行列式按行按列展开

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

$\tau(j_2 \cdots j_n)$

$$= a_{11} \sum_{1 j_2 \cdots j_n} (-1)^{\tau(1 j_2 \cdots j_n)} a_{2j_2} \cdots a_{2j_n} + a_{12} \sum_{2 j_2 \cdots j_n} (-1)^{\tau(2 j_2 \cdots n)} a_{2j_2} \cdots a_{nj_n}$$

$$+ \cdots + a_{1n} \sum_{n j_2 \cdots j_n} (-1)^{\tau(n j_2 \cdots j_n)} a_{2j_2} \cdots a_{nj_n}$$

$$(-1)^{(n+1)} (-1)^{\tau(j_2 \cdots j_n)} \qquad (-1) \sum_{j_2 \cdots j_n} (-1)^{\tau(j_2 \cdots j_n)} a_{2j_2} \cdots a_{nj_n}$$

引入：代数余子式：

n阶行列式中，划掉第 i 行第 j 列剩下的 $(n-1)^2$ 个元素按原来次序构成 $(n-1)$ 阶

行列式：称为第 i 行第 j 列元素的余子式 M_{ij}

称 $(-1)^{i+j}M_{ij}$ 为其代数余子式

记作 A_{ij}

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Note：代数余子式 A_{ij} 与第 i 行第 j 列位置有关

与 a_{ij} 取值无关。

例：$D = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix}$ 求 A_{11}、A_{12}、A_{13}

$$A_{12} = -\begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} = -12.$$

例2：求 $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -2 \\ -1 & 2 & 2 & 2 \\ 3 & -7 & 0 & -2 \end{vmatrix}$

$$= 1\cdot\begin{vmatrix} 1 & 1 & -2 \\ 2 & 2 & 2 \\ -7 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ -7 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 7 & -16 \end{vmatrix}$$

$$= 2 \times (-21) = -42$$

例:
$$\begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & 3 & -1 & 2 \\ 2 & 1 & 7 & 2 & 5 \\ 0 & 0 & -2 & 3 & 1 \\ 0 & 0 & -4 & -1 & 4 \\ 0 & 0 & 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 3 & 7 & -2 & -4 & 2 \\ -1 & 2 & 3 & -1 & 3 \\ 2 & 5 & 1 & 4 & 5 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 5 & 0 & 0 & 0 \\ 3 & -2 & -4 & 2 \\ -1 & 3 & -1 & 3 \\ 2 & 1 & 4 & 5 \end{vmatrix} = -10 \begin{vmatrix} -2 & -4 & 2 \\ 3 & -1 & 3 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= -10 \begin{vmatrix} 0 & 0 & 2 \\ 6 & 5 & 3 \\ 6 & 14 & 5 \end{vmatrix} = -20 \begin{vmatrix} 6 & 5 \\ 6 & 14 \end{vmatrix} = -1080$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix} = \sum_{j_1\cdots j_n} (-1)^{\tau(j_1\cdots j_n)} a_{1j_1} a_{2j_2}\cdots a_{nj_n}$$

$$= a_{i1} \sum_{j_1\cdots j_{i-1}\, 1\, j_{i+1}\cdots j_n} (-1)^{\tau(j_1\cdots j_{i-1}\, 1\, j_{i+1}\cdots j_n)} a_{1j_1}\cdots a_{i-1\,j_{i-1}}\, a_{i+1\,j_{i+1}}\cdots a_{nj_n}$$

$$(-1)^{\tau(j_1\cdots j_{i-1}\, 1\, j_{i+1}\cdots j_n)} = (-1)^{i+1}(-1)^{\tau(j_2\cdots j_{i-1}\, j_{i+1}\cdots j_n)}$$

$$+ a_{i2} \sum_{j_1\cdots j_{i-1}\, 2\, j_{i+1}\cdots j_n} (-1)^{\tau(j_1\cdots j_{i-1}\, 2\, j_{i+1}\cdots j_n)} a_{1j_1}\cdots a_{i-1\,j_{i-1}}\, a_{i+1\,j_{i+1}}\cdots a_{nj_n}$$

$$(-1)^{\tau(j_1\cdots j_{i-1}\, 2\, j_{i+1}\cdots j_n)} = (-1)^{i+2}(-1)^{\tau(j_2\cdots j_{i-1}\, j_{i+1}\cdots j_n)}$$

$$+\cdots+ a_{in} \sum_{j_1\cdots j_{i-1}\, n\, j_{i+1}\cdots j_n} (-1)^{\tau(j_1\cdots j_{i-1}\, n\, j_{i+1}\cdots j_n)} a_{1j_1}\cdots a_{i-1\,j_{i-1}}\, a_{i+1\,j_{i+1}}\cdots a_{nj_n}$$

$$(-1)^{\tau(j_1\cdots j_{i-1}\, n\, j_{i+1}\cdots j_n)} = (-1)^{i+n}(-1)^{\tau(j_2\cdots j_{i-1}\, j_{i+1}\cdots j_n)}$$

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

Note: 对列也成立

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

例1:
$$\begin{vmatrix} 1 & 0 & -1 & 2\\ -2 & 1 & 3 & 1\\ 0 & 2 & 0 & -2\\ 1 & 3 & 4 & -2\end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 & 2\\ -2 & 1 & 3 & 2\\ 0 & 2 & 0 & 0\\ 1 & 3 & 4 & 1\end{vmatrix} = -2\begin{vmatrix} 1 & -1 & 2\\ -2 & 3 & 2\\ 1 & 4 & 1\end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0\\ -2 & 1 & 6\\ 1 & 5 & -1\end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 6\\ 5 & -1\end{vmatrix} = 62.$$

例:
$$\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ a & \cdots & & & x \end{vmatrix}_n \xrightarrow{\text{法2:}} \begin{vmatrix} 1 & a & a & a & \cdots & a \\ 0 & x & a & a & \cdots & a \\ 0 & a & x & a & \cdots & a \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & a & a & a & \cdots & x \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & & & \\ -1 & & x-a & & \\ \vdots & & & \ddots & \\ -1 & & & & x-a \end{vmatrix}_{n+1} = \begin{vmatrix} 1+\frac{na}{x-a} & a & a & \cdots & a \\ 0 & x-a & & & \\ 0 & & x-a & & \\ \vdots & & & \ddots & \\ 0 & & & & x-a \end{vmatrix}_{n+1}$$

$$= \left(1+\frac{na}{x-a}\right)(x-a)^n = (x-a+na)(x-a)^{n-1}.$$

例:
$$\begin{vmatrix} a & b & & & \\ & a & b & & \\ & & \ddots & \ddots & \\ & & & a & b \\ b & & & & a \end{vmatrix} = a \begin{vmatrix} a & b & & \\ & a & b & \\ & & \ddots & b \\ & & & a \end{vmatrix} + b(-1)^{n+1} \begin{vmatrix} b & & & \\ a & b & & \\ & a & \ddots & \\ & & a & b \end{vmatrix}$$

$$= a^n + (-1)^{n+1} b^n.$$

例:
$$D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & & & & \\ 1 & \alpha+\beta & \alpha\beta & & & \\ & 1 & \alpha+\beta & \alpha\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \alpha+\beta & \alpha\beta \\ & & & & 1 & \alpha+\beta \end{vmatrix}$$

$$= (\alpha+\beta) D_{n-1} - \alpha\beta \begin{vmatrix} 1 & \alpha\beta & & & \\ & \alpha+\beta & \alpha\beta & & \\ & 1 & \alpha+\beta & \alpha\beta & \\ & & \ddots & \ddots & \\ & & & 1 & \alpha+\beta \end{vmatrix}$$

$D_n = (\alpha+\beta) D_{n-1} - \alpha\beta D_{n-2}$ $\qquad D_1 = \alpha+\beta$

$$D_2 = \begin{vmatrix} \alpha+\beta & \alpha\beta \\ 1 & \alpha+\beta \end{vmatrix} = \alpha^2+\alpha\beta+\beta^2$$

当 $n \geqslant 3$ 时： $D_n - \alpha D_{n-1} = \beta D_{n-1} - \alpha\beta D_{n-2}$

$$= \beta (D_{n-1} - \alpha D_{n-2}).$$

$$D_n - \alpha D_{n-1} = \beta^{n-2}(D_2 - \alpha D_1) = \beta^n$$

当 $\beta = 0$ 时： $D_n = \alpha D_{n-1} = \alpha^2 D_{n-2} = \cdots = \alpha^{n-1} D_1 = \alpha^n$

当 $\beta \neq 0$ 时： $\dfrac{D_n}{\beta^n} - \dfrac{\alpha}{\beta} \dfrac{D_{n-1}}{\beta^{n-1}} = 1.$

当 $\alpha = \beta$ 时： $\dfrac{D_n}{\beta^n} - \dfrac{D_{n-1}}{\beta^{n-1}} = 1$

$$\frac{D_n}{\beta^n} = \frac{D_1}{\beta} + (n-1)\,1 \qquad D_n = (n+1)\beta^n.$$

当 $\alpha \neq \beta$. $\underbrace{\boxed{\dfrac{D_n}{\beta^n}}}_{b_n} = \dfrac{\alpha}{\beta} \underbrace{\left(\dfrac{D_{n-1}}{\beta^{n-1}}\right)}_{b_{n-1}} + 1.$

$$b_n + A = \frac{\alpha}{\beta}(b_{n-1} + A) \qquad \left(\frac{\alpha}{\beta} - 1\right) A = 1$$

$$A = \frac{\beta}{\alpha-\beta}$$

$$\frac{D_n}{\beta^n} + \frac{\beta}{\alpha-\beta} = \left(\frac{\alpha}{\beta}\right)^{n-1} \left(\frac{D_1}{\beta} + \frac{\beta}{\alpha-\beta}\right)$$

$$\frac{D_n}{\beta^n} + \frac{\beta}{\alpha-\beta} = \left(\frac{\alpha}{\beta}\right)^{n-1} \frac{\alpha^2}{\beta(\alpha-\beta)}$$

$$\frac{D_n}{\beta^n} = \frac{\alpha^{n+1}}{\beta^n(\alpha-\beta)} - \frac{\beta}{(\alpha-\beta)}$$

$$D_n = \frac{\alpha^{n+1}}{\alpha-\beta} - \frac{\beta^{n+1}}{\alpha-\beta} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

$$D_n = \begin{cases} (n+1)\beta^n & \alpha = \beta \\ \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & \alpha \neq \beta \end{cases}$$

例:

$$D_n = \begin{vmatrix} x & b & b & \cdots & b \\ c & x & b & \cdots & b \\ c & c & x & \cdots & b \\ \vdots & & & & \\ c & c & c & \cdots & x \end{vmatrix}$$

§4:

$$D_n = \begin{vmatrix} x-c & b & b & \cdots & b \\ 0 & x & b & \cdots & b \\ 0 & c & x & \cdots & b \\ \vdots & \vdots & & & \\ 0 & c & c & \cdots & x \end{vmatrix} + \begin{vmatrix} c & b & b & \cdots & b \\ c & x & b & \cdots & b \\ c & c & x & \cdots & b \\ \vdots & \vdots & & & \\ c & c & c & \cdots & x \end{vmatrix}$$

$$= (x-c) D_{n-1} + \begin{vmatrix} c & b & b & \cdots & b \\ 0 & x-b & 0 & \cdots & 0 \\ 0 & c-b & x-b & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & c-b & c-b & \cdots & x-b \end{vmatrix}$$

$$D_n = (x-c)D_{n-1} + c(x-b)^{n-1}$$

由对称性 $D_n = (x-b)D_{n-1} + b(x-c)^{n-1}$

$$\begin{cases} D_n = (x-c)D_{n-1} + c(x-b)^{n-1} \\ D_n = (x-b)D_{n-1} + b(x-c)^{n-1} \end{cases}$$

当 $b \neq c$ 时 $(c-b)D_n = c(x-b)^n - b(x-c)^n$

$$D_n = \frac{c(x-b)^n - b(x-c)^n}{c-b}$$

当 $b=c$ 时 会做

例 范德蒙行列式：

$a_1, a_2 \cdots a_n \in \mathbb{R}$.

$$f(a_1, a_2 \cdots a_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & a_3^{n-2} & & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & & a_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2-a_1 & a_3-a_1 & \cdots & a_n-a_1 \\ & a_2^2-a_1a_2 & a_3^2-a_1a_3 & \cdots & a_n^2-a_1a_n \\ 0 & a_2^{n-2}-a_1a_2^{n-3} & a_3^{n-2}-a_1a_3^{n-3} & \cdots & a_n^{n-2}-a_1a_n^{n-3} \\ 0 & a_2^{n-1}-a_1a_2^{n-2} & a_3^{n-1}-a_1a_3^{n-2} & \cdots & a_n^{n-1}-a_1a_n^{n-2} \end{vmatrix}$$

$$= (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & & a_n \\ a_2^2 & a_3^2 & & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_2^{n-2} & a_3^{n-2} & & a_n^{n-2} \end{vmatrix}_{n-1}$$

$$= (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)$$

$$(a_3-a_2)(a_4-a_2)\cdots(a_n-a_2)\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_3 & a_4 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_3^{n-3} & a_4^{n-3} & \cdots & a_n^{n-3} \end{vmatrix}$$

$$= \cdots\cdots = \prod_{1\le j<i\le n}(a_i-a_j)$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 2^3 & 3^3 & 4^3 & 5^3 \end{vmatrix} = (3-2)(4-2)(5-2)$$

$$(4-3)(5-3)(5-4) = 12$$