Parametric Study of Granny and Reef Knots
MAE259B Final Report

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***Abstract*— Knots are a complex topological pattern of self-contact composed of slender elastic structures. Found all throughout everyday life, they are used for their ability to fasten objects. In this work, we focus on studying two knots in particular: the granny and reef knot. These knots are almost identical in structure and vary by the over/under order of a single crossing. Although these two knots are strikingly similar at a glance, one has a vastly higher knot strength over the other. We use an accurate physical simulator capable of simulating elastic knots using Discrete Elastic Rods for rod simulation and Implicit Contact Model for frictional contact handling. Using results from the simulator, we show both we wish to to study the underlying mechanics behind granny and reef knots through a parametric study.**

I. Introduction

The study of the mechanics of knots has been receiving increased attention due to their various implications in manufacturing, medicine (knotting of DNA), and more. The deformation and nonlinearity of elastic rods combined with the sheer number of types of knots result in a highly nontrivial problem to study. Previous works have studied the mechanics of overhand knots where Audoly et al. derived a theoretical model relating geometric and mechanical properties for trefoil (single crossing overhand) and cinquefoil (double crossing overhand) knots [1]. The work of Jawed et al. [2] expanded upon this by producing an analytical theoretical model for overhand knots for all crossing numbers. More recently, Patil et al. produced methods for predicting the mechanical stability of various knots based on topological observables [3]. Regardless, many questions surrounding the mechanics of knots still remain.

In this work, we study granny and reef (square) knots as shown in Fig. 1. The reef knot is known famously as the one used for tying shoelaces. The granny knot differs from the reef knot by a single over/under crossing order as shown by the circled regions in Fig. 1. Despite this stark similarity in structure, the reef knot possesses significantly higher "knot strength", i.e. the knot stays fastened from self-friction when the ends are pulled. While reef knots tend to stay fastened until breaking when pulled on, granny knots will simply slip and unravel. We propose to study and analyze this mechanical phenomenon by utilizing physically accurate simulation frameworks.

II. Methodology

Below, we lay out a brief outline of the steps that were accomplished for this course project.

1) Implementing a working version of Discrete Elastic Rods (DER).

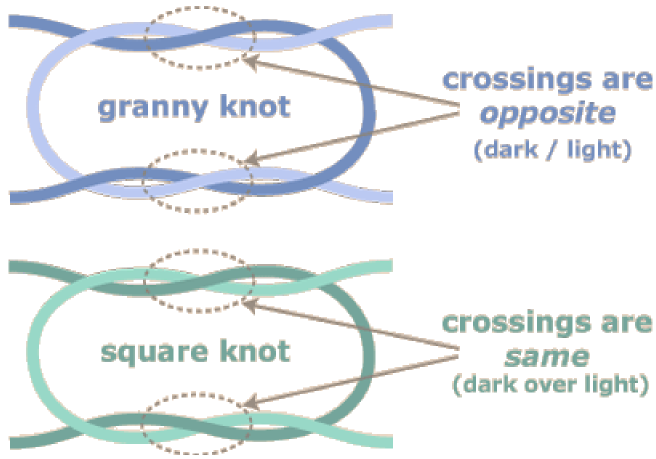

Fig. 1. Visualization of granny and reef (square) knots. The difference between them is simply a result of the circled crossings.

2) Incorporating IMC into DER for frictional contact.
3) Figuring out the necessary boundary conditions to tie both granny and reef knots.
4) Conduct a knot strength comparison between both knots.
5) Conducting a parametric study of both knots.

As the course is focused primarily on teaching DER [4], [5], we will leave out details of the DER framework itself. We will also leave out details regarding IMC [6], [7] as details regarding the frictional contact framework has been covered extensively in our midterm progress report. This contact algorithm is categorized as a penalty method [3], [8], [6], [7], a contact formulation that uses a barrier-like contact energy to enforce non-penetration. In the rest of our final report, we will go over how boundary conditions were applied to conduct our studies, as well as all quantitative results achieved from the simulation framework.

III. Experiments and Results

A. Boundary Condition Sequence

To tie a granny or reef knot, we must first obtain the necessary boundary condition sequence to obtain the desired geometric configuration. Rather than using two elastic rods as shown in Fig. 2(a), we can tie granny and reef knots using a single elastic knot by simply starting from an overhand knot as shown in Fig. 2(b). The sequence of boundary conditions necessary to tying a granny knot is shown in Fig. 3.

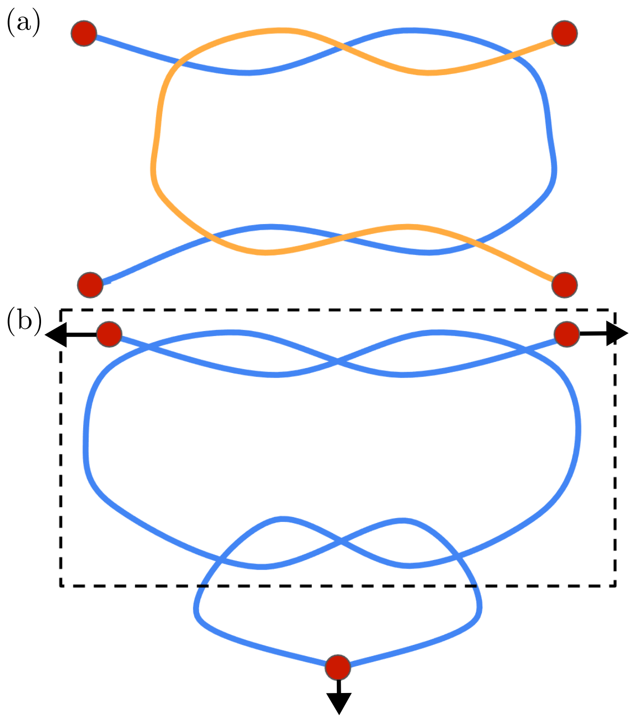

Fig. 2. (a) A traditional reef / granny knot shown tied using two separate elastic rods. (b) A reef / granny knot tied using a single elastic rod. This can be achieved by tying an initial overhand knot. Note that the portion of the knot in the boxed region is a reef / granny like the one in (a). Boundary conditions for the tying sequence are also shown in red.

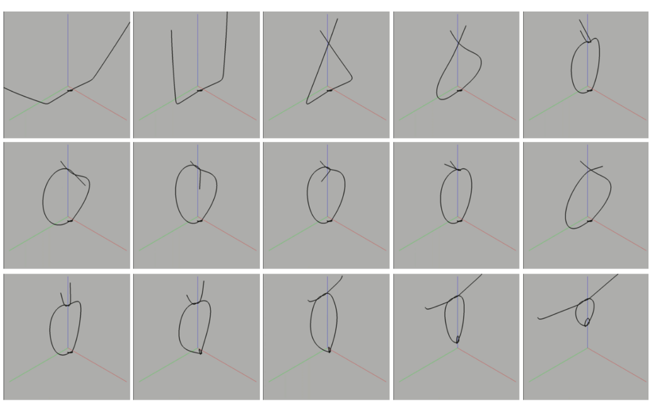

Fig. 3. Visualization of the boundary condition sequence for tying a granny knot. To tie a reef knot, the order of the overhand is simply reversed.

B. Boundary Conditions during Tying

Once the initial geometric configuration of a reef / granny knot is achieved, the boundary conditions for tying are relatively simple. First, boundary conditions are set on the ends at which pulling will occur, which constitutes the first and last edges of the knot. These ends will then be pulled in opposite directions and their experienced forces will be recorded as the traction forces F.

In addition to the pull ends, we apply a third static boundary condition to the bottom of the initial overhand knot as shown in Fig. 2(b). Interestingly, the recorded traction forces for both granny and reef knots without this final boundary condition are equivalent. We conclude that this is a result of the knot degenerating to a simple overhand knot case as the initial bottom overhand unravels on its own, resulting in the topology of the granny / reef knot being lost. Therefore, this static boundary condition is crucial to maintaining our desired knot topology.

C. Knot Strength Evaluation

We evaluate the knot strength as the normalized traction force Fh^2/EI experienced when pulling a granny and reef knot taut, where h is the rod radius and EI is the bending stiffness. We hypothesize that the knot strength of reef knots should be monotonically higher than granny knots for all stages of tightening unless both are taut. To test this theory, we conduct two sets of tightening simulations for both granny and reef knots. The first set of experiments involves light friction, $\mu = 0.1$, while the second pair involves high friction, $\mu = 0.5$. The mutual parameters that were used were Young's Modulus $E = 0.18$MPa, rod radius $h = 2$mm, rod length $L = 1$m, density $\rho = 1180$ kg/m^3, pull speed (both ends) $u = 10$cm/s, number of nodes $N = 301$, and time step $dt = 2.5$ms.

The traction forces for $\mu = 0.1$ can be seen in Fig. 4(a) as a function of pull time (both knots have identical end-to-end shortenings at each pull time). Furthermore, a sequence of tightening sequence can be seen visualized in Fig. 7(a-b). Here, we see that the reef knot's normalized traction force is monotonically higher than the granny knot at all stages of the tying process aside from the first five seconds. The values being the same in the first five seconds is caused by negligible friction forces as both knots slip considerably. Once the knot loop becomes tightened, the effects of friction become more prevalent ($t > 5$s) and the traction forces start to diverge.

In comparison, for $\mu = 0.5$, the normalized traction forces diverge immediately once pulling occurs as the friction forces are non-negligible from the start as shown in Fig. 4(b). We also note that due to the high friction, the granny and reef knots become taut much sooner as sticking becomes much more prevalent than sliding as shown in Fig. 7(c-d). This occurs roughly at $t \approx 12.5$s as illustrated by the vertical dashed red line in Fig. 4(b). The early closing of the knot loop is also a result of the traction forces for $\mu = 0.5$ becoming equivalent with enough pulling. Overall, results for both $\mu = 0.1$ and $\mu = 0.5$ excellently match the common notion that reef knots possess higher knot strength than granny knots.

D. Parametric Study for μ

A parametric study is conducted comparing the effects of μ on the normalized traction forces. To compare the forces between different μ, the average of the recorded normalized traction forces was computed. A total pulling time of 13 seconds was used and μ was varied between values of $[0.1, 0.2, 0.3, 0.4, 0.5]$. All the rest of the parameters used were the same as the ones listed in Sec. III-C.

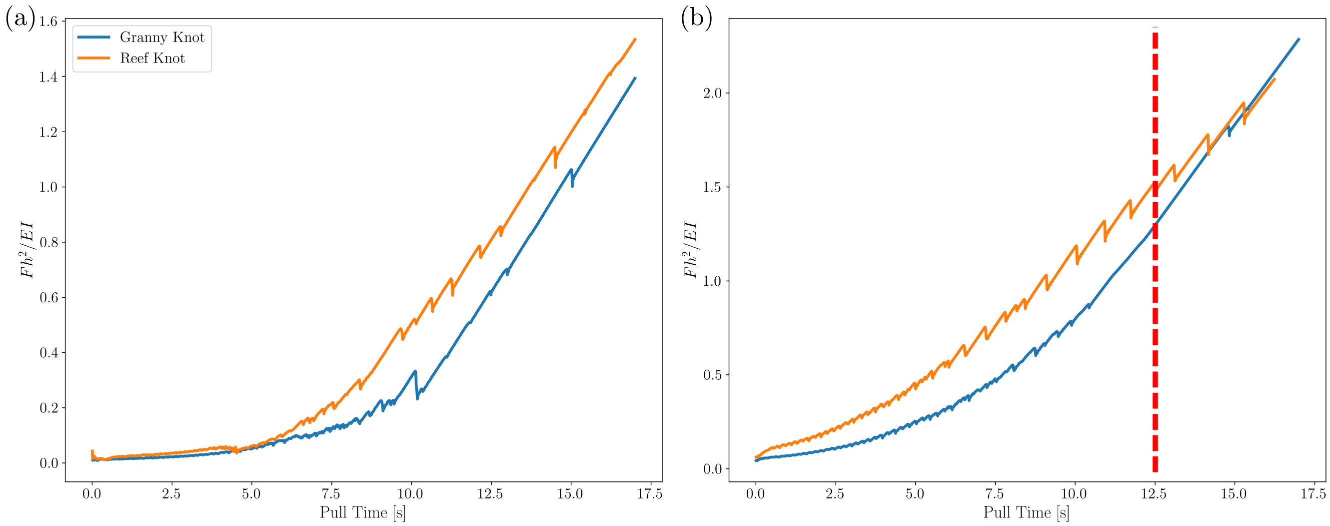

Fig. 4. Normalized traction force comparison between granny and reef knots for (a) $\mu = 0.1$ and (b) $\mu = 0.5$. The horizontal red line in (b) refers to the moment at which the knots start to become taut.

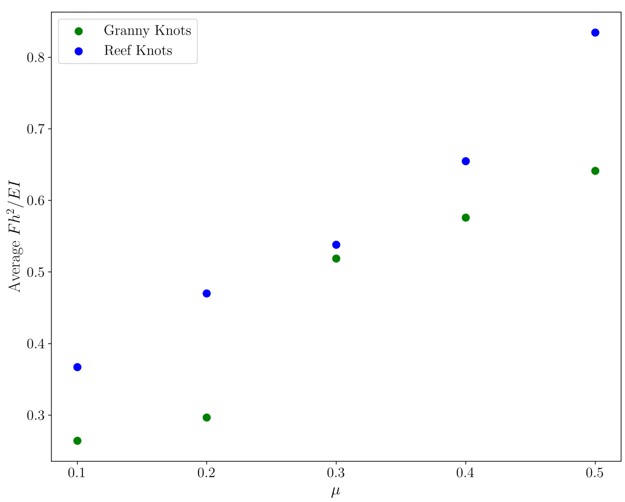

Fig. 5. Study comparing the average normalized traction forces for friction coefficients of $\mu \in [0.1, 0.2, 0.3, 0.4, 0.5]$. Note that reef knots posses a higher average traction force regardless of μ.

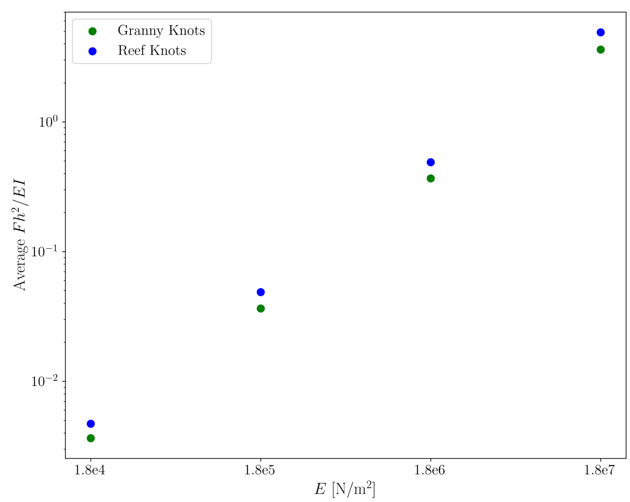

Fig. 6. Study comparing the average normalized traction forces for Young's Moduli of $E \in [1.8e4, 1.8e5, 1.8e6, 1.8e7]$ Pa.

Results for both the granny and reef knots can be seen in Fig. 5. As expected, we can see that reef knots possess a higher average normalized traction force than granny knots for all values of μ. Furthermore, the traction forces for both knots monotonically increase as μ increases. This matches our intuition as increasing μ increases the magnitude of the friction forces themselves.

E. Parametric Study for E

A parametric study comparing the effects of E on the average normalized traction force is also conducted. A total pulling time of 6.4 seconds was used and E was varied between values of $[1.8e4, 1.8e5, 1.8e6, 1.8e7]$. All the rest of the parameters used were the same as the ones listed in Sec. III-C aside from a lower time step of $dt = 1$ms.

Results for both the granny and reef knots can be seen in Fig. 6. Once again, we can see that reef knots possess a higher average normalized traction force than granny knots for all values of E. Furthermore, similar to μ, the traction forces for both knots monotonically increase as E increases. This also matches our intuition as increasing E increases the stiffness of the material, which will indirectly increase the magnitude of the friction forces given the high curvatures of the knot loop.

IV. Conclusion

Overall, our results excellently match our hypotheses concerning the comparative knot strengths of reef and granny knots. This preliminary work opens up several avenues for future research including comparison of simulation results with real world experiments and further parametric studies

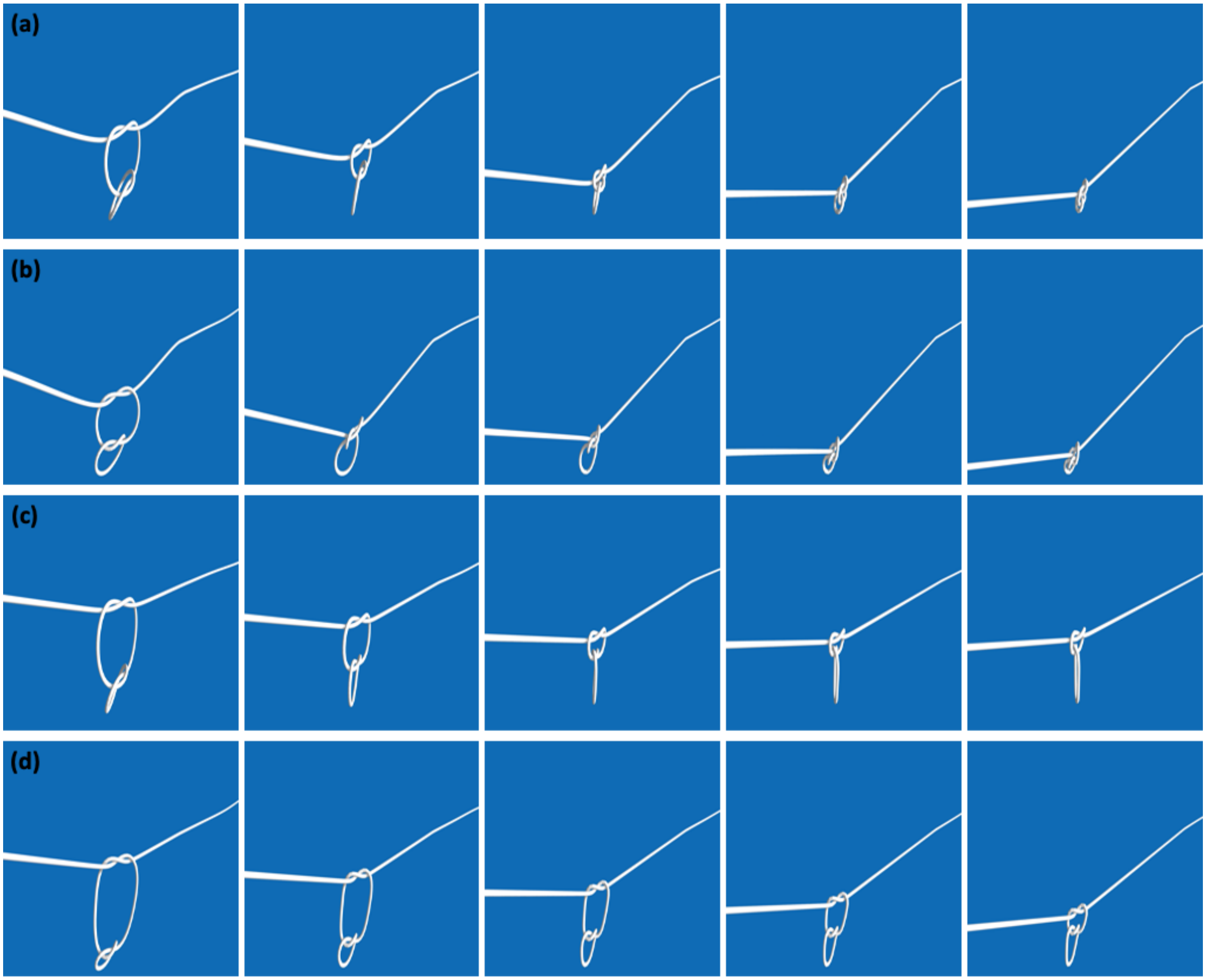

Fig. 7. Snapshots for (a) granny knot with $\mu = 0.1$, (b) reef knot with $\mu = 0.1$, (c) granny knot with $\mu = 0.5$, and (d) reef knot with $\mu = 0.5$. Each row indicates the type of knot and μ. Each column indicates a snapshot after pulling for 2, 6.7, 11.3, 16, and 20.7 seconds from left to right.

(e.g. studying the effects of rod radius h). In addition to this, simulations for the reef and granny knot can be redone with geometric configurations that are more representative of shoelace knots.

References

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