第4章 向量空间 $\mathbb{R}^n$

空间(Space)：具有一定结构的集合。

$$\mathbb{R}^n = \{(x_1, x_2 \cdots x_n)^T \mid x_1 \cdots x_n \in \mathbb{R}\} = M_{n,1}(\mathbb{R})$$

$\forall \alpha, \beta, \gamma \in \mathbb{R}^n$. $\forall k, l \in \mathbb{R}$

① $\alpha + \beta = \beta + \alpha$ ② $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

③ $\alpha + \theta = \alpha$ ④ $\alpha + (-\alpha) = \theta$.

⑤ $1 \cdot \alpha = \alpha$ ⑥. $k \cdot (l\alpha) = (kl)\alpha$

⑦ $k(\alpha+\beta) = k\alpha + k\beta$. ⑧ $(k+l)\alpha = k\alpha + l\alpha$.

线性空间：V 非空集合. K 数域 定义两种运算

"加法" $\oplus$ "数乘" $\circ$ $\forall \alpha, \beta, \gamma \in V$ $\forall k, l \in K$

若满足 ① $\alpha \oplus \beta = \beta \oplus \alpha$

② $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$

③ $\exists \theta \in V$ $\alpha \oplus \theta = \alpha$

④ $\forall \alpha \in V$ $\exists \delta \in V$ s.t $\alpha \oplus \delta = \theta$

⑤ $\exists 1 \in K$ $1 \circ \alpha = \alpha$

⑥ $k \circ (l \circ \alpha) = (kl) \circ \alpha$

⑦ $k \circ (\alpha \oplus \beta) = k \circ \alpha \oplus k \circ \beta$

⑧ $(k+l) \circ \alpha = k \circ \alpha \oplus l \circ \alpha$ 成立

$(\mathbb{R}^+, \oplus, \circ)$

$a \oplus b = ab$

$k \circ a = a^k$.

则称 $(V, \oplus, \circ, K)$ 为一个线性空间

- $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$AX = b$$

$$x_1 \underset{\alpha_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + x_2 \underset{\alpha_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \cdots + x_n \underset{\alpha_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}} = \underset{\beta}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \beta$$

$A = [\alpha_1 \cdots \alpha_n]$

1. 线性组合：$\alpha_1 \cdots \alpha_s \in \mathbb{R}^n$

$k_1 \cdots k_s \in \mathbb{R}$

称 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s$ 为 $\alpha_1 \cdots \alpha_s$ 的一个线性组合.

(linear combination)

若 $\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s$ 则称 β 可由 $\alpha_1 \cdots \alpha_s$ 线性表出.

Note: ① 零向量可由任何向量组线性表出.

$$\theta = 0 \cdot \alpha_1 + 0 \cdot \alpha_2 + \cdots + 0 \cdot \alpha_n.$$

② $\alpha_1 \cdots \alpha_s$ 可表出自己

$$\alpha_1 = 1\cdot\alpha_1 + 0\cdot\alpha_2 + 0\cdot\alpha_3 + \cdots + 0\cdot\alpha_s.$$

③ $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \to i \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$\forall \alpha \in \mathbb{R}^n.\quad \alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \alpha = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

④ $\alpha_1 \cdots \alpha_s$ 可由 $\beta_1 \cdots \beta_t$ 线性表出.

$\alpha_i, \beta_j \in \mathbb{R}^n$

$$\alpha_1 = c_{11}\beta_1 + c_{21}\beta_2 + \cdots + c_{t1}\beta_t$$

$$\alpha_2 = c_{12}\beta_1 + c_{22}\beta_2 + \cdots + c_{t2}\beta_t$$

$$\vdots$$

$$\alpha_s = c_{1s}\beta_1 + c_{2s}\beta_2 + \cdots + c_{ts}\beta_t$$

记 $A = [\alpha_1 . \alpha_2 \cdots \alpha_s] \in M_{n,s}(\mathbb{R})$

$B = [\beta_1 . \beta_2 \cdots \beta_t] \in M_{n,t}(\mathbb{R})$

$$\underset{n\times s}{[\alpha_1, \alpha_2 \cdots \alpha_s]} = \underset{n\times t}{\underline{[\beta_1\ \beta_2 \cdots \beta_t]}} \underset{t\times s}{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & & \\ c_{t1} & c_{t2} & \cdots & c_{ts} \end{bmatrix}}$$

$$A = BC$$

2、 $AX=0$ 有非零解 $\quad A_{m,n}$

$\Leftrightarrow r(A)<n \Leftrightarrow \exists x_1, x_2 \cdots x_n$ 不全为0 使 $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}=0$

$\Leftrightarrow \exists$ 不全为零之 $x_1 \cdots x_n$ 使 $x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n=\theta$.

线性相关：$\alpha_1, \alpha_2 \cdots \alpha_s \in \mathbb{R}^n$.

若存在 $k_1、k_2 \cdots k_s \in \mathbb{R}$ 不全为零 满足

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_s\alpha_s=\theta$$

则称 $\alpha_1 \cdots \alpha_s$ 线性相关 (linearly dependent)

否则称 $\alpha_1 \cdots \alpha_s$ 线性无关.

$AX=0$ 有非零解 $\Leftrightarrow$ A 之列向量组 $\alpha_1, \alpha_2 \cdots \alpha_n$ 线性相关.

$AX=0$ 只有零解 $\Leftrightarrow$ A 之列向量组 $\alpha_1 \cdots \alpha_n$ 线性无关

$AX=\beta$ 有解 $\Leftrightarrow$ β 可由 A 之列向量组 $\alpha_1 \cdots \alpha_n$ 线性表出

例1：$\alpha_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \alpha_2=\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \alpha_3=\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$\underline{-2}\alpha_1+\underline{}\alpha_2+\underline{0}\cdot\alpha_3=\theta \quad \alpha_1, \alpha_2, \alpha_3$ 线性相关.

$\alpha_1 \cdots \alpha_s$ 线性无关 $\Longleftrightarrow$ $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = \theta$

$\Leftrightarrow k_1 = k_2 = \cdots = k_s = 0$

例2: $A \in M_n(\mathbb{R})$ $\alpha \in \mathbb{R}^n$. $A^{m-1}\alpha \neq \theta$

$A^m \alpha = \theta$

证明: $\alpha, A\alpha, A^2\alpha, \cdots, A^{m-1}\alpha$ 线性无关

证: 设 $k_1\alpha + k_2 A\alpha + k_3 A^2\alpha + \cdots + k_m A^{m-1}\alpha = \theta$

左乘 A. $k_1 A\alpha + k_2 A^2\alpha + k_3 A^3\alpha + \cdots + k_{m-1}A^{m-1}\alpha$

$+ k_m A^m \alpha = \theta$

继续左乘 A ……, $k_1 \underline{A^{m-1}\alpha} = \theta$. $\Rightarrow$ $k_1 = 0$.

同理 $k_2 = \cdots = k_m = 0$

性质: ① 含有零向量的向量组线性相关.

$\alpha_1, \alpha_2 \cdots \alpha_s, \theta$.

$0\alpha_1 + 0\alpha_2 + \cdots + 0\cdot\alpha_s + 1\cdot\theta = \theta$

② 向量中两个向量对应成比例. 则向量组相关

③ $\alpha_1 \cdots \alpha_s$ 中有部分向量组 $\alpha_{i_1}\ \alpha_{i_2} \cdots \alpha_{i_r}$

线性相关 $\Longrightarrow$ $\alpha_1 \cdots \alpha_s$ 相关.

证：若 $\alpha_{i_1}\cdots\alpha_{i_r}$ 线性相关 则存在不全为0的数 $k_1\cdots k_r$

$$k_1\alpha_{i_1}+k_2\alpha_{i_2}+\cdots+k_r\alpha_{i_r}=\theta$$

则 $k_1\alpha_{i_1}+k_2\alpha_{i_2}+\cdots+k_r\alpha_{i_r}+0\cdot\alpha_{i_{r+1}}+0\cdot\alpha_{i_{r+2}}+\cdots+0\,\alpha_{i_s}=\theta$

部分相关 $\Rightarrow$ 整体相关

整体无关 $\Rightarrow$ 部分无关

④

$$A=\begin{array}{c}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ 0 & a_{22} & \cdots & a_{2s} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{ss} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times s} \begin{array}{l} \\ \\ \\ \rightarrow s\text{行} \\ \\ \\ \end{array} \\ \begin{array}{cccc} \alpha_1 & \alpha_2 & \quad & \alpha_s \end{array}\end{array}$$

$a_{ii}\neq 0$ 阶梯型矩阵.

$r(A)=s$

$AX=0$ 只有零解 $\Leftrightarrow$ $r(A)=s$

$\Leftrightarrow$ $x_1\alpha_1+x_2\alpha_2+\cdots+x_s\alpha_s=\theta$ 只有零解

$\Leftrightarrow$ $\alpha_1\cdots\alpha_s$ 线性无关.

即 阶梯型矩阵列向量组是线性无关的.

3. 线性相关与线性表出之关系

定理：$\alpha_1 \cdots \alpha_s \in \mathbb{R}^n$ 线性相关

则 ∃向量可由其余向量线性表出.

证：$\alpha_1 \cdots \alpha_s$ 线性相关 存在不全为零的 $k_1 \cdots k_s$

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = \theta$$

不妨设 $k_1 \neq 0$

$$\alpha_1 = -\frac{k_2}{k_1}\alpha_2 - \frac{k_3}{k_1}\alpha_3 - \cdots - \frac{k_s}{k_1}\alpha_s$$

即 α_1 可由 $\alpha_2, \alpha_3 \cdots \alpha_s$ 线性表出

Note: ∃向量 不是 ∀向量

定理：$\alpha_1 \cdots \alpha_s$ 线性无关. $\alpha_1 \cdots \alpha_s, \beta$ 线性相关

则 β 可由 $\alpha_1 \cdots \alpha_s$ 唯一表出.

证：$\alpha_1, \alpha_2 \cdots \alpha_s, \beta$ 线性相关. 存在不全为零的数

$k_1, k_2, \cdots, k_s, k_{s+1}$ 使得

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s + k_{s+1}\beta = \theta$$

若 $k_{s+1} = 0$ 则 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = \theta$

由于 $\alpha_1 \cdots \alpha_s$ 无关. 故 $k_1 = k_2 = \cdots = k_s = 0$ 矛盾.

故 $k_{s+1} \neq 0$.

$$\beta = -\frac{k_1}{k_{s+1}}\alpha_1 - \frac{k_2}{k_{s+1}}\alpha_2 - \cdots - \frac{k_s}{k_{s+1}}\alpha_s$$

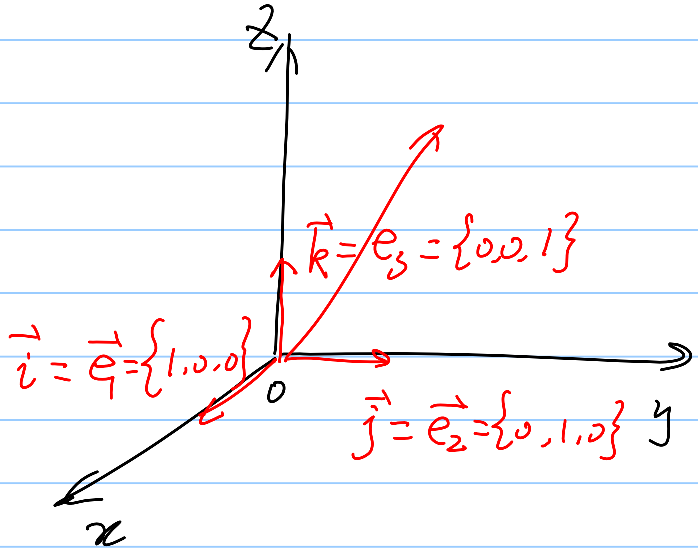
即 β 可由 $\alpha_1, \ldots, \alpha_s$ 表出.

若 $\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s$
$= l_1\alpha_1 + l_2\alpha_2 + \cdots + l_s\alpha_s$

$\Rightarrow \quad (k_1 - l_1)\alpha_1 + (k_2 - l_2)\alpha_2 + \cdots + (k_s - l_s)\alpha_s = \theta.$

$\alpha_1 \ldots \alpha_s$ 线性无关 故 $k_1 - l_1 = 0$, $k_2 - l_2 = 0 \ldots k_s - l_s = 0$

即 $k_1 = l_1$. $k_2 = l_2$ $\ldots$ $k_s = l_s$



4. 与方程组的关系. $A = [\alpha_1, \alpha_2 \cdots \alpha_s]$

$AX = 0$ 有非零解 $\Leftrightarrow$ $\alpha_1 \cdots \alpha_s$ 线性相关

特别地. $A \in M_n(\mathbb{R})$

$AX=0$ 有非零解 $\Leftrightarrow$ A 不可逆 $\Leftrightarrow |A|=0$

$\Leftrightarrow$ $r(A) < n$ $\Leftrightarrow$ A 之列向量线性相关.

A 可逆 $\Leftrightarrow$ $|A| \neq 0$ $\Leftrightarrow$ $AX=0$ 只有零解 $\Leftrightarrow$ $r(A)=n$

$\Leftrightarrow$ A 列向量组线性无关 $\Leftrightarrow$ A^T 可逆

$\Leftrightarrow$ A 行向量组线性无关

特别地、$\alpha_1 \cdots \alpha_n$ 为 n 个 n 维向量.

$\alpha_1 \cdots \alpha_n$ 线性相关 $\underset{\text{矩阵}}{\Leftrightarrow}$ $|\alpha_1, \alpha_2 \cdots \alpha_n| = 0$.

$\Downarrow$

∃向量可由剩余向量线性表出.

不妨设 $\alpha_1 = k_2\alpha_2 + k_3\alpha_3 + \cdots + k_n\alpha_n$

$\longrightarrow |\alpha_1, \alpha_2 \cdots \alpha_n|$

$= |k_2\alpha_2 + k_3\alpha_3 + \cdots + k_n\alpha_n, \alpha_2, \alpha_3, \cdots, \alpha_n|$

$= \cdots = 0$

例：$\mathbb{R}^n$ 中任何 $(n+1)$ 个向量均线性相关

证：$\alpha_1, \alpha_2 \cdots \alpha_n, \alpha_{n+1} \in \mathbb{R}^n$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n + x_{n+1}\alpha_{n+1} = \theta$$

$$\underbrace{[\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_{n+1}]}_{A_{n\times(n+1)}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = \theta.$$

$$\underline{\underline{r(A)}} \le n < \underline{\underline{(n+1)}}$$

$r(A) < n+1$，$AX = 0$ 有非零解 $\Leftrightarrow \alpha_1 \cdots \alpha_{n+1}$ 线性相关.

例：$\alpha_1 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 1 \\ -1 \\ a \\ -2 \end{bmatrix}$

线性相关 求 a

解：$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = \theta$ 有非零解

$$[\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \theta$$

记 $A = [\alpha_1, \alpha_2, \alpha_3]$ 知 $r(A) < 3$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 6 & 2 & a \\ 2 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & -10 & a-6 \\ 0 & -5 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & -10 & a-6 \\ 0 & 0 & 0 \end{bmatrix}$$

$a-6=-8 \quad a=-2$