秩： $A \in M_{m,n}(\mathbb{R})$

Note Title 2017/11/24

A 存在一个 r 阶子式不为零

所有 r+1 阶子式均为零

则称 A 之秩为 r.

① $r(A) \le \min\{m, n\}$

② $A = 0 \iff r(A) = 0$

③ $r(A) \le s \iff$ A 所有 s+1 阶子式均为零

$r(A) \ge s \iff$ A 存在一个 s 阶子式不为零

④. $r(A) = r(A^T)$

⑤ A 可逆 $\iff |A| \ne 0 \iff AX = 0$ 只有零解

$\iff r(A) = n$

⑥ $r\begin{bmatrix} A & \\ & B \end{bmatrix} = r(A) + r(B)$

⑦ $r\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \ge r(A) + r(B)$

⑧ $r(A), r(B) \le r\begin{bmatrix} A \\ B \end{bmatrix} \le r(A) + r(B)$

⑨ $A \in M_{m,n}$ $B \in M_{n,s}$

$$r(A) + r(B) - n \le r(AB) \le \begin{matrix} r(B) \\ r(A), \end{matrix}$$

⑩ $r(A)=n$ 时. $r(AB)=r(B)$

$r(B)=n$ 时. $r(AB)=r(A)$

1) $AB=AC$ $r(A)=n$ 则消去律 $\Rightarrow$ $B=C$

$A(B-C)=0$ $r(B-C)=r(A(B-C))=r(O)=0$

$B-C=0 \Rightarrow B=C$.

2) $AX=0$ $r(A)=n$, 则消去律 $\Rightarrow$ $X=0$

$AX=0$ 齐次方程组

有唯一解 $\Leftrightarrow$ $r(A)=n$

有无穷多解 $\Leftrightarrow$ $r(A)<n$.

⑪ 若 $AB=O$ 时: $r(A)+r(B)\leq n$.

例: $A\in M_n(\mathbb{R})$.

$$r(A^*)=\begin{cases} n, & r(A)=n \\ 1 & r(A)=n-1 \\ 0 & r(A)\leq n-2 \end{cases}$$

1. 例1: $\begin{bmatrix} 2 & -3 & 1 \\ 1 & a & 1 \\ 5 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 & 1 \\ 0 & a+\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{15}{2} & \frac{1}{2} \end{bmatrix}$

$a+\frac{3}{2} = \frac{15}{2} \qquad a = \frac{12}{2} = 6$

3. $r(A) + r(B) \leq n \qquad 1$

5. $A_{3\times 2} \quad B_{2\times 3} \qquad r(AB) \leq r(A) \leq 2$

$(AB)_{3\times 3}$

$A = \underset{3\times 1}{\alpha}\, \beta_{1\times 3} \qquad r(A) = r(\alpha\beta) \leq r(\alpha) \leq 1$

7. $r(AB-B) = r((A-I)B) \qquad A-I = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & -1 \\ 2 & 4 & 4 \end{bmatrix}$

$= r(B)$

8. $r(A) + r(B) \leq 3$

$AB + B = (A+I)\,B$

12. $\begin{vmatrix} 1 & a & \cdots & a \\ a & 1 & a \cdots & a \\ \vdots & & \ddots & \vdots \\ a & a & a & 1 \end{vmatrix} = 0 \qquad = [1+(n-1)a] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & 1 & a \cdots & a \\ \vdots & & & \\ a & a \cdots & a \cdots & 1 \end{vmatrix}$

$= [1+(n-1)a] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & 1-a & & 0 \\ & 0 & 1-a & \\ & & & \ddots \; 1-a \end{vmatrix}$

$= (1+(n-1)a)(1-a)^{n-1}$

13. $r(AB) < r(A) \Rightarrow r(B) < 3$

$r(AB) < r(B) \Rightarrow r(A) < 3$

14 $\begin{vmatrix} 1 & 0 & k \\ k & 1 & 5 \\ 1 & 1 & -1 \end{vmatrix} = 0$.

1.

$$A\underline{\underline{(A^{-1}+B^{-1})}}B = (I + AB^{-1})B$$

$$= B + A = A + B$$

$$(A^{-1}+B^{-1}) = A^{-1}(A+B)B^{-1}.$$

$$(A^{-1}+B^{-1})^{-1} = B(A+B)^{-1}A$$

$$\frac{1}{\left(\frac{1}{a}+\frac{1}{b}\right)} = \frac{ab}{a+b}$$

~~$f(x)$ 满足 $f(0)=0$. $f'(0)=0$. 求 $\lim\limits_{x\to 0} \frac{f(\sin x)-\cdots}{x}$~~

$$|AB| = |A| \cdot |B|$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4. $$\big| |A| B \big| = |A|^n |B|$$

$$\big| |B| A \big| = |B|^n |A|$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^T = A.$$

$$(A^2)^T = (A \cdot A)^T = A^T \cdot A^T = A \cdot A = A^2$$

$$(A^*)^{-1} = \frac{A}{|A|}.$$

$$AA^* = A^* A = |A| I$$

$$\frac{A}{|A|} A^* = A^* \cdot \frac{A}{|A|} = I$$

$$(A^*)^{-1} = \frac{A}{|A|}$$

$$\left((A^{-1})^T\right)^{-1} = \left((A^T)^{-1}\right)^{-1} = A^T.$$

$$\left((A^T)^{-1}\right)^T = \left((A^{-1})^T\right)^{-1} = A^{-1}$$

8.

$A+B=(\alpha_1+\beta_1,\ 2\alpha_2,\ 2\alpha_3,\ 2\alpha_4)$

$|A+B| = 8\,|\alpha_1+\beta_1,\ \alpha_2,\ \alpha_3,\ \alpha_4|$

$A\cdot A = I \qquad A^{-1}=A$

$AA^{*}=A^{*}A=|A|\,I$

$|A\cdot A| = 1 \qquad |A|=\pm 1$

$A^{-1}=\dfrac{A^{*}}{|A|}$

$|A|\cdot|A|=1$

$\text{diag}\ A = \dfrac{A^{*}}{|A|}$

$A^2 - I = 0$

$(A-I)(A+I)=0$

$|A-I|\cdot|A+I|=0$

10. $(AB)^2=I \qquad AB\cdot AB = I$

$\underline{AB}AB = I \qquad B^{-1}=ABA$

$B^{-1}A^{-1}=(AB)^{-1}=AB$

$$ABAB = I$$

$$BAB \cdot A = I \qquad (BA)^2 = I.$$

$$AB = BC = CA = I$$

$$A = B^{-1} = C \qquad A^2 = I$$

$$A_{41} + A_{42} + A_{43} + A_{44} = \begin{vmatrix} 3 & 7 & 9 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 4 & 9 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$r(A^TA)_{4\times4} \le r(A) \le 3$$

$$A^T = A \qquad a_{ij} = a_{ji} \qquad a_{ij} \in \mathbb{R}$$

$$A^2 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = O$$

$$a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 = 0$$

$$A^T A = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = O$$

23. $AX = 0 \implies A^T A X = A^T 0 = 0$

$A^T A X = 0 \implies X^T A^T A X = 0$

$$(AX)^T AX = 0$$

$$AX = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad [x_1 \ x_2 \ \cdots \ x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \implies x_1 = x_2 = \cdots = x_n = 0.$$

$$AX = 0$$

补充：① A 是 n 阶方阵 $A^2=A$.

证明： $r(A)+r(A-I)=n$

证明： $A^2-A=0$　　$A(A-I)=0$

$$r(A)+r(A-I)\le n$$

$$\underbrace{r((A-I)-A)}_{=r(-I)=n}\le r(A-I)+r(A)$$

$r(A-B)\le r(A)+r(B)$

② $A.B$ 为 n 阶方阵 ∵ $AB=2A-B$

求证： $AB=BA$.

$$AB-2A+B=0$$

$$(A+I)(B-2I)=\underbrace{AB-2A+B}-2I$$

$$=-2I$$

$$(A+I)\frac{(B-2I)}{-2}=I$$

$$(A+I)^{-1} = \frac{B-2I}{-2}$$

$$\frac{B-2I}{-2} \cdot (A+I) = I$$

$$(B-2I)(A+I) = -2I$$

$$BA - 2A + B - 2I = -2I$$

$$BA = 2A - B$$

ts $AB = BA$