Discrete Elastic Rods Algorithm

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***Abstract*— For homework 2, Discrete Elastic Rods (DER) algorithm for 3D simulation of elastic rods was introduced. Using the algorithm, 3D simulation for the deformation of a rod under gravity was carried out.**

I. Introduction

In the second homework of MAE 259B class, students do a 3d simulation for rod under gravity. Bending, stretching, and twisting are considered for a complete physically based simulation of a rod.

II. Discrete elastic rods algorithm

The DOF vector of a rod with N nodes and (N-1) edges has a size of (4N-1) and defined as

$$q = [x_1, \theta^1, x_2, \theta^2, \cdots, x_{N-1}, \theta^{N-1}, x_N]$$

where x_k (k=1,…,N) are the nodal coordinates and θ^K is the twist angle at edge $e^K = x_{k+1} - x_k$. The equations in 3D scenario to march from $t = t_j$ to $t = t_{j+1} = t_j + \Delta t$ is

$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{j+1}) - q_i(t_j)}{\Delta t} - \dot{q}_i(t_j)\right] + \frac{\partial E_{elastic}}{\partial q_i} - f_i^{ext} = 0 \quad (1)$$

where i=1,…,4N-1, the old DOF $q_i(t_j)$ and velocity $\dot{q}_i(t_j)$ are known. $E_{elastic}$ is the elastic energy evaluated at $q_i(t_{j+1})$, f_i^{ext} is the external force (or moment for twist angles). Gravity and m_i is the lumped mass at each DOF. $m_i = \frac{1}{2}\Delta m r_0^2$ (Moment of Inertia) for twist angle where Δm is the mass of the edge and r_0 is the cross sectional area (with the assumption of a solid circular cross section). The Jacobian for Equation (1) is

$$J_{ij} = \frac{\partial f_i}{\partial q_j} = J_{ij}^{inertia} + J_{ij}^{elastic} + J_{ij}^{ext} \quad (2)$$

where

$$J_{ij}^{inertia} = \frac{m_i}{\Delta t^2}\delta_{ij} \quad (3)$$

$$J_{ij}^{elastic} = \frac{\partial^2 E_{elastic}}{\partial q_i \partial q_j}\delta_{ij} \quad (4)$$

$$J_{ij}^{ext} = -\frac{\partial f_i^{ext}}{\partial q_j} \quad (5)$$

The (4N-1) equations of motion in equation (12) can be solved to obtain the new DOF $q(t_{j+1})$. The new velocity is simply

$$\dot{q}(t_{j+1}) = \frac{q(t_{j+1}) - q(t_j)}{\Delta t} \quad (6)$$

The total elastic energy of an elastic rod is

$$E_{elastic} = \sum_{k=1}^{N-1} E_k^s + \sum_{k=1}^{N-1} E_k^b + \sum_{k=1}^{N-1} E_k^t \quad (7)$$

where
E_k^s is stretching energy, E_k^b is bending energy, and E_k^t is twist energy.

$$E_k^s = \frac{1}{2}EA\left(\frac{|x_{k+1} - x_k|}{|\overline{e^k}|} - 1\right)^2 |\overline{e^k}| \quad (8)$$

$$E_k^b = \frac{1}{2}EI(|k_k - k_k^0|)\frac{1}{\overline{l_k}} \quad (9)$$

$$E_k^s = \frac{1}{2}GJ({\tau_k}^2)\frac{1}{\overline{l_k}} \quad (10)$$

where
EA is stretching stiffness, EI is bending stiffness, GJ is twist stiffness, $|\overline{e^k}|$ is length of the edge e^k in undeformed state, x_k is node, k_k is the curvature vector at node x_k, k_k^0 is the natural (undeformed) curvature at the same node, and τ_k is the integrated twist at node x_k.

For the simulation, an elastic rod with total length $l = 20cm$ that naturally curved with radius $R_n = 2cm$ is considered with the physical parameters of density $\rho = 1000kg/m^3$, cross sectional area $r_0 = 1mm$,Young's modulus $E = 10MPa$, shear modulus $G = E/3$, and the gravitational acceleration $-9.81m/s^2$ in z direction.

A. *The deformation of the rod under gravity from t = 0 to t = 5 s.*

The graph below shows the final deformation of the rod when the rod is exposed to the gravity for 5s.

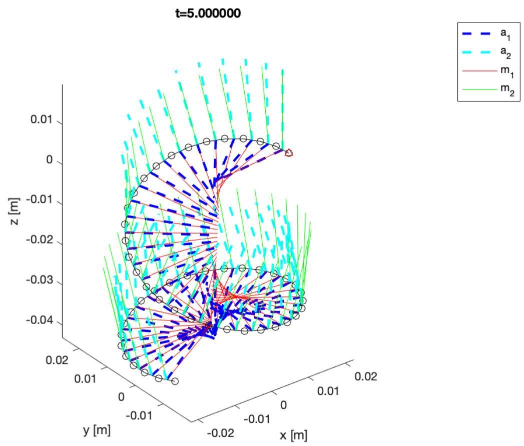

Figure 1. Deformation of the rod under gravity

B. *Z coordinate of the last node with time*

The figure below shows the z coordinate of the last node (X_N) as a function of time for t = 0 to t = 5s.

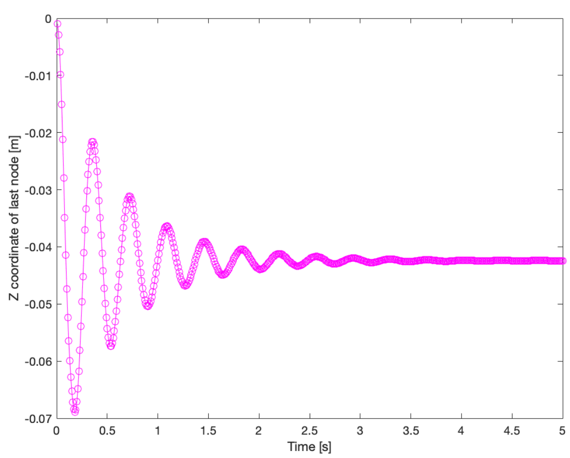

Figure 2. Z coordinate of the last node with time

III. Conclusion

In this homework, students did 3d simulation to show the final deformation of an elastic rod under gravity for 5 seconds. Also, they showed the z-coordinate location of the last node as a function of time. Through this homework, students got more familiar with DER algorithm and prepared to use the algorithm for the final project.

Acknowledgment

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References

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