Parametric Study of Granny and Reef Knots
Frictional Contact Formulation
MAE259B Midterm Report

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***Abstract*— Knots are a complex topological pattern of self-contact composed of slender elastic structures. Found all throughout everyday life, they are used for their ability to fasten objects. In this work, we focus on studying two knots in particular: the granny and reef knot. These knots are almost identical in structure and vary by the over/under order of a single crossing. Although these two knots are strikingly similar at a glance, one has a vastly higher knot strength over the other. We propose to implement an accurate physical simulator capable of simulating elastic rods along with contact and friction. Using this simulator, we wish to to study the underlying mechanics behind granny and reef knots through a parametric study.**

I. Brief Recap of Project Outline

Below, we lay out a brief outline of the project necessary for completing our final parametric study of the granny and reef knots. This outline was first mentioned in our project proposal.

1) Implementing a working version of Discrete Elastic Rods (DER). *done*
2) Incorporating IMC into DER for frictional contact. *done*
3) Developing an intuitive interface for applying boundary conditions.
4) Figuring out the necessary boundary conditions to tie both granny and reef knots.
5) Conducting a parameteric study of both knots.

To even start to study knots, accurate frictional contact must be incorporated into the DER algorithm. For our midterm report, we will give a concise formulation of the frictional contact framework we plan on using, Implicit Contact Model (IMC) [1], [2]. This contact algorithm is categorized as a penalty method [3], [4], [1], [2], a contact formulation that uses a barrier-like contact energy to enforce non-penetration. In the following section, we show that the IMC contact algorithm is formulated in a way that naturally integrates into the DER solving scheme.

II. Contact Formulation

The Discrete Elastic Rods (DER) algorithm [5], [6] simulates an elastic rod by solving its elastic energies. However, not included in this formulation is the enforcement of contact. In other words, using DER alone, when an elastic rod encounters self-contact, penetration occurs, which violates physical realism.

As knots are simply elastic rods in complex patterns of self-contact, it is trivial to see that contact is absolutely necessary for any sort of meaningful study of knots so as to enforce non-penetration. In addition to non-penetration, friction must be also be incorporated as these forces are the primary influence behind the strength of a knot [1].

Despite this, accurately simulating contact and friction is a nontrivial task as both processes are highly nonlinear in nature. Therefore, we use smooth approximations to model frictional contact. In this report, we go over the formulations used to compute the minimum distance between edges Δ, contact energy $E(\Delta)$, and friction.

A. Computing Min-distance between edges

The minimum distance between two edges $(\mathbf{x}_i, \mathbf{x}_{i+1})$ and $(\mathbf{x}_j, \mathbf{x}_{j+1})$ can be formulated as the constrained optimization problem

$$\begin{aligned}\Delta = \min_{\beta_i,\beta_j} &\|\mathbf{x}_i + \beta_i(\mathbf{x}_{i+1} - \mathbf{x}_i) - (\mathbf{x}_j + \beta_j(\mathbf{x}_{j+1} - \mathbf{x}_j))\| \\ &\ni 0 \leq \beta_i, \beta_j \leq 1,\end{aligned} \tag{1}$$

where β_i and β_j represent the contact point ratios along the respective edges. The minimum distance between two edges, Δ, can be categorized into three different cases: point-to-point, point-to-edge, and edge-to-edge. For each case, Δ must be derived using different equations. In the point-to-point case, both β constraints are active, and Δ can easily be solved using the Euclidean distance formula,

$$\Delta^{PP} = \|\mathbf{x}_a - \mathbf{x}_b\|, \tag{2}$$

where $\mathbf{x}_a$ and $\mathbf{x}_b$ are the nodes for first and second edges in contact. In the point-to-edge case, one β constraint is active and Δ can be solved as follow:

$$\Delta^{PE} = \frac{\|(\mathbf{x}_a - \mathbf{x}_b) \times (\mathbf{x}_b - \mathbf{x}_c)\|}{\|\mathbf{x}_a - \mathbf{x}_b\|}, \tag{3}$$

where $\mathbf{x}_a$ and $\mathbf{x}_b$ are the nodes of the edge where the minimum distance vector is contacted on a point and $\mathbf{x}_c$ is the node of the edge which the minimum distance vector is contacted on an end. Finally, for the edge-to-edge case, there are no active constraints, and the distance for the i-th and j-th edges can be solved as

$$\begin{aligned}\mathbf{u} &= (\mathbf{x}_{i+1} - \mathbf{x}_i) \times (\mathbf{x}_{j+1} - \mathbf{x}_j), \\ \Delta^{EE} &= |\mathbf{x}_i - \mathbf{x}_j \cdot \hat{\mathbf{u}}|,\end{aligned} \tag{4}$$

where $\hat{}$ indicates a unit vector. These three cases fully define the calculation for the minimum distance between edges and are all twice differentiable.

B. Contact Energy

In the ideal setting, contact energy must satisfy two properties: (1) it is zero for any distance $\Delta > 2h$ (where h is the rod radius) and (2) it is non-zero at exactly distance $\Delta = 2h$. A unit step function can describe these properties, but this function is non-continuous and sudden change in value occurs. Thus, Heaviside step function can't be used for root-finding algorithms such as Newton's method.

Instead, we choose to smoothly approximate a step function by the following formulation:

$$E(\Delta) = \begin{cases} (2h-\Delta)^2 & \Delta \in (0, 2h-\delta] \\ (\frac{1}{K_1}\log(1+... & \\ \exp(K_1(2h-\Delta))))^2 & \Delta \in (2h-\delta, 2h+\delta) \\ 0 & \Delta \geq 2h+\delta \end{cases} \tag{5}$$

where $K_1 = 15/\delta$ represents the stiffness of the energy curve and δ is the distance tolerance that represents the distance above the contact surface for which non-zero contact forces are experienced.

With this formulation, the model is susceptible to penetration, but a sufficient contact stiffness k remedies this issue. Furthermore, this formulation squares the energy formulation from [1] so that the contact gradient grows exponentially instead of linearly, further enforcing non-penetration.

C. Friction

Friction is modeled according to Coulomb's friction law, which defines the necessary conditions for two bodies to transition between sticking and sliding states. It states that F^{fr} is (1) equal to μF^n during sliding, (2) is in the region of [0, μF^n) when sticking, and (3) is independent of the magnitude of velocity. Here, μ is the friction coefficient and F^n is the normal force experienced by the body.

Similar to elastic energy, the contact force and its Jacobian can be derived as follows: $\mathbf{F}^c \equiv k\nabla_{\mathbf{x}}E$, $\mathbf{J}^c \equiv k\nabla^2_{\mathbf{x}}E$. Following this, we can obtain the normal force on the i-th and $i+1$-th nodes for a contact pair as $\mathbf{F}^n_i = \|\mathbf{F}^c_i\|$ and $\mathbf{F}^n_{i+1} = \|\mathbf{F}^c_{i+1}\|$. Contact norm vector can then simply be derived as

$$\mathbf{n}_i = \frac{\mathbf{F}^c_i + \mathbf{F}^c_{i+1}}{\|\mathbf{F}^c_i + \mathbf{F}^c_{i+1}\|}. \tag{6}$$

The direction of friction is along the tangential relative velocity between edges i and j. To calculate this, the relative velocities of the edges at the point of contact need to be calculated, which can be done using $\beta_i, \beta_f \in [0,1]$ as shown below:

$$\begin{aligned} \mathbf{v}^e_i &= (1-\beta_i)\mathbf{v}_i + \beta_i\mathbf{v}_{i+1}, \\ \mathbf{v}^e_j &= (1-\beta_j)\mathbf{v}_j + \beta_j\mathbf{v}_{j+1}, \\ \mathbf{v}^{rel} &= \mathbf{v}^e_i + \mathbf{v}^e_j, \end{aligned} \tag{7}$$

where $\mathbf{v}_i$, $\mathbf{v}_{i+1}$, $\mathbf{v}_j$, and $\mathbf{v}_{j+1}$ are the velocities of the i-th, $i+1$-th, j-th, and $j+1$-th nodes. Now, the direction of fiction at edge i with respect to edge j can then be derived as

$$\mathbf{v}^{Trel} = \mathbf{v}^{rel} - (\mathbf{v}^{rel} - (\mathbf{v}^{rel}\cdot\mathbf{n}_i)\mathbf{n}_i) \tag{8}$$

where $\hat{\mathbf{v}}^{Trel} = \mathbf{v}^{Trel}/\|\mathbf{v}^{Trel}\|$ is the friction direction. Now, the contact model should be capable of simulating the transition between sticking and sliding.

Coulomb's law tells us that $\|\mathbf{v}^{Trel}\| = 0$ for static friction and $\|\mathbf{v}^{Trel}\| > 0$ for sliding friction. Sticking occurs up until the tangential force threshold μF^n. is surpassed, after which sliding begins. This relation can also be described by a unit step function. For the same reasons as before, this step function is replaced by another smooth approximation:

$$\gamma(\|\mathbf{v}^{Trel}\|, \nu) = \frac{2}{1+\exp(-K_2\|\mathbf{v}^{Trel}\|)} - 1 \tag{9}$$

where $K_2(\nu) = 15/\nu$ is the stiffness parameter and ν (m/s) is the desired slipping tolerance (analogous to the distance tolerance δ for contact energy). Finally, the friction experienced by a node i contacting against node j can be described as

$$\mathbf{F}^{fr}_i = -\mu\gamma\hat{\mathbf{v}}^{Trel}F^n_i. \tag{10}$$

Overall, these smooth approximations allow for the simulation of physically realistic contact and friction in a way that is solvable by Newton's method.

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