Note Title 2017/10/27

$A \in M_n(K)$ 若 $\exists B$. $AB = BA = E$

B 是 A 之逆矩阵 记为 A^{-1}

1. 可逆之充要条件. $A \in M_n(K)$

$$A \text{ 可逆} \Longleftrightarrow |A| \neq 0$$

证: 若 A 可逆. $AX = b$

$$A^{-1}AX = A^{-1}b$$

$$X = A^{-1}b \quad \text{唯一解}$$

由 Cramer 法则 $AX = b$ 有唯一解 $\Longleftrightarrow |A| \neq 0$.

故 A 可逆 $\Longrightarrow |A| \neq 0$

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} 0 & i \neq j \\ |A| & i = j \end{cases}$$

$$[a_{i1} \ a_{i2} \cdots a_{in}]\begin{bmatrix} A_{j1} \\ A_{j2} \\ \vdots \\ A_{jn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}}_{A \text{之伴随矩阵 } A^*} = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A|E$$

$$AA^* = |A|E$$

若 $|A| \neq 0$ $\quad A \cdot \frac{A^*}{|A|} = E$

反之 $\frac{A^*}{|A|}A = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$= \cdots = E$

有 A 可逆 $\quad \underline{(1)} \quad A^{-1} = \frac{A^*}{|A|}$

Note: 1. $AA^* = A^*A = |A|E$

2. 若 A 可逆则，A^* 也可逆. $(A^*)^{-1} = \frac{A}{|A|}$.

$\frac{A}{|A|}A^* = A^* \cdot \frac{A}{|A|} = E$.

3. $AB = AC \quad A \in M_n(K) \quad A$ 可逆

$A^{-1}AB = A^{-1}AC$

$\Rightarrow B = C$

4. $AX = 0$. A 可逆时,

$A^{-1}AX = A^{-1}0 \Rightarrow X = 0$.

5. $A \in M_n(K)$ 则

A 可逆 $\Leftrightarrow |A| \neq 0 \Leftrightarrow AX = 0$ 只有零解.

例：当 $ad-bc\neq 0$ 时 求 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$.

解：$A^{-1} = \dfrac{A^*}{|A|}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

性质：① 若 A 可逆 则 A^{-1} 也可逆，$(A^{-1})^{-1} = A$

② 若 $A, B \in M_n(K)$ 可逆 则 AB 也可逆

且 $(AB)^{-1} = B^{-1}A^{-1}$.

证：$(AB)\cdot(B^{-1}A^{-1}) = AEA^{-1} = E$

$(B^{-1}A^{-1})(AB) = E$.

推论：$A_1, A_2 \cdots A_s$ 为 n 阶可逆矩阵

则 $A_1A_2\cdots A_s$ 也可逆 且

$$(A_1A_2\cdots A_s)^{-1} = A_s^{-1}A_{s-1}^{-1}\cdots A_2^{-1}\cdot A_1^{-1}$$

③ $k\neq 0$. A 可逆 则 kA 也可逆 $(kA)^{-1} = \frac{1}{k}A^{-1}$

④ A 可逆 则 A^T 也可逆 且 $(A^T)^{-1} = (A^{-1})^T$.

$AA^{-1} = E$. $A^{-1}A = E$.

两边转置 $(A^{-1})^T\cdot A^T = E$. $A^T\cdot(A^{-1})^T = E$

⑤ 若A可逆 $(A^{-1})^K = \underbrace{A^{-1}\cdot A^{-1}\cdots\cdots A^{-1}}_{K个}$

$K\in N^*$

$$A^K\cdot(A^{-1})^K = \underbrace{A\cdot A\cdots\cdots A}_{K个}\cdot \underbrace{A^{-1}\cdot A^{-1}\cdots A^{-1}}_{K个}$$

$$= E$$

同理 $(A^{-1})^K\cdot A^K = E$

即 $(A^K)^{-1} = (A^{-1})^K$.

定义 $A^{-K} = (A^{-1})^K$ 其中A可逆

$K\in N^*$ 规定 $A^0 = E$.

到此为止. 矩阵运算讲了 数乘. 加法. 减法.
乘法. 求逆
乘方. 负整数次方.

例1: A, B, C 可逆方阵 $ABC=E$

则 $BCA=E$. . $CAB=E$

例2: A是n阶方阵. 满足 $A^2+A-2E=O$

证明: A 与 $A-2E$ 均可逆. 并求其逆

证: $A^2+A-2E=0$

$A(A+E) = 2E$ $\quad A\dfrac{(A+E)}{2}=E$

$(A+E)A = 2E$ $\quad \dfrac{A+E}{2}\cdot A=E$

故 A 可逆 $A^{-1} = \frac{A+E}{2}$

$$(A+E)(A-E)$$
$$= A^2 - \underline{AE} + \overline{EA} - E^2$$
$$= A^2 - E$$
$$\parallel$$
$$(A-E)(A+E)$$

$$A^2 + A - 2E = 0$$

$$(A-2E)(A+3E)$$
$$= A^2 - 2EA + 3AE - 6E^2$$
$$= A^2 + A - 6E = -4E$$

反之 $(A+3E)(A-2E) = -4E$.

$$\frac{(A+3E)}{-4} \cdot (A-2E) = E.$$

$(A-2E) \cdot \dfrac{A+3E}{-4} = E$ 即 $A-2E$ 可逆

$$(A-2E)^{-1} = \frac{A+3E}{-4}$$

例: A^* 为 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ 的伴随矩阵.

$A^*B = A$. 求 B 矩阵. 第3行元素的和.

解: $A^*B = A$.

$$\Rightarrow \quad AA^*B = A^2.$$

$$|A|\,B = A^2 \qquad |A| = 2$$

$$B = \frac{1}{2}A^2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} & & \\ 2 & 1 & 1 \end{bmatrix}$$

B 之第 3 行元素之和 2

例4: $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ 求 $(A^*)^{-1} = \frac{A}{|A|}$

$$= -A$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

定理: $A \in M_n(K)$. 若 $\exists B \in M_n(K)$ s.t

$AB = E$ 则 A 可逆，且 $A^{-1} = B$.

证明: 若 $\exists B \in M_n(K)$. s.t $AB = E$

$$BX = 0 \implies ABX = A \cdot 0 = 0.$$

$BX=0$ 之解 均为 $ABX=0$ 之解

又由于 $ABX=0$ 即 $EX=0$ 即 $X=0$

故 $BX=0$ 只有零解 $\implies$ B 可逆

$$AB = E \implies AB \cdot B^{-1} = EB^{-1} \implies A = B^{-1}$$

从而 $A^{-1} = B$. $BA = E$

$$\begin{cases} ① \ A, B \in M_n(K) \\ ② \ AB = E \\ ③ \ BA = E. \end{cases}$$

彼此不独立. 由其中任何两个可以推出第3个

- 初等变换与矩阵乘法:

$AX=b$ ①倍法变换 ②交换变换 ③消去变换

1)
$$\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \lambda a_{i1} & \cdots & \lambda a_{in} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{array}{c} \\ i \\ j \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ \cdots & 0 & \cdots & 1 & \cdots \\ \cdots & 1 & \cdots & 0 & \cdots \\ & & & & 1 \end{bmatrix}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{array}{c} \\ \\ \to i \\ \\ \to j \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ i \\ j \\ \\ \end{array}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & k & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1}+ka_{i1} & a_{j2}+ka_{i2} & \cdots & a_{jn}+ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

结论: 初等行(列)变换相当于矩阵左(右)乘.

左乘(右) 把单位矩阵做相应变换之矩阵.

初等矩阵：$E_i(\lambda) = \begin{array}{c}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\end{array}\begin{array}{l} \\ \\ \to i \\ \\ \\ \end{array}$ $\lambda \neq 0$.

$$E_{ij} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}\begin{array}{l} \\ \\ \to i \\ \\ \to j \\ \\ \\ \end{array}$$

$$E_{i,j}(\lambda) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \lambda \quad \ddots & 1 & \\ & & & & \ddots \quad 1 \end{bmatrix}\begin{array}{l} \\ \\ \to i \\ \to j \\ \\ \end{array}$$

初等矩阵的性质：　　$|A| = \det A$

$\lambda \neq 0$, (1) $\left(E_i(\lambda)\right)^{-1} = E_i\left(\frac{1}{\lambda}\right)$

$$E_i\left(\tfrac{1}{\lambda}\right) \cdot E_i(\lambda)\ E = E$$

$$E_i\left(\tfrac{1}{\lambda}\right) \cdot E_i(\lambda) = E.$$

(2) $\left(E_{ij}\right)^{-1} = E_{ij}$

(3) $\left(E_{ij}(\lambda)\right)^{-1} = E_{i,j}(-\lambda)$

$$\det\left(E_i(\lambda)A\right) = \lambda \det A = \det E_i(\lambda) \cdot \det A$$

$$\det\left(E_{i,j}A\right) = (-1)\det A = \det E_{ij} \det A$$

$$\det\left(E_{ij}(\lambda)A\right) = \det A = \det E_{ij}(\lambda) \det A.$$

对于初等矩阵 P　　$|PA| = |P| \cdot |A|$