CSCI 4/5576: Project Checkpoint
The Random Logistic Map

Amy Le (5576)
Long Tat (4576)
Emily Bertelson (4576)
Kristina Entzel (4576)

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1 Introduction

The purpose of this project is to characterize the Random Logistic Map. In particular, we will be studying the stability of the map, which includes locating fixed points and generating bifurcation diagrams. The two main goals are:

1. Find the expected number of order p periodic orbits for a the random map ($p = 1, 2, 3, ...$)
   (a) For an initial starting value $x_0 \in [0, 1]$ and a specific random function $R_0(x)$, iterate until you find the fixed point(s), x_i^* associated with $R_0(x)$. We may implement a numerical scheme to find this fixed point quickly (e.g. Steffensen's Algorithm or Aitken's Δ^2 Method. Both give quadratic convergence).
   (b) Classify the fixed points in terms of a period p orbit.
   (c) Each processor should take a different initial x_0 and report whether the initial condition led to finding a unique stable orbit (many initial conditions may converge to the same orbit or different orbits, while others may never converge). The processors should be properly load balanced.
   (d) Repeat the above steps for a large number of different random maps $R_i(x)$, $i = 0, 1, 2, ...\bar{N}$ in order to find the expected number of order p periodic orbits for the random map.
2. Create a set valued bifurcation diagram [1]
   (a) For many values of $r \in [0, 4]$, and a fixed random function $R_0(x)$, plot the locations of the periodic orbits as a function of r. A period p orbit will have p corresponding x values as its orbit locations (e.g. a period 1 orbit will have 1 fixed point, a period 2 orbit will have 2 fixed points, and so on).

As the map has an element of randomness to it, many simulations (a large $\bar{N}$) would be required for statistical analysis. This project is a subset of a larger work in progress and requires optimization. A serial implementation has been developed in MATLAB. The parallel implementation will be in C++.

2 Description and Background

2.1 Deterministic Logistic Map

The Logistic map is a popularly studied topic in nonlinear dynamics. The Logistic function is described as follows:

$$\hat{f}(x) = rx(1-x) \tag{1}$$

where r is a parameter that takes on values in [0,4]. To get the Logistic Map, one simply discretizes the function by replacing $\hat{f}(x)$ with x_{n+1} and x with x_n:

$$x_{n+1} = rx_n(1-x_n)$$

Fixed points are located where the line $y = x$ intersects with the Logistic Map. As the parameter r is increased or decreased over the interval [0,4], the map undergoes period doubling. Around $r = 3.6$, the map begins to display chaotic behavior.

2.2 Random Logistic Map

The Random Logistic function is a special case of the Logistic function, which is not as popularly studied. In the random case, r becomes a value dependent on x. Essentially, the Random Logistic Map introduces the notion of spatial randomness to the deterministic function (1). The Random Logistic function and map are as follows:

$$f(x) = R(x)x(1-x)$$

$$x_{n+1} = R(x_n)x_n(1-x_n) \tag{2}$$

$R(x)$ consists of a Fourier series whose coefficients are independent (but not identical) random variables drawn from a uniform distribution whose bounds depend on the Fourier mode.

$$\begin{aligned}
\ln(R(x)) &= \xi(x) \\
\xi(x) &= \ln(r) + 2\sum_{n=1}^{N} a_n \cos(2\pi n x) - b_n \sin(2\pi n x) \\
a_n, b_n &\sim Unif(-M_n, M_n) \\
M_n &= \sqrt{1.5 S_n} \\
S_n &= \alpha e^{-L|n|} \\
\alpha &= \sigma^2 \tanh(L/2) \\
\sigma &< \ln(4/r)\frac{\tanh(L/4)}{\sqrt{1.5\tanh(L/2)}}
\end{aligned}$$

Where $L \in (0,1)$ represents the correlation length (and is fixed for each simulation) and $r \in [0,4]$ is also fixed for each simulation.

An initial serial implementation has been developed for this project and has been mildly tested in MATLAB. Our project would consist of converting the existing code to C++ and implementing parallelization through MPI with proper load balancing. Other possible improvements include IO optimization and memory optimization (possible improvements through HDF5).

2.3 Application to engineering

This project has potential applications to civil engineering in terms of treating contaminated groundwater [2]. Essentially, the distribution of rocks underground can be viewed as a random field distributed spatially, through which treatment solution must mix with contaminated groundwater. Understanding the problem of spatial randomness in one dimension can be generalized to two dimensions, with the end goal of successfully simulating chaotic advection and reaction in two dimensions.

3 Analysis

The analysis of our simulation will encompass the following:

1. Compare the MFLOP/s (using profiling from psrun) for a variable number of simulation sizes.

2. Measure the speedup after implementing MPI.

3. Analyze the Karp-Flatt Metric to see how the code performs in parallel.

4. Compare the performance of our simulation in terms of GFLOP/s to the bite:flop ratio in the Roofline plot

3.1 Testing

Even though the nature of this project is random, it is possible to compare the output of the parallel implementation to the serial implementation for accuracy. The randomness lies in the function $R(x)$, so using the same values of $R(x)$ for the serial and parallel code should output the same results.

References

[1] Jeroen S.W. Lamb, Martin Rasmussen, and Christian S. Rodrigues. Topological bifurcations of minimal invariant sets for set-valued dynamical systems. *Proceedings of the American Mathematical Society*, 2013.

[2] Roseanna M. Neupauer, James D. Meiss, and David C. Mays. Chaotic advection and reaction during engineered injection and extraction in heterogeneous porous media. *Water Resources Research*, Volume 50, 2014.