1. 矩阵之相抵关系：矩阵 $B$ 可以由 $A$ 经过一系列初等变换得到，则称 $A$ 与 $B$ 相抵或等价

$A, B$ 相抵 $\Longleftrightarrow$ $\exists$ 一系列初等矩阵 $P_1, P_2 \cdots P_n$, $Q_1, Q_2 \cdots Q_m$

$$B = P_n P_{n-1} \cdots P_1 A Q_1 Q_2 \cdots Q_m$$

① 反身性. $A \in M_{m,n}(K)$ $\quad A = I_m A I_n$

② 对称性 $A$ 与 $B$ 等价 则 $B$ 与 $A$ 等价

$$A = P_1^{-1} \cdots P_{n-1}^{-1} P_n^{-1} B Q_m^{-1} Q_{m-1}^{-1} \cdots Q_1^{-1}$$

③ 传递性: $A$ 与 $B$ 等价 $B$ 与 $C$ 等价 则 $A$ 与 $C$ 等价

$$B = P_n P_{n-1} \cdots P_1 A Q_1 Q_2 \cdots Q_m$$

$$C = R_s \cdots R_2 R_1 B T_1 T_2 \cdots T_r$$

$$C = R_s R_{s-1} \cdots R_1 P_n P_{n-1} \cdots P_1 A Q_1 Q_2 \cdots Q_m T_1 \cdots T_r$$

$$A_{m\times n} \xrightarrow{\text{初等行变换}} \begin{bmatrix} c_{11} & \cdots & c_{1i_2} & \cdots & c_{1i_r} & \cdots & c_{1n} \\ 0 & \cdots & c_{2i_2} & \cdots & c_{2i_r} & \cdots & c_{2n} \\ \vdots & & \vdots & & \vdots & & \\ 0 & \cdots & 0 & \cdots & c_{ri_r} & \cdots & c_{rn} \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$c_{11}, c_{2i_2} \cdots c_{ri_r} \neq 0$

$$\xrightarrow{\text{初等列变换}} \begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ & c_{2i_2} & \cdots & 0 \\ & & \ddots & \\ 0 & & & c_{ri_r} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \; 1 \end{matrix} & O \\ O & 0 \end{bmatrix}$$

定理：任何一个矩阵都与形如 $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ 之矩阵相抵

$$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & -2 & -1 & -3 \\ 2 & -1 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{列} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{列} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note：$A.B$ 相抵 $\Longleftrightarrow$ $r_A = r_B$.

$r_A$. $r_B$ 分别为 $A.B$ 化阶的标准型矩阵后. 非零行数

称 $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ 为相抵标准型

Note：$\forall A \in M_{m.n}(K)$ $\exists P \in M_m(K).\ Q \in M_n(K)$ （P.Q 可逆）

s.t. $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

2. 初等变换与矩阵求逆.

① $A \in M_n(K)$ 若 $A$ 可逆 则 $|A| \neq 0$.

$$P_t \cdots P_3 P_2 P_1 A = \begin{bmatrix} c_{11} & & * \\ & c_{22} & \\ & & \ddots \\ & & & c_{nn} \end{bmatrix}$$

由 $|A| \neq 0$ 知 $\begin{vmatrix} c_{11} & & * \\ & c_{22} & \\ & & \ddots \\ & & & c_{nn} \end{vmatrix} \neq 0$ 即 $c_{11} \cdot c_{22} \cdots c_{nn} \neq 0$

对可逆方阵，可以仅作行变换将其化成上三角形且对角元均不为零

$$Q_n \cdots Q_2 Q_1 P_t \cdots P_2 P_1 A = \begin{bmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

$$\boxed{R_s \cdots R_1 Q_n \cdots Q_2 Q_1 P_t \cdots P_2 P_1} A = I$$

即 $A^{-1} = R_s R_{s-1} \cdots R_1 Q_n \cdots Q_2 Q_1 P_t \cdots P_1 I$

初等变换法求逆矩阵：$[A \mid I] \xrightarrow{行} [I \mid A^{-1}]$

$$\begin{bmatrix} A \\ \hline I \end{bmatrix} \xrightarrow{列} \begin{bmatrix} I \\ \hline A^{-1} \end{bmatrix}$$

例：$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$ 求 $A^{-1}$.

解：$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}.$$

Note: $A$ 可逆 $\quad \underbrace{P_t \cdots P_2 P_1} A = I$

從而 $\quad A^{-1} = P_t P_{t-1} \cdots P_2 P_1$

$$A = (P_t P_{t-1} \cdots P_2 P_1)^{-1}$$

$$= P_1^{-1} P_2^{-1} \cdots P_{t-1}^{-1} P_t^{-1}$$

由于初等矩阵之逆矩阵仍是初等矩阵

$A$ 可逆 $\Longleftrightarrow$ $A$ 可以写成一组初等矩阵之乘积。

例： $\underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}}_{A \text{ 可逆}} X = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$ 求 $X_{3\times 2}$

$$X = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$$

$\underline{P_t \cdots P_2 P_1 A} = I$

$A^{-1} = P_t P_{t-1} \cdots P_2 P_1$

$A$ 可逆 $AX = B$

$X = A^{-1} B$

$= P_t P_{t-1} \cdots P_2 P_1 B$.

$$\left[\begin{array}{ccc|cc} 2 & 3 & -1 & 2 & 1 \\ 1 & 2 & 0 & -1 & 0 \\ -1 & 2 & -1 & 3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{c|c} I & X \end{array}\right]$$

3. $A, B \in M_n(K)$ $\quad |AB| = |A|\cdot|B|$

若 $B$ 可逆时：B 可以写成一系列初等矩阵乘积

$B = P_1 P_2 \cdots P_t$ $\quad P_1 \cdots P_t$ 为初等矩阵

$$AB = A P_1 P_2 \cdots P_t$$

$$\begin{aligned}
|AB| &= |A P_1 P_2 \cdots P_{t-1} P_t| \\
&= |A P_1 \cdot P_2 \cdots P_{t-1}| \cdot |P_t| \\
&= \cdots = |A|\,|P_1|\,|P_2| \cdots |P_{t-1}|\,|P_t| \\
&= |A|\,|P_1||P_2| \cdots |P_{t-2}|\,|P_{t-1} P_t| \\
&= |A| \cdot |P_1 P_2 \cdots P_{t-1} P_t| = |A|\,|B|
\end{aligned}$$

若 $B$ 不可逆时：$BX = 0$ 有非零解

$BX = 0 \Longrightarrow ABX = A0 = 0$

即 $BX=0$ 的解一定是 $ABX=0$ 的解

即 $ABX=0$ 有非零解 $\quad AB$ 不可逆 $\Rightarrow |AB| = 0$

此时 $\quad |AB| = |A|\,|B| = 0$

例1: $|A| = 5$ 求 $|A^{-1}| =$

$$AA^{-1} = I$$

两边取行列式 $|AA^{-1}| = 1$

$$|A|\cdot|A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{5}$$

Note: 若 A 可逆. 则 $|A^{-1}| = \frac{1}{|A|}$.

例2: 证明: $|A^*| = |A|^{n-1}$.

证明. ① 当 A 可逆时 $AA^* = |A|I$

两边取行列式 $|AA^*| = \big||A|I\big| = |A|^n$

$$|A|\cdot|A^*| = |A|^n$$

$$|A^*| = |A|^{n-1}.$$

② 当 A 不可逆时. $|A| = 0$. 下证 $A^*X = 0$ 有非零解

~~若A非零~~ $A^*A = |A|I = O$

$$A^*\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 \cdots & 0 \\ 0 & 0 \cdots & 0 \\ \vdots & & \\ 0 & 0 \cdots & 0 \end{bmatrix}$$

若 $A = O$. 则 $A^* = O$

从而 $|A^*| = 0$

$|A| = 0$

$|A^*| = |A|^{n-1}$

$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$ 均为 $A^*X=0$ 之解

又由于 A 非零矩阵 故 $A^*X=0$ 存在非零解

故 $A^*$ 不可逆 $|A^*|=0$ 也满足

$$|A^*| = |A|^{n-1} = 0$$

~~例3~~：$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ 求 $|A^{-1}+A^*|$

解：$A^{-1} = \dfrac{A^*}{|A|}$ $\quad |A|=-4$

故 $A^{-1} = \dfrac{A^*}{-4}$

$$|A^{-1}+A^*| = \left|-\frac{1}{4}A^* + A^*\right|$$
$$= \left|\frac{3}{4}A^*\right| = \left(\frac{3}{4}\right)^3 |A^*|$$
$$= \left(\frac{3}{4}\right)^3 \times |A|^2 = \frac{3^3}{4^3} \times (-4)^2 = \frac{27}{4}.$$

例4：$A, B \in M_3(K)$

$|A|=2$. $|B|=3$

求 ① $\left|\left(-\frac{1}{2}\right)A^{-1} - 2A^*\right|$ ② 求 $|2A^*B^{-1}|$

$= 8|A^*||B^{-1}|$

$= 8 \cdot 4 \times \frac{1}{3} = \frac{32}{3}$

$$A^{-1} = \frac{A^*}{|A|} = \frac{A^*}{2} \qquad A^* = 2A^{-1}$$

$$\left|\left(-\tfrac{1}{2}\right)A^{-1} - 2A^*\right| = \left|\left(-\tfrac{1}{2}\right)A^{-1} - 4A^{-1}\right|$$

$$= \left|\left(-\tfrac{9}{2}\right)A^{-1}\right| = \left(-\tfrac{9}{2}\right)^3 \frac{1}{|A|}$$

$$= \frac{-9^3}{2^4} = -\frac{729}{16}.$$

例5: $\alpha_1, \alpha_2, \alpha_3$ 是3维列向量. $|\alpha_1, \alpha_2, \alpha_3| = 1$

求 $|\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3| = ?$

解: 所求 $= |\alpha_1,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3|$

$$+ |\alpha_2,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3|$$

$$+ |\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3|$$

$$= |\alpha_1,\ 2\alpha_2+4\alpha_3,\ 3\alpha_2+9\alpha_3|$$

$$+ |\alpha_2,\ \alpha_1+4\alpha_3,\ \alpha_1+9\alpha_3|$$

$$+ |\alpha_3,\ \alpha_1+2\alpha_2,\ \alpha_1+3\alpha_2|$$

$$= |\alpha_1,\ 2\alpha_2,\ 3\alpha_2+9\alpha_3| + |\alpha_1,\ 4\alpha_3,\ 3\alpha_2+9\alpha_3|$$

$$+ |\alpha_2,\ \alpha_1,\ \alpha_1+9\alpha_3| + |\alpha_2,\ 4\alpha_3,\ \alpha_1+9\alpha_3|$$

$$+ |\alpha_3,\ \alpha_1,\ \alpha_1+3\alpha_2| + |\alpha_3,\ 2\alpha_2,\ \alpha_1+3\alpha_2|$$

$$= 18|\alpha_1, \alpha_2, \alpha_3| - 12|\alpha_1, \alpha_2, \alpha_3| - 9|\alpha_1, \alpha_2, \alpha_3|$$

$$+ 4|\alpha_1, \alpha_2, \alpha_3| + 3|\alpha_1, \alpha_2, \alpha_3| - 2|\alpha_1, \alpha_2, \alpha_3|$$

$$= 2.$$

法2: (快!)

$$(\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3)$$

$$= [\alpha_1, \alpha_2, \alpha_3]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$|\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3|$$

$$= \left| [\alpha_1, \alpha_2, \alpha_3]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \right| = \underbrace{\cancel{|\alpha_1, \alpha_2, \alpha_3|}}_{1} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= (2-1)(3-1)(3-2)$$

$$= 2$$