$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix} = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

共有 $n!$ 项.

例1: $g(x) = \begin{vmatrix} 2 & -1 & x & 2x\\ 1 & 1 & x & -1\\ 0 & x & 2 & 0\\ x & 0 & -1 & 3+x \end{vmatrix}$ 求 x^4 的系数
求常数项、

x^4: $2x \cdot x \cdot x \cdot x \, (-1)^{\tau(4321)} = 2x^4$

常数项: $2 \times 1 \times 2 \times 3 \, (-1)^{\tau(1234)} + (-1) \times 1 \times 2 \times 3 \, (-1)^{\tau(2134)}$

$= 12 + 6 = 18$

例2: $\begin{vmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1}\\ b_n & & & & a_n \end{vmatrix}$

当 $n=1$ 时 $|a_1| = a_1$ (注意: 不是绝对值).

当 $n \geq 2$ 时 $j_1=1,\ j_2=2 \cdots\ j_n=n,\quad a_1a_2\cdots a_n(-1)^{\tau(12\cdots n)}$

$j_1=2\ \ j_2=3 \cdots\ \ j_{n-1}=n,\ j_n=1$

$b_1b_2\cdots b_n(-1)^{\tau(23\cdots n\,1)}$

行列式 $= a_1a_2\cdots a_n + (-1)^{n-1} b_1b_2\cdots b_n$

例3：证明：
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}b^{1-1} & a_{12}b^{1-2} & \cdots & a_{1n}b^{1-n} \\ a_{21}b^{2-1} & a_{22}b^{2-2} & \cdots & a_{2n}b^{2-n} \\ \vdots & & & \\ a_{n1}b^{n-1} & a_{n2}b^{n-2} & \cdots & a_{nn}b^{n-n} \end{vmatrix}$$
$(b \neq 0)$.

$$右 = \sum_{j_1\cdots j_n} (-1)^{\tau(j_1 j_2\cdots j_n)} (a_{1j_1}b^{1-j_1})(a_{2j_2}b^{2-j_2})\cdots(a_{nj_n}b^{n-j_n})$$

$$= \sum_{j_1 j_2\cdots j_n} (-1)^{\tau(j_1\cdots j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n} \underbrace{b^{(1+2+\cdots+n)-(j_1+j_2+\cdots+j_n)}}_{\parallel\; b^0=1}$$

$= 左$

§3：行列式的性质

性质1：行列式中某行为零，行列式的值为零

性质2：行列式中行列互换，其值不变。

证：$|A| = \sum_{j_1 j_2\cdots j_n} (-1)^{\tau(j_1\cdots j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n}$

$1 \quad 2 \quad \cdots \quad n \qquad \tau(12\cdots n)=0$

$j_1 \quad j_2 \qquad\quad j_n \qquad \tau(j_1\cdots j_n)$

$2 \quad 1 \quad 3 \cdots n$

$j_2 \quad j_1 \quad j_3 \cdots j_n$

在某一对中
即 行标逆序与列标逆序总和不变

例如: $a_{12}\,a_{23}\,a_{31}\,a_{44}$ $\qquad \tau(2314)=2$.

$a_{31}\,a_{12}\,a_{23}\,a_{44}$ $\qquad \tau(3124)=2$

$$|A|=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1\cdots j_n)}a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

$$=\sum_{i_1\cdots i_n\ j_1\cdots j_n}(-1)^{\tau(i_1\cdots i_n)+\tau(j_1\cdots j_n)}a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$$

$$=\sum_{i_1\cdots i_n}(-1)^{\tau(i_1i_2\cdots i_n)}a_{i_11}a_{i_22}\cdots a_{i_nn}$$

$$=\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{vmatrix}$$

Note:

对行成立之性质, 对列也成立

(3) 行列式中某行有公因子k. 可以提到行列式外

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & & & \\ ka_{i1} & ka_{i2} & \cdots & ka_{in}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1\cdots j_n)}a_{1j_1}a_{2j_2}\cdots(ka_{ij_i})\cdots a_{nj_n}$$

$$=k\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1\cdots j_n)}a_{1j_1}a_{2j_2}\cdots a_{ij_i}\cdots a_{nj_n}$$

$$=k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$$

(4) 加行拆分

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

(5). 对换行列式中的两行，其值变为原来的相反数.

$$\begin{array}{c} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{array} \begin{array}{c} \\ \to i \\ \\ \to j \\ \\ \end{array} = \sum_{k_1k_2\cdots k_n} (-1)^{\tau(k_1\cdots k_i\cdots k_j\cdots k_n)} a_{1k_1}\cdots a_{ik_i}\cdots a_{jk_j}\cdots a_{nk_n}$$

$$+$$

$$\begin{array}{c} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{array} \begin{array}{c} \\ \to i \\ \\ \to j \\ \\ \end{array} = \sum_{k_1\cdots k_n} (-1)^{\tau(k_1\cdots k_j\cdots k_i\cdots k_n)} a_{1k_1}\cdots a_{jk_j}\cdots a_{ik_i}\cdots a_{nk_n}$$

$$\|$$

$$0$$

推论：① 行列式中两行相同．值为 0

② — — — — — — — — 对应成比例．值为 0

(6)
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1}+ka_{i1} & a_{j2}+ka_{i2} & \cdots & a_{jn}+ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & & a_{in} \\ \vdots & & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{0}$$

某行乘以常数加到其他行，行列式之值不变

例1:
$$\begin{vmatrix} 1 & 2 & -2 & 3 \\ -1 & -2 & 4 & -2 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 2 & -5 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3(-1)^{\tau(1324)} = -3$$

例2:
$$\begin{vmatrix} a_1+b_1 & b_1+c_1 & c_1+a_1 \\ a_2+b_2 & b_2+c_2 & c_2+a_2 \\ a_3+b_3 & b_3+c_3 & c_3+a_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

求 m.

解: 左 $= \begin{vmatrix} a_1 & b_1+c_1 & c_1+a_1 \\ a_2 & b_2+c_2 & c_2+a_2 \\ a_3 & b_3+c_3 & c_3+a_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_1+c_1 & c_1+a_1 \\ b_2 & b_2+c_2 & c_2+a_2 \\ b_3 & b_3+c_3 & c_3+a_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1+c_1 & c_1 \\ a_2 & b_2+c_2 & c_2 \\ a_3 & b_3+c_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & c_1+a_1 \\ b_2 & c_2 & c_2+a_2 \\ b_3 & c_3 & c_3+a_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

$m = 2$

例3: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$ 求 $\begin{vmatrix} 3a_{31} & a_{11}+2a_{21} & a_{11} \\ 3a_{32} & a_{12}+2a_{22} & a_{12} \\ 3a_{33} & a_{13}+2a_{23} & a_{13} \end{vmatrix} = ?$

$$= 6 \begin{vmatrix} a_{31} & a_{21} & a_{11} \\ a_{32} & a_{22} & a_{12} \\ a_{33} & a_{23} & a_{13} \end{vmatrix} = -12$$

例4: $D_n = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & & & & \\ b & b & b & \cdots & a \end{vmatrix}$ $n=1$ 时 $D_n = a$

当 $n \geq 2$ 时 $D_n = \left((n-1)b+a\right) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \\ b & b & b & \cdots & a \end{vmatrix}$

$$= \left[(n-1)b+a\right] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & a-b & & 0 \\ & & a-b & \\ 0 & & & \ddots \\ & & & a-b \end{vmatrix} = \left[(n-1)b+a\right](a-b)^{n-1}$$

例5: a, b, c 为 $x^3+2x-4=0$ 之根 求 $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

解: $x^3+2x-4 = (x-a)(x-b)(x-c)$

$= x^3-(a+b+c)x^2+ \cdots\cdots$

故 $a+b+c=0$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ \cdots & \cdots & \cdot \\ \cdots & \cdots & \cdots \end{vmatrix} = 0$$

例6: $n+1$ 阶行列式 ($a_i \neq 0$, $i=1, 2 \cdots n$)

$$\begin{vmatrix} a_0 & \underline{b_1} & b_2 & \cdots & b_n \\ c_1 & a_1 & & & \\ c_2 & & a_2 & & \\ \vdots & & & \ddots & \\ c_n & & & & a_n \end{vmatrix}$$

列2 $\times(-\frac{c_1}{a_1})$ + 列1

$$\begin{vmatrix} a_0-\frac{b_1c_1}{a_1} & b_1 & b_2 & \cdots & b_n \\ 0 & a_1 & & & \\ c_2 & & a_2 & & \\ \vdots & & & \ddots & \\ c_n & & & & a_n \end{vmatrix}$$

$$= \left(a_0 - \sum_{i=1}^{n} \frac{b_i c_i}{a_i}\right) a_1 a_2 \cdots\cdots a_n$$

例7: $x_1, \cdots, x_n \neq 0$

求 $D_n = \begin{vmatrix} 1+x_1 & 2 & 3 & \cdots & n \\ 1 & 2+x_2 & 3 & \cdots & n \\ 1 & 2 & 3+x_3 & \cdots & n \\ \vdots & \vdots & \vdots & & \\ 1 & 2 & 3 & \cdots & n+x_n \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2+x_2 & 3 & \cdots & n \\ \vdots & \vdots & & & \\ 1 & 2 & 3 & \cdots & n+x_n \end{vmatrix} + \begin{vmatrix} x_1 & 2 & 3 & \cdots & n \\ 0 & 2+x_2 & 3 & \cdots & n \\ \vdots & \vdots & & & \\ 0 & 2 & 3 & \cdots & n+x_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & 0 \cdots & 0 \\ 0 & 0 & 0 & 0 \cdots & x_n \end{vmatrix} + x_1 \begin{vmatrix} 2+x_2 & 3 & \cdots & n \\ 2 & 3+x_3 & \cdots & n \\ \vdots & & & \\ 2 & 3 & \cdots & n+x_n \end{vmatrix}$$

递推.

例8. 证明: 奇数阶反对称行列式值为0

$(a_{ij} = -a_{ji})$

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{vmatrix} \xlongequal{\text{行列互换}} \begin{vmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{vmatrix}$$

例9：求n阶行列式

$$D_n=\begin{vmatrix} 2a & a^2 & & & \\ 1 & 2a & a^2 & & \\ & 1 & 2a & a^2 & \\ & & \ddots & \ddots & \ddots \\ & & 1 & 2a & a^2 \\ & & & 1 & 2a \end{vmatrix}$$

当 $n=1$ 时

$D_n=2a$

当 $n\geq 2$ 时，

$$D_n=\begin{vmatrix} 2a & 0 & & & & \\ 1 & \frac{3}{2}a & a^2 & & & \\ 0 & 1 & 2a & a^2 & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & 1 & 2a & a^2 \\ 0 & & & & 1 & 2a \end{vmatrix}$$

$$=\begin{vmatrix} 2a & 0 & & & \\ 1 & \frac{3}{2}a & 0 & & \\ 0 & 1 & \frac{4}{3}a & a^2 & \\ & & \ddots & 1\quad 2a & a^2 \\ & & & 1 & 2a \end{vmatrix}$$

$$=\cdots\cdots=\begin{vmatrix} 2a & 0 & & \\ 1 & \frac{3}{2}a & 0 & \\ 0 & 1 & \frac{4}{3}a & \\ & & \ddots & \\ & & 1 & \frac{n+1}{n}a \end{vmatrix}=(n+1)a^n$$