第七章.

Note Title 2018/1/3

一. 内积空间. $(\mathbb{R}^n, +, \cdot, \mathbb{R})$ 线性空间.

定义: $\varphi: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ 满足:

① $\varphi(\alpha, \beta) = \varphi(\beta, \alpha)$

② $\varphi(\alpha+\beta, \gamma) = \varphi(\alpha, \gamma) + \varphi(\beta, \gamma)$

③ $\varphi(k\alpha, \beta) = k\varphi(\alpha, \beta)$

④ $\varphi(\alpha, \alpha) \geq 0$ $\quad \varphi(\alpha,\alpha)=0 \Leftrightarrow \alpha = \theta$.

则称 φ 为 $\mathbb{R}^n$ 上之内积.

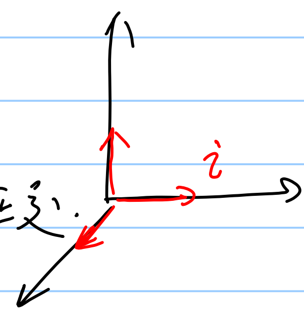
例: $\mathbb{R}^n$ 中 $\alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\beta = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$\varphi(\alpha, \beta) = \alpha^T \beta = [x_1 \ x_2 \cdots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

$(\mathbb{R}^n, +, \cdot, (\cdot,\cdot))$ 内积空间. Euclidean 空间.

2. 标准正交基:



① 若 $(\alpha, \beta) = 0$ 称 $\alpha \perp \beta$ 或 α 与 β 正交.

定义 $\cos\theta = \dfrac{(\alpha, \beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}}$.

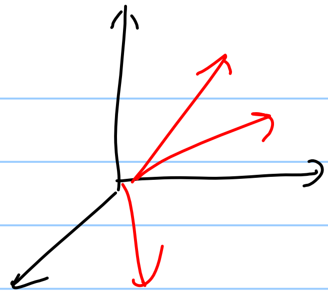
$\|\alpha\| = \sqrt{(\alpha, \alpha)}$

$= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

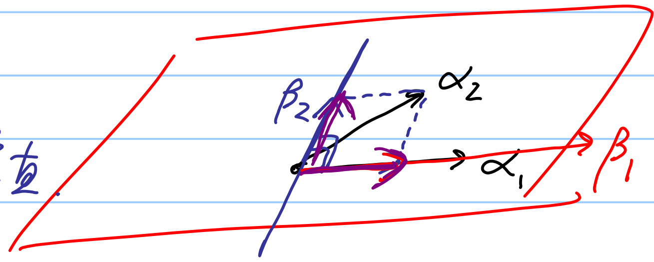
② 目标：$\alpha_1 \cdots \alpha_n$ 为 $\mathbb{R}^n$ 一组基.

改造 $\alpha_1 \cdots \alpha_n$ 变成两两垂直一组基.

且模长为 1 （标准正交基）



令 $\beta_1 = \alpha_1$



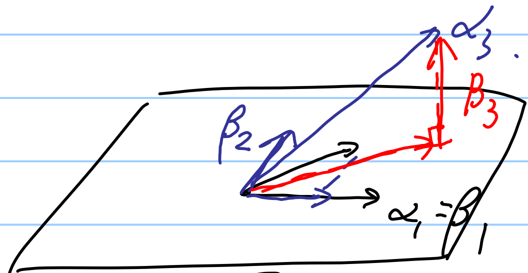
$\alpha_2 = k\alpha_1 + \beta_2$ 希望 β_2 与 α_1 垂直

$(\alpha_2, \alpha_1) = k(\alpha_1, \alpha_1) + \cancel{(\beta_2, \alpha_1)}$.

$$k = \frac{(\alpha_2 . \alpha_1)}{(\alpha_1, \alpha_1)}$$

$$\boxed{\beta_2 = \alpha_2 - \frac{(\alpha_2 . \alpha_1)}{(\alpha_1 . \alpha_1)}\alpha_1 = \alpha_2 - \frac{(\alpha_2 . \beta_1)}{(\beta_1, \beta_1)}\beta_1}$$

$\alpha_3 = \beta_3 + k_1\beta_1 + k_2\beta_2$



希望 β_3 与 β_1、β_2 垂直

$(\alpha_3, \beta_1) = \cancel{(\beta_3, \beta_1)} + k_1(\beta_1 . \beta_1) + k_2\cancel{(\beta_2, \beta_1)}$

$k_1 = \frac{(\alpha_3 . \beta_1)}{(\beta_1 . \beta_1)}$ 同理 $k_2 = \frac{(\alpha_3 . \beta_2)}{(\beta_2 . \beta_2)}$.

$$\beta_3 = \alpha_3 - \frac{(\alpha_3 . \beta_1)}{(\beta_1 . \beta_1)}\beta_1 - \frac{(\alpha_3 . \beta_2)}{(\beta_2 . \beta_2)}\beta_2$$

施密特正交化方法. $\alpha_1, \cdots \alpha_n$ 为 $\mathbb{R}^n$ 一组基

① 正交化. $\beta_1 = \alpha_1$.

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2$$

$$\vdots$$

$$\beta_n = \alpha_n - \frac{(\alpha_n, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_n, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \cdots - \frac{(\alpha_n, \beta_{n-1})}{(\beta_{n-1}, \beta_{n-1})}\beta_{n-1}$$

② 单位化. $\eta_i = \frac{\beta_i}{\|\beta_i\|}$ $i = 1, 2 \cdots n)$.

例: $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

利用 Schmidt 正交化方法. 将 $\alpha_1, \alpha_2, \alpha_3$ 化为一组标准正交基.

解: 令 $\beta_1 = \alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2$$

$$= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

② $\eta_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\end{bmatrix}$

$\eta_2 = \frac{\beta_2}{\|\beta_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix}$

$\eta_3 = \frac{\beta_3}{\|\beta_3\|} = \begin{bmatrix}-\frac{\sqrt{6}}{6}\\+\frac{\sqrt{6}}{3}\\-\frac{\sqrt{6}}{6}\end{bmatrix}$

3.

$\eta_1, \eta_2 \cdots \eta_n$ 是 $\mathbb{R}^n$ 标准正交基.

令 $A = [\eta_1, \eta_2 \cdots \eta_n]$ $\quad A^T = \begin{bmatrix}\eta_1^T\\\eta_2^T\\\vdots\\\eta_n^T\end{bmatrix}$

$$A^TA = \begin{bmatrix}\eta_1^T\\\eta_2^T\\\vdots\\\eta_n^T\end{bmatrix}[\eta_1, \eta_2, \cdots, \eta_n] = \begin{bmatrix}1 & 0 & \cdots & 0\\0 & 1 & \cdots & 0\\\vdots & & & \\0 & 0 & \cdots & 1\end{bmatrix} = I$$

定义：若 $A^TA = I$ 则称 A 为正交矩阵

性质：① 正交矩阵列向量组（行向量组）是 $\mathbb{R}^n$ 一组标准正交基.

② $|A| = \pm 1$ 证：$A^TA = I$

$|A^T| \cdot |A| = 1 \Rightarrow |A|^2 = 1.$

③ $A^{-1} = A^T.$

④ A、B 为正交阵，则 AB 也为正交阵.

证：$A^{-1} = A^T$ $B^{-1} = B^T.$

$(AB)^{-1} = B^{-1}A^{-1} = B^T \cdot A^T = (AB)^T$

即 AB 是正交阵.

§7.2. 实对称阵的对角化

$A \in M_n(\mathbb{R})$ $A^T = A$

① 实对称矩阵特征值是实数

② 实对称矩阵属于不同特征值的特征向量正交.

③ 实对称矩阵一定可以正交相似于对角形.

即 ∃正交矩阵 Q. $Q^TAQ = Q^{-1}AQ = \Lambda.$

证：①：设 $\lambda \in \mathbb{C}$ 为 A 的特征值.

$X \in \mathbb{C}^n$ 为 A 的对应于 λ 的特征向量.

$AX = \lambda X$ 两边取共轭

$A\bar{X} = \bar{\lambda}\bar{X}$ ① $X^TA\bar{X} = \bar{\lambda}X^T\bar{X}$

‖数

$$(\bar{X})^T A^T X$$

$$\parallel$$

$$(\bar{X})^T A X = (\bar{X})^T \lambda X$$

即 $\bar{\lambda} X^T \bar{X} = \lambda \underbrace{(\bar{X})^T X}_{\text{数}} = \lambda X^T \bar{X}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$ $\quad X^T \bar{X} = \underline{[x_1 . x_2 \cdots x_n]} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$

$= x_1 \bar{x}_1 + x_2 \bar{x}_2 + \cdots + x_n \bar{x}_n$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 > 0$

（特征向量非零）

$$\Rightarrow \bar{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}$$

② $A \in M_n(\mathbb{R})$ $\quad A^T = A.$

$\underline{A\xi_1 = \lambda_1 \xi_1}.$ $\quad \underline{A\xi_2 = \lambda_2 \xi_2}.$ $\quad \lambda_1 \neq \lambda_2.$ $\xi_1 \neq 0.$ $\xi_2 \neq 0.$

证:

$$\xi_2^T A \xi_1 = \lambda_1 \xi_2^T \xi_1$$

$$\parallel$$

$$\xi_1^T A \xi_2 = \lambda_2 \xi_1^T \xi_2$$

$$\lambda_1 \xi_2^T \xi_1 = \lambda_2 \xi_1^T \xi_2 = \lambda_2 \xi_2^T \xi_1$$

$(\lambda_1 - \lambda_2)\, \xi_2^T \xi_1 = 0 \qquad \lambda_1 \neq \lambda_2$

$\Rightarrow \xi_2^T \xi_1 = 0$ 即 $\xi_1 \perp \xi_2$

例: $A=\begin{bmatrix}0&-2&-1\\-2&3&2\\-1&2&0\end{bmatrix}$ 求 A^{100}.

解: $|\lambda I-A|=\begin{vmatrix}\lambda&2&1\\2&\lambda-3&-2\\1&-2&\lambda\end{vmatrix}=\begin{vmatrix}0&2+2\lambda&1-\lambda^2\\0&\lambda+1&-2-2\lambda\\\boxed{1}&-2&\lambda\end{vmatrix}$

$=\begin{vmatrix}2+2\lambda&1-\lambda^2\\\lambda+1&-2-2\lambda\end{vmatrix}=(\lambda+1)^2\begin{vmatrix}2&1-\lambda\\1&-2\end{vmatrix}$

$=(\lambda+1)^2(\lambda-5)$ $\lambda_1=\lambda_2=-1$、$\lambda_3=5$.

$\lambda_1=\lambda_2=-1$ 时 $(-I-A)X=0$ $\begin{bmatrix}-1&2&1\\2&-4&-2\\1&-2&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

基础解系: $\alpha_1=\begin{bmatrix}2\\1\\0\end{bmatrix}$ $\alpha_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

$\lambda_3=5$ 时: $(5I-A)X=0$ $\begin{bmatrix}5&2&1\\2&2&-2\\1&-2&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\rightarrow\begin{bmatrix}1&-2&5\\0&12&-24\\0&6&-12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

基础解系 $\alpha_3=\begin{bmatrix}-1\\2\\1\end{bmatrix}$

② $\beta_1=\alpha_1=\begin{bmatrix}2\\1\\0\end{bmatrix}$. $\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{2}{5}\begin{bmatrix}2\\1\\0\end{bmatrix}$

β_1, β_2 都为 -1 的特征向量 $= \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

单位化: $\eta_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}$

$$\eta_2 = \frac{\beta_2}{\|\beta_2\|} = \begin{bmatrix} -\frac{\sqrt{30}}{30} \\ -\frac{\sqrt{30}}{15} \\ -\frac{\sqrt{30}}{6} \end{bmatrix} \qquad \eta_3 = \frac{\beta_3}{\|\beta_3\|} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

令 $Q = [\eta_1, \eta_2, \eta_3] = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{30}}{30} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{30}}{6} & \frac{\sqrt{6}}{6} \end{bmatrix}$

$$Q^T A Q = \underline{\underline{Q^{-1} A Q}} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix}$$

$$A = Q\begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix}Q^{-1} = Q\begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix}Q^T$$

$$A^{100} = Q\begin{bmatrix} (-1)^{100} & & \\ & (-1)^{100} & \\ & & 5^{100} \end{bmatrix}Q^T$$

$$= \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{30}}{30} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{30}}{6} & \frac{\sqrt{6}}{6} \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 5^{100} \end{bmatrix}\begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ \frac{\sqrt{30}}{30} & -\frac{\sqrt{30}}{15} & -\frac{\sqrt{30}}{6} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix}$$