Exercise 7

Task 1

a)

We want to show that $A_h^1 v_k = \lambda_k v_k$. We can then write for the i-th component:

$$\begin{aligned}\lambda_k v_k^i &= 2v_k^i - v_k^{i-1} - v_k^{i+1}\\ (2-2\cos(k\pi h))\sin(k\pi ih) &= 2\sin(k\pi ih) - \sin(k\pi(i-1)h) - \sin(k\pi(i+1)h)\end{aligned}$$

We can write $\sin(k\pi(i-1)h)$ as:

$$\begin{aligned}\sin(k\pi(i-1)h) &= \sin(k\pi ih - k\pi h)\\ &= \sin(k\pi ih)\cos(k\pi h) - \sin(k\pi h)\cos(k\pi ih)\end{aligned}$$

Analogously, it holds:

$$\begin{aligned}\sin(k\pi(i+1)h) &= \sin(k\pi ih + k\pi h)\\ &= \sin(k\pi ih)\cos(k\pi h) + \sin(k\pi h)\cos(k\pi ih)\end{aligned}$$

Thus for the right side we get:

$$\begin{aligned}2\sin(k\pi ih) - \sin(k\pi(i-1)h) - \sin(k\pi(i+1)h) =& 2\sin(k\pi ih)\\ &- \sin(k\pi ih)\cos(k\pi h) + \sin(k\pi h)\cos(k\pi ih)\\ &- \sin(k\pi ih)\cos(k\pi h) - \sin(k\pi h)\cos(k\pi ih)\\ &= 2\sin(k\pi ih) - 2\sin(k\pi ih)\cos(k\pi h)\\ &= \lambda_k v_k^i\end{aligned}$$

Similarly, we can show that this holds for the special cases $i = 1$ and $i = N$.

Example $i = 1$:
We have

$$\begin{aligned}\lambda_k v_k^1 &= 2v_k^1 - v_k^2\\ (2-2\cos(k\pi h))\sin(k\pi h) &= 2\sin(k\pi h) - \sin(2k\pi h)\\ &= 2\sin(k\pi h) - 2\sin(k\pi h)\cos(k\pi h)\end{aligned}$$

b)

We now want to show that $A_h^2 v_{kl} = \lambda_{kl} v_{kl}$. Similarly, we can then write for the $i \cdot N + j$-th component of v_{kl}

$$v_{kl}^{i,j} = \sin(k\pi ih)\sin(l\pi jh)$$

We then have

$$
\begin{aligned}
4v_{kl}^{i,j} - v_{kl}^{i+1,j} - v_{kl}^{i-1,j} - v_{kl}^{i,j+1} - v_{kl}^{i,j-1} &= \lambda_{kl} v_{kl}^{i,j} \\
&= (4 - 2\cos(k\pi h) - 2\cos(l\pi h)) \sin(k\pi i h) \sin(l\pi j h)
\end{aligned}
$$

From before, we know that

$$
\sin(k\pi(i-1)h) + \sin(k\pi(i+1)h) = 2\sin(k\pi i h)\cos(k\pi h)
$$

We have

$$
\begin{aligned}
-v_{kl}^{i+1,j} - v_{kl}^{i-1,j} &= -\sin(k\pi(i-1)h)\sin(l\pi j h) - \sin(k\pi(i+1)h))\sin(l\pi j h) \\
&= -\sin(l\pi j h)(\sin(k\pi(i-1)h) + \sin(k\pi(i+1)h)) \\
&= -\sin(l\pi j h) \cdot 2\sin(k\pi i h)\cos(k\pi h)
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
-v_{kl}^{i,j+1} - v_{kl}^{i,j-1} &= -\sin(l\pi(j-1)h)\sin(k\pi i h) - \sin(l\pi(j+1)h))\sin(k\pi i h) \\
&= -\sin(k\pi i h)(\sin(l\pi(j-1)h) + \sin(l\pi(j+1)h)) \\
&= -\sin(k\pi i h) \cdot 2\sin(l\pi j h)\cos(l\pi h)
\end{aligned}
$$

Putting everything together, we obtain

$$
\begin{aligned}
4v_{kl}^{i,j} - v_{kl}^{i+1,j} - v_{kl}^{i-1,j} - v_{kl}^{i,j+1} - v_{kl}^{i,j-1} =& 4\sin(k\pi i h)\sin(l\pi j h) \\
&- 2\sin(l\pi j h) \cdot \sin(k\pi i h)\cos(k\pi h) \\
&- 2\sin(k\pi i h) \cdot \sin(l\pi j h)\cos(l\pi h) \\
&= (4 - 2\cos(k\pi h) - 2\cos(l\pi h))\sin(k\pi i h)\sin(l\pi j h) \\
&= \lambda_{kl} v_{kl}^{i,j}
\end{aligned}
$$

As before, we can show that this holds for the special cases for $i = 1 \lor i = N \lor j = 1 \lor j = N$.

Example $i = 1, j = 1$:

We have

$$
\begin{aligned}
4v_{kl}^{1,1} - v_{kl}^{1,2} - v_{kl}^{2,1} &= \lambda_{kl} v_{kl}^{1,1} \\
&= (4 - 2\cos(k\pi h) - 2\cos(l\pi h))\sin(k\pi h)\sin(l\pi h)
\end{aligned}
$$

Expanding the left side, we get

$$
\begin{aligned}
4v_{kl}^{1,1} - v_{kl}^{1,2} - v_{kl}^{2,1} =& 4\sin(k\pi h)\sin(l\pi h) \\
&- \sin(k\pi h))\sin(2l\pi h) \\
&- \sin(l\pi h)\sin(2k\pi h)) \\
=& 4\sin(k\pi h)\sin(l\pi h) \\
&- \sin(k\pi h))2\sin(l\pi h)\cos(l\pi h) \\
&- \sin(l\pi h)2\sin(k\pi h)\cos(k\pi h) \\
=& \sin(k\pi h)\sin(l\pi h)\big(4 - 2\cos(l\pi h) - 2\cos(k\pi h)\big) \\
&= \lambda_{kl} v_{kl}^{1,1}
\end{aligned}
$$