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Serial production systems with random yield and rigid demand: A heuristic

Tal Ben-Zvi[a], Abraham Grosfeld-Nir[b,*]

[a] *Faculty of Management, Tel-Aviv University, Tel-Aviv 69978, Israel*
[b] *Academic College of Tel-Aviv-Yaffo, 29 Melchet, Tel-Aviv 61560, Israel*

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Abstract

We consider a heuristic for serial production systems with random yields and rigid demand: all usable units exiting a stage move forward. We calculate optimal lots and corresponding expected costs for binomial, interrupted-geometric, and all-or-nothing yields. Our method is that it makes it easy to analyze large systems.

Keywords: Random yield; Binomial distribution; Interrupted-geometric distribution; Rigid demand; Serial system

1. Introduction

We consider serial systems (Fig. 1) in an environment where production is in lots requiring fixed (setup) and variable production costs. We assume that production yields are random and that demand for final goods is rigid (i.e., demand is to be met in its entirety). For each possible configuration of intermediate inventories a production policy must specify at which stage to produce next and the lot size to be processed. The objective is to minimize the total of setup and variable production costs.

A close examination of even the most elementary system, i.e., a two-stage system, reveals that the problem is hard to solve [7]. Pentico [20] proposed a simple and effective heuristic instructing that all usable units exiting a stage will be processed at the next stage. Pentico also proposed a dynamic program to determine the optimal lot to enter the first stage. However, at the same time, he noted that "optimization by dynamic programming is not really practical" and that cost expressions get "extremely messy". Consequently, for binomial yield, he suggested several rules for selecting the lot to enter the first stage.

One difficulty with any heuristic (specifying actions to be taken at all circumstances) is in the need to calculate the corresponding expected cost. At the present time this could be done by simulation or via using the fixed-policy-theorem introduced in [5,7].

* Corresponding author. Tel.: +972 3 5268111.
E-mail addresses: talbenz@post.tau.ac.il (T. Ben-Zvi), agn@mta.ac.il (A. Grosfeld-Nir).

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Surprisingly, for binomial, interrupted-geometric (IG) and all-or-nothing (A–N) yields, the expected cost associated with Pentico's method can be calculated by a simple formula. Moreover, this formula makes it possible to treat the lot to enter the first stage as a decision variable and thus to efficiently obtain its optimal size.

2. Motivation and related literature

The modeling of manufacturing systems with random yields attracted the attention of many researchers; see [23] for a comprehensive literature review. Two variants of demand have been addressed in the literature: (i) "rigid demand", where an order must be satisfied in its entirety, possibly necessitating multiple production runs; see [11] for a literature review of such models; and (ii) "non-rigid demand", where there is only one production run and a penalty for a shortage; see [23] and Section 9.4.8 of [25] for a description of such single-attempt scenarios.

The review article [23] includes over 120 references and describes many of the suggested applications. Also, [24,11] point out that rigid demand problems frequently arise in high tech industries, when orders are for small quantities and products are custom-made. An interesting application in fiber optics has been studied in [19], while [22] was motivated by wafer fabrication for integrated circuits.

2.1. Modeling yield uncertainty

Three primary discrete yield distributions have been suggested to be highly practical: binomial; IG; and A–N. For a general discussion concerning yield distributions see [11,23].

Binomial yield represents a scenario where the quality of different products is independent of each other, with identical success probability. Such would be the case, for example, when a lot receives thermal or chemical treatment. IG yield represents a scenario where a machine processing a run could go "out of control" during production; all units produced prior to this event conform, and all units produced thereafter are defective. A–N represents a scenario where a facility processing a lot could go out of control during production; if it goes out of control (e.g., temperature or humidity rises or falls beyond allowed limits) all units are defective; otherwise all units are conforming.

In addition to the above, there is a family of distributions referred to as stochastically proportional (SP) that has been used intensively in the context of random yield (e.g., [8,16,18]). SP yield fits situations where production lots and yield are continuous: let X_Q be the random yield corresponding to a continuous lot Q. Then, $X_Q = QX$, where X is a random variable independent of Q.

2.2. Multistage production systems

The literature concerning multistage systems with random yield is quite limited. Yano and Lee [23] believe that the reason for this is that problems with random yields become enormously complex, both theoretically and computationally, as the size and complexity of the underlying manufacturing system increase. They also conjecture that most realistic problems cannot be solved optimally, and thus they emphasize the need for heuristic solution procedures that are computationally inexpensive and easy to implement.

Lee and Yano [18] studied a serial system with non-rigid demand, zero setup costs with linear holding and shortage costs, and SP yields. In [3] this model is considered with (i) binomial yields and the possibility of procurement of unfinished goods and (ii) rigid demand. However, it is assumed that additional production runs do not require setup costs. A somewhat similar rigid-demand model was suggested by Wein [22]; instead of procurement of unfinished goods she assumed that defective units could be perfectly reworked, thus necessitating at most two production runs at each stage. Grosfeld-Nir and Gerchak [10] expanded Wein's model to include the possibility of unlimited rework at each stage.

In [6,14] the authors studied "single-bottleneck systems" (SBNS): multistage systems where all setup costs except one (the BN) are zero. In [7,13] the authors studied a two-stage binomial system. Pentico [20] was the first to propose a heuristic for a serial system with rigid demand. In [15] the authors considered a two-echelon assembly system: a system where the first echelon consists of parallel stages whose output is assembled by a single (final) stage. In [5] the authors studied multistage serial and assembly systems; the main result is that the expected cost of any heuristic can be evaluated by solving a finite set of linear equations.

2.3. The single stage

The single stage under rigid demand has been analyzed since the mid-1950s, often under the label of "reject allowance" in conjunction with binomial yield [4,21], discrete uniform yield [1], and IG yield [2,24].

Consider a single stage facing a rigid order for D units. We denote α and β as the fixed (setup) and variable (per unit) production costs, respectively. We denote $p(x, N)$ the probability to obtain x conforming units from a lot N. We define the following cost functions:

> V_D is the optimal (minimal) expected cost required to fulfill an order D.
>
> $V_D(N)$ is the expected cost required to fulfill an order D, if the lot is N whenever the remaining demand is D and an optimal lot is used whenever the demand is less than D.

Therefore, $V_D = \min_N\{V_D(N)\}$, and

$$V_D(N) = \alpha + \beta N + p(0, N)V_D(N) + \sum_{x=1}^{D-1} p(x, N)V_{D-x}.$$

This becomes

$$V_D(N) = \frac{\alpha + \beta N + \sum_{x=1}^{D-1} p(x, N)V_{D-x}}{1 - p(0, N)}, \tag{1}$$

with $V_D \equiv 0$, $D \leqslant 0$. Thus, the optimal lot and optimal expected cost can be calculated recursively in D, via a search over N.

We refer to a stage as "binomial machine", with success probability θ, if

$$p(x, N) = \binom{N}{x} \theta^x (1-\theta)^{N-x}.$$

For this case Beja [4] proved that $V_D(N)$ is quasi-convex in N and that the optimal lot strictly increases in D.

We refer to a stage as "IG machine", with success probability θ, if

$$p(x, N) = \begin{cases} \theta^x(1-\theta), & 0 \leqslant x \leqslant N-1, \\ \theta^N, & x = N. \end{cases}$$

For this case it is clear that the optimal lot does not exceed the demand. In [9] the authors provide an example showing that the optimal lot may decrease in the demand, while in [2] it is proved that the optimal lot size is uniformly bounded.

We refer to a stage as "A–N machine", with success probability θ, if

$$p(0, N) = 1 - \theta, \quad p(N, N) = \theta \text{ and,}$$
$$\text{otherwise}, \quad p(x, N) = 0.$$

For this case it is clear that the optimal lot equals the demand.

Before proceeding we wish to make the following comment. Let X_N be the random number of conforming units corresponding to a lot N. Then, clearly, by definition, $p(x, N) = P(X_N = x)$. Note that for IG yield, $P(X_N \geqslant x) = \theta^x$, $x = 1, \ldots, N$. (We use this characterization of IG later.)

3. The model

We refer to a system as binomial (IG, A–N) if all machines are binomial (IG, A–N). We assume that stages work independently of each other. This assumption has been used by all the authors mentioned above and is also commonly used when "push" and "pull" systems (or systems with blocking) are considered. See [12,17] for example.

Referring to Fig. 1, we denote $\alpha_k + \beta_k N$ the cost of processing a lot N at stage k, and by θ_k the corresponding (binomial, IG or A–N) success probability. The system faces a rigid demand for D units.

Pentico [20] proposed the following heuristic: a lot N (a decision variable) is processed at M_1; all usable units exiting M_1 enter M_2; all usable units exiting M_2 enter M_3 and so on. If the number of usable units exiting M_S is short of D, further production runs are initiated, as necessary.

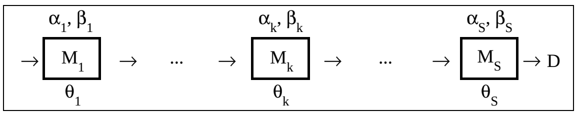

Fig. 1. An S-stage serial production system.

We will refer to a production policy fitting the above description as a "P-Policy". Note that any P-Policy is completely characterized by the lots N_D, $D\geqslant 1$, to enter M_1 whenever the demand is D. The objective is to find optimal P-Policies, i.e., to find the optimal lots N_D^*, $D=1$, to enter M_1 whenever the demand is D.

3.1. Formulating the P-Policy problem

Consider a P-Policy and let N be the lot to enter M_1. We denote by X_k^N the random number of usable units exiting M_k. The following theorem, specifying the distribution of X_k^N, for binomial, IG and A–N yields, is key in solving P-Policies.

Theorem 1. *For a binomial (IG, A–N) system the random variable X_k^N is binomial (IG, A–N) with N the number-of-trials and success probability=$\theta_1\theta_2\ldots\theta_k$. Hence, in particular, for binomial yield $P(X_k^N=0)=[1-\theta_1\theta_2\ldots\theta_k]^N$, and for the IG and A–N yield $P(X_k^N=0)=[1-\theta_1\theta_2\ldots\theta_k]$. Also, for binomial and A–N $E(X_k^N)=N\theta_1\ldots\theta_k$, while for the IG $E(X_k^N)=(\theta_1\ldots\theta_k[1-(\theta_1\ldots\theta_k)^N])/(1-\theta_1\ldots\theta_k)$.*

Proof. For A–N yields the proof is trivial. For binomial yield simply note that for a unit exiting M_k to be conforming it must be successfully processed at M_k and all previous stages, independent of all other units.

(Another way of arriving at the above conclusion is by observing that X_k^N, the number of usable units exiting M_k, would remain unchanged if all N units, including the defective units, would be processed at all stages. Thus, clearly, X_k^N is binomial.)

For IG yield assume that all N units, including the defective units, are processed, in the same order, at all stages. Recall that the probability that the machine at stage k will perform at least x successful operations is $(\theta_k)^x$, $x=1,\ldots,N$. Thus, the probability that at least x conforming units will exit stage k is

$$P(X_k^N\geqslant x)=(\theta_1\cdots\theta_k)^x,\quad x=1,\ldots,N. \quad \square$$

Theorem 2. *Consider a P-Policy. Let N_D, $D\geqslant 1$, be the lot to enter M_1 when the demand is D and let U_D, $D\geqslant 1$, be the corresponding expected cost. Suppose that $U_1,\ldots,U_{D-1}$ are known. Then, the expected cost U_D can be calculated by (writing, simply, N instead of N_D)*

$$U_D=\frac{\alpha_1+\beta_1 N+\sum_{k=1}^{S-1}\left[\alpha_{k+1}\cdot P(X_k^N>0)+\beta_{k+1}E(X_k^N)\right]+\sum_{t=1}^{D-1}U_{D-t}\cdot P(X_S^N=t)}{1-P(X_S^N=0)}. \tag{2}$$

Proof. (i) Clearly, the cost of activating M_1 is $\alpha_1+\beta_1 N$.

(ii) The cost of activating $M_{k+1}, k=1,\ldots,S-1$, is $\alpha_{k+1}\cdot 1(X_k^N>0)+\beta_{k+1}X_k^N$, where $1(X_k^N>0)$ is the indicator of the event $X_k^N>0$ (later, in taking expectation, recall that $E[1(X_k^N>0)]=P(X_k^N>0)$).

(iii) If $X_S^N=t$, $1\leqslant t<D$, the future expected cost is U_{D-t}; if $X_S^N=0$ the future expected cost is U_D.

This leads to

$$\begin{aligned}U_D&=\alpha_1+\beta_1 N+\sum_{k=1}^{S-1}\Big[\alpha_{k+1}\cdot P(X_k^N>0)\\&\quad+\beta_{k+1}E(X_k^N)\Big]+\sum_{t=1}^{D-1}U_{D-t}\cdot P(X_S^N=t)\\&\quad+U_D\cdot P(X_S^N=0). \quad \square\end{aligned}$$

Theorems 1 and 2 provide an efficient method to calculate the expected cost corresponding to any specific P-Policy. The calculation is recursive in D and is easy to perform even for large systems.

3.2. Optimal P-Policies

For P-Policies we define the following cost functions:

> F_D is the optimal (minimal) expected cost required to fulfill an order D.
>
> $F_D(N)$ is the expected cost required to fulfill an order D, if the lot to enter M_1 is N whenever the demand is D and an optimal lot is used whenever the demand is less than D.

Therefore, $F_D=\min_N F_D(N)\equiv F_D(N_D^*)$, where N_D^* is an optimal lot to enter M_1 for demand D.

Theorem 3. *The expected cost $F_D(N)$ can be calculated by*

$$F_D(N) = \frac{\alpha_1 + \beta_1 N + \sum_{k=1}^{S-1}\left[\alpha_{k+1} \cdot P(X_k^N > 0) + \beta_{k+1} E(X_k^N)\right] + \sum_{t=1}^{D-1} F_{D-t} \cdot P(X_S^N = t)}{1 - P(X_S^N = 0)}. \quad (3)$$

Proof. Omitted as it is similar to the proof of Theorem 2. □

Theorem 3 provides an effective method to calculate the optimal lots, N_D^*, $D \geqslant 1$, of P-Policies. The calculation is recursive in D, via a search over N. However, a rule to end the search is required. For this purpose we will calculate an upper bound for N_D. Also, in order to evaluate the effectiveness of P-Policies, we will calculate a lower bound for the (global) optimal expected cost.

3.3. Some interesting observations

Clearly, for A–N yield, the optimal lot, corresponding to a P-Policy, is $N_D = D$. However, while such a policy may appear globally optimal, this is not always true. To see why consider a two-stage system where the first stage has a relative large setup cost. Then it may be optimal to process a lot larger than the demand on the first stage, so that several production runs can be attempted on the second stage. Also, one may be tempted to think that if intermediate inventory exists in a two-stage system, and it is optimal to process units on the second stage, multiples of the demand should be used. To understand why this is not necessarily true see [6].

Clearly, for IG yield, the optimal lot, corresponding to a P-Policy, satisfies $N_D \leqslant D$: it is helpful to think that all units, including the defective units are processed on all machines, in the same order. Note that then; units in excess of the demand will always be discarded (conforming or defective). However, using a lot $N_D \leqslant D$ is not necessarily globally optimal. To see why consider again a two stage system where the first stage has a relative large setup cost. Then, again, it may be optimal to process a lot larger than the demand on the first stage.

3.4. Binomial systems: bounds for optimal P-Policies

Finding the optimal lot requires a search over N, thus a rule to end the search is required. Since for A–N $N_D = D$, and for IG $N_D \leqslant D$, we will explain how to obtain workable bounds for binomial yield.

We start calculating an upper bound for the optimal lot, N_D^{UB}, for a binomial system. Let H_D be the expected cost required to satisfy the demand, of some specific P-Policy. For example, $N_D = D$ is a heuristic specifying that the lot to enter M_1 is equal to the demand. Note that H_D can be calculated via (2); also, clearly, $F_D \leqslant H_D$.

Proposition 1. *Let H_D be the expected cost of some P-Policy and let N_D^{UB} be calculated by*

$$N_D^{\mathrm{UB}} = \frac{H_D - \sum_{i=1}^{S}\alpha_i - \sum_{j=2}^{S}\beta_j D}{\beta_1}. \quad (4)$$

Then, N_D^{UB} is an upper bound for the optimal lot to enter M_1 whenever the demand is D.

Proof. Note that to satisfy the demand, any production policy requires, at least once, the activation of each stage, and the total number of units crossing each stage is at least D. Thus, using $N_D > N_D^{\mathrm{UB}}$ will result in an expected cost exceeding H_D (and F_D). □

Next we calculate a lower bound on the (global) optimal expected cost for a binomial system. Note that any such serial system would be easy to solve, *optimally*, if all setup costs, except for one, were zero. Then, an optimal solution, based upon an SBNS, can be used (see [6]). As of now optimal solutions for A–N or IG SBNS's do not exist in the literature.

Proposition 2. *Referring to Fig. 1, let G_D be the global optimal expected cost and let $G_D(\alpha_1 \neq 0)$ be the optimal expected cost, if all setups except α_1 were zero. Then, $G_D^{\mathrm{LB}}(\alpha_1 \neq 0)$, a lower bound for G_D, based upon $G_D(\alpha_1 \neq 0)$ is given by*

$$G_D^{\mathrm{LB}}(\alpha_1 \neq 0) = G_D(\alpha_1 \neq 0) + \sum_{k=2}^{S}\alpha_k. \quad (5)$$

Proof. Simply observe that $G_D^{\mathrm{LB}}(\alpha_1 \neq 0)$ represents a situation where setups for $M_2, \ldots, M_S$ will be incurred only once. □

Similarly, a lower bound $G_D^{\mathrm{LB}}(\alpha_2 \neq 0)$ can be obtained, and so on. Finally, the tightest lower bound, G_D^{LB}, is the upper lower bound. That is,

$$G_D^{\mathrm{LB}} = \max_j \{G_D^{\mathrm{LB}}(\alpha_j \neq 0)\}. \tag{6}$$

In performing numerical tests, the lower bound G_D^{UB}, calculated via (6), is used to evaluate the quality of optimal P-Policies. Note that N_D^{UB}, which is calculated via (4), is used merely to limit the search for the optimal (P-Policy) lot to enter M_1.

4. Numerical tests

In this section we present the results of several numerical experiments. These results help in assessing the effectiveness of optimal P-Policies.

The suggested algorithm is very fast: we performed a considerable number of experiments; one should bear in mind that when we report about $D=20$ and $S=10$ (10 stages), also all problems for $D \leqslant 20$ need to be solved. The total amount of running time of all experiments in Section 4.1 below was less than 2 min. For this reason we do not report running times of individual experiments.

4.1. Binomial systems

In this section we compare between optimal P-Policies and (a) heuristic methods suggested in [20], (b) a lower bound, and (c) (global) optimal results for an SBNS.

To calculate the initial lot Pentico [20] used four variations: EXP (Expected Value), ORL (Orlicky), MAR (Marginal Analysis), and SSN (Sepheri-Silver-New), each variation is either "based on D": the lot to enter M_1 is based upon a certain single stage and D; or "based on $s_{i-1}(x)$": the lot to enter M_1 is calculated recursively. Two of the eight resulting lots coincide to obtain seven different rules. Using (2) we calculated the expected costs corresponding to these rules, for a four-stage binomial system. Numerical results exhibited in Table 1 affirm the statement made in [20] that the MAR D and SSN D rules are either best or close to being best (we shadowed the optimal rule(s) for each value of D).

Next, in Table 2, we compare between Pentico's best rule and the optimal P-Policy. As both Pentico's best rule and the optimal P-Policy are heuristics we also include the (global) lower bound calculated in Section 3.4. The comparison to the lower bound proves the optimal P-Policy to be effective.

As a further numerical illustration, we solve serial binomial systems, for $D=5$, with the number of stages being $S=1,\ldots,10$. Each stage k, $k=1,\ldots,S$,

Table 1
Four stages: lots and expected cost corresponding to Pentico's rules

D	Four-stage system: $\alpha_k = 40$; $\beta_k = 1$; $\theta_k = 0.8$; $k=1,2,3,4$.													
	EXP D		MAR $s_{i-1}(x)$		MAR D		ORL $s_{i-1}(x)$		ORL D		SSN $s_{i-1}(x)$		SSN D	
	Cost	Lot	Cost	Lot	Cost	Lot	Cost	Lot	Cost	Lot	Cost	Lot	Cost	Lot
1	229.6	2	213.2	18	186.2	8	263.3	35	188.1	9	207.3	16	186.2	8
2	248.4	5	230.9	24	198.6	12	286.9	43	200.4	13	219.1	20	200.4	13
3	275.5	7	242.7	28	208.6	15	304.6	49	210.1	16	230.9	24	210.1	16
4	275.8	10	257.4	33	219.7	19	316.5	53	219.7	19	248.6	30	219.7	19
5	295.7	12	266.3	36	229.0	22	331.2	58	229.0	22	260.4	34	230.8	23
6	296.0	15	281.0	41	238.3	25	343.0	62	238.3	25	263.5	35	240.0	26
7	313.3	17	287.0	43	249.1	29	354.8	66	246.3	27	278.2	40	249.1	29
8	314.1	20	301.7	48	258.1	32	363.7	69	254.6	29	287.0	43	258.2	32
9	329.9	22	313.5	52	267.1	35	372.5	72	263.4	32	301.8	48	267.2	35
10	345.1	24	325.3	56	276.1	38	384.4	76	271.9	34	304.8	49	276.1	38

Table 2
Four stages: a comparison between Pentico's best rule and optimal P-Policies

D	Four-stage system: $\alpha_k = 40$; $\beta_k = 1$; $\theta_k = 0.8$; $k = 1, 2, 3, 4$.					
	Pentico's best rule		Optimal P-Policy		Lower bound	Gap (%)
	Cost	Lot	Cost	Lot	Cost	
1	186.2	8	184.9	6	175.2	5.5
2	198.6	12	197.1	10	184.8	6.7
3	208.6	15	207.7	14	193.8	7.2
4	219.7	19	217.6	17	202.1	7.7
5	229.0	22	227.1	20	210.5	7.9
6	238.3	25	236.4	23	218.8	8.0
7	246.3	27	245.5	26	226.8	8.2
8	254.6	29	254.3	28	234.8	8.3
9	263.4	32	263.1	31	242.9	8.3
10	271.9	34	271.7	34	250.7	8.4

Gap is between the optimal P-Policy and the lower bound.

Table 3
Numerical results for systems with $S = 1, \ldots, 10$ stages; $D = 5$; and $\alpha_k = 40$, $\beta_k = 1$, $\theta_k = 0.8$

Stages S	Expected cost			Gap (%)	Lot to enter M_1	
	Best Pentico	Optimal P-Policy	Lower bound		Best Pentico	Optimal P-Policy
1	49.9	49.9	49.9	0.0	9	9
2	104.8	104.3	100.7	3.6	13	12
3	164.2	163.3	153.9	6.1	17	16
4	229.0	227.1	210.5	7.9	22	20
5	296.8	296.7	270.6	9.6	25	25
6	374.1	373.1	335.6	11.2	29	31
7	468.3	457.8	405.7	12.8	32	38
8	572.0	552.4	482.7	14.4	58	47
9	685.9	658.9	568.4	15.9	72	57
10	813.0	780.1	664.0	17.5	88	70

Gap is between the optimal P-Policy and the lower bound.

Table 4
Five-stage SBNS: a comparison between Pentico's best rule, optimal P-Policies and the optimal solution

D	Five-stage SBNS: $\alpha_j = 0$; $j = 1, 2, 4, 5$; $\alpha_3 = 100$; $\beta_k = 5$; $\theta_k = 0.8$; $k = 1, \ldots, 5$.					
	Pentico's best rule		Optimal P-Policy		Global optimum	Gap (%)
	Cost	Lot	Cost	Lot	Cost	
1	208.1	4	208.1	4	191.9	8.4
2	283.0	6	279.0	7	255.6	9.2
3	346.1	9	342.2	10	315.5	8.5
5	463.9	15	461.0	16	430.5	7.1
10	743.4	31	742.2	30	706.5	5.1
15	1016.4	46	1014.0	44	974.9	4.0
20	1285.5	61	1281.7	58	1240.7	3.3

Gap is between the optimal P-Policy and the global optimum.

Table 5
Selected numerical results for binomial yield

D	S	α_k $(k=1,\ldots,S)$	θ_k $(k=1,\ldots,S)$	Cost	N_D
1	5	1	0.9	13.9	1
5	5	1	0.9	45.8	7
10	5	1	0.9	82.0	15
20	5	1	0.9	152.9	30
1	10	1	0.9	37.2	2
5	10	1	0.9	122.5	11
10	10	1	0.9	219.9	23
20	10	1	0.9	410.6	47
1	5	80	0.9	424.8	5
5	5	80	0.9	466.3	14
10	5	80	0.9	509.7	25
20	5	80	0.9	590.7	44
1	10	80	0.9	875.3	9
5	10	80	0.9	991.5	26
10	10	80	0.9	1112.0	43
20	10	80	0.9	1334.5	76
1	5	1	0.6	46.9	5
5	5	1	0.6	175.0	38
10	5	1	0.6	326.6	81
20	5	1	0.6	626.0	175
1	10	1	0.6	495.3	28
5	10	1	0.6	2181.3	299
10	10	1	0.6	4247.8	742
20	10	1	0.6	8366.2	1785
1	5	80	0.6	510.1	35
5	5	80	0.6	685.9	103
10	5	80	0.6	870.2	176
20	5	80	0.6	1211.8	312
1	10	80	0.6	1810.1	239
5	10	80	0.6	3882.3	889
10	10	80	0.6	6159.7	1636
20	10	80	0.6	10508.7	3105

has the following parameters:

$$\alpha_k = 40, \quad \beta_k = 1, \quad \theta_k = 0.8.$$

Table 3 demonstrates that for large systems Pentico's best rule sometimes selects excessive lots. We note that the expected costs of the optimal P-Policies are reasonably close to the lower bound.

Finally, in Table 4, we solve a five-stage single bottleneck system (SBNS). Thus, Pentico's best rule and the optimal P-Policy can be compared to the optimal solution. This comparison shows the optimal P-Policy to be very effective for SBNS problems, especially as demand increases.

4.2. Sensitivity analysis

To further explore the quality of the results obtained by P-Policies we conducted a numerical test for the binomial, IG and A–N yields, for $D=1, 2, \ldots, 20$, as follows (for all systems β_k, $k=1,\ldots,S$, is kept fixed, i.e., $\beta_k=1$):

(a) For $S=5$ and $S=10$, we altered α_k $(k=1,\ldots,S)$ to assume the values 1, 10, 20, 40 and 80. Similarly, we altered θ_k to assume the values 0.6, 0.8, 0.9 and 0.97. Thus, the number of experiments conducted for each yield is 800.

(b) For $S=10$, we altered θ_k $(k=1,\ldots,S)$ to assume the values 0.7 and 0.9, and formed four sets of tests: (i) increasing α : $\alpha_1=1$; $\alpha_2=10$; $\alpha_j=\alpha_{j-1}+10$

Table 6
Selected numerical results for IG yield

D	S	α_k $(k=1,\ldots,S)$	θ_k $(k=1,\ldots,S)$	Cost	N_D
1	5	1	0.9	17.5	1
5	5	1	0.9	57.1	5
10	5	1	0.9	109.7	4
20	5	1	0.9	215.0	4
1	10	1	0.9	66.9	1
5	10	1	0.9	261.1	3
10	10	1	0.9	508.0	3
20	10	1	0.9	1002.0	3
1	5	80	0.9	565.4	1
5	5	80	0.9	1502.4	5
10	5	80	0.9	2689.3	10
20	5	80	0.9	5080.4	10
1	10	80	0.9	1542.6	1
5	10	80	0.9	5584.2	5
10	10	80	0.9	10663.7	6
20	10	80	0.9	20822.6	6
1	5	1	0.6	258.4	1
5	5	1	0.6	1264.8	2
10	5	1	0.6	2523.4	2
20	5	1	0.6	5040.6	2
1	10	1	0.6	41189.5	1
5	10	1	0.6	205614.4	2
10	10	1	0.6	411146.2	2
20	10	1	0.6	822210.0	2
1	5	80	0.6	2600.8	1
5	5	80	0.6	12282.4	3
10	5	80	0.6	24399.5	3
20	5	80	0.6	48633.7	3
1	10	80	0.6	73654.9	1
5	10	80	0.6	367159.7	2
10	10	80	0.6	734042.8	2
20	10	80	0.6	1467809.0	2

$(j=3,\ldots,10)$; (ii) decreasing $\alpha:\alpha_1=100$; $\alpha_j=\alpha_{j-1}-10$ $(j=2,\ldots,9)$; $\alpha_{10}=1$; (iii) "convex" in α : $\alpha_1=\alpha_{10}=80$; $\alpha_2=\alpha_9=\alpha_{40}$; $\alpha_3=\alpha_8=20$; $\alpha_4=\alpha_7=10$; $\alpha_5=\alpha_6=1$; and (iv) "concave" in $\alpha:\alpha_1=\alpha_{10}=1$; $\alpha_2=\alpha_9=10$; $\alpha_3=\alpha_8=20$; $\alpha_4=\alpha_7=40$; $\alpha_5=\alpha_6=80$. Thus, the number of experiments conducted for each yield is 160.

(c) For $S=10$, we altered α_k $(k=1,\ldots,10)$ to assume the values 80 and 1, and formed four sets of tests: (i) increasing $\theta:\theta_1=0.6$; $\theta_j=\theta_{j-1}+0.05$ $(j=2,\ldots,8)$; $\theta_9=0.97$; $\theta_{10}=1$; (ii) decreasing $\theta:\theta_1=1$; $\theta_2=0.97$; $\theta_3=0.95$; $\theta_j=\theta_{j-1}-0.05$ $(k=4,\ldots,10)$; (iii) "convex" in $\theta:\theta_1=\theta_{10}=0.95$; $\theta_2=\theta_9=0.9$; $\theta_3=\theta_8=0.8$; $\theta_4=\theta_7=0.7$; $\theta_5=\theta_6=0.6$; and (iv) "concave" in $\theta:\theta_1=\theta_{10}=0.6$; $\theta_2=\theta_9=0.7$; $\theta_3=\theta_8=0.8$; $\theta_4=\theta_7=0.9$; $\theta_5=\theta_6=0.95$. Thus, the number of experiments conducted for each yield is 160.

Running time for all 800 experiments for the binomial (IG, A–N) yield in (a) was 50 min (7 s, 4 s). We believe the rather large gap in running times between the binomial and IG yields can be ascribed to the search over N which is required for binomial yield, and to the fact that our algorithm with binomial yield requires the import of binomial probabilities from Excel. Similarly, running time for all 320 experiments for the binomial (IG, A–N) yield in (b) and (c) was 40 min (7 s, 4 s). Conducting the tests we find (numerically): (i) For the binomial and the A–N yields, N_D is strictly increasing in D. For IG N_D may decrease in D. (ii) For the binomial yield, N_D is gradually increasing with α and S, and decreasing in θ. For IG yield, N_D is gradually increasing with α and θ, and decreasing with S. We report some selected results of the binomial and IG yields in Tables 5 and 6, respectively.

5. Conclusion

It is interesting that binomial (IG, A–N) yield at each machine implies that the distribution of X_k^N is likewise binomial (IG, A–N). However, this property does not hold true for independent stages in general. For example, it does not apply to discrete uniform yield. We believe the reason for this is that with binomial (IG, A–N) yield the probability that the first unit, second unit, etc., within a lot will conform is independent of the lot size.

Future research could consider an improvement of the above P-Policies: if the number of usable units exiting M_k is too small a new lot is processed on M_1; and if this number is too large some units do not proceed to the next stage.

Future research could also consider heuristics based upon P-Policies to solve assembly, distribution and general tree systems. Such methods would require additional ad hoc rules at nodes. Some initial experiment that we have made for these problems leads to reasonable results.

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