HPC sheet 2

Timo Hauser 7417403
Nicolas Schumann 7426357

ex. 1

ass.: Let $\Omega \subset \mathbb{R}^3$, $u, f, g: \Omega \to \mathbb{R}$.

boundary value problem: $-\Delta u = f$ in Ω

$u = g$ on $\partial\Omega$

a) derive a 7 point star

Approximation of $-\Delta u$ with Taylorentwicklung

In x_1-direction:

$$(\mathrm{I})\quad u(x_1+h, x_2, x_3) = u(x_1, x_2, x_3) + h\frac{\partial u(x_1,x_2,x_3)}{\partial x_1} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} + \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_1^3} + \mathcal{O}(h^4)$$

$$(\mathrm{II})\quad u(x_1-h, x_2, x_3) = u(x_1, x_2, x_3) - h\frac{\partial u(x_1,x_2,x_3)}{\partial x_1} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} - \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_1^3} + \mathcal{O}(h^4)$$

$$(\mathrm{I})+(\mathrm{II}):\quad u(x_1+h,x_2,x_3) + u(x_1-h,x_2,x_3) = 2u(x_1,x_2,x_3) + h^2\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} + \mathcal{O}(h^4)$$

$$\Leftrightarrow\quad -\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} = \frac{1}{h^2}\Big(-u(x_1-h,x_2,x_3) + 2u(x_1,x_2,x_3) - u(x_1+h,x_2,x_3)\Big) + \mathcal{O}(h^2)$$

In x_2-Richtung:

$$(\mathrm{I})\quad u(x_1, x_2+h, x_3) = u(x_1, x_2, x_3) + h\frac{\partial u(x_1,x_2,x_3)}{\partial x_2} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} + \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_2^3} + \mathcal{O}(h^4)$$

$$(\mathrm{II})\quad u(x_1, x_2-h, x_3) = u(x_1, x_2, x_3) - h\frac{\partial u(x_1,x_2,x_3)}{\partial x_2} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} - \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_2^3} + \mathcal{O}(h^4)$$

$$(\mathrm{I})+(\mathrm{II}):\quad u(x_1,x_2+h,x_3) + u(x_1,x_2-h,x_3) = 2u(x_1,x_2,x_3) + h^2\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} + \mathcal{O}(h^4)$$

$$\Leftrightarrow\quad -\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} = \frac{1}{h^2}\Big(-u(x_1-h,x_2,x_3) + 2u(x_1,x_2,x_3) - u(x_1+h,x_2,x_3)\Big) + \mathcal{O}(h^2)$$

In x_1-Richtung:

$$\text{(I)}\quad u(x_1,x_2,x_3+h) = u(x_1,x_2,x_3) + h\frac{\partial u(x_1,x_2,x_3)}{\partial x_3} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2} + \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_3^3} + \mathcal{O}(h^4)$$

$$\text{(II)}\quad u(x_1,x_2,x_3-h) = u(x_1,x_2,x_3) - h\frac{\partial u(x_1,x_2,x_3)}{\partial x_3} + \frac{h^2}{2}\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2} - \frac{h^3}{6}\frac{\partial^3 u(x_1,x_2,x_3)}{\partial x_3^3} + \mathcal{O}(h^4)$$

$$\text{(I)}+\text{(II)}:\quad u(x_1,x_2,x_3+h) + u(x_1,x_2,x_3-h) = 2u(x_1,x_2,x_3) + h^2\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2} + \mathcal{O}(h^4)$$

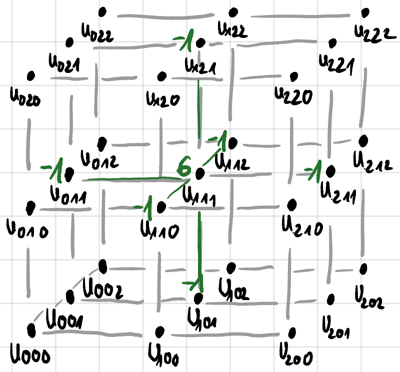
$$\Longleftrightarrow\quad -\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2} = \frac{1}{h^2}\Big(-u(x_1-h,x_2,x_3) + 2u(x_1,x_2,x_3) - u(x_1+h,x_2,x_3)\Big) + \mathcal{O}(h^2)$$

All 3 directions together result in:

$$-\Delta u(x_1,x_2,x_3) \approx \frac{1}{h^2}\Big(-u(x_1+h,x_2,x_3) - u(x_1-h,x_2,x_3) - u(x_1,x_2+h,x_3) - u(x_1,x_2-h,x_3) - u(x_1,x_2,x_3+h) - u(x_1,x_2,x_3-h) + 6u(x_1,x_2,x_3)\Big)$$

We get:

$$-\Delta_h^{(7)} := \frac{1}{h^2}$$



b) matrix free multiplication for 7-point-star

Let N^3 be number of nodes, s our 7-point-star and r a vector with N entries. Let $\varphi: N^3 \to \{0, \dots, N^3-1\}$.

Algo: Input: N, r, s

Output: y

```
// start with inner points
for i = 2 to N-1
    for j = 2 to N-1
        for k = 2 to N-1
            y[φ(i,j,k)] = 6r[φ(i,j,k)] - r[φ(i-1,j,k)] - r[φ(i+1,j,k)]
                          - r[φ(i,j-1,k)] - r[φ(i,j+1,k)] - r[φ(i,j,k-1)] - r[φ(i,j,k+1)]
        end
    end
end
// (inner points of) bottom side
for i=2 to N-1:
    for k=2 to N-1:
        y[φ(i,0,k)] = 6r[φ(i,0,k)] - r[φ(i-1,0,k)] - r[φ(i+1,0,k)]
                      - r[φ(i,1,k)] - r[φ(i,0,k-1)] - r[φ(i,0,k+1)]
    end
end
// front edge, bottom side
for i=2 to N-1:
    y[φ(i,0,0)] = 6r[φ(i,0,0)] - r[φ(i-1,0,0)] - r[φ(i+1,0,0)]
                  - r[φ(i,1,0)] - r[φ(i,0,1)]
end
```

// left corner, front edge, bottom side

$$y[\varphi(0,0,0)] = 6r[\varphi(i,0,0)] - r[\varphi(1,0,0)] - r[\varphi(0,1,0)] - r[\varphi(0,0,1)]$$

// other corners, edges and sides ...

⋮