Measurements of the Rydberg Constant for Infinite Mass and Deuteron-Proton Mass Ratio

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Abstract

The use of a spectrometer to measure the spectra of the balmer series for hydrogen and deuterium, and our use of the quantum theory of hydrogen allowed us to measure the Rydberg constant for infinite mass for our hydrogen and deuterium data sets, and also measure the deuteron-proton mass ratio (referred to as the deuterium-hydrogen mass ratio in the report). For the Rydberg constant for infinite mass, we measured $R_\infty = 10975100 \pm 591\ m^{-1}$ for our hydrogen sample, and $R_\infty = 10975000 \pm 701\ m^{-1}$ for our deuterium sample. The measurement that we got for our deuteron-proton mass ration was $\frac{m_D}{m_H} = 2.00 \pm 0.02$. Although our measurement of the Rydberg constant for infinite mass was not in agreement with the accepted value of $R_\infty = 10973731.568160(21)\ m^{-1}$ [1], we were still able to validate the accuracy of the Bohr postulates since our measurement for the deuteron-proton mass ratio was in agreement with the accepted value of $\frac{m_D}{m_H} = 1.99900750139(11)$[1]. It is important to note however, that we only validated the accuracy of the Bohr and Einstein postulates for both the hydrogen and deuterium atoms.

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I. INTRODUCTION: HYDROGEN SPECTROSCOPY

Atomic spectra are a very important object when it comes to quantum mechanics. The observation of unique wavelengths for different types of atoms has made spectroscopy a useful tool in astronomy, and spectroscopy was a major reason behind the development of quantum mechanics. One of the most well studied atomic spectra is the hydrogen spectrum due to its simple structure.

The hydrogen spectrum is a highly studied object and has contributed a lot to the development of quantum mechanics. One of the major contributions came from the work done by Johannes Rydberg [2, pp 97-98], who found an expression for the Balmer series of the hydrogen spectrum given by the following expression:

$$k = \frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{n^2} \right) \tag{1}$$

This expression then gave rise to the development of Bohr's postulates which predicts the qunatization of energy for electrons inside of an atom, as well as the fact that an electron emits electromagnetic radiation when it is transitioning from a higher energy state to a lower energy state. Another important prediction is that the energy that is emitted in the form of electromagnetic radiation is equal to the difference in energies between the initial state of the electron and its final state.

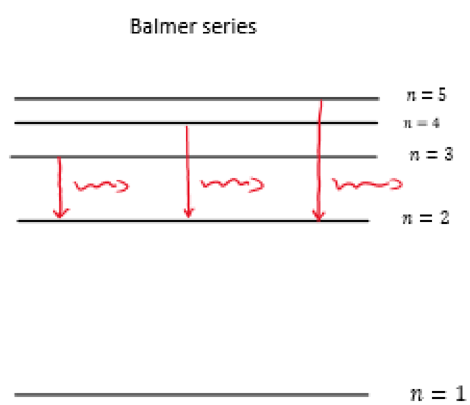

FIG. 1. The Balmer series for the hydrogen spectrum up until n=5.

There have been many experiments done with the hydrogen spectrum, many of which gave precise measurements of many fundamental constants. Amongst those fundamental values is the Rydberg constant for an atom with a nucleus of infinite mass, which is one of the most precisely measured fundamental constants and the main focus of this experiment.

Since we do not have a nucleus with infinite mass, we will use the hydrogen spectrum as well as the deuterium spectrum in order to measure the Rydberg constant for infinite mass. We will do this by measuring the wavelengths of light released by excited hydrogen and deuterium atoms which allows us to measure the energy released by a transition.

Assuming that Bohr's postulates gives us exact values, the energy released by a transition is given by the equation:

$$\Delta E_{i\to f} = hcR_N\left(\frac{n_i^2 - n_f^2}{n_i^2 n_f^2}\right) \tag{2}$$

Since the energy of a photon is related to its wavelength by the expression $\lambda = \frac{\Delta E}{hc}$, we can express the wavelength of light emitted in terms of the Rydberg constant for appropriate mass.

$$\lambda_{i\to f} = \frac{1}{R_N \xi_{i\to f}} \tag{3}$$

$$\frac{1}{\lambda_{i\to f}} = R_N \xi_{i\to f} \tag{4}$$

Where $\xi_{i\to f} = \frac{n_i^2 - n_f^2}{n_i^2 n_f^2}$. Using equation (4), we can find Rydberg's constant for infinite mass since $R_N = \frac{R_\infty}{F_N}$, here $F_N \equiv 1 + \frac{m_e}{m_N}$, which for Hydrogen it is measured to be $F_H = 1.000544617021487(33)$ [1], thus we have:

$$\frac{1}{\lambda_{i\to f}} = 1.000544617021487(33) R_\infty(\xi_{i\to f}) \tag{5}$$

There are some important things to note however; for equation (4), plotting $\frac{1}{\lambda_{i\to f}}$ vs $\xi_{i\to f}$ will result in a light that should have a slope intercept of 0; secondly, equation (5) applies to both hydrogen and deuterium but we must use different masses. For hydrogen, we use the electron-proton mass ratio, $\frac{m_e}{m_p}$ and for deuterium we use the electron-deuteron mass ratio, $\frac{m_e}{m_d}$.

The second goal of this experiment is to measure the mass ratio between deuterium and hydrogen. Using equation (3), we can find deuterium-hydrogen ratio to be given by the following expressions:

$$\frac{m_D}{m_H} = \frac{\lambda_H - \lambda\infty}{\lambda_D - \lambda_\infty} \tag{6}$$

$$\frac{m_D}{m_H} = \frac{A_{i\to f}}{A_{i\to f} - \Delta\lambda_{i\to f}} \tag{7}$$

Out of the two expressions, we will be using equation (7) in order to measure the deuterium-hydrogen ratio. This is because we cannot measure λ_∞, but we can calculate $A_{i\to f} = \lambda_H - \lambda_\infty$ by using the following equation.

$$A_{i\to f} = \frac{1}{R_\infty \xi_{i\to f}} \left(\frac{m_e}{m_p}\right) \tag{8}$$

II. EXPERIMENT

For this experiment, we used the Horiba 1250M-II to gather our data. The Horiba 1250M-II has four slits and it uses the *Czerny-Turner configuration*; in this configuration, light enters through an entrance slit (S1 or S3), passes through a concave mirror M1 which then goes to a reflecting grating G. From the grating, the light goes to another mirror M2 and then out an exit slit (S2 or S4).The way that we get specific wavelength is by using the rotatable grating which disperses different wavelengths to different angles, this allows us to focus a specific set of wavelengths onto a detector at one of the exit slits.

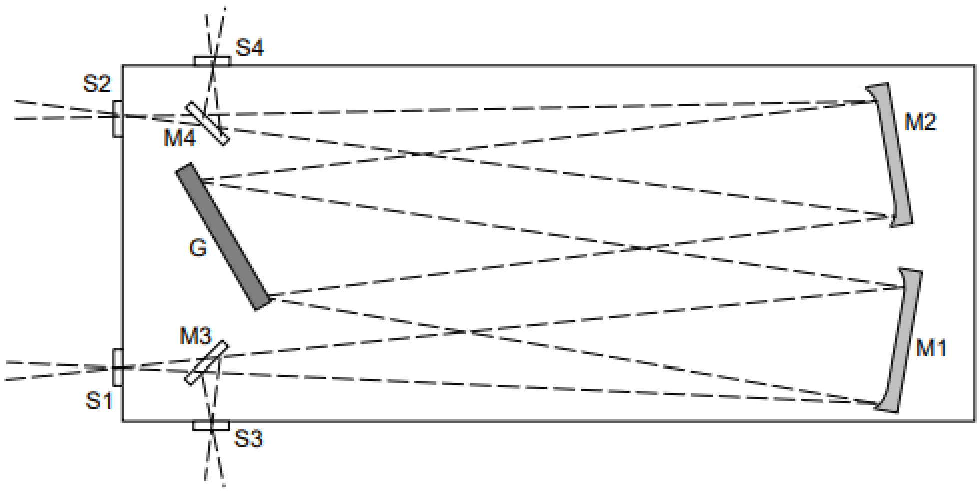

FIG. 2. The Horiba 1250M-II layout and set up. M3 and M4 are optional mirrors, M1 and M2 are focusing mirrors, S1 and S3 are entrance slits and S2 and S4 are exit slits. G is the diffraction grating which is allowed to rotate. Figure was taken from the Lab Guide [3, pp 1]

In our setup, we initially used an 8 micron slit width, a deuterium lamp that contains 25% hydrogen, and a PMT voltage of 800 V.Throughout the experiment, we changed the PMT voltage, integration time, and step size appropriately so that we can get data that was not noisy and had clear peaks; the range for the scans were also changed appropriately depending on which initial state we were starting at (i.e. for H_ϵ we had a range from 3970 to 3973 Å).

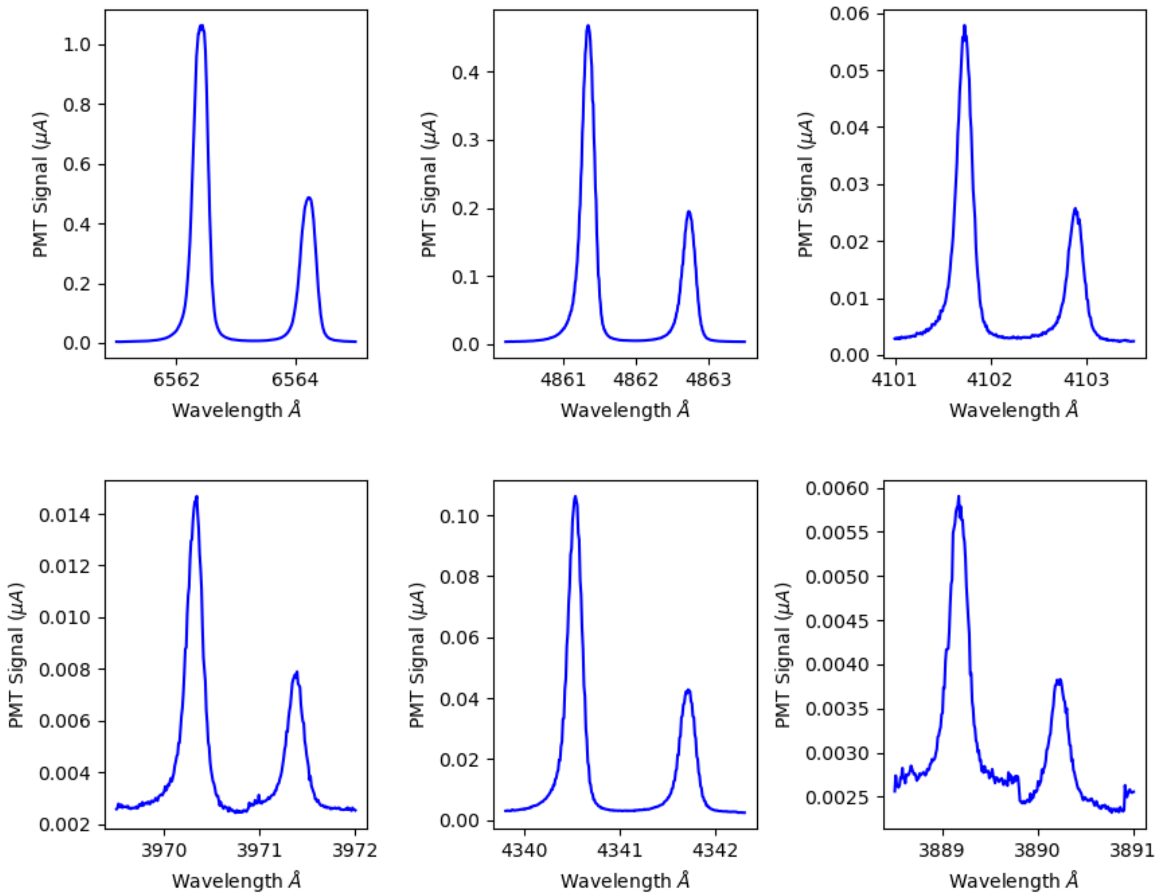

FIG. 3. The data that we got from our experiments. (Top left) H_α taken with 8 microns slit width and PMT voltage of 800 V, (Top middle) H_β taken with 12 micron slit width, (Top right) H_γ taken with PMT voltage of 850 V, (Bottom left) H_δ taken with PMT voltage of 1000 V, (Bottom middle) H_ϵ, (Bottom right) H_ς. This plot was made by using the matplotlib and pandas libraries in Python.

A. Offset

The measurement for the offset that we expect to find for a peak wavelength near 500 nm was done by looking at multiple measurements for the peak wavelength of a variety of atoms. The important thing to note is that the offset was not measured for individual wavelengths in this experiment and that we only used the expected offset for the 500 nm wavelength.

The first thing we did was measure peak wavelengths for helium, mercury, and sodium samples which spanned a range from 200 nm to about 600 nm. We then calculated the uncertainty within our measurements and the offset by using the accepted wavelength values from the National Institute of Standards and Technology.

Interferometric λ (nm)	Measure λ (nm)	σ_{λ_M} (nm)	$\lambda_I - \lambda_M$ (nm)
253.6521	253.7780	0.0033	-0.1259
404.6559	404.7850	0.0012	-0.1291
435.8323	435.9670	0.0040	-0.1347
546.0731	546.2070	0.0020	-0.1339
313.1555	313.2850	0.0016	-0.1295
313.1844	313.3150	0.0013	-0.1306
626.3110	626.4700	0.0015	-0.1590
626.3688	626.5280	0.0089	-0.1592
587.56251	587.6860	0.0009	0.0130
667.81516	667.9420	0.0022	0.0150

TABLE I. Table containing measurements of peak wavelengths, the FWHM, uncertainty, and offset from accepted values

After making the measurements for the peak wavelengths and the offsets, we plotted the offset as a function of the measured wavelengths and then made a best fit line by using the least-squares method. This resulted in the following expression:

$$x_{offset} = -2 * 10^{-5} x - 0.1231 \tag{9}$$

We then calculated $x_{offset}(500) = -0.13 \pm 0.02$ nm. This was the offset we used for all of our measurements in this experiment, but it is worth noting that this might not be the actual offset for our measurements, which adds error to our calculations and measurements.

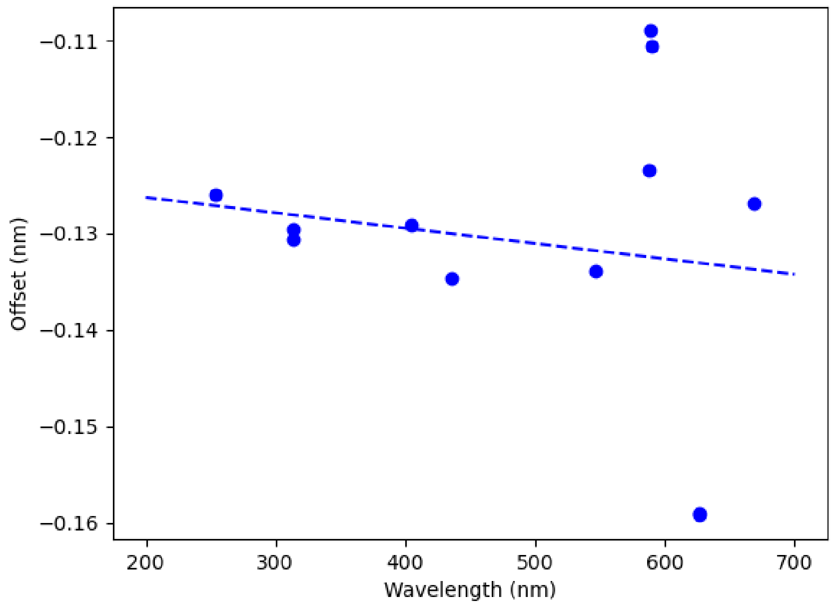

FIG. 4. Plot of measured wavelength as a function of offset and its best fit line. Plot was made using the matplotlib and pandas libraries in python.

III. RESULTS

Once we got our measurements for the wavelength in a vacuum for both the deuterium and hydrogen samples, we tabulated our results into two separate tables for the hydrogen and deuterium spectra.

λ_{air} (nm)	σ_{λ_m} (nm)	$\lambda_{corrected}$ (nm)	σ_{λ_c} (nm)	λ_{vac} (nm)	$\sigma_{\lambda_{vac}}$ (nm)
656.423	0.00146	656.292	0.020	656.473	0.020
486.273	0.00067	486.142	0.020	486.276	0.020
434.171	0.00054	434.040	0.020	434.160	0.020
410.289	0.00044	410.158	0.020	410.271	0.020
397.139	0.00019	397.008	0.020	397.118	0.020
389.023	0.00002	388.892	0.020	388.999	0.020

TABLE II. Table containing the measurements for the hydrogen Balmer series.

λ_{air} (nm)	σ_{λ_m} (nm)	$\lambda_{corrected}$ (nm)	σ_{λ_c} (nm)	λ_{vac} (nm)	$\sigma_{\lambda_{vac}}$ (nm)
656.244	0.00481	656.113	0.021	656.294	0.021
486.134	0.00199	486.003	0.020	486.137	0.020
434.053	0.00063	433.922	0.020	434.042	0.020
410.173	0.00115	410.042	0.020	410.155	0.020
397.034	0.0003	396.903	0.020	397.012	0.020
388.917	0.00009	388.786	0.020	388.893	0.020

TABLE III. Table containing the measurements for the deuterium Balmer series.

IV. DISCUSSION

Using our tabulated data, we used equation (4) to find the Rydberg constants for hydrogen and deuterium. Since we had multiple measurements for the wavelengths in the balmer series spectra for hydrogen and deuterium, we tabulated our measurements made with equation (4) and then used least-squares fit to find the Rydberg constants for hydrogen and for deuterium [4, Chapter 3]. For hydrogen we got $R_H = 10969100\pm, 590$ m^1 and $R_D = 10972000 \pm 701$ m^1.

As stated earlier, equation (4) applies to both hydrogen and deuterium, but we need to use different mass ratios when using equation (4). For hydrogen, we got a measurement of $R_\infty = 10975100 \pm 591$ m^1 ; for our deuterium sample, we needed to use the electron-deuteron mass ratio value in order to calculate F_N which will allow us to calculate R_∞ by using our deuterium measurements. For the electron-deuteron mass ration, I used $\frac{m_e}{m_d} = 0.0002724437107462(96)$ [1] which means that our value for $F_N = 1.0002724437107462(96)$; this resulted in a measurement of $R_\infty = 10975000 \pm 701$ m^{-1} for the deuterium measurements.

Finally, using equation (7), we measured the deuterium-hydrogen mass ratio to be $\frac{m_D}{m_H} = 2.00 \pm 0.02$ which are weighted values [4, Chapter 7].

Hydrogen		Deuterium	
$\xi_{i\to f}$	$\frac{1}{\lambda_{i\to f}}$ nm^{-1}	$\xi_{i\to f}$	$\frac{1}{\lambda_{i\to f}}$ nm^{-1}
0.13889	0.0015233	0.13889	0.0015237
0.18750	0.0020564	0.18750	0.0020570
0.21000	0.0023033	0.21000	0.0023039
0.22222	0.0024374	0.22222	0.0024381
0.22959	0.0025181	0.22959	0.0025188
0.23438	0.0025707	0.23438	0.0025714

TABLE IV. Table of measurements for the inverse wavelength of our hydrogen and deuterium Balmer series data and our calculated values for $\xi_{i\to f}$

$\xi_{i\to f}$	$A_{i\to f}$ (nm)	λ_H (nm)	λ_D (nm)	$\Delta\lambda$ (nm)	$\sigma_{\Delta\lambda}$ (nm)	m_D/m_H	σ_r
0.13889	0.358	656.47	656.30	0.17	0.00503	1.90	0.050
0.18750	0.265	486.28	486.14	0.14	0.00210	2.00	0.036
0.21000	0.236	434.16	434.04	0.12	0.00083	2.00	0.015
0.22222	0.223	410.27	410.16	0.11	0.00123	2.00	0.021
0.22959	0.216	397.12	397.01	0.11	0.00036	2.00	0.007
0.23438	0.212	389.00	388.89	0.11	0.00009	2.10	0.002
Weighted Average			2.00				
Weighted Uncertainty			0.02				

TABLE V. Table containing the measurements for λ_H, λ_D, $\Delta\lambda$ and m_D/m_H and their uncertainties along with the calculated values for $\xi_{i\to f}$ and $A_{i\to f}$.

The first focus on this discussion will be on the measurements we made for R_∞. For the hydrogen Balmer series, we measured $R_\infty = 10975058.11 \pm 590.48\ m^{-1}$ and for our deuterium Balmer series, we measured $R_\infty = 10974980.22 \pm 701.18\ m^{-1}$. In comparison to the accepted value, $R_\infty = 10973731.568160(21)\ m^{-1}$ [1], both of our measured values for R_∞ were not in agreement with the accepted value.

The fact that we didn't get R_∞ values that agreed with the accepted value could be due to the way that we calculated the offset for the wavelengths that we got for all of our measurements. Basically, we used data that we gathered for different atoms and their peak wavelengths, and then used accepted values for their peak wavelengths to calculate an offset value. We then used least-squares to come up with a trend that can give us the offset value at any given wavelength, for this experiment we found the expected offset value for a wavelength of 500 nm. Any inaccuracy in our measurement for the offset value would have definitely made our other measurements inaccurate which can explain the disagreement in our measurements for the Rydberg constant for infinite mass.

Secondly, our deuterium-hydrogen mass ratio measurement was $\frac{m_D}{m_H} = 2.00 \pm 0.02$. In comparison to the accepted value of $\frac{m_D}{m_H} = 1.99900750139(11)$[1], our measured value was in agreement with the accepted value. This means that our assumption that the Bohr postulates give us exact values for transition energy and its relationship to its wavelength was correct which verifies both Bohr's postulates and Einstein's postulates for the hydrogen and deuterium atoms.

It is important to note that the accuracy of all of our measured values was impacted by our measurement for the offset value at 500 nm. We could have gotten more accurate results if we measured the offset values for each of the wavelengths as opposed to doing it for only one wavelength.

V. CONCLUSION

In conclusion, the measured values that we got for the Rydberg constant for infinite mass were not in agreement with the accepted value; however our value for the deuterium-hydrogen mass ratio was in agreement with the accepted value. This means that we could not fully validate Bohr's postulates which connects Rydberg's constant to the transition energies and wavelengths, but we were able to somewhat validate that Bohr's postulates gives us exact values for the hydrogen and deuterium atoms and as a result, validate our approach which used Bohr's postulates and Einstein's postulates.

Appendix A: Derived Equations

Derivation of Hydrogenic Atom Energy Levels

The energy levels for an hydrogen-like atom is given by the following equation [2, pp 100-107]:

$$E_n = -\frac{1}{2}\mu c^2 (Z\alpha)^2 \frac{1}{n^2} \tag{A1}$$

Since we are working with hydrogen, $Z = 1$ which results in the following expression:

$$E_n = \left(-\frac{\mu c^2 \alpha^2}{2} \right) \frac{1}{n^2} \tag{A2}$$

The $(-\frac{\mu c^2 \alpha^2}{2})$ term is a term composed out of fundamental constants (With the exception of μ). Here, we have α which is the fine structure constant, c which is the speed of light, and μ which is the reduced mass. We can actually express μ and α as the following expressions:

$$\mu = \frac{m_e m_N}{m_e + m_N} \tag{A3}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0} \frac{1}{\hbar c} \tag{A4}$$

We can use equations (A3) and (A4) in order to further reduce equation (A2). We begin first by substituting equation (A4) and equation (A3) into equation (A2).

$$E_n = \left(-\frac{c^2}{2} \right) \left(\frac{m_e m_N}{m_e + m_N} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \left(\frac{1}{\hbar c} \right)^2 \frac{1}{n^2} \tag{A5}$$

We can then reorganize equation (A5).

$$E_n = \frac{m_e c^2}{(\hbar c)^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \left(\frac{\hbar c}{2\hbar c} \right) \left(1 + \frac{m_e}{m_N} \right)^{-1} \frac{-1}{n^2} \tag{A6}$$

$$E_n = \frac{m_e c^2}{4\pi(\hbar c)^3} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \left(1 + \frac{m_e}{m_N} \right)^{-1} \left(\frac{-hc}{n^2} \right) \tag{A7}$$

In equation (A7), notice the $\frac{m_e c^2}{4\pi(\hbar c)^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$, this is known as the Rydberg Constant for an infinite mass nucleus, R_∞, which is a fundamental constant. The other term that came out from the reduced mass constant, $\left(1 + \frac{m_e}{m_N}\right)^{-1}$ is also an important term. R_∞ only describes atoms with an infinite mass nucleus, but we can add a corrective term so that we can describe an atom that does not have an infinite mass nucleus. This term is often denoted as $F_N \equiv 1 + \frac{m_e}{m_N}$. In order to get the Rydberg constant for the appropriate mass nucleus, R_N, we must only divide R_∞ by our corrective term F_n:

$$R_N = \frac{R_\infty}{F_N} = \frac{m_e c^2}{4\pi(\hbar c)^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\left(1 + \frac{m_e}{m_N}\right)^{-1} \tag{A8}$$

Using equation (A8), we can further reduce equation(A7).

$$E_n = -\frac{hcR_N}{n^2} \tag{A9}$$

Transition Energies and Wavelengths

In order to find the energy that an electron releases when it transitions from its initial state, n_i, to a final state, n_f, we must only look at the energy differences between those two states. In other words, we must only subtract the electron's energy in the final state by the electron's energy in its initial state.

$$\Delta E_{i\to f} = -E_{n_f} - (-E_{n_i}) = E_{n_i} - E_{n_f} \tag{A10}$$

Substituting our expression for the energy levels, we get:

$$\Delta E_{i\to f} = hcR_N\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) = hcR_n\left(\frac{n_i^2 - n_f^2}{n_i^2 n_f^2}\right) \tag{A11}$$

Since n_i and n_f are only principle quantum numbers, we can express them as a constant, $\xi_{i\to f}$, which is a function of the initial and final states of the electron. Substituting $\xi_{i\to f}$ into equation (A11) gives us our final expression for $\Delta E_{i\to f}$.

$$\Delta E_{i\to f} = hcR_n\xi_{i\to f} \tag{A12}$$

The way that an electron releases energy when it transition from a higher state to a lower state is by releasing a photon with that same energy. This allows us to relate the wavelength of light that the electron releases with the transition energy:

$$\Delta E_{i \to f} = h\nu = \frac{hc}{\lambda_{i \to f}} \tag{A13}$$

$$\lambda_{i \to f} = \frac{hc}{\Delta E_{i \to f}} = \frac{hc}{hcR_N \xi_{i \to f}} = \frac{1}{R_N \xi_{i \to f}} \tag{A14}$$

Using equation (A12), we were able to express the wavelength in terms of the Rydberg constant for appropriate mass; which also allows us to express the wavelength in terms of the Rydberg constant for infinite mass.

$$\lambda_{i \to f} = \frac{F_N}{R_\infty \xi_{i \to f}} \tag{A15}$$

Deuterium and Hydrogen Mass Ratio

Using equation (A15), we can derive an expression for the mass ratio of deuterium and hydrogen.

$$\lambda_H - \lambda_\infty = \frac{1}{R_\infty \xi_{i \to f}}(F_H - 1) \tag{A16}$$

$$\lambda_D - \lambda_\infty = \frac{1}{R_\infty \xi_{i \to f}}(F_D - 1) \tag{A17}$$

Dividing the hydrogen expression by the deuterium expression results in the following:

$$\frac{\lambda_H - \lambda_\infty}{\lambda_D - \lambda_\infty} = \frac{F_H - 1}{F_D - 1} = \frac{1 - 1 + m_e/m_H}{1 - 1 + m_e/m_D} = \frac{m_D}{m_H} \tag{A18}$$

$$\frac{\lambda_H - \lambda_\infty}{\lambda_D - \lambda_\infty} = \frac{m_D}{m_H} \tag{A19}$$

Equation (A19) is a good expression for the mass ratio, however, we cannot measure λ_∞ so we must create some sort of workaround. Assuming that equations (A15) gives us an exact answer, we can calculate $\lambda_H - \lambda_\infty$, and then come up with an expression for $\lambda_D - \lambda_\infty$ in terms of our calculated value.

Let $A_{i \to f} = \lambda_H - \lambda_\infty$

$$\lambda_D - \lambda_\infty = \lambda_D - \lambda_\infty + \lambda_H - \lambda_H = \lambda_D - \lambda_H - (\lambda_\infty - \lambda_H) = (\lambda_H - \lambda_\infty) - (\lambda_D - \lambda_H) \tag{A20}$$

$$\lambda D - \lambda_\infty = A_{i\to f} - \Delta\lambda_{i\to f} \tag{A21}$$

We can now make a ratio the same we did for equation (A19) to get the following:

$$\frac{\lambda_H - \lambda_\infty}{\lambda_D - \lambda_\infty} = \frac{A_{i\to f}}{A_{i\to f} - \Delta\lambda_{i\to f}} \tag{A22}$$

It is important to note that we can't measure $A_{i\to f}$, but we can calculate it.

$$A_{i\to f} = \lambda_H - \lambda_\infty \tag{A23}$$

$$A_{i\to f} = \frac{F_H}{R_\infty \xi_{i\to f}} - \frac{1}{R_\infty \xi_{i\to f}} \tag{A24}$$

$$A_{i\to f} = \frac{1}{R_\infty \xi_{i\to f}}(F_H - 1) \tag{A25}$$

$$A_{i\to f} = \frac{1}{R_\infty \xi_{i\to f}}\left(\frac{m_e}{m_p}\right) \tag{A26}$$

Using equation (A26), we can calculate values for $A_{i\to f}$, which will help us get values for $\frac{m_D}{m_H}$ without needing to measure λ_∞.

Appendix B: Error Propagation, and Weighted Averages

1. Weighted Averages and Error Propagation

Since our uncertainties weren't entirely connected, we could not just sum up our uncertainties when when measuring the difference in the wavelengths in the hydrogen Balmer series and the deuterium Balmer series. As a result of this, we had to use the following equation in order to calculate the uncertainties [4, Chapter 7].

$$\sigma_c = \sqrt{\sigma_a^2 + \sigma_b^2} \tag{B1}$$

This of course leads into another issue when it came to calculating the deuterium-hydrogen mass ratio, since the uncertainty for our ratio involved an uncertainty that would change for different transitions, we decided to use weighted averages and weighted uncertainties. The theory behind weighted averages is pretty simple, since every data point has a different uncertainty, we use that unique uncertainty to determine a weight for our data point.

$$w_i = \frac{1}{\sigma_i^2} \tag{B2}$$

Once we find a weight, we then calculate a weighted data point by simply multiplying the weight value by that data point. In order to find the best weighted average, we just use the following equation:

$$x_{best} = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i} \tag{B3}$$

$$\sigma_{best} = \frac{1}{\sqrt{\sigma_{i=1}^{N} w_i}} \tag{B4}$$

[1] E. Tiesinga, P. J. Mohr, D. B. Newell, and B. N. Taylor, *The 2018 CODATA Recommended Values of the Fundamental Physical Constants (Web Version 8.1)* (National Institute of Standards and Technology, Gaithersburg, MD 20899, 21 January 2020).

[2] R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles* (John Wiley & Sons, 1985).

[3] C. Conover, *Introduction to Optical Spectrsocopy* (2020).

[4] J. R. Taylor, *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements*, 2nd ed. (University Science Books, 1997).