## 二次型：（二次齐次式）

Note Title 2018/1/5

$z=ax^2+bxy+cy^2$ a.b.c 常数

$$= [x\ \ y]\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

① 判断符号：$ax^2+bx+c = [x,\ 1]\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix}$

$a>0$ $\Delta = b^2-4ac<0$ $ax^2+bx+c$ 恒正.

$a>0$ $\begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} = ac-\frac{b^2}{4} = \frac{4ac-b^2}{4} > 0$

② $xy=1$

坐标轴旋转公式：$\begin{cases} x_0 = x_1\cos\alpha - y_1\sin\alpha \\ y_0 = x_1\sin\alpha + y_1\cos\alpha \end{cases}$ $\alpha=\frac{\pi}{4}$

旧：$x_0y_0=1$

$$\frac{\sqrt{2}}{2}(x_1-y_1)\cdot\frac{\sqrt{2}}{2}(x_1+y_1)=1$$

$$x_1^2-y_1^2=2$$

$xy=1$

$\parallel$

$[x,\ y]\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$A=\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^2-\frac{1}{4}=0$

$\lambda_1=\frac{1}{2},\ \lambda_2=-\frac{1}{2}$

$A$ 对称矩阵：∃正交矩阵

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{bmatrix}$$

$$A = Q\begin{bmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{bmatrix}Q^T$$

$$xy=1 \Rightarrow [x \ \ y]\underbrace{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}_{= A}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$[x, y]\, Q \begin{bmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{bmatrix} Q^T \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

令 $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Q^T\begin{bmatrix} x \\ y \end{bmatrix}$

$[x_1, y_1] = [x, y]\, Q$.

$$[x_1, y_1]\begin{bmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 1$$

$$\frac{1}{2}x_1^2 - \frac{1}{2}y_1^2 = 1$$

$$x_1^2 - y_1^2 = 2$$

1. 二次型定义：

$$Q(x_1, x_2 \cdots x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$ 称为 $n$ 元二次型

$a_{ij} = a_{ji}$.

$$Q(x_1 \cdots x_n) = [x_1 \ x_2 \cdots x_n]\, A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A^T = A$$

$A$ 称为二次型之矩阵

2 化二次型为标准型（无交叉项）

例1: $Q(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2+4x_1x_2+4x_1x_3+4x_2x_3$

利用正交变换将 Q 化为标准型

解: $Q(x_1,x_2,x_3)=\underline{[x_1\ \ x_2\ \ x_3]}\underbrace{\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$|\lambda I-A|=\begin{vmatrix}\lambda-1&-2&-2\\-2&\lambda-1&-2\\-2&-2&\lambda-1\end{vmatrix}=(\lambda-5)\begin{vmatrix}1&1&1\\-2&\lambda-1&-2\\-2&-2&\lambda-1\end{vmatrix}$$

$$=(\lambda-5)\begin{vmatrix}1&1&1\\0&\lambda+1&0\\0&0&\lambda+1\end{vmatrix}=(\lambda-5)(\lambda+1)^2$$

$\lambda_1=5.\ \lambda_2=\lambda_3=-1$

$\lambda_1=5$ 时 $(5I-A)X=0$. $\begin{bmatrix}4&-2&-2\\-2&4&-2\\-2&-2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}2&-1&-1\\1&-2&1\\1&1&-2\end{bmatrix}\to\begin{bmatrix}1&1&-2\\0&-3&3\\0&-3&3\end{bmatrix}\to\begin{bmatrix}1&1&-2\\0&1&-1\\0&0&0\end{bmatrix}$$

基础解系 $\xi_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\lambda_2=-1$ 时 $(-I-A)X=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\quad \begin{bmatrix}-2&-2&-2\\-2&-2&-2\\-2&-2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

基础解系: $\xi_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\quad \xi_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\xi_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad \eta_2=\xi_2-\dfrac{(\xi_2,\eta_1)}{(\eta_1,\eta_1)}\eta_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$$\eta_3=\xi_3-\frac{(\xi_3,\eta_1)}{(\eta_1,\eta_1)}\eta_1-\frac{(\xi_3,\eta_2)}{(\eta_2,\eta_2)}\eta_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

令 $\varepsilon_1=\dfrac{\eta_1}{\|\eta_1\|}=\left[\frac{\sqrt3}{3},\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right]^T$ $\quad \varepsilon_2=\dfrac{\eta_2}{\|\eta_2\|}=\begin{bmatrix}-\frac{\sqrt2}{2}\\\frac{\sqrt3}{2}\\0\end{bmatrix}$

$$\varepsilon_3=\frac{\eta_3}{\|\eta_3\|}=\begin{bmatrix}-\frac{\sqrt6}{6}\\-\frac{\sqrt6}{6}\\+\frac{\sqrt6}{3}\end{bmatrix}$$

令 $P=[\varepsilon_1,\varepsilon_2,\varepsilon_3]=\begin{bmatrix}\frac{\sqrt3}{3}&-\frac{\sqrt2}{2}&-\frac{\sqrt6}{6}\\\frac{\sqrt3}{3}&\frac{\sqrt2}{2}&-\frac{\sqrt6}{6}\\\frac{\sqrt3}{3}&0&\frac{\sqrt6}{3}\end{bmatrix}$

则 $P^{-1}AP=P^TAP=\begin{bmatrix}5&&\\&-1&\\&&-1\end{bmatrix}\Rightarrow A=P\begin{bmatrix}5&&\\&-1&\\&&-1\end{bmatrix}P^T$

$$Q(x_1,x_2,x_3)=\underline{[x_1,x_2,x_3]}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$= [x_1, x_2, x_3]\, P \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix} P^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

令 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

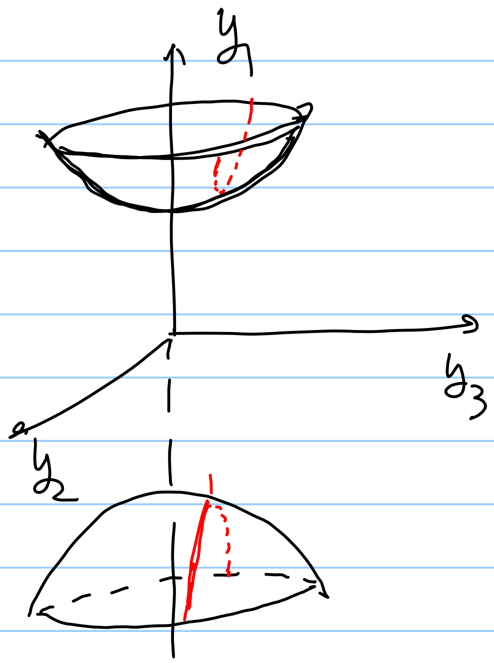
故 $Q = [y_1, y_2, y_3] \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= 5y_1^2 - y_2^2 - y_3^2$$

又问: $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 = 1$ 表示什么几何图形 $Q(x_1, x_2, x_3) = 0$ 表示什么几何图形

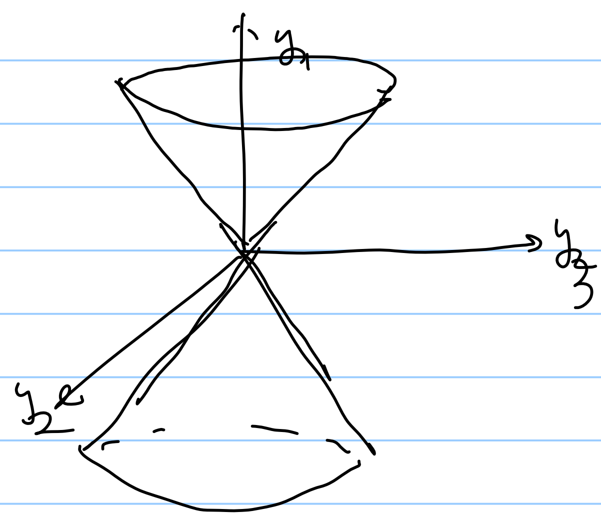
$Q(x_1, x_2, x_3) = 1 \Rightarrow 5y_1^2 - y_2^2 - y_3^2 = 1$

双叶双曲面.




$Q(x_1, x_2, x_3) = 0$

$5y_1^2 = y_2^2 + y_3^2$ 锥面.



## 2. 正定二次型及判定.

① $Q(x_1 \cdots x_n)$ 称正定二次型 若

$\forall x_1 \cdots x_n \in \mathbb{R}$ $x_1 \cdots x_n$ 不全为零 均有 $Q(x_1 \cdots x_n) > 0$

② $Q(x_1 \cdots x_n) = X^T A X \quad A^T = A$

$Q$ 正定 也称对称矩阵 $A$ 正定.

③ 判定方法: $Q$ 是正定二次型 $\Leftrightarrow$ $A$ 的各阶顺序主子式均大于零. $\Leftrightarrow$ $A$ 特征值均大于零.

例: $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2t\,x_1x_2 - 2t\,x_1x_3 - 2t\,x_2x_3$

正定二次型. 求 $t$ 的范围

$$f = [x_1, x_2, x_3]\begin{bmatrix} 2 & -t & -t \\ -t & 2 & -t \\ -t & -t & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\underline{2 > 0}$

$\underbrace{\begin{vmatrix} 2 & -t \\ -t & 2 \end{vmatrix} = 4 - t^2 > 0.}$ $\Downarrow$ $-2 < t < 2$

$\underbrace{\begin{vmatrix} 2 & -t & -t \\ -t & 2 & -t \\ -t & -t & 2 \end{vmatrix} > 0.}$

$(2-2t)\begin{vmatrix} 1 & 1 & 1 \\ -t & 2 & -t \\ -t & -t & 2 \end{vmatrix} > 0$

$(2-2t)\underline{(t+2)^2} > 0$

$t < 1 \quad t \neq -2$

$\Rightarrow -2 < t < 1$

内积：数量积，点积

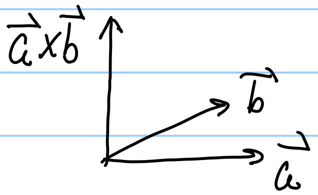
$$\alpha=(a_1,a_2,\cdots,a_n)^T \in \mathbb{R}^n \qquad \alpha\cdot\beta = a_1b_1+a_2b_2+\cdots+a_nb_n$$
$$\beta=(b_1,b_2,\cdots,b_n)^T$$

$$\|\alpha\| = \sqrt{(\alpha,\alpha)} = \sqrt{a_1^2+a_2^2+\cdots+a_n^2}$$

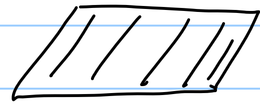
$$\alpha\cdot\beta = \|\alpha\|\|\beta\|\cos\theta$$

外积：向量积，叉乘

$\vec{a}\times\vec{b}$ 是一个向量：大小 $|\vec{a}||\vec{b}|\sin\langle\vec{a},\vec{b}\rangle$

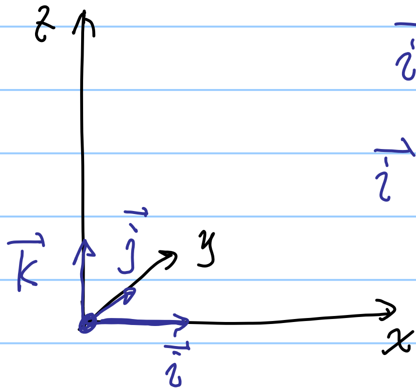



方向：$\vec{a},\vec{b},\vec{a}\times\vec{b}$ 成右手系

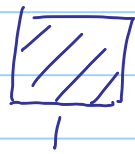


且 $\vec{a}\times\vec{b}$ 与 $\vec{a},\vec{b}$ 均垂直

Note: $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$




$\vec{i}\times\vec{j}=\vec{k} \quad \vec{j}\times\vec{k}=\vec{i} \quad \vec{k}\times\vec{i}=\vec{j}$



$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$

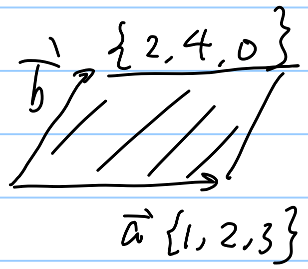
$$\alpha = a_1\vec{i}+a_2\vec{j}+a_3\vec{k} \qquad \beta = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$

$$\alpha\times\beta = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$$

$$= a_1 b_2 \vec{k} - a_1 b_3 \vec{j} - a_2 b_1 \vec{k} + a_2 b_3 \vec{i} + a_3 b_1 \vec{j} - a_3 b_2 \vec{i}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



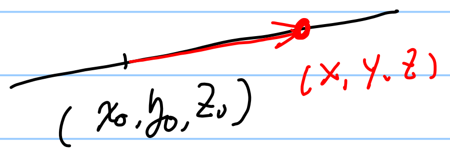

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 4 & 0 \end{vmatrix}$$

$$= -12\vec{i} + 6\vec{j} + 0\vec{k}$$

面积 $= |\vec{a} \times \vec{b}| = \sqrt{144+36} = \sqrt{180} = 6\sqrt{5}$.

① 直线的方程：

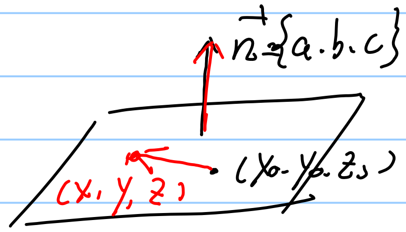
$\vec{\ell} = \{a, b, c\}$




$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

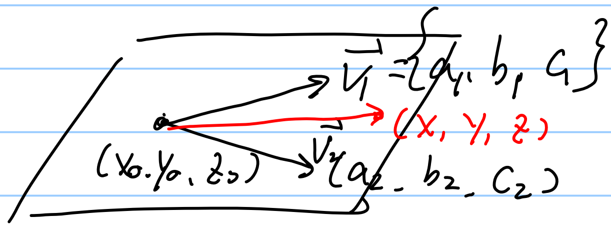
② 平面方程

1) 点法式方程.



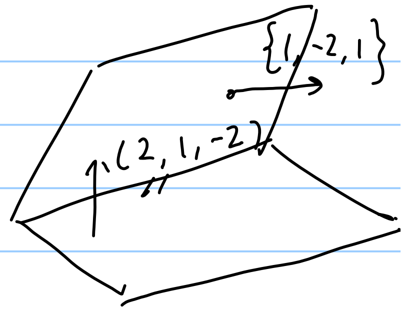

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

2) 点向式方程：



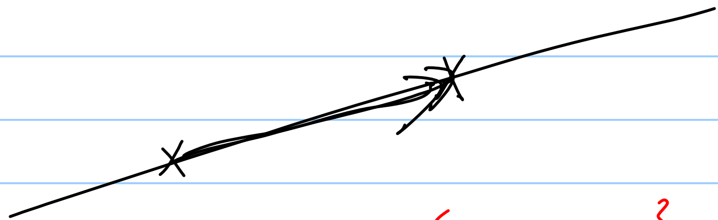

$$\begin{vmatrix} x-x_0 & a_1 & a_2 \\ y-y_0 & b_1 & b_2 \\ z-z_0 & c_1 & c_2 \end{vmatrix} = 0 .$$

例：求直线 $\begin{cases} x-2y+z=1 \\ 2x+y-2z=-2 \end{cases}$ 的方向向量




法1：$\vec{n_1}\times\vec{n_2}=\{1,-2,1\}\times\{2,1,-2\}$

法2：$\begin{cases} x-2y+z=1 \\ 2x+y-2z=-2. \end{cases}$



$z=0$ $\begin{cases} x-2y=1 \\ 2x+y=-2 \end{cases}$ $\begin{cases} 2x-4y=2 \\ 2x+y=-2 \end{cases}$ $y=-\frac{4}{5}$, $x=-\frac{3}{5}$.

$\left(-\frac{3}{5}, -\frac{4}{5}, 0\right)$ $(0, 0, 1)$

$z=1$ $x=2y.$ $2x+y=0$ $x=0, y=0$

$$\vec{v}=\left(\frac{3}{5}, \frac{4}{5}, 1\right)$$