Rapid Prototyping for Elastic Beam Falling in Viscous Flow and Elastic Beam Bending

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***Abstract*— For homework 1, rapid prototyping MATLAB was carried out to simulate three different systems with the conservative force and potential energy: The motion of three connected spheres falling inside viscous fluid and its generalized case (N nodes), and the deformation of elastic beams.**

I. Introduction

In the first two homework of MAE 259B class, students do rapid prototyping a software that can be useful for the final project of the course. The first homework asks to simulate three different scenarios: The motion of three connected spheres falling inside viscous fluid and its generalized case (N spheres), and the deformation of elastic beams. Throughout the procedure, students would get more familiar with the analysis of an elastic beam.

II. Simulation of the Motion of Three Connected Spheres Falling Inside Viscous Fluid

Assume a system with DOF vector q of size N has a potential energy, $E_{ponential}$. The motion of the system can be written as the following equation:

$$M\ddot{q} + \frac{\partial E_{potential}}{\partial q} = 0 \tag{1}$$

Where M represents a lumped mass matrix.
The assumption that a damping force, $-c\dot{q}$, where c is related to viscous damping, is acting on q updates (1) as follows:

$$M\ddot{q} + \frac{\partial E_{potential}}{\partial q} + c\dot{q} = 0 \tag{2}$$

For the simulation, discrete version of the above equation will be used. For implicit method, the equation will be expressed as follow:

$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1} - t_k)}{\Delta t} - \dot{q_i}(t_k)\right] + \frac{\partial E_{potential}}{\partial q} + c_i \frac{q_i(t_{k+1} - t_k)}{\Delta t} = 0 \tag{3}$$

Where f means discrete function, m represents mass, Δt means time difference

For explicit method, the equation will be expressed as follow:

$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1} - t_k)}{\Delta t} - \dot{q_i}(t_k)\right] + \frac{\partial E_{potential}}{\partial q} + c_i \dot{q_i}(t_k) = 0 \tag{4}$$

Implicit method requires Jacobian:

$$J_{ij} = \frac{\partial f_i}{\partial q_j} = J_{ij}^{inertia} + J_{ij}^{potential} + J_{ij}^{viscous} \tag{5}$$

Where J means Jacobian and

$$J_{ij}^{inertia} = \frac{m_i}{\Delta t^2}\delta_{ij} \tag{6}$$

$$J_{ij}^{potential} = \frac{\partial^2 E_{potential}}{\partial q_i q_j} \tag{7}$$

$$J_{ij}^{viscous} = \frac{c_i}{\Delta t}\delta_{ij} \tag{8}$$

Let's consider three rigid spheres on an elastic beam that is falling under gravity in a viscous fluid. The viscosity of the fluid is μ = 1000 $Pa \cdot s$. The radius of the spheres are R1 = 0.005 m, R2 = 0.025 m, and R3 = 0.005 m. The density of the spheres is $\rho_{metal} = 7000 kg/m^3$ whereas the fluid density is $\rho_{fluid} = 1000 kg/m^3$. The elastic beam has a length, l = 0.10 m, cross-sectional radius, r_0 = 0.001 m, and Young's modulus, E = 1.0 × 10^9 Pa. This corresponds to a stretching stiffness of EA = $E\pi {r_0}^2$and bending stiffness of EI = $E\pi {r_0}^4/4$. For implicit approach, the time stamp for the simulation is chosen to be 10^{-2}s whereas the timestamp is chosen to be 10^{-5}s for explicit method. The simulation time is 10s.

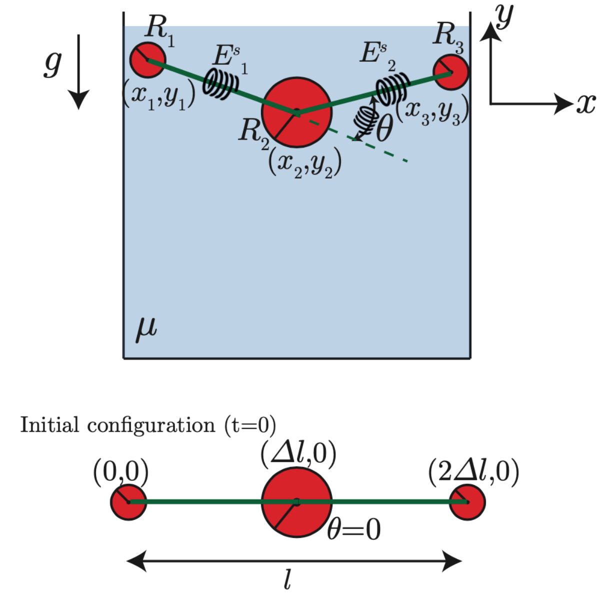

Figure 1. Three rigid spheres attached to an elastic beam falling in a viscous fluid

A. Shape of the structure at t = {0, 0.01, 0.05, 0.10, 1.0, 10.0} s

The graphs for the shape of the structure at different time are derived by simply changing the simulation time in the code. The graphs are derived as follows:

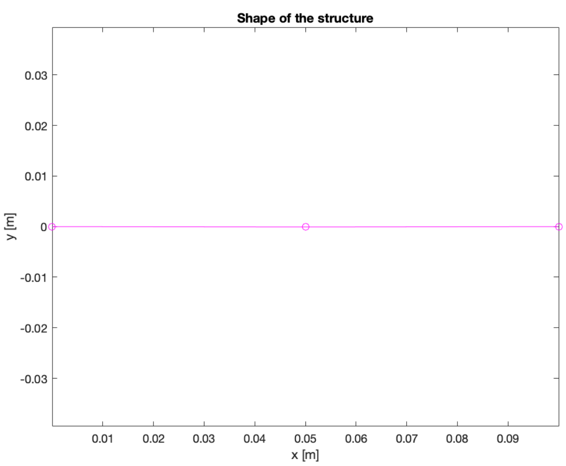

Figure 2. Shape of the structure at t = 0s

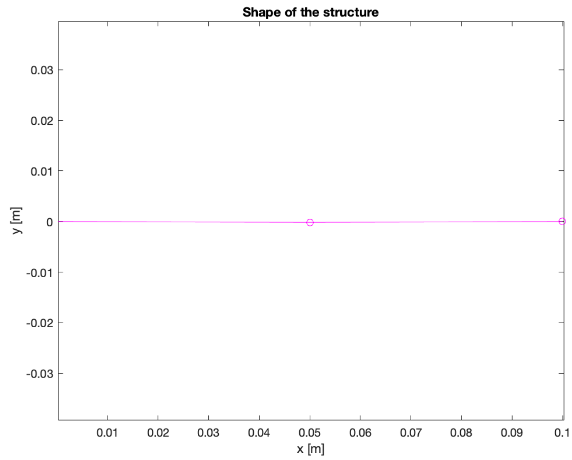

Figure 3. Shape of the structure at t = 0.01s

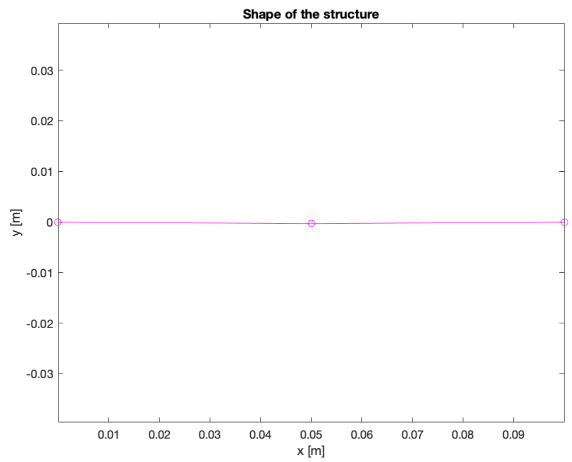

Figure 4. Shape of the structure at t = 0.05s

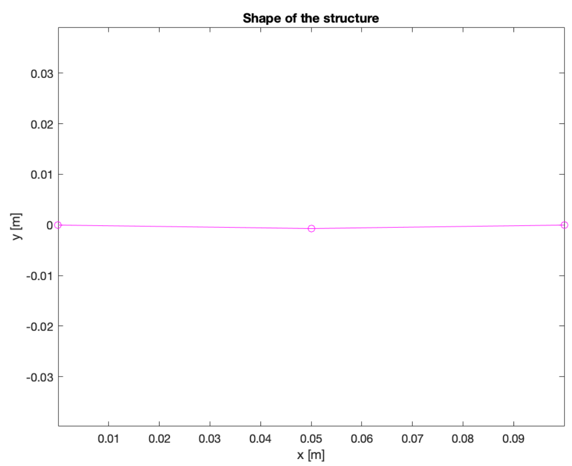

Figure 5. Shape of the structure at t = 0.1s

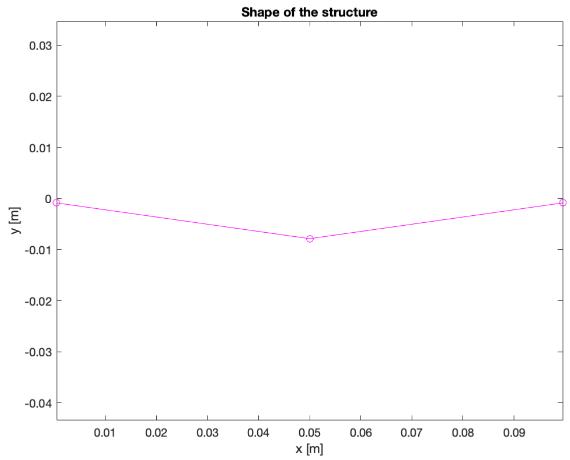

Figure 6. Shape of the structure at t = 1.0s

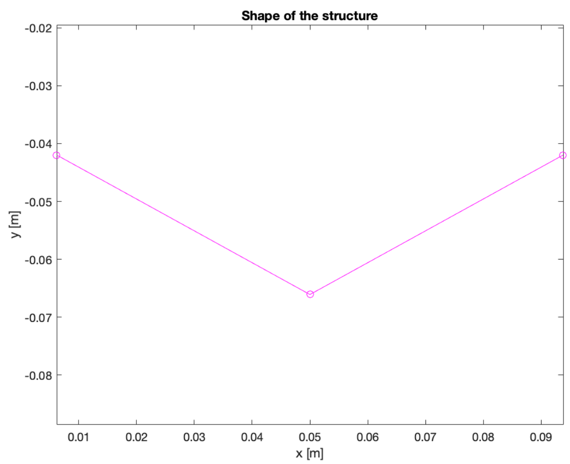

Figure 7. Shape of the structure at t = 10.0s

B. The position and velocity (along y-axis) of R_2 as a function of time.

Now, the position and velocity of the sphere in the middle node are plotted as a function of time.

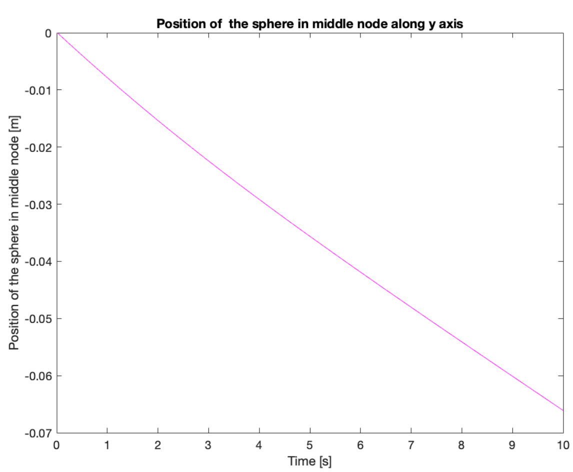

Figure 8. Vertical position of the middle node along y axis

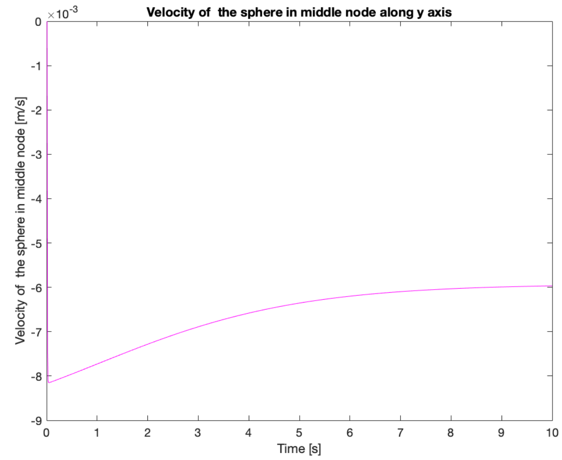

Figure 9. Velocity of the middle node along y axis

C. Terminal velocity (along y-axis) of the system

The derived data from Matlab shows that the system has -0.0060m/s terminal velocity along y-axis. The velocity has negative value because the system moves toward -y direction. The value can be verified with the graph in Figure 9. Since the simulation time is 10s, the velocity of the system at t=10s is the terminal velocity of the system.

D. The turning angle when all the radius (R1, R2,R3) are the same

I believe the turning angle would remain as 0, which is the same as with its initial condition where three rigid spheres are placed horizontally, because all the spheres would have the same weight.

The simulation for the situation that all nodes have the spheres with the same radius has done with the assumption that all the spheres have the same radius as r = 0.005m. The derived graph shows that the terminal shape of the structure is horizontal, and it means that the turning angle remains as 0 degree. Thus, the simulation agrees with the intuition.

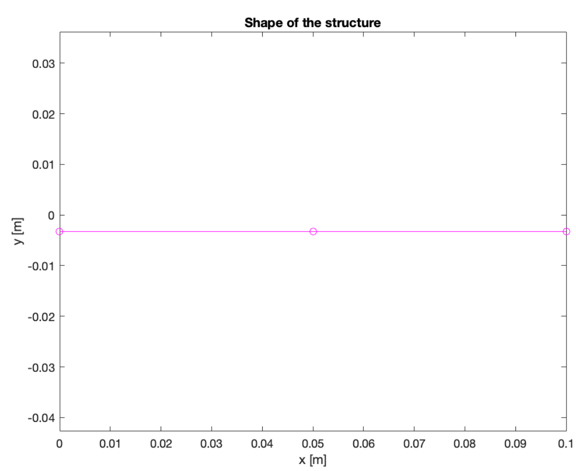

Figure 10. Shape of the structure when all spheres have the same radius (r = 0.005m)

E. Benefits and drawbacks of the explicit and implicit approach.

The explicit method calculates the state of the system at future time based on the state of the system at current time whereas the implicit method calculates the state of the system at future time from the state of the system and both present and future time. Therefore, for implicit simulation, Newton-Raphson method is used to find out the root of the calculation, and it increases the simulation time. The code for implicit approach is more complicated than explicit method, and it reduces the speed of a simulation. Also, the complex code can increase the computer specifications to run the simulation, which means that the simulation might not work properly on a computer with a low specification. However, implicit method is more accurate than explicit method since implicit method includes more feedback from the numerical solution path. When a time stamp is large, explicit approach will generate a significant numerical error.

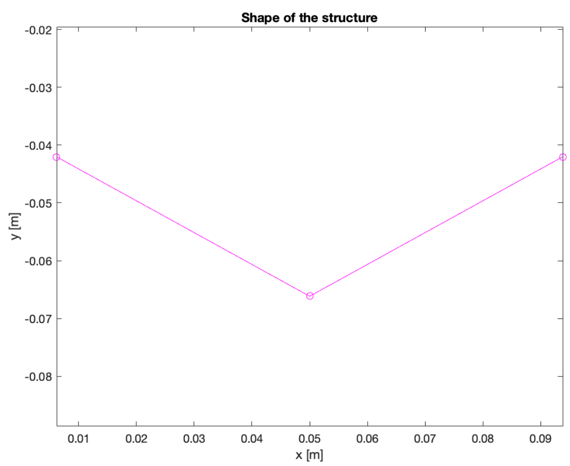

Figure 11. Shape of the structure

The graph above shows the shape of the structure derived from the explicit approach simulation with timestamp $\Delta t = 10^{-5} s$. It shows that the timestamp for explicit method needs to be much smaller than the one for implicit method to get a correct result. When the timestamp value has changed into different values such as $\Delta t = 10^{-4}$, $\Delta t = 10^{-3}$, $\Delta t = 10^{-2}$, Matlab failed to draw a graph for the shape of the structure. The position of each node is shown as NAN, which shows a numerical error happened during the simulation. When the timestamp is set to be 1s, a graph for the shape of the structure was drawn, but incorrect result was derived from the simulation.

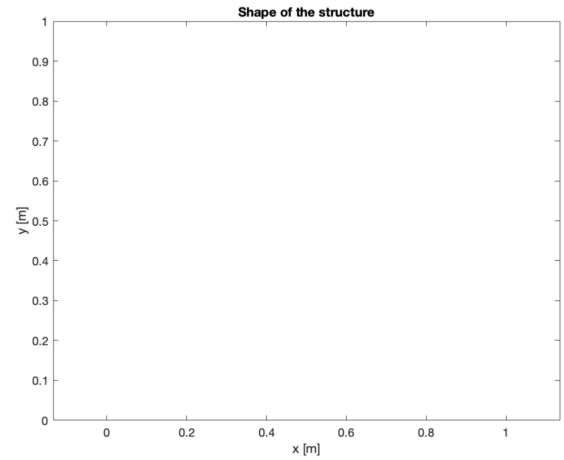

Figure 12. Shape of the structure with timestamp 1e-3 s

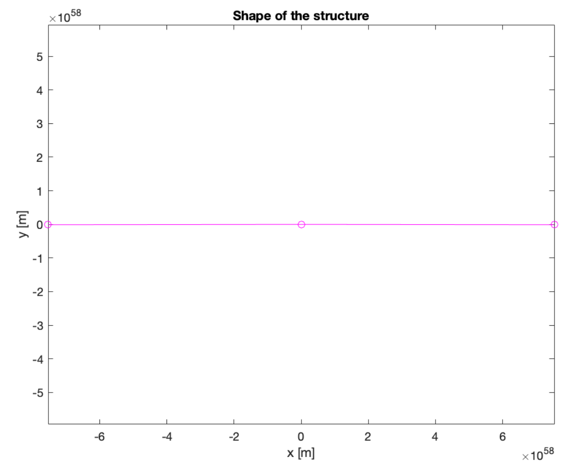

Figure 13. Shape of the structure with timestamp 1s

It shows once again that explicit approach has the benefit of convenience (simulation speed is fast, code is simple, and straightforward analysis) but it has the drawback of inaccuracy. On the other hand, implicit method has the drawback of complexity (more calculation, complicated code, longer simulation time) but it derives an accurate result.

III. Simulation of Generalized Case of Elastic Beam Falling in Viscous Flow

Now, let's generalize the case. Let's consider the system with N nodes.

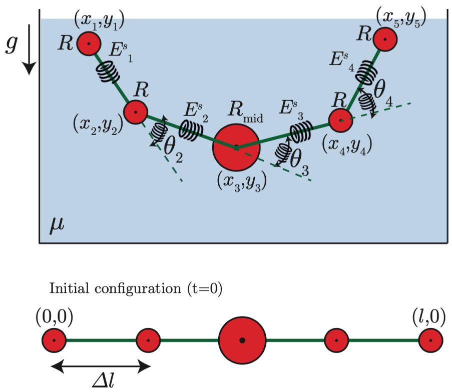

Figure 14. N rigid spheres attached to an elastic beam falling in a viscous fluid (N=5)

There are N-1 stretching springs and N-2 bending springs. Thus, the elastic energy can be calculated by the following equation:

$$E^{elastic} = \sum_{j=1}^{N-1} E_j^s + \sum_{j=2}^{N-1} E_j^b \tag{9}$$

where E_j^b is associated with turning angle θ_j

For the simulation, let's consider a system with 21 nodes (N-21). The sphere at the middle node has radius r = 0.025m and all the other spheres have a radius r = $\Delta l/10$ where $\Delta l = l/(N-1)$ is the length of each discrete segment. Implicit method is used for the simulation, and the timestamp is chosen to be 10^{-2}s. The simulation time is 50s.

A. *The vertical position and velocity of the middle node with time*

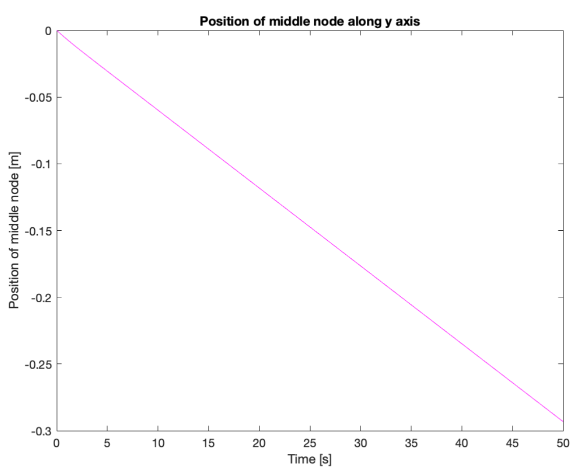

Figure 15. Vertical position of middle node along y axis

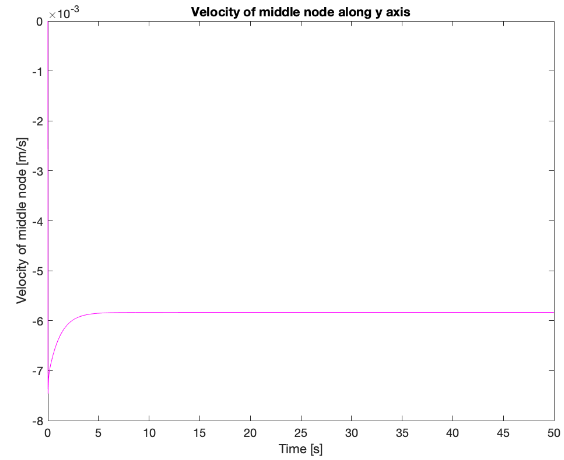

Figure 16. Velocity of the middle node along y axis

The derived graph shows that the terminal velocity of the structure is -0.00583 m/s. The velocity has negative value because the system moves toward -y direction. The value can be verified with the graph in Figure 16. Since the simulation time is 50s, the velocity of the system at t=50s is the terminal velocity of the system.

B. *The final deformed shape of the beam*

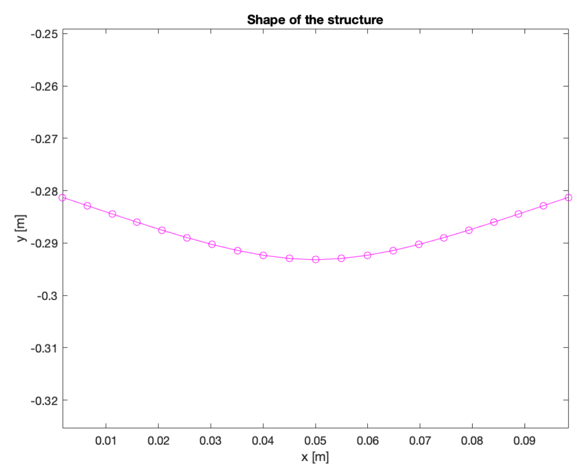

Figure 17. Final deformed shape of the beam

C. *The significance of spatial discretization (N) and temporal discretization (Δt)*

The sufficient N and Δt will provide the correct result, which means that the result from the discrete simulation has a high accuracy when compared with the actual result. Smaller temporal discretization will generate higher number of simulations, and it will draw a precise result. However, if a system is not sufficiently discretized, for example, Δt is not small enough or N is too small, and it generates only a few times of simulation, the result will not be accurate. The graphs below show how the terminal velocity of the system change for different value of N and Δt.

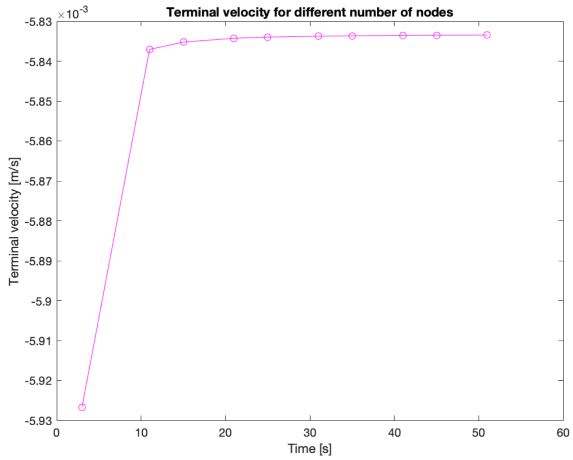

Figure 18. Terminal velocity of the system for different N

The system was simulated with different values of N where N = {3, 11, 15, 21, 25, 31, 35, 41, 45, 51};

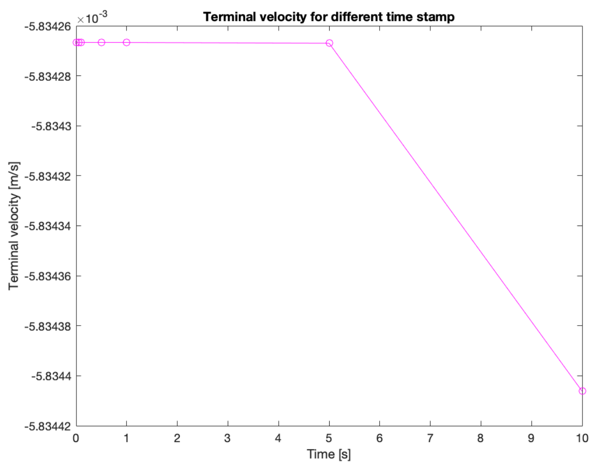

Figure 19. Terminal velocity of the system for different Δt

The system was simulated with different values of Δt where Δt = {0.01 0.05 0.1 0.5 1 5 10};

When N has small value, such as 3, or Δt has large value, such as Δt=10s, the derived terminal velocity value has a big error percentage showing a big difference with other data on the graphs.

When N=3, the terminal velocity about -0.005833 m/s, and it has about 1.59 % error percentage

$$e = \frac{0.005927 - 0.005833}{0.005927} \cdot 100 \approx 1.59 \tag{10}$$

When Δt=10s, the terminal velocity is about -0.005834 m/s, and it has about 0.014%

$$e = \frac{0.00583427 - 0.00583348}{0.00583427} \cdot 100 \approx 0.014\% \tag{11}$$

IV. Simulation of the Deformation of Elastic Beams and Comparison with Euler-Bernoulli Beam Theory

The discrete system simulation is now used to simulate the beam deformation. The simulation result will be compared with the theoretical expectation from Euler Bernoulli beam theory. For the simulation, a simply supported aluminum beam subjected to a single point load will be considered as shown as the figure below.

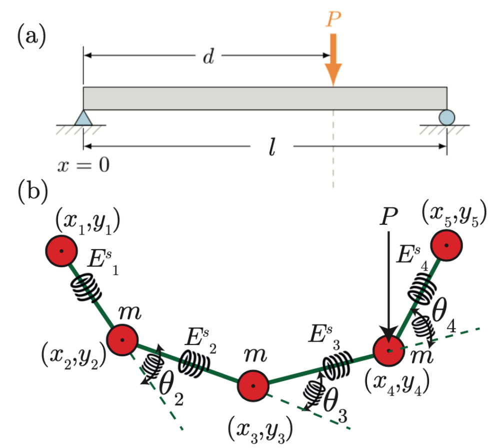

Figure 20. (a) Elastic beam and (b) its discrete representation

The bar has length l=1m with a constant circular-tube cross section with outer radius R = 0.013 m and inner radius r = 0.011 m. An external force, P = 2000 N, is applied 0.75 m away from the left-hand edge. The modulus of elasticity, E, is 70 GPa for aluminum. The moment of inertia, I, of the cross section can be calculated by the following equation:

$$I = \frac{\pi(R^4 - r^4)}{4} \tag{12}$$

The beam is represented as mass spring system with a mass m located at each node where m is

$$m \equiv \frac{\pi(R^2 - r^2) l\rho}{N-1} \tag{13}$$

where the density of aluminum, ρ, is 2700 kg/m^3. The simulation is 1s.

A. The maximum vertical displacement, y_{max}

Below is the graph of the maximum vertical displacement as a function of time. It has negative value because the mass spring system gets deformed through negative y direction

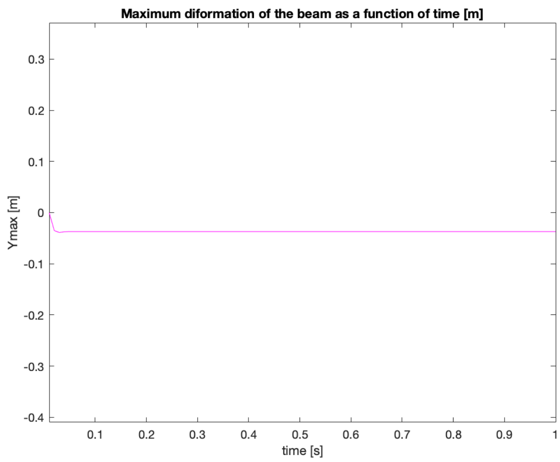

Figure 21. Maximum deformation of the beam as a function of time

Figure 21 shows that the maximum deformation of the beam, y_{max}, reaches a steady value

B. The accuracy of the simulation against the theoretical prediction

Theoretical prediction for the maximum vertical displacement of a beam can be calculated as follow:

$$y_{max} \equiv \frac{Pc(l^2 - c^2)^{1.5}}{9\sqrt{3}EIl} \tag{14}$$

Where P = external force applied on the beam, E = Elasticity, l = length of the beam, I = moment of Inertia, c = min(l, d-l) where d = distance from the origin to the point

With equation 14, theoretical prediction is derived as:

$$y_{max} = \frac{Pc(l^2 - c^2)^{1.5}}{9\sqrt{3}EIl} = \frac{2000N \cdot 0.25m \cdot (1m^2 - 0.25m^2)^{1.5}}{9\sqrt{3} \cdot 70GPa \cdot 1.0933 \cdot 10^{-8}m^4 \cdot 1m} = 0.038m \tag{15}$$

where

$$c = \min(l, d - l) = d - l = 0.25m \tag{16}$$

$$I = \frac{\pi \cdot (R^4 - r^4)}{4} = \frac{\pi \cdot (0.013m^4 - 0.011m^4)}{4} = 1.0933 \cdot 10^{-8}m^4 \tag{17}$$

y_{max} is derived to be 0.037m from the simulation, and it has about 2.63% error percentage.

$$e = \frac{0.038 - 0.037}{0.038} \cdot 100 \approx 2.63\% \tag{18}$$

2.63% of error percentage is acceptable, and it is reasonable to say that the accuracy of the simulation is acceptable against the prediction from Euler beam theory.

C. The benefit of simulation over the predictions from beam theory

Euler beam theory is valid for a small deformation. Therefore, the theory can generate a wrong result for a large deformation. Thus, when high load is applied for a beam, and the beam undergoes high deformation, analysis using a simulation will provide much more precise result than the beam theory. It shows that the simulation covers much more usage than the theory

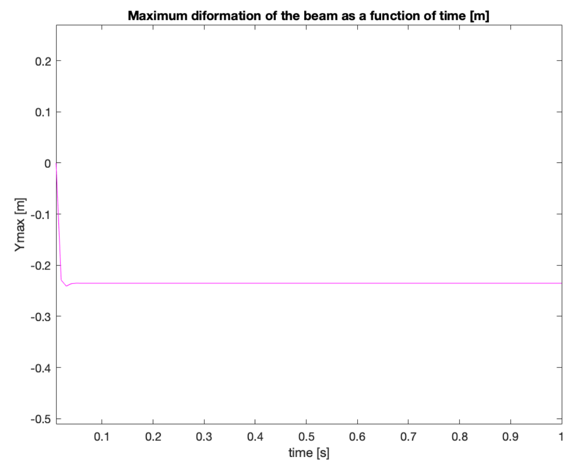

Figure 22. Maximum deformation of the beam as a function of time

Figure 22 shows the maximum deformation of the beam when P = 20000N is applied on the system. The simulation shows that the maximum deformation of the beam should be 0.2353m but the prediction from Euler theory shows that it needs to be 0.3804m. It has about 61.67% of error percentage

$$e = \frac{0.3804 - 0.2353}{0.2353} \cdot 100 \approx 61.67\% \tag{18}$$

It shows that the prediction from Euler beam theory is no longer valid.

V. Conclusion

In this homework, students compared implicit and explicit method through rapid prototyping. Also, they learned how to analysis an elastic beam deformation by representing it as a mass-spring system with a mass located at each node. The analysis of beam deformation also showed the benefits of using simulation with the comparison with a theoretical expectation from Euler beam theory.

Acknowledgment

I would like to express my gratitude to Professor Khalid, who guided students this homework during the class. I would also like to thank Dr. Andrew Choi who helped me to have a deep insight into the homework.

References

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[4] M. Khalid Jawed hessEb.m

[5] M. Khalid Jawed hessEs.m

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[7] M. Khalid Jawed gradEs.m