.

We have that $f_n \to f$ uniformly, so there exists an N such that $d(f_n(t), f(t)) < \frac{\epsilon}{3}$ for all $n \geq N$ and $t \in X$.

We now look at f_N. This is continuous so that δ such that $d(x, y) < \delta$ implies that $d(f_N(x), f_N(y)) < \frac{\epsilon}{3}$.

So, suppose $d(x, y) < \delta$. Then, by the triangle inequality,

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

as required. ■

Definition 6.5. We say that a sequence $f_n : X \to Y$ is **uniformly Cauchy** if for all $\epsilon > 0$ there exists N such that for all $n, m \geq N$ we have

$$d(f_n(x), f_m(x)) < \epsilon$$

for all $x \in X$/

Theorem 6.6. *Let X and Y be metric spaces and assume Y is complete. The sequence of functions $f_n : X \to Y$ is uniformly Cauchy if and only if (f_n) is uniformly convergent.*

Proof. ($\Longleftarrow$) (This does not require completeness). Say $f_n \to f$ uniformly. Let $\epsilon > 0$ and pick N such that $d(f_n(x), f(x)) < \epsilon$ whenever $n \geq N$ and for all $x \in X$. Then, if $n, m \geq N$ we have, by the triangle inequality, that

$$d(f_n(x), f_m(x)) \leq d(f_n(x), f(x)) + d(f(x), f_m(x)) < 2\epsilon \qquad \forall x \in X$$

($\Longrightarrow$) Assume that (f_n) is uniformly Cauchy. Then, for all $x \in X$ the sequence $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, for each $x \in X$ the sequence $(f_n(x))$ converges, say to $a_x \in Y$.

Put $f(x) = a_x$. We already have that $f_n(x) \to f(x)$; that is, we have pointwise convergence and the challenge is proving that the convergence is uniform. Since we have pointwise convergence, for all $\epsilon > 0$ and for all $x \in X$, there exists an M_x such that for all $m \geq M_x$ we have $d(f_m(x), f(x)) < \epsilon$.

Since (f_n) is uniformly Cauchy, for all $\epsilon > 0$ there exists N such that $\forall n, m \geq N$ and $\forall x$ we have $d(f_n(x), f_m(x)) < \epsilon$. Take ϵ, extract N (temporarily fix $x \in X$) and let $n \geq N$. Pick $m \geq \max(N, M_x)$. Then,

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) \\ &< \epsilon + \epsilon \qquad (*) \\ &= 2\epsilon \end{aligned}$$

where follows $(*)$ since the first ϵ arises by being uniformly Cauchy, and the second arises by the choice of $m \geq M_x$. Since $d(f_n(x), f(x)) < 2\epsilon$ for all $n \geq N$ and for all $x \in X$, this implies that $(f_n) \to f$ uniformly. ■

Theorem 6.7. ***Weierstrass M-test.*** *Let $f_n : X \to Y$ where Y is a complete normed vector space. Suppose there exists constants $M_n \in \mathbb{R}$ such that $\|f_n(x)\|_Y \leq M_n$ for all x and for all n and*

$$\sum_{n=1}^{\infty} M_n < \infty$$

Then,

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly.

Proof. Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Note that $(S_n(x))$ is a sequence in Y. Then,

$$\begin{aligned} d(S_n(x), S_m(x)) &= \|S_n(x) - S_m(x)\| \\ &= \left\| \sum_{k=m+1}^{n} f_k(x) \right\| \qquad \text{WLOG take } n > m \\ &\leq \sum_{k=m+1}^{n} \|f_k(x)\| \\ &\leq \sum_{k=m+1}^{n} M_k \\ &< \epsilon \end{aligned}$$

if n and m are big enough since

$$\sum M_k$$

converges independent of the choice of x. Hence $(S_n(x))$ is a uniformly Cauchy and Y is complete. So (S_n) converges uniformly by the theorem above. ■

Theorem 6.8. ***Dini's Theorem.*** *Suppose K is compact and $f_n : K \to \mathbb{R}$ converges pointwise to f on K. If f_n and f are continuous and the convergence is monotone, namely,*

$$f_{n+1}(x) \leq f_n(x) \quad \forall x \in K, \forall n$$

then $f_n \to f$ uniformly.

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then, we want to show that $g_n \to 0$ uniformly. We know that $g_n \to 0$ pointwise (because $f_n \to f$ pointwise); g_n is continuous for all n, $g_n(x) \geq 0$ at all x (from monotonicity of f_n), and the g_n's are monotonic themselves.

Let $\epsilon > 0$. We want to show that there exists N such that for all $n \geq N$, $0 \leq g_n(x) < \epsilon$. Since $g_n \to 0$ pointwise and for all $t \in K$ there exists N_t such that $0 \leq g_n(t) < \frac{\epsilon}{2}$ implying that $0 \leq g_{N_t}(t) < \frac{\epsilon}{2}$.

Since g_{N_t} is continuous, we have that $\exists \delta_t > 0$ such that $d(t.y) < \delta_t$ implies that $|g_{N_t}(t) - g_{N_t}(y)| < \frac{\epsilon}{2}$.

We look at $B(t, \delta_t) \subseteq K$ for each $t \in K$. These form an open cover of K. Take a finite subcover

$$B(t_1, \delta_{t_1}), B(t_2, \delta_{t_2}), \ldots, B(t, \delta_{t_r})$$

Now suppose that $x \in B(t_i, \delta_{t_i})$, meaning that $d(x, t_i) < \delta_{t_i}$. Thus, $|g_{N_{t_i}}(t_i) - g_{N_{t_i}}(x)| < \frac{\epsilon}{2}$. Also, $g_{N_{t_i}}(t_i) < \frac{\epsilon}{2}$, which implies that

$$g_{N_{t_i}}(x) < \epsilon \qquad (*)$$

Now, let $N = \max(N_{t_1}, \ldots, N_{t_r})$. We claim this is the desired N for uniform convergence. Let $n \geq N$ and take $x \in K$. Using our finite subcover, $\exists i$ such that $x \in B(t_i, \delta_{t_i})$. By monotonicity of g_n and the choice of N, we have

$$0 \leq g_n(x) \leq g_N(x) \leq g_{N_{t_i}}(x) < \epsilon$$

using $(*)$. Since this holds for every $x \in K$ and for each $\epsilon > 0$, we have that $g_n \to 0$ uniformly. ■

6.2 The metric space $C(X)$

Definition 6.9. Here is a notational remark. We write the following:

$$\begin{aligned}
C(X) &= \{f : X \to \mathbb{R} : \text{where } f \text{ is continuous}\} \\
C_b(X) &= \{f : X \to \mathbb{R} : \text{where } f \text{ is continuous and bounded}\} \\
\|f\| &= \sup_{x \in X} |f(x)| \qquad \text{for } f \in C_b(X) \\
d(f, g) &= \|f - g\|
\end{aligned}$$

Remark. If K is compact, $C(K) = C_b(K)$. Furthermore, it is easy to see that $\|f\|$ satisfies the properties of a norm. The norm defined above is sometimes called the "sup-norm" or "uniform norm". The metric induced by the norm is the greatest distance between the functions. This is also the standard metric on $C_b(X)$.

Example 6.10. Another metric on $C([0, 1])$ is $d_1(f, g) = \int_0^1 |f - g|$ which corresponds to the area between f and g. Note that $d_1(f, g) \leq d(f, g)$.

Example 6.11. We arm ourselves to show that the function space $(C_b(X), d)$ is complete. Let $f_n, f \in C_b(X)$. To say that $f_n \to f$ uniformly on X means that $\forall \epsilon > 0$, there exists an N such that $\forall n \geq N$ we have $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X$. We have the following sequence of equivalences:

$$\begin{aligned}
f_n \to f \text{ uniformly} &\iff \forall \epsilon > 0 \sup_{x \in X} |f_n(x) - f(x)| < \epsilon, \forall n \geq N \\
&\iff \|f_n - f\| \leq \epsilon \quad n \geq N \\
&\iff d(f_n, f) \leq \epsilon \quad \forall n \geq N \\
&\iff f_n \to f \text{ in the metric space } C_b(X)
\end{aligned}$$

That is convergence is precisely uniform convergence in $C_b(X)$. Similarly (f_n) in $C_b(X)$ is Cauchy if and only if (f_n) is uniformly Cauchy. ■

Theorem 6.12. *The metric space $(C_b(X), d)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence in $C_b(X)$. That means it is a uniformly Cauchy sequence of continuous functions. By Theorem 3.6, $f_n \to f$ uniformly for some continuous function f.

We claim that $f \in C_b(X)$. For some suitably large N, we have that $|f_N(x) - f(x)| \le 1$ for all $x \in X$. Thus

$$\sup_x \le \sup (1 + |f_N(x)|) \le 1 + \|f_N\| < \infty$$

which implies that f is bounded, namely, $f \in C_b(X)$. Thus, since $f_n, f \in C_b(X)$ and $f_n \to f$ uniformly, this implies that $f_n \to f$ in the $C_b(X)$ space, which is thus complete. ■

Remark. $C_b(X)$ is a complete normed vector space. Drum-roll...

Definition 6.13. A **Banach space** is a complete, normed vector space.

Example 6.14. We now talk about what the balls look like in $C_b(X)$. We have

$$B(f, r) = \{g \in C_b(X) : d(f, g) < r\} = \{g \in C_b(X) : \|f - g\| < r\}$$

Let $X = [0, 1]$ Since my LaTeX graphics skills suck, I won't provide a picture, but just a description. Suppose we have a nice wobbly function f on X and we want $B(f, r)$; then the ball is simply the envelope of size r around the graph of the function f.

Example 6.15. We explore an interesting function space. Let $U = \{f \in C[0, 1] : f(x) > 0, \forall x \in [0, 1]\}$. Note that by the Extreme Value Theorem, U is open (we can bound below by some $\epsilon > 0$, which may potentially be the minimum value of f and then consider balls).

Then $U^c = \{f \in C[0, 1] : f(x) \le 0 \text{ at some } x\}$ is closed. Let's explore the accumulation point characterisation of being closed. Suppose f is an accumulation point of U^c. Thus, there exists a sequence $f_n \to f$ where $f_n \in U^c$ (this convergence is, moreover, uniform). We know, thus, that $\exists x_n \in [0, 1]$ such that $f_n(x_n) \le 0$. Thus, by the Bolzano-Weierstrass theorem, there exists a subsequence $x_{n_k} \to x_0$. Notice that $f(x_{n_k}) \to f(x_0)$. We see that $f(x_0) \le \epsilon$ for all $\epsilon > 0$. If we can do that, then we are done, since then $f(x_0) \le 0$ and thus $f \in U^c$. Since $f_n \to f$ uniformly, it is the case that $f_{n_k} \to f$ uniformly, meaning that for large enough n_k, $d(f_{n_k}, f) < \epsilon$. Thus

$$\sup_x |f_{n_k}(x) - f(x)| < \epsilon \quad \Longrightarrow \quad |f_{n_k}(x_{n_k} - f(x_{n_k})| < \epsilon$$

So $f(x_{n_k}) < f_{n_k}(x_{n_k}) + \epsilon \le \epsilon$ and we are done.

Example 6.16. Let $E = \{x^n : n = 1, 2, 3, \dots\} \subseteq C[0, 1]$. This set is closed, bounded (and even complete). However, it is not compact, since no sequence has convergent subsequences.

Example 6.17. Let $E = \{\frac{x^2}{x^2+(1-nx)^2} : n = 1, 2, 3, \dots\} \subseteq C[0, 1]$. It is closed, bounded and not compact.

Chapter 7

The Arzela-Ascoli Theorem

Definition 7.1. Let $E \subseteq C_b(X)$. We say that E is **equicontinuous** if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $\forall f \in E$ and $\forall x, y \in X$ such that $d(x, y) < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Example 7.2. If $E = \{f\}$, then E is equicontinuous if and only if f is uniformly continuous.

Example 7.3. Let $E = \{f_1, f_2, \ldots, f_n\}$. Then E is equicontinuous if each f_i is uniformly continuous. To show this, fix ϵ and take $\delta_i > 0$ from the definition of uniform continuity for each f_i. Pick $\delta = \min\{\delta_i\} > 0$. This δ works for the definition of equicontinuity.

Example 7.4. The set $E = \{x^n : n = 1, 2, 3, \ldots\}$ is not equicontinuous even though each f_i is uniformly continuous. Note that not equicontinuous means that

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in X \text{ with } d(x, y) < \delta \text{ and some } f \in E \text{ with } |f(x) - f(y)| \geq \epsilon$$

Take $\epsilon = 0.5$. Take any $\delta > 0$, $x = 1$, and $y = 1 - \frac{\delta}{2}$. Choose n such that $y^n < \frac{1}{2}$. Thus,

$$|f_n(x) - f_n(y)| = x^n - y^n > 1 - \frac{1}{2} = \frac{1}{2} = \epsilon$$

proving that E is not equicontinuous.

Note: No proofs today because of a midterm. Sorry.

Theorem 7.5. *Suppose X is compact and $f_n \in C(X)$ for $n = 1, 2, 3, \ldots$. If $f_n \to f$ uniformly, then $E = \{f_n : n \in \mathbb{N}\}$ is equicontinuous.*

Definition 7.6. We say that $E \subseteq C_b(X)$ is **pointwise bounded** if for all $x \in X$ there exists M_x such that $|f(x)| \leq M_x$ for all $f \in E$.

Remark. Clearly, uniformly bounded implies pointwise bounded, but the converse is not true.

Theorem 7.7. *Let X be compact and assume $E \subseteq C(X)$ is pointwise bounded and equicontinuous. Then E is uniformly bounded.*

Theorem 7.8. *Let X be compact. Assume that the set $\{f_n : n = 1, 2, 3, \ldots\}$ is a pointwise bounded, equicontinuous family. Then there exists a subsequence of $(f_n)_{n=1}^{\infty}$ which converges uniformly.*

Lemma 1. Any compact metric space is separable (has a countable dense subset).

Proof of Lemma 1. Consider the balls $\{B\left(x, \frac{1}{n}\right), x \in X\}$. This is an open cover of X. Since X is compact, there is a finite subcover of this set, say $B\left(x_1^{(n)}, \frac{1}{n}\right), B\left(x_2^{(n)}, \frac{1}{n}\right), \ldots, B\left(x_{r_n}^{(n)}, \frac{1}{n}\right)$. Let, $K_n = \{x_1^{(n)}, \ldots x_{r_n}^{(n)}\}$ and put

$$K = \bigcup_{n=1}^{\infty} K_n$$

K is clearly countable, since it is the countable union of finite sets. We show that K is dense. So let $x \in X$ and we will show that $\forall \epsilon > 0$ there exists $k \in K$ such that $x \in B(k, \epsilon)$. Pick n such that $\frac{1}{n} < \epsilon$. Notice that

$$\bigcup_{j=1}^{k_n} B(x_j^{(n)}, \frac{1}{n}) = X$$

Thus, there exists a choice of j such that $x \in B(x_j^{(n)}, \frac{1}{n}) \subseteq B(x_j^{(n)}, \epsilon)$. This completes the proof of the lemma.

Lemma 2. Let K be a countable set, and let $\{f_n : K \to \mathbb{R} \,|\, n \in \mathbb{N}\}$ be a pointwise bounded family. Then, there exists a subsequence of $(f_n)_{n=1}^{\infty}$ that converges pointwise at every $k \in K$ (that is, $(f_n(k))_{n=1}^{\infty}$ converges.

Proof of Lemma 2. Let $K = \{x_j\}_1^{\infty}$. Start with $(f_n(x_1))_{n=1}^{\infty}$. This is a bounded sequence of real numbers, since $\{f_n\}$ is pointwise bounded. Thus, by the Bolzano-Weierstrass Theorem for real numbers, there exists a converging subsequence $\left(f_{n_j}(x_1)\right)_{n=1}^{\infty}$. To avoid sub-index and LaTeX hell, we rename f_{n_j} as $f_j^{(1)}$. So, $\left(f_j^{(1)}(x_1)\right)_{j=1}^{\infty}$ converges.

Now, we look at $(f_j^{(1)}(x_2))$. This is a bounded sequence of real numbers; again, by the Bolzano-Weierstrass Theorem for real numbers, it has a convergent subsequence. Call is $(f_j^{(2)}(x_2))_{j=1}^{\infty}$. Notice also that $(f_j^{(2)}(x_1))_{j=1}^{\infty}$ also converges, since it is a subsequence of $(f_j^{(1)})$. We shall repeatedly do this[1].

Theorem 7.9. ***Arzela-Ascoli Theorem.*** *Suppose X is compact. Then $E \subseteq C(X)$ is compact if and only if E is pointwise bounded, closed, and equicontinuous.*

($\Longrightarrow$) This does not use the theorem above. Let E be compact. Then E is closed and bounded. Then E is uniformly bounded and hence pointwise bounded.

We check that E is equicontinuous. Arguing by contradiction, suppose it is not equicontinuous. Then we have a failing ϵ. More formally, $\exists \epsilon > 0$ such that $\forall \delta = \frac{1}{n}$ (for all $n \in \mathbb{N}$) there exists $x_n, y_n \in C$ such that $d(x_n, y_n) < \frac{1}{n}$ and $\exists f_n \in E$ with $|f_n(x_n) - f_n(y_n)| \geq \epsilon$. Since E is compact, by the Bolzano-Weierstrass Theorem, there exists a (uniformly) converging subsequence $(f_{n_k})_{k=1}^{\infty}$. By a previous theorem, the set $\{f_{n_k} : k = 1, 2, \ldots\}$ is equicontinuous. Thus, for this choice of ϵ, there is a good δ (one that works).

Pick n_k so large that $\frac{1}{n_k} < \delta$. Then $d(x_{n_k}, y_{n_k}) < \frac{1}{n_k} < \delta$. Hence, it must be the case that $|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < \epsilon$. This is a contradiction, hence E is equicontinuous.

($\Longleftarrow$) We shall verify the Bolzano-Weierstrass characterisation of compactness. That is, that every sequence from E has a convergent subsequence with limit in E.

Suppose E is pointwise bounded, closed, and equicontinuous. Take a sequence $(f_n)_{n=1}^{\infty}$ in E. Since E is pointwise bounded, so is the set $\{f_n : n = 1, 2, \ldots\}$; likewise, since E is equicontinuous, so is the set of functions f_n. We have now satisfied the conditions from the theorem above, thus the sequence $(f_n)_{n=1}^{\infty}$ has a converging subsequence; since E is closed, the limit, say f, is in E. Since every sequence has a convergent subsequence in E, E is compact. $\blacksquare$

[1]This proof is, obviously, incomplete. I'll wait for Zachary's notes to fill in the details

Chapter 8

The Stone-Weierstrass Theorem

Theorem 8.1. ***Weierstrass Theorem.*** *Let $f : [0,1] \to \mathbb{R}$ be continuous and let $\epsilon > 0$. Then, there is a polynomial p such that $\|p - f\| < \epsilon$. So the set of polynomials is dense in $C[0,1]$.*

Remark. In fact, the Bernsteir polynomial

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

converge uniformly to f. An intuitive explanation for why this is the case is using a binomial distribution with and payoff random variable and then applying the weak law of large numbers to arrive at convergence in probability (which is pointwise convergence of the proposed Bernstein polynomials).

Proof. First, some technical calculations. By the Binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

which can be viewed as a function of x. Taking the derivative with respect to x, we obtain

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}$$

Differentiating again,

$$n(n-1)(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} y^{n-k}$$

We can multiply the expressions above to obtain:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{8.1}$$

$$xn(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^k y^{n-k} \tag{8.2}$$

$$x^2 n(n-1)(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^k y^{n-k} \tag{8.3}$$

We now put

$$r_k(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{8.4}$$

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) \tag{8.5}$$

We evaluate (6.1), (6.2), and (6.3) at $y = 1 - x$ and get

$$1 = \sum_{k=0}^{n} r_k(x)$$

and

$$nx = \sum_{k=0}^{n} k r_k(x)$$

and

$$x^2 n(n-1) = \sum_{k=0}^{n} k(k-1) r_k(x) = \sum_{k=0}^{n} \left(k^2 r_k(x) - k r_k(x)\right) = \sum_{k=0}^{n} \left(k^2 r_k(x)\right) - nx$$

So,

$$\begin{aligned}
\sum_{k=0}^{n} (k - nx)^2 r_k(x) &= \sum_{k=0}^{n} (k^2 - 2knx + n^2x^2) r_k(x) \\
&= nx + x^2 n(n-1) - 2nx \times nx + n^2 x^2 - 1 \\
&= nx - x^2 n \\
&= nx(1-x) \qquad (*)
\end{aligned}$$

Which is our desired identity. Now we are equipped to prove the theorem.

Since f is continuous on $[0,1]$, it is bounded; that is, there exists an M such that $|f(x)| \leq M$ for all $x \in [0,1]$. Also, f is uniformly continuous, so that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. We want to prove that $\forall \epsilon > 0$, $\exists N$ such that $|p_n(x) - f(x)| < \epsilon$ for all $x \in [0,1]$ and $\forall n \geq N$.

Fix $\epsilon > 0$. Take N such that $\frac{2M}{\delta^2 N} < \epsilon$. Let $n \geq N$ and $x \in [0,1]$. Then,

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) - f(x) \right| \\
&= \left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) - f(x) \sum_{k=0}^{n} r_k(x) \right| \qquad \text{since } \sum_{k=0}^{n} r_k(x) = 1 \\
&\leq \left| \sum_{k=0}^{n} \left| \left(f\left(\frac{k}{n}\right) - f(x) \right) r_k(x) \right| \right|
\end{aligned}$$

We shall break this down into two cases:

1. Case A: k such that $|\frac{k}{n} - x| < \delta$
2. Case B: The other k's

Case 1.

We have that $|\frac{k}{n} - x| < \delta$ implies that $|f(\frac{k}{n}) - f(x)| < \epsilon$. Then[1],

$$\begin{aligned}
\left| \sum_{k \in A} \left| \left(f\left(\frac{k}{n}\right) - f(x) \right) r_k(x) \right| \right| &\leq \sum_{k \in A} \epsilon r_k(x) \\
&\leq \sum_{k=0}^{n} \epsilon r_k(x) \\
&= \epsilon \qquad \text{since } \sum_{k=0}^{n} r_k(x) = 1
\end{aligned}$$

[1] Careful, there is an error here...

Case 2.

Here $|\frac{k}{n} - x| \geq \delta$ if and only if $|k - nx| \geq \delta n$. Then,

$$\begin{aligned}
\sum_{k \in B} |f(x) - f(\frac{k}{n})| r_k(x) &\leq \sum_{\{k : |k - nx| \geq n\delta\}} \left(|f(x)| + |f(\frac{k}{n})| \right) r_k(x) \\
&\leq 2M \sum_{\{k : |k - nx| \geq n\delta\}} r_k(x) \\
&= 2M \sum_{\{k : |k - nx| \geq n\delta\}} \frac{(k - nx)^2)}{(k - nx)^2)} r_k(x) \\
&\leq \frac{2M}{(n\delta)^2} \sum_{k=0}^{n} (k - nx)^2 r_k(x) \\
&\leq \frac{2M}{(n\delta)^2} nx(1 - x) \qquad \text{by } (*) \\
&\leq \frac{2M}{n\delta^2} \\
&\leq \frac{2M}{N\delta^2} \\
&< \epsilon
\end{aligned}$$

by our choice of N. Combining cases, we obtain

$$|p_n(x) - f(x)| \leq \sum_{k \in A} + \sum_{k \in B} \leq \epsilon + \epsilon = 2\epsilon$$

for all $n \geq N$ and $x \in [0, 1]$. Thus $p_n \to f$ uniformly. ■

Definition 8.2. A **trigonometric polynomial** (of degree N) is a function of the form

$$\sum_{n=0}^{N} a_n \cos(nx) + b_n \sin(nx) = \sum_{n=0}^{N} c_n e^{inx}$$

We say that these functions are 2π-periodic if $f(x) = f(x + 2\pi)$ for all x. Of note, when $f \in C[-\pi, \pi]$, we have $f(-\pi) = f(\pi)$.

Example 8.3. We consider the inner product on $C[-\pi, \pi]$ given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g}(x) dx$$

which induces the norm

$$\|f - g\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f - g|^2 \right)^{1/2}$$

We observe that $\{e^{inx}\}_{n=-\infty}^{\infty}$ is an orthonormal set. Thus, we can form the best approximation to f from a degree N trigonometric polynomial in the sense of this inner product. This approximation is:

$$f_N(x) = \sum_{n=-N}^{N} \langle f, e^{inx} \rangle e^{inx}$$

In particular, we call the following the **n-th Fourier coefficient of** f:

$$\langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \hat{f}(n)$$

In fact, we have that $f_N \to f$ in the norm induced from the inner product (which will be proven in PMATH 450). However, we do not have that $f_n \to f$ pointwise, much less uniformly. We prove this below.

Define

$$K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int}$$

and put

$$p_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_N(x-t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt$$

We are now ready for the next theorem.

Theorem 8.4. *Using the notation from the example above, p_N are trigonometric polynomials of degree N and $p_N \to f$ uniformly as $N \to \infty$.*

Proof. First we show that the p_N's are trigonometric polynomials.

$$p_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left(\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int} \right) dt$$

If we let $u = x - t$ and $t = x - u$, we obtain,

$$\begin{aligned}
p_N(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left(\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int} \right) dt \\
&= -\frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(u) \left(\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{in(x-u)} \right) du \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx)} e^{-inu} du \\
&= \frac{1}{2\pi} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx)} \int_{-\pi}^{\pi} f(u) e^{-inu} du \\
&= \sum_{n=-N}^{N} \hat{f}(n) \left(1 - \frac{|n|}{N+1}\right)
\end{aligned}$$

where the last line is a trigonometric polynomial of degree N. For the second part of the proof, we need the following facts (which we state without proof):

1. $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) dt = 1$
2. $K_n(t) = \frac{1}{n+1} \left(\frac{\sin^2(\frac{n+1}{2} t)}{\sin^2(t/2)} \right) \geq 0$
3. Fix $0 < \delta < \pi$. Then,

$$\lim_{N \to \infty} \frac{1}{2\pi} \int_{|t| > \delta} K_n(t) dt = 0$$

The latter one has the following heuristic justification. If $|t| > \delta$, then $|K_n(t)| \leq \frac{1}{n+1} \times \frac{1}{\sin^2(\delta/2)}$. Then,

$$\left| \int_{|t| > \delta} K_n(t) \right| \leq \int_{|t| < \delta} \frac{C}{n+1} dt \leq \frac{C}{n+1} 2\delta \to 0$$

as $n \to \infty$.

We are now ready to show that $p_N \to f$ uniformly. Let $\epsilon > 0$. Get δ from uniform continuity of f so that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. Since f is bounded (it is continuous over a compact set), there exists an M such that, for all x, $|f(x)| \leq M$. We use fact (3) to pick an N_0 such that for all $N \geq N_0$ we have

$$\frac{1}{2\pi}\int_{|t|>\delta} K_N(t)dt < \frac{\epsilon}{M} \qquad \forall N \geq N_0 \qquad (*)$$

Then,

$$\begin{aligned}
|p_N(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\left(f(x-t) - f(x)\right)dt\right| \\
&\leq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|\left|\left(f(x-t) - f(x)\right)\right|dt\right| \\
&= \frac{1}{2\pi}\left(\int_{|t|\leq\delta} + \int_{|t|>\delta}\right)
\end{aligned}$$

where we break up the integral into two parts A and B, in the order they appear. In A, we have $|(x-t)-x| = |t| \leq \delta$ and $|f(x-t) - f(x)| < \epsilon$ by its uniform continuity. Then,

$$\begin{aligned}
A &\leq \frac{1}{2\pi}\int_{|t|\leq\delta} \epsilon|K_N(t)|dt \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \epsilon \\
&= \epsilon
\end{aligned}$$

where the last step follows by fact (1). For B,

$$\begin{aligned}
B &= \int_{|t|>\delta} |K_N(t)||f(x-t) - f(x)|dt \\
&\leq 2M\frac{1}{2\pi}\int_{|t|>\delta} K_N(t) \qquad \text{by the triangle inequality} \\
&\leq 2M\frac{\epsilon}{M} \qquad \forall N \geq N_0 \\
&= 2\epsilon
\end{aligned}$$

where the penultimate step follows from $(*)$. Hence,

$$|p_N(x) - f(x)| \leq A + B < \epsilon + 2\epsilon = 3\epsilon$$

whenever $N \geq N_0$, for all x in our domain. This shows that

$$\|p_N - f\| \leq 3\epsilon \qquad \forall N \geq N_0$$

which allows us to conclude that $p_N \to f$ uniformly. ■

8.1 The Proof of the Stone-Weierstrass Theorem

This theorem is so special it deserves its own section. There are a bunch of lemmas that precede the main theorem.

Definition 8.5. An **algebra** is a family of functions that is closed under addition, subtraction, multiplication, and multiplication by a scalar.

Definition 8.6. We say that a set **separates** points if whenever $x \neq y \in X$ then $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 8.7. *If $f, g \in \mathcal{A}$, then $\max(f,g), \min(f,g) \in \mathcal{A}$.*

Theorem 8.8. *If $f \in C(X)$, $x_0 \in X$, and $\epsilon > 0$, then there exists a function $g \in \overline{\mathcal{A}}$ such that $g(x_0) = f(x_0)$ and $g(z) \leq f(z) + \epsilon$ for all $z \in X$.*

Theorem 8.9. ***Stone-Weierstrass Theorem.*** *Let X be compact. Let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and contains the constants. The $\mathcal{A}$ is dense in $C(X)$.*

Proof. First, we show that for all $f \in C(X)$ and $\epsilon > 0$ there exists a $g \in \overline{\mathcal{A}}$ such that $\|g - f\| < \epsilon$. Once we have this then we get $h \in \mathcal{A}$ such that $\|g - h\| < \epsilon$ and then

$$\|f - h\| \leq \|f - g\| + \|g - h\| < 2\epsilon$$

That proves that $\mathcal{A}$ is dense.

By a lemma above, $\forall x \in X$ there exists a function $g_x \in \overline{\mathcal{A}}$ such that $g_x(x) = f(x)$ and

$$g_x(z) \leq f(z) + \epsilon \qquad (*)$$

for all $z \in X$. The function $f - g_x$ is continuous and is zero at x. Let's get the number $\delta_x > 0$ (which, of course depends on g_x), such that $\forall y \in B(x, \delta_x)$ we have that $|f(y) - g_x(y)| < \epsilon$ (this is just straightforward continuity).

We look at $B(x, \delta_x)$ as x ranges across the whole compact metric space X. Since this family forms an open cover of X and X is compact, this cover permits a finite subcover, say $B(x_1, \delta_{x_1}), \ldots, B(x_n, \delta_{x_n})$. We put $g = \max(g_{x_1}, \ldots, g_{x_n}) \in \overline{\mathcal{A}}$ by a lemma above. Let $y \in X$. Then, there exists an index i such that $y \in B(x_i, \delta_{x_i})$, implying that

$$|f(y) - g_{x_i}(y)| < \epsilon$$

so that $g_{x_i}(y) > f(y) - \epsilon$. In particular $g(y) - \epsilon < g_{x_i}(y) \leq \max(g_i) = g(y) = g_{x_j}(y) \leq f(y) + \epsilon$, for some suitable index j and via $(*)$. Altogether, this means that

$$|g(y) - f(y)| \leq \epsilon \qquad \forall y \in X$$

and $\|g - f\| \leq \epsilon$ ■

8.2 Applications of the Stone-Weierstrass Theorem

Example 8.10. We say that $f : X \to \mathbb{C}$ is continuous at $x \in X$ if, whenever $x_n \to x$ then $f(x_n) \to f(x)$ where we say $z_n, z \in \mathbb{C}$, $z_n \to z$ if $|z_n - z| \to 0$. We can define a function as a sum of its real and imaginary part, in the usual way and $\bar{f}(x) = \overline{f(x)} = Re(f) - iIm(f)$. We can extend the Stone-Weierstrass Theorem to complex-valued functions.

Theorem 8.11. ***Stone-Weierstrass for Complex Valued Functions****. Let X be compact. Assume that $\mathcal{A}$ is an algebra in $C(X, \mathbb{C})$, the continuous complex functions with complex scalars. Suppose the algebra separates points and contains the constants. Suppose, furthermore, that $\mathcal{A}$ is closed under conjugation, meaning that if $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$.*

Proof. Denote $\mathcal{A}_{\mathbb{R}}$ denote all the real-valued functions in $\mathcal{A}$. If $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$ so that $f \pm \overline{f} \in \mathcal{A}$, implying that $Re(f), Im(f) \in \mathcal{A}$ and $Re(f), Im(f) \in \mathcal{A}_{\mathbb{R}}$. Then, we have that $\mathcal{A}_{\mathbb{R}}$ is an algebra over $\mathbb{R}$ in $C(X)$. If we take $x \neq y$, then there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$ (since $\mathcal{A}$ separates points). This means that either $Re(f(x)) \neq Re(f(y))$ or $Im(f(x)) \neq Im(f(y))$, so $\mathcal{A}_{\mathbb{R}}$ separates points.

By the Stone-Weierstrass Theorem for real-valued functions, $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X)$. Given $f \in C(X, \mathbb{C})$ and $\epsilon > 0$, we have $Re(f), Im(f) \in C(X)$ so there exists $g, h \in \mathcal{A}_{\mathbb{R}}$ such that $\|Re(f) - g\| < \epsilon$ and $\|Im(f) - h\| < \epsilon$. Take $k(x) = g(x) + ih(x) \in \mathcal{A}$. Then,

$$\begin{aligned} f(x) - k(x)| &= |(Re(f(x)) - g(x)) + i\,(Im(f(x)) - h(x))| \\ &\leq |Re(f(x)) - g(x)| + |Im(f(x)) - h(x)| \\ &\leq 2\epsilon \end{aligned}$$

Ta-da! ■

Example 8.12. Let $X = \{z \in \mathbb{C} : |z| = 1\}$, the complex numbers of modulus one. Note that $\forall z \in X$, we have $z = e^{i\theta}$. We look at the functions

$$\mathcal{A} = \left\{ \sum_{n=-N}^{N} a_n z^n \,:\, a_n \in \mathbb{C} \right\} = \left\{ \sum_{n=-N}^{N} a_n e^{in\theta} \,:\, a_n \in \mathbb{C} \right\}$$

We can identify X with the interval $[-\pi, \pi]$, in the usual way and observe that the functions in $\mathcal{A}$ are 2π-periodic. We can think about the set $C(X, \mathbb{C})$ as the 2π-periodic, continuous complex-valued functions. Since $\mathcal{A}$ is the set of trigonometric polynomials, it is an algebra, it separates points, it contains all the constants (if we let $N = 0$), and it is closed under conjugation. Hence $\mathcal{A}$ is dense in $C(X, \mathbb{C})$. ■

Example 8.13. A corollary of the Stone-Weierstrass Theorem is that $C[a, b]$ is separable. This is a very nice property for functional analysis. We prove this in Assignment 7 Question 6.

Example 8.14. If $f \in C[0, 1]$ and $\int_0^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \ldots$, then $f(x) = 0$ for all $x \in [0, 1]$.

Proof. Since $\int_0^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \ldots$ we certainly have that $\int_0^1 f(x)p(x)dx$ where $p(x)$ is a polynomial. By the Stone-Weierstrass Theorem, there exists a sequence of polynomials (p_n) such that $p_n \to f$ uniformly. We claim that

$$\int_0^1 f(x)p_n(x)dx \to \int_0^1 (f(x))^2 dx$$

and assuming this we have, from baby Calculus, that $\int_0^1 f^2 = 0 \implies f = 0$. We prove this claim:

$$\begin{aligned} \left| \int_0^1 f(x)(p_n(x) - f(x))dx \right| &\leq \int_0^1 |f(x)||p_n - f|dx \\ &\leq \|f\| \int_0^1 |p_n - f|dx \\ &\leq \|f\|\,\|p_n - f\| \int_0^1 dx \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

since $p_n \to f$ uniformly. And we are done! ■

We would love to give more examples, but Mr Baire and Mr Category[2] are waiting.

[2]If you were born yesterday, you may not realise that Mr Category is an imaginary friend.

Chapter 9

Baire Category Theorem

Definition 9.1. We say that $A \subseteq X$ is **nowhere dense** if $int(\overline{A}) = \emptyset$.

Example 9.2. The set $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Example 9.3. The set $\mathbb{Q}$ is *not* nowhere dense in $\mathbb{R}$.

Definition 9.4. We say that $A \subseteq X$ is **first category** if

$$A = \bigcup_{n=1}^{\infty} V_n$$

where each V_n is nowhere dense. We say that $A \subseteq X$ is **second category** if A is not first category.

Example 9.5. $\mathbb{Q}$ is first category since $\mathbb{Q} = \bigcup_{n=1}^{\infty}\{q_n\}$ where each $\{q_n\}$ is nowhere dense in $\mathbb{R}$ as they are already closed.

Theorem 9.6. *If A is nowhere dense then A^c is dense.*

Proof. We want to use the following characterisation: a set S is dense if and only if S intersects every open set. We prove this little lemma first.

($\Longrightarrow$) Let U be any nonempty open set and say $x \in U$. Then $x \in \overline{S}$ and hence $U \cap S \neq \emptyset$ (since x is either in S or it is an accumulation point by density).

($\Longleftarrow$) Let $x \in X$ and consider $B(x, \epsilon) = U$. By assumption, $U \cap S \neq \emptyset$ so that $x \in \overline{S}$. ■

Now to prove the fact we shall prove that $A^c \cap U \neq \emptyset$ for all nonempty open set U. Arguing by contradiction, suppose $U \cap A^c = \emptyset$, so that $U \subseteq A$. This means that $U \subseteq A^\circ$ and $A^\circ \neq \emptyset$, thus implying that $(\overline{A})^\circ \neq \emptyset$, contradicting the assumption that A nowhere dense. ■

Theorem 9.7. *A is closed and nowhere dense if and only if $A^c$is open and dense.*

Proof. Left as exercise in class. It is essentially the same idea as the theorem above. ■

Theorem 9.8. *The metric space X is second category if and only if the intersection of every countable family of open dense sets in X is non-empty.*

Proof. X is first category if and only if X can be written as

$$X = \bigcup_{n=1}^{\infty} X_n$$

where X_n are closed and nowhere dense. We can write this without loss of generality, since taking the big union of closures adds nothing to the original universe (which we already had!).

($\Longrightarrow$) Suppose X is first category. Let $\{G_n\}$ be a countable family of open dense sets. Suppose, for contradiction

$$\bigcap_{n=1}^{\infty} G_n = \emptyset$$

Then

$$\bigcup_{n=1}^{\infty} G_n^c = \left(\bigcap_{n=1}^{\infty} G_n\right)^c = \emptyset^c = X$$

where G_n^c are closed and nowhere dense (since the G_n's where assumed to be open and dense). This implies that X is first category, a contradiction.

($\Longrightarrow$) Arguing by contrapositive, suppose X is not second category. Then X is first category. Hence

$$X = \bigcup_{n=1}^{\infty} X_n$$

where X_n are closed and nowhere dense. Taking complements,

$$\emptyset = X^c = \left(\bigcup_{n=1}^{\infty} X_n\right)^c = \bigcap_{n=1}^{\infty} X_n^c$$

where X_n^c are open and dense, thus contradicting the right half of the proposition, completing the contrapositive. ■

Theorem 9.9. ***Baire Category Theorem.*** *A non-empty complete metric space is second category. Namely, X is not the countable union of closed nowhere dense sets.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be open and dense and $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$[1]. Let $x_1 \in A_1$. Since A_1 is open, there is a ball $B(x_1, r_1) = U_1 \subseteq A_1$. Since A_2 is dense, it hits every open set, in particular, it hits U_1 and $A_2 \cap U_1 \neq \emptyset$, so say $x_2 \in A_2 \cap U_1 \subseteq A_2 \cap A_1$, which is open. So there exists a set $V_2 = B(x_2, r_2)$ such that $V_2 \subseteq A_2 \cap U_1$ and without loss of generality let $r_2 \leq \frac{r_1}{2}$. Put $U_2 = B\left(x_2, \frac{r_2}{2}\right)$. Then $\overline{U}_2 \subseteq V_2 \subseteq A_2 \cap U_1$ and $\text{diam}\overline{U}_2 \leq \frac{1}{2}\text{diam}\overline{U}_1$.

Proceed inductively to get x_n, contained in open sets in the form $x_n \in U_n$ such that

$$U_n \subseteq \bigcap_{k=1}^{n} A_k$$

with $\overline{U}_n \subseteq U_{n-1}$ and $\text{diam}U_n \to 0$. We now check that $(x_n))_{n=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$ and pick an N so large that $\text{diam}U_N < \epsilon$. If $n, m \geq N$, we have that $x_n, x_m \in U_N$ since the sequence (U_n) is nested. Hence $d(x_n, x_m) \leq \text{diam}U_N < \epsilon$. So (x_n) is Cauchy and X is complete so $(x_n) \to x \in X$. Furthermore, since $x_n \in \overline{U}_N$ for all $n \geq N$, this implies that $x \in \overline{U}_N \subseteq U_{N-1} \subseteq \cap_{j=1}^{N-1} A_j$ for all N. This implies that $x \in A_j$ for all j which allows us, then, to conclude

$$x \in \bigcap_{j=1}^{\infty} A_j$$

which is us repeating the argument for Cantor's intersection theorem! What a great theorem, we like-y! ■
We proceed with some applications of the Baire Category Theorem.

Theorem 9.10. *The set of functions which are continuous everywhere but nowhere differentiable are second category.*

Proof. I missed this class, so I'll fill in the details later. ■

[1]We like this proof because it resembles Cantor's intersection theorem.

Chapter 10

Banach Contraction Mapping Principle

Theorem 10.1. *Let X be a complete metric space and let $T : X \to X$ which is a contraction; namely, there exists a number r with $0 \leq r < 1$ such that*

$$d(T(x), T(y)) \leq rd(x, y) \qquad \forall x, y \in X$$

Then T is continuous and has a unique fixed point; that is, there exists a unique $z \in X$ such that $T(z) = z$.

Proof. This was done in Assignment 4 and it is exactly the same as the proof for (R^n, d), which was done in MATH 247. I'll leave out the details. ■

This is a super cool theorem, because it has super cool applications. The first one is to fractal geometry.

Theorem 10.2. *Let $K = \{E \subseteq \mathbb{R}^d, E \neq \emptyset, E \text{is compact}\}$ and let d_H be the Hausdorff metric. Then (K, d_H) is a complete metric space.*

Definition 10.3. Suppose $S_1, \dots, S_k : E \to E$ with $E \in K$ are contractions. We say that $F \in K$ is **invariant** if $F = \bigcup_{j=1}^k S_j(F)$.

Example 10.4. The Cantor set is invariant. Let $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$. Define $S : K \to K$ by

$$S(Y) = \bigcup_{j=1}^{k} S_j(Y)$$

We note that S is a contraction on (X, d_H). By the Banach Contraction Mapping Principle, it has a unique fixed point; namely, there is a unique F with $S(F) = F$. The proof of the contraction mapping principle provides an algorithm to obtain F as the limit of iterations of some F_0. If we let $F_0 = [0, 1]$ and apply S repeatedly, we obtain the Cantor set.

10.1 An application: Existence and Uniqueness of ODEs

We begin with an example of an easy ODE.

Example 10.5. Solve the differential equation

$$y' = 1 + x - y$$

for $x \in [-0.5, 0.5]$ with $y(0) = 1$. That is, let $y = f(x)$ and solve the initial value problem

$$f'(x) = 1 + x - f(x) \qquad f(0) = 1$$

We convert this to an integral equation by the fundamental theorem of calculus (or what applied mathematicians call the method of successive approximations).

$$\begin{aligned} f(x) &= \int_0^x f'(t)dt + f(0) \\ &= \int_0^x (1+t-f(t))dt + 1 \\ &= x + \frac{x^2}{2} + 1 - \int_0^x f(t)dt \end{aligned}$$

We define an operator T on $C[-0.5, 0.5]$ by

$$T(f(x)) = 1 + x + \frac{x^2}{2} - \int_0^x f(t)$$

where $T(f) \in C[-0.5, 0.5]$ and $T : C[-0.5, 0.5] \to C[-0.5, 0.5]$. Notice that $(Tf)' = 1 + x - f(x)$. If we can find a fixed point of T, say f, then $f(x) = T(f(x))$ implying that $f(0) = T(f(0)) = 1$ and $f' = (Tf)' = 1 + x - f(x)$ so that f solves the ODE.

Conversely, if fr solves the ODE, then $f'(x) = 1 + x - f(x) = (Tf)'(x)$ for all $x \in [-0.5, 0.5]$. But $f(0) = 1 = T(f(1))$ implying that $f = T(f)$ so any solution to the ODE is a fixed point of T. Thus, we have that the solution to the IVP exists and is unique.

Thus we check that T is a contraction. We want there to exist an $r < 1$ such that $d(Tf, Tg) \leq rd(f, g)$ for all $f, g \in C[-0.5, 0.5]$. This is equivalent to showing

$$\|Tf - Tg\| \leq r\,\|f - g\| \qquad \forall f, g \in C[-0.5, 0.5]$$

We perform the calculation:

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| -\int_0^x f + \int_0^x g \right| \\ &= \left| \int_0^x (g - f) \right| \\ &\leq \|g - f\| \left| \int_0^x dx \right| \\ &= \|g - f\|\,|x| \qquad x \in [-0.5, 0.5] \\ &\leq \frac{1}{2}\,\|g - f\| \end{aligned}$$

Thus T is a contraction. By the Banach Contraction Mapping Principle, T has a unique fixed point f. Thus, our ODE has a unique solution.
In fact $f = \lim_{n\to\infty} T^n f_0$ for some initial continuous function f_0. For example, take $f_0 = 1$, then using induction, we find,

$$T^n f_0 = \sum_{k=0;k\neq 1}^{\infty} \frac{(-1)^n}{k!} x^k \to e^{-x} + x = f(x)$$

as $n \to \infty$. ■

Definition 10.6. Let $\Phi : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous. We say that Φ is **Lipschitz in the second variable** if there exists an L such that

$$|\Phi(x, y) - \Phi(x, z)| \leq L|y - z| \qquad \forall x \in [a, b], \text{ and } y, z \in \mathbb{R}$$

Theorem 10.7. ***The Global Picard Theorem.*** *Suppose* $\Phi : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and is Lipschitz in the second variable. Then the differential equation*

$$F'(x) = \Phi(x, F(x)) \qquad F(a) = c$$

has a unique solution.

Proof idea. Define $T : C[a,b] \to C[a,b]$ by

$$Tf(x) = c + \int_a^x \Phi(t, f(t))dt$$

As in the example above, we check that F solves the differential equation if and only if $F = TF$. We then check that T is a contraction, which holds when either $[a,b]$ is small or using the Lipschitz condition. If we are lucky that this is the case, then we invoke the Banach Contraction Mapping Principle, if not a similar argument can be applied. ∎

Remarks about the final exam:

1. Office hours are on Tuesday July 31st between 3-4, and Wednesday August 1st between 10:30 and 12.
2. The exam will not ask about cardinality, except for countability.
3. The exam will not contain content on the Banach Contraction Mapping Principle.
4. The exam won't expect us to prove the Arzela-Ascoli Theorem, nor the Stone-Weierstrass Theorem, but it might ask for a lemma along the way.
5. The exam will not ask about Bernstein polynomial or the proof about the convergence of trigonometric polynomials.
6. We are not responsible for any PMATH 450 topics.
7. We are not responsible for any of the definitions that only arose on the homework.
8. We should know all the definitions and the statements of all the theorem studied in class.
9. Things that were excluded on the midterm which are not excluded now are no longer excluded.