例: $A=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $B=\begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix}$ $C=\begin{bmatrix} -7 & 1 \\ 1 & 2 \end{bmatrix}$

Note Title 2017/10/25

求 AB. AC

$$AB=\begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix} \qquad AC=\begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix}$$

$AB=AC$ 但 $B\neq C$

矩阵 $0\cdot A=O$. $A\cdot O=O$

$$IA=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_m \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$=A$$

$$A\cdot I=A$$

- $A(B+C)=AB+AC$
- $(B+C)A=BA+CA$
- $A(BC)=(AB)\cdot C$
- $(AB)^T=B^TA^T$.

证明: 设 $A_{m\times n}$ $B_{n\times s}$ $C_{s\times t}$

$A(BC)$ m行 t列

$(AB)C$ m行 t列

在 $A(BC)$ 中 第i行, 第j列元素.

$$a_{i1}l_{1j}+a_{i2}l_{2j}+\cdots+a_{in}l_{nj}=\sum_{k=1}^{n}a_{ik}l_{kj}$$

其中 l_{kj} 是 BC 的第 k 行第 j 列之元素，

$$l_{kj}=b_{k1}c_{1j}+b_{k2}c_{2j}+\cdots+b_{ks}c_{sj}=\sum_{p=1}^{s}b_{kp}c_{pj}$$

即 $A(BC)$ 第 i 行第 j 列之元为

$$\sum_{k=1}^{n}a_{ik}l_{kj}=\sum_{k=1}^{n}a_{ik}\sum_{p=1}^{s}b_{kp}c_{pj}$$

$$=\sum_{k=1}^{n}\sum_{p=1}^{s}a_{ik}b_{kp}c_{pj}$$

$(AB)C$ 的第 i 行第 j 列之元

$$w_{i1}c_{1j}+w_{i2}c_{2j}+\cdots+w_{is}c_{sj}=\sum_{p=1}^{s}w_{ip}c_{pj}$$

w_{ip} 是 AB 的第 i 行第 p 列

$$w_{ip}=a_{i1}b_{1p}+a_{i2}b_{2p}+\cdots+a_{in}b_{np}=\sum_{k=1}^{n}a_{ik}b_{kp}$$

$(AB)C$ 的第 i 行第 j 列之元为

$$\sum_{p=1}^{s}\sum_{k=1}^{n}a_{ik}b_{kp}c_{pj}$$

从而 $A(BC)=(AB)C$

下证: $(AB)^T = B^T A^T$

$A_{m\times n}$ $B_{n\times s}$ $B^T_{s\times n}$ $A^T_{n\times m}$

$(AB)^T$ 第i行第j列元素为 AB 第j行第i列, 即

$$\sum_{k=1}^{n} a_{jk} b_{ki}$$

B^TA^T 第i行第j列元素为 $\sum_{k=1}^{n} b'_{ik} a'_{kj}$

$$= \sum_{k=1}^{n} b_{ki} a_{jk}$$

例: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 & -1 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

求 $(AB)^T$

解: $(AB)^T = B^T A^T = \begin{bmatrix} 1 & 4 & 2 \\ 7 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 17 \\ 14 & 13 \\ -3 & 10 \end{bmatrix}$$

Note: (反)对称矩阵之乘积、不一定是对称阵 (反)

$A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$$A=\left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad B=\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$$AB=\begin{bmatrix} 0 & 0 & 1 \\ & & \\ 0 & & \end{bmatrix}$$ 不是反对称矩阵

例：A, B 均为 n 阶 对称矩阵

若 $AB=BA$，则 AB 是对称阵。

证明：$(AB)^T = B^T A^T = BA = AB$

故 AB 是对称阵

Note：上(下)三角阵乘积仍是上(下)三角阵.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \\ 0 & & & b_{nn} \end{bmatrix} = \begin{bmatrix} \diagdown & \\ ? & \diagdown \end{bmatrix}$$

Note：
$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \cdots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \cdots & \lambda_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_m a_{m1} & \lambda_m a_{m2} & \cdots & \lambda_m a_{mn} \end{bmatrix}$$

初等行变换 相当于 矩阵 左乘.

列　　　　　　　　　　　　　　右

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix} =$$

1. 矩阵的乘方：　$A^n = \overbrace{A \cdot A \cdots A}^{n个}$

(方阵)

设 $f(x)$ 为多项式　$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$(a_n \neq 0)$

借用 $f(x)$ 记号：

$$f(A) == a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$$

矩阵多项式

Note: 1. $A^k \cdot A^l = A^{k+l}$　　$(A^k)^l = A^{kl}$

$k, l \in N^*$.

2. 一般地　$A, B \in M_n(K)$.　$AB \neq BA$.

$$(AB)^k = \underbrace{(AB)(AB) \cdots\cdots (AB)}_{k个} \neq A^k B^k$$

当 $AB = BA$ 时　$(AB)^k = \overbrace{(AB)(AB) \cdots (AB)(AB)}^{k个}$

$= A^k B^k$

一般地 $f(A)$ 是 A 的矩阵多项式　$g(B)$ 是 B 的矩阵多

$$f(A)\cdot g(B) \neq g(B)\cdot f(A)$$

当 $AB=BA$. 有 $f(A)g(B) = g(B)f(A)$

特别地 $f(A)g(A) = g(A)f(A)$

例
$$\begin{cases} (A+B)(A-B) = A^2 - AB + BA - B^2 \\ (A-B)(A+B) = A^2 + AB - BA - B^2 \end{cases}$$

但 $(A+I)(A-I) = A^2 - A + A - I$

$(A-I)(A+I) =$

Note: 方阵的行列式:

① $|A+B| \neq |A| + |B|$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$A+B = O \qquad |A+B| = |O| = 0$

但 $|A| + |B| = -1 - 1 = -2$.

② $|kA| = k^n |A|$

特别地 $\big|\, |A| B \,\big| = |A|^n |B|$

例1: $\alpha = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$. $A = E - \alpha\alpha^T$

$B = E + 2\alpha\alpha^T$.

$\alpha_{4\times 1}$ 求 AB.

解: $AB = (E - \alpha\alpha^T)(E + 2\alpha\alpha^T)$

$$= E + 2\alpha\alpha^T - \alpha\alpha^T - 2\alpha\alpha^T\alpha\alpha^T$$

$$= E + \alpha\alpha^T - 2\underset{4\times1}{\alpha}\,\underbrace{\underset{1\times4}{\alpha^T}\,\underset{4\times1}{\alpha}}_{=1}\,\underset{1\times4}{\alpha^T}$$

$\alpha^T = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ $\alpha = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$= E + \alpha\alpha^T - 2\alpha\alpha^T$$

$$= E - \alpha\alpha^T = E - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} [\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}]$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \ldots$$

例2: $\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\beta = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$ $A = \underset{3\times1}{\alpha}\,\underset{1\times3}{\beta^T}$ 求 A^4

$$A^4 = \alpha\,\boxed{\beta^T\alpha}\,\boxed{\beta^T\alpha}\,\boxed{\beta^T\alpha}\,\beta^T$$

$\beta^T = [1 \ \frac{1}{2} \ 0]$ $\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$= 8\alpha\beta^T = 8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ \frac{1}{2} \ 0]$$

$$= 8 \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 \\ 16 & 8 & 0 \\ 24 & 12 & 0 \end{bmatrix}$$

例3: $A=\begin{bmatrix}2&4&6\\1&2&3\\1&2&3\end{bmatrix}$ 求 A^n. $(n\in N^*)$

解: $A=\begin{bmatrix}2\\1\\1\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}$

$$A^n=A\cdot A\cdots\cdots A=\begin{bmatrix}2\\1\\1\end{bmatrix}\underline{\underline{\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}2\\1\\1\end{bmatrix}}}\begin{bmatrix}1&2&3\end{bmatrix}\cdots\begin{bmatrix}2\\1\\1\end{bmatrix}\begin{bmatrix}1,2,3\end{bmatrix}$$

$$=7^{n-1}\begin{bmatrix}2\\1\\1\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}$$

$$=7^{n-1}A=\begin{bmatrix}7^{n-1}\cdot 2&7^{n-1}\,4&7^{n-1}6\\7^{n-1}&7^{n-1}\cdot 2&7^{n-1}3\\7^{n-1}&7^{n-1}\cdot 2&7^{n-1}3\end{bmatrix}$$

$(\beta\beta^T)^T=(\beta^T)^T\beta^T=\beta\beta^T$

例4: β是3×1矩阵 $\beta\beta^T=\begin{bmatrix}1&-1&-2\\-1&1&2\\-2&2&4\end{bmatrix}$ 求 $\beta^T\beta=6$.

解:

设 $\beta=\begin{bmatrix}a\\b\\c\end{bmatrix}$ 则 $\beta^T=[a, b, c]$

$$\beta\beta^T=\begin{bmatrix}a^2&ab&ac\\ab&b^2&bc\\ca&cb&c^2\end{bmatrix}$$

$$\beta^T\beta=\begin{bmatrix}a&b&c\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$
$$=\underline{\underline{a^2+b^2+c^2}}$$

例5: $\underbrace{\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}}_{1\times 3}\underbrace{\begin{bmatrix}1&0&-2\\2&4&-1\\0&-3&5\end{bmatrix}}_{3\times 3}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ 求 x_2x_3 系数.

-4

$$=\begin{bmatrix}x_1+2x_2&4x_2-3x_3&-2x_1-x_2+5x_3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$5x^2+3y^2+2z^2\underbrace{-2xy}+3yz\underbrace{-6xz}$ 二次齐次式

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 5 & -1 & -3 \\ -1 & 3 & \frac{3}{2} \\ -3 & \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

6. $A=[\gamma, \alpha_1, \alpha_2, \alpha_3]$ $B=[\beta, \alpha_1, \alpha_2, \alpha_3]$

$\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ 为4维列向量

$|A|=-1$, $|B|=-5$ 求 $|A+B|$

解: $A+B=[\beta+\gamma, 2\alpha_1, 2\alpha_2, 2\alpha_3]$

$$|A+B| = |\beta+\gamma, 2\alpha_1, 2\alpha_2, 2\alpha_3|$$
$$= 8|\beta+\gamma \ \alpha_1, \alpha_2, \alpha_3|$$
$$= 8\left[|\beta, \alpha_1, \alpha_2, \alpha_3| + |\gamma \ \alpha_1, \alpha_2, \alpha_3|\right]$$
$$= 8[-5-1] = -48$$

7. $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ 是4维列向量

若 $\underbrace{|\alpha_1, \alpha_2, \alpha_3, \gamma| = a}$, $|\beta+\gamma \ \alpha_1, \alpha_2, \alpha_3| = b$

求 $|2\beta, \alpha_1, \alpha_2, \alpha_3|$

解: $|2\beta, \alpha_1, \alpha_2, \alpha_3| = 2|\beta, \alpha_1, \alpha_2, \alpha_3|$

$$= 2\left[|\beta+\gamma, \alpha_1, \alpha_2, \alpha_3| + |-\gamma \ \alpha_1, \alpha_2, \alpha_3|\right]$$
$$= 2\left[b-|\gamma, \alpha_1, \alpha_2, \alpha_3|\right] = 2a+2b.$$

§6. 逆矩阵

1. 引入：$ax=b$　$a\neq 0$ 时

$$\frac{1}{a}\cdot ax=\frac{1}{a}\cdot b \Rightarrow x=\frac{b}{a}$$

$$\underset{n\times n}{A}\ \underset{n\times 1}{X}=b_{n\times 1}$$

2. 定义：$A\in M_n(K)$，若 $\exists B\in M_n(K)$

使得 $AB=I$，$BA=I$ 则称 A 是可逆矩阵

或非奇异矩阵。称 B 是 A 的逆矩阵，记作 A^{-1}

否则称为 A 不可逆或奇异

Note：① 唯一性：$AB=I$，$BA=I$

$AC=I$，$CA=I$。

证 $B=C$。

证：$B=BI=B(AC)=(BA)C=IC=C$

② 逆矩阵是互为的。$AA^{-1}=I$，$A^{-1}A=I$。

$$(A^{-1})^{-1}=A$$

例：

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^{-1} \qquad \lambda_i \neq 0$$

$$= \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix}$$

证明：

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$