Quantitative International Economics

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Weeks 9 and 10 - Dynamic GE Models - Part 2
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Outline

- Dynamic Models
- We will cover a series of tools to bring dynamic considerations
  - Path dependence
  - Forward looking agents

Class of Models

- Spatial Development (DONE)
  - Desmet, Nagy and Rossi-Hansberg (2018)
- Migration (TODAY)
  - Caliendo, Parro and Dvorkin (2019), Artuc, Chaudhuri, and Mclaren (2010), Allen and Donaldson (2024)
  - Pellegrina and Sotelo (2024)

Allen and Donaldson (2024)

Questions

- What does explain the high concentration of economic activity?
- How persistent are economic shocks? In other words, how fast does it take to reach the steady state equilibrium?
- Doe temporary shocks have permanent effects? Do we have path dependence? In other words, do we have multiple steady state equilibria?
- This paper develops a new framework to address these questions in the context of the US

A Model of Persistence and Path Dependence

- Counties indexed by i
- Time is discrete t
- World habited by forward-looking dynastic families
  - Each family lives 2 periods
  - 1st period $\Rightarrow$ a child is born
  - 2nd period $\Rightarrow$ child becomes and adult, migrates, receives an idiosyncratic preference for living in a region, works, consumes, gives birth to a child and dies
- Trade and production $\Rightarrow$ Armington

Productivity and Amenity

- Every location produces with TFP

$$A_{it} = \bar{A}_{it} L_{it}^{\alpha_1} L_{it-1}^{\alpha_2}$$

  - α_1 is the static agglomeration
  - α_2 is historical agglomeration

- Every location has an amenity term given by

$$u_{it} = \bar{u}_{it} L_{it}^{\beta_1} L_{it-1}^{\beta_2}$$

  - β_1 is the static agglomeration
  - β_2 is the historical agglomeration

Trade Flows

- Trade flows

$$X_{ijt} = \left(\frac{\tau_{ijt} w_{it} / A_{it}}{P_{jt}} \right)^{1-\sigma} w_{jt} L_{jt}$$

where

$$P_{jt} = \left(\sum_k \left(\tau_{kit} \frac{w_{kt}}{A_{kt}} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Migration

- Value for an adult from living in i at time t

$$\tilde{V}_{it} = \log W_{it} + \delta \mathbb{E}\left[\max_{j}\left\{\tilde{V}_{j,t+1} - \tilde{\mu}_{ij,t+1} + \varepsilon_{ij,t+1}\right\}\right]$$

  - $W_{it} = u_{it}\dfrac{w_{it}}{P_{it}}$
  - $\tilde{\mu}_{ij,t+1}$ is fixed cost of migrating from i to j
  - $\varepsilon_{ij,t+1}$ is an idiosyncratic extreme value (Gumbel) with shape paremeter θ

- That gives (Caliendo, Parro and Dvorkin, 2019; Artuc, Chaudhuri, and Mclaren, 2010)

$$\tilde{V}_{it} = \log W_{it} + \frac{\delta}{\theta}\log\left\{\sum_{k}\exp\left(\theta\left(\tilde{V}_{kt+1} - \tilde{\mu}_{ikt+1}\right)\right)\right\}$$

Migration

- We can re-write

$$\tilde{V}_{it} = \log W_{it} + \frac{\delta}{\theta} \log \left\{ \sum_k \exp\left(\theta\left(\tilde{V}_{kt+1} - \tilde{\mu}_{ikt+1}\right)\right) \right\}$$

as

$$V_{it} = W_{it} \Pi_{i,t+1}^{\delta}$$

- where $\Pi_{i,t} \equiv \left(\sum_k \left(V_{kt+1}/\mu_{ikt+1}\right)^{\theta}\right)^{\frac{1}{\theta}}$ and $V_{it} \equiv \exp\left(\tilde{V}_{it}\right)$
- The value of living in a location i depends on present consumption W_{it} and future utility $\Pi_{i,t+1}$
  - If $\delta = 0$, individuals are myopic

Migration

- When an adult is choosing where to live, the value of moving from i to j is

$$\tilde{V}_{ijt} = \tilde{V}_{jt} - \tilde{\mu}_{ij,t} + \varepsilon_{ijt}$$

- They choose

$$\max_j \left\{\tilde{V}_{ijt}\right\} = \max_j \left\{\tilde{V}_{jt} - \tilde{\mu}_{ij,t} + \varepsilon_{ijt}\right\}$$

- The share of families moving from i to j is

$$\lambda_{ijt} = \frac{exp\left(\theta\left(\tilde{V}_{jt} - \tilde{\mu}_{ij,t}\right)\right)}{\sum_k exp\left(\theta\left(\tilde{V}_{jt} - \tilde{\mu}_{ij,t}\right)\right)}$$

$$\lambda_{ijt} = \frac{\left(V_{jt}/\mu_{ijt}\right)^{\theta}}{\left(\Pi_{i,t}\right)^{\theta}}$$

$$\lambda_{ijt} = \frac{\left(W_{it}/\mu_{ijt}\right)^{\theta}\left(\Pi^{\delta}_{i,t+1}\right)^{\theta}}{\left(\Pi_{i,t}\right)^{\theta}}$$

General Equilibrium

- Total sales equals total payments to labor

$$w_{it}L_{it} = X_{it} \tag{1}$$

- Trade flows

$$X_{ijt} = \sum_j \left(\frac{\tau_{ijt} w_{it}/A_{it}}{P_{jt}} \right)^{1-\sigma} X_{jt} \qquad P_{jt}^{1-\sigma} = \sum_k \left(\tau_{kit} \frac{w_{kt}}{A_{kt}} \right)^{1-\sigma} \tag{2}$$

- Population in a location equals the one arriving there $L_{it} = \sum_j L_{jit}$

$$L_{it} = \sum_j \left(\frac{V_{it}/\mu_{jit}}{\Pi_{jt}} \right)^{\theta} L_{jt-1}, \qquad \Pi_{jt}^{\theta} = \sum_j (V_{jt}/\mu_{ijt})^{\theta} \tag{3}$$

- Agents are forward looking

$$V_{it} = W_{it}\Pi_{i,t+1}^{\delta} \qquad W_{it} = u_{it}\frac{w_{it}}{P_{it}} \tag{4}$$

- Productivities and amenities are given by

$$A_{it} = \bar{A}_{it} L_{it}^{\alpha_1} L_{it-1}^{\alpha_2} \qquad u_{it} = \bar{u}_{it} L_{it}^{\beta_1} L_{it-1}^{\beta_2} \tag{5}$$

Definition: Given a sequence of productivity and amenity shifters $\{\bar{A}_{it}, \bar{u}_{it}\}$, trade costs and migration costs $\{\tau_{ijt}, \mu_{ijt}\}$, preferences $\{\sigma\}$, migration elasticity $\{\theta\}$, agglomeration parameters $\{\alpha_1, \alpha_2\}$, congestion parameters $\{\beta_1, \beta_2\}$, and a vector of initial conditions $\{L_{i0}\}$, an equilibrium is a sequence of wages $\{w_i\}$ that satisfies equations (1)-(5).

Steady State Equilibrium

- To think about path dependence, we look at the steady state.

- How does the equilibrium conditions look like in steady state?

Steady State Equilibrium

- Total sales equals total payments to labor
$$w_i L_i = X_i \tag{6}$$
- Trade flows
$$X_{ij} = \sum_j \left(\frac{\tau_{ij} w_i / A_i}{P_j} \right)^{1-\sigma} X_j \qquad P_j^{1-\sigma} = \sum_k \left(\tau_{ki} \frac{w_k}{A_k} \right)^{1-\sigma} \tag{7}$$
- Population in a location equals the one arriving there $L_i = \sum_j L_{ji}$
$$L_i = \sum_j \left(\frac{V_i / \mu_{ji}}{\Pi_j} \right)^{\theta} L_j, \qquad \Pi_j^{\theta} = \sum_j (V_j / \mu_{ij})^{\theta} \tag{8}$$
- Agents are forward looking
$$V_i = W_i \Pi_i^{\delta} \qquad W_i = u_i \frac{w_i}{P_i} \tag{9}$$
- Productivities and amenities are given by
$$A_i = \bar{A}_i L_i^{\alpha_1 + \alpha_2} \qquad u_i = \bar{u}_i L_i^{\beta_1 + \beta_2} \tag{10}$$

Definition: Given productivity and amenity shifters $\{\bar{A}_i, \bar{u}_i\}$, trade costs and migration costs $\{\tau_{ij}, \mu_{ij}\}$, preferences $\{\sigma\}$, migration elasticity $\{\theta\}$, agglomeration parameters $\{\alpha_1, \alpha_2\}$, and congestion parameters $\{\beta_1, \beta_2\}$, an equilibrium is a vector of wage $\{w_i\}$ and labor allocation $\{L_i\}$ that satisfies equations (6)-(10).

Steady State Equilibrium

- Notice what happened to the agglomeration and congestion forces. Before we had

$$A_{it} = \bar{A}_{it} L_{it}^{\alpha_1} L_{it-1}^{\alpha_2} \qquad u_{it} = \bar{u}_{it} L_{it}^{\beta_1} L_{it-1}^{\beta_2}$$

- Now

$$A_i = \bar{A}_i L_i^{\alpha_1+\alpha_2} \qquad u_i = \bar{u}_i L_i^{\beta_1+\beta_2}$$

- In the sequential equilibrium, L_{it-1} is given, so the existence of equilibrium in period t only depend on α_1 and β_1. In steady state, the existence of equilibria now depends also on α_2 and β_2.

An Algorithm to Solve for the Dynamic Equilibrium

- Set a grid of time $t = 1, ..., T$ – sufficiently large so that the economy converges to SS if there exists one
- Step 1: Solve for the steady state, recover Π_i^{SS}. Set $\Pi_{i,T} = \Pi_i^{SS}$
- Step 2: Guess a sequence of wages w_{it}^g, i.e., a guess of wage for every $i = 1, ..., I$ and $t = 1, ..., T$
- Step 3: Using backward induction, solve for the sequence of V_{it}, using the final condition $\Pi_{i,T}$
- Step 4: Using forward induction, solve for the sequence of L_{it}, using the initial condition L_{i0}
- Step 5: Solve the static equilibrium for every period given L_{it}, recover equilibrium wage w_{it}
  - If $\max_{it} \{w_{it} - w_{it}^g\} < \epsilon$ then stop algorithm
  - Otherwise, update w_{it} and go back to step 3
    - Increase w_{it} if $w_{it} - w_{it}^g > 0$
    - Decrease w_{it} if $w_{it} - w_{it}^g < 0$

Calibration

- Outer-loop that ensures that shifters are such that the model-implied value match the data
  - Share of workers in every location in every period
  - Share of gross output produced by every location in every period

Condition for Existence and Uniqueness

- In the paper, they provide several theorems that can be used to study the properties of the equilibrium

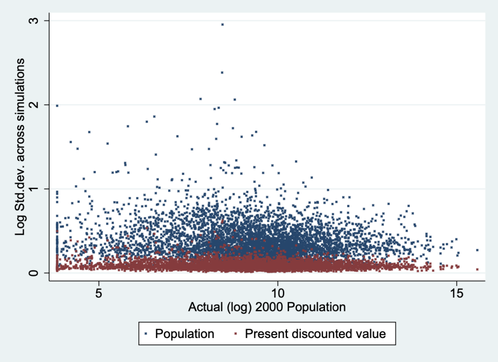
- We can re-write all the equations as a general dynamic system of nonlinear equations

$$x_{i,h,t} = \sum_{j=1}^{N} K_{ij,h,t} \prod_{h'=1}^{H} \left(x_{j,h',t}\right)^{\varepsilon_{h,h'}^{j,t}} \left(x_{j,h',t+1}\right)^{\varepsilon_{h,h'}^{j,t+1}} \left(x_{i,h',t+1}\right)^{\varepsilon_{h,h'}^{i,t+1}} \left(x_{j,h',t-1}\right)^{\varepsilon_{h,h'}^{j,t-1}} \left(x_{i,h',t-1}\right)^{\varepsilon_{h,h'}^{i,t-1}},$$

- Spectral radius of the system becomes a key element in this analysis — computed based on the exponents of the right hand side

Results

Figure 4: How resilient are locations to historical shocks?

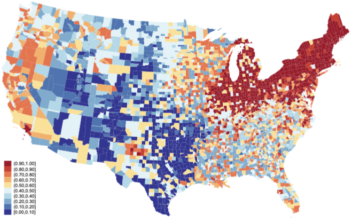


Notes: This figure plots the standard deviation of log population $\ln L_{i,2000}^{(b)}$ (and log present discounted value $\ln V_{i,2000}^{(b)}$) in the year 2000, across 100 different simulations b of alternative historical conditions, against each location's actual year 2000 log population $\ln L_{i,2000}$. Each simulated history randomly shuffles the realized exogenous productivity $\overline{A}_{i,1900}^{(b)}$ in the year 1900 between all pairs of locations, where pairs are assigned to locations with the closest 1900 populations.

Results

Figure 6: How lucky were different locations?

(a) Population



Notes: This figure compares the actual distribution of economic activity in the year 2000 to 100 different simulations of alternative historical conditions. Each simulated history b randomly shuffles the realized exogenous productivity $\overline{A}_{i,1900}^{(b)}$ in the year 1900 between all pairs of locations, where pairs are assigned to locations with the closest 1900 populations. For each simulated history, we calculate the rank of each location (in terms of its population in panel (a) or PDV in panel (b)) relative to all other locations. The two panels of the figure show the fraction of simulated histories for which each location exceeds that rank in its actual history.

Results

- In their words:

  A particularly rich region of the model's parameter space—and one that our application to the U.S. from 1800 onwards suggests is very much a possibility—is where equilibria are unique and easy to solve for, persistence lasts many centuries, and minor perturbations in historical conditions can lead the economy towards distinct steady-states with substantial differences in overall efficiency.

Taking Stock

- We have covered two types of dynamic spatial equilibrium models
  - Without anticipatory effects
  - With anticipatory effects
- Next two classes we will work on the computation of these models and one application (mine!)