ENGR 6141: Nonlinear Systems Project

A Summary and Application of Local Stabilization of Switched Affine Systems [1]

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Abstract

This report presents a summary and an application of the publication Local Stabilization of Switched Affine Systems [1]. The work presented in [1] demonstrates the ability to develop a locally asymptotic stabilization control law for switched affine systems using nonlinear stabilization techniques. The method makes use of convex combinations of Filippov set-value maps, Lyapunov nonlinear stability theory, and sliding dynamics for the design of state dependent switching laws to guarantee local asymptotic stability of switched affine systems.

This paper is written in fulfillment of Concordia University's ENGR 6141: Nonlinear Systems course, whereby [1] was chosen as the topic of the project. This paper first presents the importance of [1], which is the first development of state dependent switching laws specifically designed for local asymptotic stability of switched affine systems. Following this, an in-depth summary of [1] is presented with necessary supporting materials. Finally, the theory from [1] is applied to two different systems to demonstrate the ability to design a stabilization control law for the reformulated nonlinear system, and to demonstrate the development of switching laws based on various Lyapunov functions.

Contents

1 INTRODUCTION

Switched linear systems are a category of hybrid systems. They have a switching rule that selects through an available list of subsystems in order to achieve global asymptotic stability. An affine system is a system that is linear in the input u, $x = f(x) + G(x)u$. A switched affine system is therefore a system of the form $x = A_\sigma x + b_\sigma$, where $x(t) \in \mathbb{R}^n$ is the state, $\sigma(\cdot)$ is the switching strategy and $u(t) \in \mathbb{R}^m$ is the input, linearly related to b_σ. Not all switched affine systems can be stabilized globally. Until the paper of [1], if a switched affine system could not be stabilized globally, it could not be stabilized at all.

In their work [1], they show a way in which they stabilize a switched affine system in a region close to the origin even if the system cannot be stabilized globally. They show that if there exists a classical continuous feedback $k^c(x)$that renders a switched affine system locally or globally stable, then there also exists a local discontinuous stabilizer $k(x)$ that selects subsystems from a set and renders the affine system locally stable.

In this report, the theory developed in [1] will be applied to control a power converter. These devices are widely used in power electronics. They are circuits controlled by transistors and diodes to adjust the electrical energy of a power source to meet the requirements of a load. They are controlled through Pulse Width Modulation technique and therefore these devices are good candidates to be modeled by non-linear switched systems.

The report gathers and analyses data from the literature in the Literature Review section 2, gives further details about the problem in the Problem Formulation section 3, further expands the problem statement in section 4 and presents analysis and simulation results in section 5, respectively.

2 LITERATURE REVIEW

Stabilization of switched affine systems: An application to the buck-boost converter
In their paper, authors Corona, Buisson, Schutter and Giua present a technique to design an asymptotic switching law to stabilize switched systems. It is called the Switched Table Procedure (STP) and it consists of 3 steps. In the first step, it is decided if a switch from the current dynamics should occur based on some cost function. The second step states that there exists a maximum number of switches after which there is limited improvement that can be achieved. The third step obtains an augmented system by adding a stable dummy dynamic, and so if the solution of the optimal control problem for the augmented system does not contain the label associated to the dummy dynamics, the table found is also a solution of the initial system.

As an example, the STP is used to derive a switching law for the Buck-Boost converter, successfully. When compared to deriving a switching law from a unique Lyapunov function derived from the converter model, the STP stabilizes the system at a reduced cost.

Switched Affine Systems Control Design with Application to DC DC Converters
A paper that is mentioned multiple times when searching for stabilization of switched affine systems is that of Deaecto, Geromel, Garcia and Pomilio. Their goal is to calculate the set of attainable equilibrium points that can be reached from any initial condition. In order to do so, they design a switching rule that can take any trajectory of the system to a desired point inside a set by minimizing a cost function.

The switched systems presented are of the form (1)

$$\begin{aligned} \dot{x} &= A_\sigma x + b_\sigma u \qquad x(0) = x_0 \\ z &= C_\sigma x \end{aligned} \tag{1}$$

where $x(t)$ is the state, $u(t) = u$ is the input, constant for all time, $z(t)$ is the controlled output and $\sigma(\cdot)$ is the switching strategy.

They provide two theorems to find switching laws but only one is simple enough to be implemented in practice because it has a linear switching function.
Theorem 2: Consider the switched affine system (1) with constant input $u(t) = u$ for all $t \geq 0$ and let $x_e \in \mathbb{R}^n$ be given. If there exists $\lambda \in \Lambda$, and a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such

that

$$A_i^T P + PA_i + Q_i < 0 \tag{2}$$

$$A_\lambda x_e + B_\lambda u = 0 \tag{3}$$

for all $i \in \mathbb{K}$, then the switching strategy

$$\sigma(x) = \texttt{arg}\min_{i\in\mathbb{K}} \xi P(A_i x_e + B_i u) \tag{4}$$

where $\xi = x - x_e$ makes the equilibrium point $x_e \in \mathbb{R}^n$ globally asymptotically stable and the guaranteed cost (5) holds.

$$\int_0^\infty (z - C_\sigma x_e)^T (z - C_\sigma x_e) dt < (x_0 - x_e)^T P (x_0 - x_e) \tag{5}$$

The paper goes on to apply this theorem to Buck, Boost and Buck-Boost converters. For each of the converters, the state trajectories start from the 0 initial condition and reach equilibrium points, thus successfully applying the switching strategy of Theorem 2 to power systems.

On Control Design of Switched Affine Systems with Application to DC-DC Converters Mainardi Junior, Teixeira, Cardim, Moreira, Assunacao and Yoshimura present a theory that constructs a switching rule for switched affine systems to achieves asymptotic stability for a known equilibrium point. It applies to switched affine systems with constant input and given equilibrium point x_r. If there exists $\lambda \in \Lambda$, symmetric matrices N_i, $i \in K$ and a symmetric positive definite matrix $P \in R^{n\times n}$ such that

$$A_i^T P + PA_i + Q_i - N_i < 0$$
$$A_\lambda x_r + B_\lambda w = 0$$
$$N_\lambda = 0$$

For all $i \in K$, where $Q_i = Q_i^T$, then the switching strategy.

$$\sigma(x) = \texttt{arg}\min_{i\in\mathbb{K}} \xi^T (N_i \xi + 2P(A_i x_r + B_i w))$$

where $\xi = x - x_r$, makes the equilibrium point x_r globally asymptotically stable and from

$$J = \int_0^\infty (y - C_\sigma x_r)^T (y - C_\sigma x_r) dt < (x_0 - x_r)^T P (x_0 - x_r)$$

the guaranteed cost function

$$J < (x_0 - x_r)^T P (x_0 - x_r)$$

holds.

The theory is then applied to buck, boost, buck-boost and sepic DC-DC converters. For every power system the matrices P and Ni are calculated and a switching law is found, thus stabilizing the system within reasonable time and cost.

3 PROBLEM FORMULATION

Consider a switched affine system defined in a general sense as a characterization of N vector fields as shown in (6).

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma \qquad \sigma \in \{1, ..., N\} \tag{6}$$

The switched affine system can therefore be described as a family of functions from $\mathbb{R}^N$ to $\mathbb{R}^N$ where σ is an element of the finite index set of modes of the system. This gives rise to a family of systems as described in (7).

$$\dot{x}(t) = f_\sigma(x(t)) \qquad \sigma \in \{1, ..., N\} \tag{7}$$

The problem addressed in [1] is to determine a state dependent switching law that will guarantee local asymptotic stability of the switched affine system around an equilibrium point. Furthermore, the authors of [1] claim that prior to their publication, existing solutions for the development of switching laws for switched affine systems were only applicable to systems that could be made globally asymptotically stable around an equilibrium point. Therefore, The work presented in [1] provides a solution for the development of switching laws for switched affine systems that can be made locally asymptotically stable.

3.1 Equilibrium Points

The first step of system stability analysis is to determine the equilibrium points of the system. Let I_N be the set $\{1, ..., N\}$, where the switched affine system is composed of $N \geq 2$ modes. Also, let the family of systems as defined in (7) be described by $\tilde{A}_i \in \mathbb{R}^{n \times n}$ and $\tilde{b}_i \in \mathbb{R}^n$, where the system has $n \geq 1$ states. Now, let us consider the switched affine system described by (8)

$$\dot{z}(t) = \tilde{X}(z(t)) = \tilde{A}_{s(z(t))} z(t) + \tilde{b}_{s(z(t))} \tag{8}$$

where $z(t)$ are the time-variant states of the system, and $s : \mathbb{R}^N \to I_N$ is the measurable state dependent switching law that guarantees local asymptotic stability of the switched affine system around an equilibrium point. If $\tilde{X}(z(t))$ is a locally bounded discontinuous vector field describing the switched affine system, a Filippov set-valued map of the system can be defined as shown in (9)

$$F[\tilde{X}](z) = \cap_{\epsilon > 0} \cap_{\mu(S)=0} \overline{conv} \left\{ \tilde{X}(\overline{z}) : \overline{z} \in B(z, \epsilon) \setminus S \right\} \tag{9}$$

where the specifics of Filippov solutions are described in more detail in APPENDIX III: FILIPPOV SOLUTIONS, and set-value mapping is described in APPENDIX I: SET-VALUE MAPPING. Since $\tilde{X}(z(t))$ is locally bounded and defined for all $z(t)$ locally, $F[\tilde{X}](z)$ is also locally bounded, non-empty, and compact. Furthermore, since $F[\tilde{X}](z)$ is comprised of the convex hull set of the state velocities around $z(t)$, it too is convex. Finally, since the switching law is state dependent, it is natural that the states must be measurable (and hence $F[\tilde{X}](z)$ is also measurable for all states locally), and the switching law will be designed such that it is upper semi-continuous. As described in APPENDIX III: FILIPPOV SOLUTIONS, the conditions are therefore met for the existence of at least one Filippov solution in the set-valued map $F[\tilde{X}](z)$.

Finally, a point $z^* \in \mathbb{R}^n$ is an equilibrium point of the differential inclusion defined by (9), and hence the system described by (8), if and only if (10) is satisfied.

$$\mathbf{0} \in F[\tilde{X}](z^*) \tag{10}$$

3.2 Local Asymptotic Stability

The definition for local stability and local asymptotic stability around the origin of a system described by (8) is very similar to that of nonlinear systems. A nonlinear system is said to be stable if for each $\epsilon > 0$ there exists some $\lambda > 0$ such that the condition defined in (11) is satisfied.

$$|z(0)| < \lambda \Rightarrow |z(t)| < \epsilon \;\; \forall \;\; t \in [0, \infty) \tag{11}$$

Furthermore, a system is said to be locally asymptotically stable if there exists some boundary $\eta \in (0, \infty]$ such that the condition defined in (12) is satisfied.

$$|z(0)| < \eta \Rightarrow \lim_{t \to \infty} |z(t)| = 0 \tag{12}$$

Also, a system is said to be globally asymptotically stable if (12) is satisfied for all $\eta \in (0, \infty]$.

Finally, the sufficient condition for local asymptotic stability of nonlinear systems can be applied to the notion of local asymptotic stability of discontinuous ordinary differential equations of the form (8) where $\tilde{X}(z(t))$ is locally bounded. The assessment is accomplished via the Lyapunov theory for stability of nonlinear systems, where a system is said to be locally asymptotically stable in some domain D if the conditions presented in (13) are satisfied.

$$\begin{array}{rclcrcl} V(0) & = & 0, & & V(z(t)) & > & 0; \;\; \forall z(t) \neq 0, z(t) \in D \\ \dot{V}(0) & = & 0, & & \dot{V}(z(t)) & < & 0; \;\; \forall z(t) \neq 0, z(t) \in D \end{array} \tag{13}$$

Since the Filippov set-valued map consists of all possible state velocities around the current states of the system, which are time-invariant velocities, the condition for the Lyapunov function $V(z(t))$

remains the same. However for the Lyapunov function's derivative the condition becomes as shown in (14) [2]

$$\sup_{y \in F[\tilde{X}](z(t))} \langle \nabla V(z(t)), y \rangle \quad < \quad 0; \quad \forall z(t) \in B(0, \gamma) \setminus \{0\} \tag{14}$$

where γ is some positive scalar. This is due to the fact that the Filippov set-value map contains a set of velocities, as opposed to a singleton value, and hence all elements of that set must be evaluated against the Lyapunov criteria. Therefore, if the supremum of the gradient multiplied with the velocity vectors is negative, then all other elements of the Filippov set-value mapping multiplied with the gradient must therefore be negative - and therefore the Lyapunov criteria for the derivative of $V(x(t))$ is satisfied.

4 ANALYSIS

This section will elaborate on the problem formulation described in Section 3 to define the local asymptotic stabilization solution for switched affine systems in [1]. The methodology applied is to define a nonlinear affine system as described in (15).

$$\dot{x} = f(x) + G(x)u(x) \tag{15}$$

4.1 Equilibrium Points

As was detailed in Section 3: Equilibrium Points, there exists at least one Filippov solution to (9) for a system described by (8), and the solution is an equilibrium point of the system if and only if (10) is satisfied.

A *k-simplex* is a K-dimensional polytope composed of the closed convex hull set of its $K+1$ vertices. The convex hull set of the $K+1$ affinely independent points $v_0, v_1, ..., v_k \in \mathbb{R}^N$ are linearly independent, which means that the simplex defined by those points is as defined in (16) [3].

$$C = \left\{ \delta_0 v_0 + ... + \delta_N v_N \,\middle|\, \sum_{i=0}^{N} \delta_i = 1,\ \delta_i \geq 0 \forall i \in \{0, ..., N\} \right\} \tag{16}$$

This means that each element of a convex hull can be related to the simplex. Since the Filippov set-valued map $F[X](z(t))$ is a convex hull set of the velocities around $z(t)$ for the system described by (8), if $\mathbf{0} \in F[\tilde{X}](z^*)$ then (17) must be true

$$\sum_{i=1}^{N} (\tilde{A}_i z^* + \tilde{b}_i)\delta_i^* = 0 \tag{17}$$

where δ_i^* is a barycentric coordinate of an element of the simplex of the convex hull set.

A coordinate transformation can then be applied to (8) according to $x(t) = z(t) - z^*$ to study the stabilization for $x(t)$ around the equilibrium point $x^* = 0$. Since z^* is simply a scalar equilibrium point for $\dot{z}(t)$, we can declare that $\dot{x}(t) \equiv \dot{z}(t)$ from the transformation, and we can declare that the switching law in $z(t)$ will be equivalent to the switching law in $x(t)$ such that $\sigma(x(t)) \equiv s(z(t))$. This then allows for defining $\dot{x}(t)$ according to (18)

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_{\sigma(x(t))}(x(t) + z^*) + \tilde{b}_{\sigma(x(t))} \\ &= \tilde{A}_{\sigma(x(t))}x(t) + \tilde{A}_{\sigma(x(t))}z^* + \tilde{b}_{\sigma(x(t))} \\ &= A_{\sigma(x(t))}x(t) + b_{\sigma(x(t))} \end{aligned} \tag{18}$$

where we have made the simplifications as defined in (19).

$$\begin{aligned} A_{\sigma(x(t))} &= \tilde{A}_{\sigma(x(t))} \\ b_{\sigma(x(t))} &= \tilde{A}_{\sigma(x(t))} z^* + \tilde{b}_{\sigma(x(t))} \end{aligned} \tag{19}$$

Furthermore, using the transformation and simplification as defined in (18) and (19), (17) can be redefined according to $x(t)$ as shown in (20)

$$\sum_{i=1}^{N} (\tilde{A}_i z^* + \tilde{b}_i)\delta_i^* = \sum_{i=1}^{N} b_i \delta_i^* = 0 \tag{20}$$

which is a necessary condition for $\sigma(x(t))$ such that $0 \in F[X](0)$. In simpler terms, this is the necessary condition such that the velocity of the vector field at the origin is 0.

4.2 System Reformulation

Consider the system described by (21)

$$\dot{x}(t) = X(x(t)) = A_{\sigma(x(t))} x(t) + b_{\sigma(x(t))} \tag{21}$$

which can be reformulated as shown in (22)

$$\begin{aligned} &\dot{x}(t) = \sum_{i=1}^{N} (A_i x(t) + b_i)\varphi_i(x(t)) \\ &\varphi(x(t)) = \psi_{\sigma(x(t))} \end{aligned} \tag{22}$$

where $\varphi(x(t))$ is the control law that determines which system mode is active, and $\psi_{\sigma(x(t))}$ is one of the vertices of the unit simplex. Note that, at a vertex of the unit simplex, all $\delta_i = 0$ except for one, which means that the control law will only allow for one active system mode at any given time.

The system's control law can then be translated by δ^* and have sliding dynamics applied to project it from $\mathbb{R}^N$ to an $\mathbb{R}^{N-1}$ plane, which arrives at a new control law $\tilde{\varphi}(x(t)) = \varphi(x(t)) - \delta^*$. This redefined controller is designed for controlling the error with respect to δ^* as shown in (23).

$$\begin{aligned} &\tilde{\varphi}_j(x(t)) = \varphi_j(x(t)) - \delta_j^* \qquad \forall j \in \{1, ..., N-1\} \\ &\tilde{\varphi}_N(x(t)) = -\sum_{i=1}^{N-1} \tilde{\varphi}_i(x(t)) \end{aligned} \tag{23}$$

Therefore, the reformulated system with the control law defined in (22) can be redefined as shown in (24).

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{N} (A_i x(t) + b_i)\varphi_i(x(t)) \\
&= \sum_{i=1}^{N} (A_i x(t) + b_i)(\tilde{\varphi}_i(x(t)) + \delta_i^*) \\
&= \sum_{i=1}^{N} A_i\delta_i^* x(t) + \sum_{i=1}^{N} b_i\delta_i^* + \sum_{i=1}^{N-1} (A_i x(t) + b_i)\tilde{\varphi}_i(x(t)) - (A_N x(t) + b_N) \sum_{i=1}^{N-1} \tilde{\varphi}_i(x(t)) \\
&= \sum_{i=1}^{N} A_i\delta_i^* x(t) + \sum_{i=1}^{N} b_i\delta_i^* + \sum_{i=1}^{N-1} \left((A_i - A_N)\, x(t) + (b_i - b_N) \right) \tilde{\varphi}_i(x(t))
\end{aligned}
\tag{24}
$$

Finally, since $\sum_{i=1}^{n} b_i\delta_i^* = 0$ (as shown in the previous section as a necessary condition for $x(t) = 0$ to be an equilibrium point of the system), the system is simplified according to (25)

$$
\dot{x}(t) = \sum_{i=1}^{N} A_i\delta_i^* x(t) + \sum_{i=1}^{N-1} \left((A_i - A_N)\, x(t) + (b_i - b_N) \right) \tilde{\varphi}_i(x(t)) \tag{25}
$$

which is equivalent to the nonlinear system

$$
\dot{x}(t) = f(x(t)) + G(x(t))u(x(t)) \qquad u(x(t)) \in \mathbb{R}^{N-1} \tag{26}
$$

where $f(x(t))$ and $G(x(t))$ are defined according to (27).

$$
\begin{aligned}
f(x(t)) &= \sum_{i=1}^{N} A_i\delta_i^* \\
G(x(t)) &= [g_1(x(t)), \ldots, g_{N-1}(x(t))] \\
g_i(x(t)) &= (A_i - A_N)\, x(t) + (b_i - b_N)
\end{aligned}
\tag{27}
$$

Finally, the control law can then be defined as shown in (28)

$$
u_j(x(t)) \quad = [I_{N-1\times N-1} 0_{N-1\times 1}]\tilde{\varphi}_j(x(t)) \quad \forall j \in \{1, \ldots, N\} \tag{28}
$$

where $\tilde{\varphi}_j(x(t))$ is as defined in (23).

4.3 Proof of Existence of Local Stabilization Controller Design

By Kurzweil's Converse Theorem, any locally asymptotically stable system with a continuous right-hand side has a C^∞ strict Lyapunov function [4]. The objective of this section is therefore to demonstrate that it is possible to develop a continuous locally asymptotic stabilization control law $u(x(t))$.

Since we require a continuous control law for local asymptotic stability, we can declare that the control law $u(x)$ is constrained to the set of finite vectors $V(\delta^*) = \{v_1, ..., v_N\}$. We will also note that $0 \in \overline{conv}\{V(\delta^*)\}$, which implies that the system may have no input applied to it by the controller, and hence (29) can be defined according to this circumstance.

$$\sum_{i=1}^{N} \delta_i^* v_i = 0 \tag{29}$$

Furthermore, let $e_j, j \in \{1, ..., N-1\}$ be the standard basis of the vector space $V(\delta^*)$, which can be defined as shown in (30)

$$e_j = M\psi_j \qquad j \in \{1, ..., N-1\} \tag{30}$$

where $M = [I_{N-1\times N-1} 0_{N-1\times 1}]$. Since $v_j = M(\psi_j - \delta^*), j \in \{1, ..., N\}$ from (28), (23), and (22), (29) can be redefined as shown in (31).

$$\begin{aligned} \sum_{i=1}^{N} \delta_i^* v_i &= \sum_{i=1}^{N-1} \delta_i^* v_i + \delta_N^* v_N \\ &= \sum_{i=1}^{N-1} \delta_i^* v_i - \delta_N^* M\delta^* \end{aligned} \tag{31}$$

Equating (31) to 0 as shown in (29), (32) can be defined.

$$M\delta^* = \left(\delta_N^*\right)^{-1} \sum_{i=1}^{N-1} \delta_N^* v_i \tag{32}$$

A relationship between the standard basis and the vector space can then be defined according to (33).

$$\begin{aligned} \delta_N^* e_j &= \delta_N^* M\psi_j \\ &= \delta_N^* v_j + \delta_N^* M\delta^* \\ &= \delta_N^* v_j + \sum_{i=1}^{N-1} \delta_i^* v_i \\ &= \left(\delta_N^* + \delta_j^*\right) v_j + \sum_{\substack{i=1 \\ i\neq j}}^{N-1} \delta_i^* v_i \end{aligned} \tag{33}$$

Therefore, the standard basis of the vector space $V(\delta^*)$ is a conic combination of $v_i, i \in \{1, ..., N-1\}$. Since δ^* is an element of the simplex of the convex hull set, if $\beta^+ = \delta_N > 0$ then (34) can be declared true.

$$\beta^+ e_j \in \overline{conv}\{V(\delta^*)\} \quad \forall j \in \{1, ..., N-1\} \tag{34}$$

Consequently, (35) can be declared.

$$-\beta^+ e_j = -\sum_{i=1}^{N-1} \lambda_i v_i \qquad \begin{array}{ll} \lambda_i = \delta_i^* & \forall i \neq j \\ \lambda_i = \delta_N^* + \delta_j^* & i = j \end{array} \tag{35}$$

Furthermore, using (29) and the notion that $\sum_{i=1}^{N} \delta_i^* v_i = \delta_i^* v_i + \sum_{\substack{k=1 \\ k\neq j}}^{N} \delta_j^* v_j$, (36) can be declared.

$$v_i = -\left(\delta_i^*\right)^{-1} \sum_{\substack{k=1 \\ k\neq j}}^{N} \delta_k^* v_k \tag{36}$$

which can finally be combined with (35) to arrive at (37).

$$-\beta^+ e_j = \sum_{i=1}^{N-1} \lambda_i \left(\delta_i^*\right)^{-1} \sum_{\substack{k=1 \\ k\neq j}}^{N} \delta_k^* v_k \tag{37}$$

Since $\beta^+ > 0$, and if $\delta_i^* > 0 \; \forall \; i \in \{1, ..., N\}$, then we can declare some function $\mu_i(\beta^+, \delta_I^*, \lambda_i) > 0$ such that (38) is always true.

$$-e_j = \sum_{i=1}^{N} \mu_i v_i \tag{38}$$

This derivation is sufficient for the determination that there exists a $\beta_j^- > 0$ and that $-\beta_j^- e_j \in \overline{conv}\left\{V(\delta^*)\right\}$ $\forall \, j \in \{1, ..., N-1\}$. Since $\beta^+ > 0$, and since $\beta_j^- > 0$, $\beta^+ e_j \in \overline{conv}\{V(\delta^*)\}$, and $-\beta_j^- e_j \in \overline{conv}\left\{V(\delta^*)\right\}$ for all $j \in \{1, ..., N-1\}$, there must exist a $\beta_{min} > 0$ such that $\pm\beta_{min} e_j \in \overline{conv}\left\{V(\delta^*)\right\}$ for all $j \in \{1, ..., N-1\}$. Finally, since $u(x(t)) = \sum_{j=1}^{N-1} u_j e_j$, the region of inputs $\left\{u(x(t)) \in \mathbb{R}^{N-1} : |u(x(t))| < \epsilon\right\} \in \overline{conv}\{V(\delta^*)\}$, and if $u(x(t))$ is continuous and $u(0) = 0$, then there is a sufficiently small $\gamma > 0$ such that (39) is true.

$$x(t) \in B(0, \gamma) \Rightarrow u(x(t)) \in \overline{conv}\left\{V(\delta^*)\right\} \tag{39}$$

This demonstrates that if the following two conditions are met:

1. The simplex vector $\delta^* = \left[\delta_1^*, ..., \delta_N^*\right]^T$ where $\delta_i^* > 0$ for all $i \in \{1, ..., N\}$ exists such that $\sum_{i=1}^{N} \delta_i^* b_i = 0$.
2. The system is continuously locally stabilizable around the origin by $u(x(t))$, where $u(0) = 0$.

then there exists a sufficiently small $\gamma > 0$ where (39) is satisfied.

4.4 Determination of Switching Surfaces

As was previously discussed in (13), one important condition for Lyapunov stability of a nonlinear system of the form (15) is as described in (40).

$$\langle \nabla V(x(t)), f(x(t)) + G(x(t))u(x(t)) \rangle < 0; \qquad \forall x(t) \in B(0, \eta) \setminus \{0\}, \eta > 0 \tag{40}$$

Also, from Section 4.3: Proof of Existence of Local Stabilization Controller Design, it was demonstrated that, if certain conditions have been met, for all $x(t) \in B(0,\gamma)$ there exists a state dependent simplex vector $a(x(t)) = [a_1(x(t)),\dots,a_N(x(t))]^T$ such that $u(x(t)) = \sum_{i=1}^{N} a_i(x(t))v_i$, and therefore (40) can be redefined as shown in (41).

$$\sum_{i=1}^{N} a_i(x(t)) \langle \nabla V(x(t)), f(x(t)) + G(x(t))v_i \rangle < 0; \qquad \forall x(t) \in B(0,\gamma) \setminus \{0\}, B(0,\gamma) \subseteq B(0,\eta) \tag{41}$$

Since $a(x(t))$ is a simplex vector, $a_i(x(t)) \geq 0$ for all $i \in \{1,\dots,N\}$, hence there must exist at least one index i^*such that $\left\langle \nabla V(x(t)), f(x(t)) + G(x(t))v_i^* \right\rangle < 0$. Consider then the set of indexes that minimizes the Lyapunov function's derivative as the set defined in (42)

$$J_{min} = \texttt{arg} \min_{i\in\{1,\dots,N\}} \langle \nabla V(x(t)), G(x(t))v_i \rangle \tag{42}$$

where the term $f(x(t))$ was removed due to the fact of it being independent of i. Therefore, for all $i \in J_{min}$, it can be said that $\langle \nabla V(x(t)), f(x(t)) + G(x(t))v_i \rangle < 0$. Furthermore, if $i \in J_{min}$ contains more than one element (i.e. multiple system modes will minimize the Lyapunov function by the same magnitude), which arises when the system's dynamics arrive to a switching surface, the filippov set-value map remains a convex hull set of all surrounding velocities, as described by (43).

$$F[X](x) = \overline{conv}\{A_i x + b_i, i \in J_{min}\} \tag{43}$$

This means that any choice of system $i \in J_{min}$ is adequate and the decay of the Lyapunov function is guaranteed. Therefore, a switching law can be defined according to the Lyapunov function where $\sigma(x(t)) \in J_{min}$.

Finally, the switching surfaces can be declared as shown in (44)

$$\left\langle \nabla V(x), (A_i - A_j)x + (b_i - b_j) \right\rangle = 0, i, j \in 1,\dots,N \tag{44}$$

where i and j are the two system modes on either side of the surface.

5 SIMULATION RESULTS & COMPARISON STUDY

This section will apply the previously discussed theory for the design of a local stabilization switching law as presented in [1] to two different systems that are taken from [5]. In [5] the author develops a stabilization switching law for sampled-data where the aurthors' designed switching law is a function of the system state at various sampling instants. The work presented in [5] is an important aspect in controller design since a majority of the systems operate in continuous time, whereas controllers and analysis is performed for high-frequency discretized sampling of the states. Furthermore, the solution presented in [5] drives the system to a region containing the equilibrium point in an exponential decay. For this reason, [5] was chosen for the comparative study to evaluate how the increased complexity presented in [5] could compare to the relatively simpler solution presented in [1].

It was determined to be important to briefly introduce the work applied in the solution of [5] before beginning the comparative study. However the readers are encouraged to review [5] in detail if there remains any uncertainties.

The solution of [5] has some similarities to the work presented in [1] in that the two definitions presented in (45) are applied.

$$\begin{aligned} A(\delta) &= \sum \delta_i A_i \\ b(\delta) &= \sum \delta_i A_i \end{aligned} \tag{45}$$

where δ_i forms the simplex per (16). Furthermore, the equilibrium point is determined via (46)

$$x^* = -A^{-1}(\delta)b(\delta) \tag{46}$$

at which point the authors simplify the solution of the equilibrium point by a translation such that the equilibrium point may be observed at $x^* = 0$, and hence $b(\delta) = 0$. It's important to note that this presents a the same result as shown in (20). The authors then utilize a continuous-time Lyapunov function strictly of the form presented in (47)

$$V(x(t)) = x(t)^T P x(t) \tag{47}$$

which is then related to a discretized Lyapunov function. For the sake of saving space, much of the proofs are omitted in this section, but the necessary condition for stabilizability of the system is as defined in (48) and (49), where matrices $P, U_i \in \mathbb{R}^{n\times n}$ are symmetric positive-definite matrices that must be determined.

$$\begin{bmatrix} \Omega_i^1(\delta,0) & \Omega_i^2(0) \\ * & \Omega_i^3(0) \end{bmatrix} \prec \mathbf{0} \tag{48}$$

$$\begin{bmatrix} \Omega_i^1(\delta, T_{max}) & \Omega_i^2(T_{max}) & -T_{max}\Psi_i(\delta) \\ * & \Omega_i^3(T_{max}) & T_{max}B_i^T P \\ * & * & \Omega_i^4(\delta, T_{max}) \end{bmatrix} \prec \mathbf{0} \tag{49}$$

where

$\Omega_i^1(\delta,\tau) = A^T(\delta)P + PA(\delta) + 2\gamma P + (T_{max} - \tau)A_i^T U_i A_i \; ; \gamma > 0$

$\Omega_i^2(\tau) = (T_{max} - \tau)A_i^T U_i b_i$

$\Omega_i^3(\tau) = (T_{max} - \tau)b_i^t U_i b_i - \beta T_{max}\mathbf{I} \; ; \beta > 0$

$\Omega_i^4(\delta,\tau) = -\tau U_i e^{-2\gamma T_{max}} + \tau^2 \Psi_i(\delta)$

$\Psi_i(\delta) = (A(\delta) - A_i)^T P + P(A(\delta) - A_i)$

Finally, the resulting switching law is as defined in (50)

$$\sigma(x(t)) \in \texttt{arg} \min_{i\in\{1,\dots,N\}} x(t)^T P(A_i x(t) + b_i) \tag{50}$$

5.1 Example 1

The first example for the comparitive study of the solution presented in this paper is taken from [5] Example 1. The proposed system is as defined in (51):

$$\begin{aligned}
A_1 &= \begin{bmatrix} 4.15 & -1.06 & -6.70 \\ 5.74 & 4.78 & -4.68 \\ 26.38 & -6.38 & -8.29 \end{bmatrix} & b_1 &= \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \\
A_2 &= \begin{bmatrix} -3.20 & -7.60 & -2.00 \\ 0.90 & 1.20 & -1.00 \\ 1.00 & 6.00 & 5.00 \end{bmatrix} & b_2 &= \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 5.75 & -16.48 & 2.41 \\ 9.51 & -9.49 & 19.55 \\ 16.19 & 4.64 & 14.05 \end{bmatrix} & b_3 &= \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \\
A_4 &= \begin{bmatrix} -12.38 & 18.42 & 0.54 \\ -11.90 & 3.24 & -16.32 \\ -26.50 & -8.64 & -16.60 \end{bmatrix} & b_4 &= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}
\end{aligned} \tag{51}$$

The conditions that $\sum_{i=1}^{4} b_i \delta_i^* = 0$ and $\sum_{i=1}^{4} \delta_i^* = 1$ are met for $\delta^* = \left\{\frac{3}{20}, \frac{1}{5}, \frac{3}{10}, \frac{7}{20}\right\}$. The selection of δ_i^* then leads to the reformulation of the switched affine system for $f(x)$ according to (27), as shown in (52)

$$f(x(t)) = \begin{bmatrix} -2.6255 & -0.1760 & -0.4930 \\ -0.2710 & -0.7560 & -0.7490 \\ -0.2610 & -1.3890 & -1.8385 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{52}$$

which is Hurwitz with eigenvalues $\lambda_1 = -2.9915$, $\lambda_2 = -2.0750$, and $\lambda_3 = -0.1535$. Since the reformulated system is linear and globally asymptotically stable, no stabilization controller is required to be designed.

Next it is necessary to determine a Lyapunov function that would demonstrate the stability of the system. Consider the generalization of the stable system defined as $\dot{x}(t) = A_\delta x(t)$. The Lyapunov function that was chosen was of the form $V(x(t)) = x(t)^T P x(t)$, where P is positive definite and symmetric. From this, basic Lypaunov theory can be applied to determine that $\dot{V}(x(t)) = x(t)^T \left(A_\delta^T P + P A_\delta\right) x(t)$, where $\left(A_\delta^T P + P A_\delta\right)$ must be negative definite and symmetric. A CVX [6] excerpt was taken from [7] and executed in MATLAB to perform a feasibility check on the Lyapunov candidate, and to determine a solution to the problem presented in (53)

$$\begin{aligned} \min \quad & \eta \\ \texttt{s.t.} \quad & \eta \geq 0 \\ & A_\delta^T P + P A_\delta \leq -\epsilon I_{3\times 3} \\ & P \geq I_{3\times 3} \\ & P \leq \eta I_{3\times 3} \end{aligned} \tag{53}$$

The solution to (53) yielded the P matrix as described in (54)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{54}$$

which yielded the Lyapunov function and its derivative as shown in (55):

$$\begin{aligned} V(x(t)) &= x_1(t)^2 + x_2(t)^2 + x_3(t)^2 \\ \dot{V}(x(t)) &= -5.251 x_1(t)^2 - 0.894 x_1(t) x_2(t) - 1.508 x_1(t) x_3(t) - 1.512 x_2(t)^2 \\ & \quad -4.276 x_2(t) x_3(t) - 3.677 x_3(t)^2 \end{aligned} \tag{55}$$

The Lyapunov candidate $V(x(t))$ is obviously globally positive definite with a single minimum of 0 at the equilibrium. A maximization was then performed on its derivative $\dot{V}(x(t))$ and it was confirmed that it was negative definite with a single maximum of 0 at the equilibrium point. Hence, (55) is a valid Lyapunov function that demonstrates global asymptotic stability of the system.

Following the definition of (55), the switching laws can be derived as 3-dimensional lines in $\mathbb{R}^3$ to accommodate the sliding dynamics. As previously described, the switching laws are based on the Lyapunov function, and are decided upon by the mode that most minimizes the Lyapunov function. The set of 12 switching surfaces are defined in (56), where $S_{i,j}$ denotes a change of the switched system between $\dot{x} = A_i x + b_i$ and $\dot{x} = A_j x + b_j$, and $S_{j,i} = -S_{i,j}$. Also, the notation for $x_i(t)$ was simplified to x_i.

$$\begin{aligned}
S_{1,2} &= 14.7x_1^2 + 22.76x_1x_2 + 41.36x_1x_3 - 6x_1 + 7.16x_2^2 - 32.12x_2x_3 - 4x_2 \\
&\quad -26.58x_3^2 + 4x_3 \\
S_{1,3} &= -3.2x_1^2 + 23.3x_1x_2 + 2.16x_1x_3 + 6x_1 + 28.54x_2^2 - 70.5x_2x_3 - 10x_2 \\
&\quad -44.68x_3^2 + 4x_3 \\
S_{1,4} &= 33.06x_1^2 - 3.68x_1x_2 + 91.28x_1x_3 + 4x_1 + 3.08x_2^2 + 27.8x_2x_3 \\
&\quad -12.0x_2 + 16.62x_3^2 \\
S_{2,3} &= -17.9x_1^2 + 0.54x_1x_2 - 39.2x_1x_3 + 12x_1 + 21.38x_2^2 - 38.38x_2x_3 \\
&\quad -6.0x_2 - 18.1x_3^2 \\
S_{2,4} &= 18.36x_1^2 - 26.44x_1x_2 + 49.92x_1x_3 + 10x_1 - 4.08x_2^2 + 59.92x_2x_3 - 8x_2 \\
&\quad +43.2x_3^2 - 4.0x_3 \\
S_{3,4} &= 36.26x_1^2 - 26.98x_1x_2 + 89.12x_1x_3 - 2x_1 - 25.46x_2^2 + 98.3x_2x_3 - 2x_2 \\
&\quad +61.3x_3^2 - 4x_3
\end{aligned} \tag{56}$$

The system with the determined switching laws can then be simulated in MATLAB, where a plot of the system response from the set of initial conditions $x_1(0) = 0$, $x_2(0) = 0.9$, and $x_3(0) = 0.2$ can be observed in Figure 1.

The initial conditions were chosen such that the proposed solution from [1] may be compared with [5]. However, before performing the comparison, some observation may be noted simply on the solution of [1]. Firstly, it can be observed that the state $x_1(t)$ experiences a significant decrease from it's initial condition before returning to equilibrium. Secondly, states $x_2(t)$ and $x_3(t)$ exhibit an initially smooth decrease towards equilibrium. Finally, as $x_1(t)$ and $x_2(t)$ approach their settling state, the third state appears to experience a significant drop in magnitude, at which point it finally stabilizes at its equilibrium around the same time as the other two states. These plots are now compared with the results from [5], which performed the same experiment, however their derivation of a switching law (also dependent the same style of Lyapunov function as used in this simulation), uses a different P matrix. The results from [5] are shown in Figure 2.

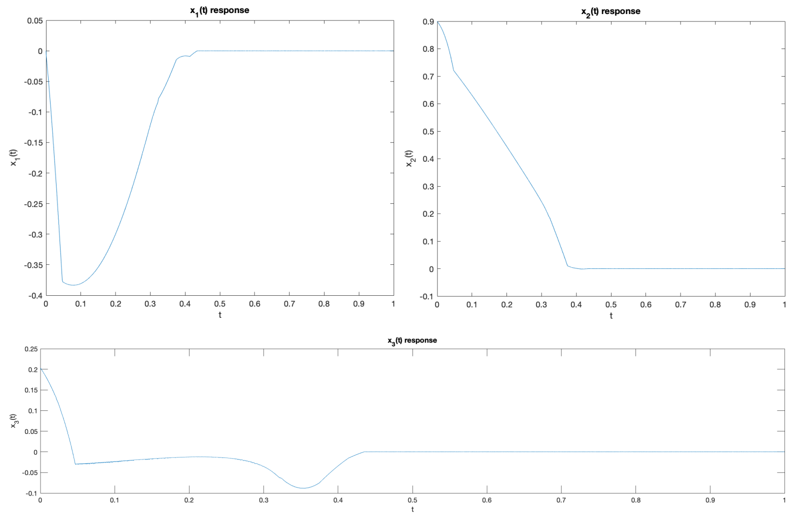


Figure 1: Simulation results for (51) with switching laws per (56)

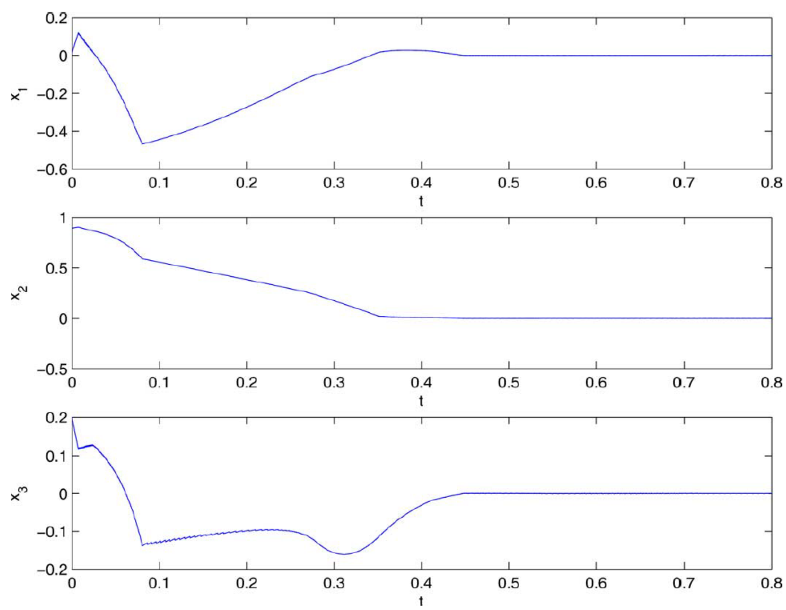


Figure 2: Simulation results for (51) from [5]

Comparing the results obtained in our simulation with those of [5] yield some interesting observations. Firstly, the system responses are generally similar and the systems net settling time to the equilibrium point is relatively similar (a minor note can be made that $x_2(t)$ does stabilize slightly more quickly in the solution presented in [5]). However, the first observation to be made is that in the solution presented in Figure 1, state $x_1(t)$ does not spike upwards as is shown in the results of [5], nor does it drop in magnitude as much as in [5]. Also, the third state in Figure 1 exhibited a smooth decline to slightly below it's equilibrium, whereas in [5] the third state exhibits a sudden jump in magnitude shortly following the start of the simulation, and it's decline is far more significant than shown in Figure 1. Following these observation, a decision was made to evaluate the solution presented in [1] with switching laws dependent on the same Lyapunov function as used in [5]. The newly defined P matrix is as defined in (57).

$$P = \begin{bmatrix} 0.1 & -0.02 & 0 \\ -0.02 & 0.15 & 0.2 \\ 0 & 0.02 & 0.11 \end{bmatrix} \tag{57}$$

In an effort to reduce on space, the newly defined switching laws are omitted. The resulting plots can be observed in Figure 3.

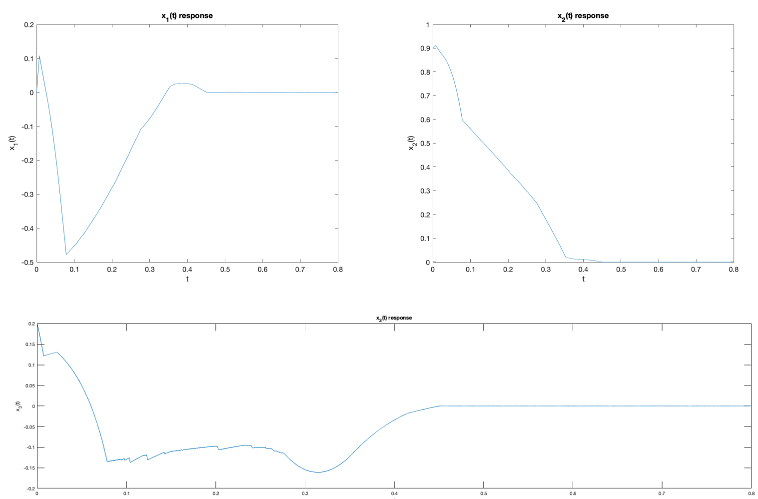


Figure 3: Simulation results for (51) with the same Lyapunov function as [5] Example 1

The newly established set of switching laws yields very similar results to the solution proposed in [5], however one noticeable difference is the non-smooth characteristics for state x_3 as it increases in magnitude between 0.1 and 0.3 seconds. It is the opinion of the authors of this paper that when the solution presented in [1] is applied to this particular system given the Lyapunov function presented in [5], the system yields less desirable results. However, using the Lyapunov function derived from the CVX optimization, the results appear to demonstrate minor improvements.

5.2 Example 2

The first example for the comparitive study of the solution presented in this paper is taken from [5] Example 3. The proposed system is as defined in (58).

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -3 & -6 & 3 \\ 2 & 2 & -3 \\ 1 & 0 & -2 \end{bmatrix} \qquad & b_1 &= \begin{bmatrix} -35 \\ 0 \\ 0 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix} \qquad & b_2 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
A_3 &= A_1 \qquad & b_3 &= -b_1
\end{aligned} \tag{58}
$$

The conditions that $\sum_{i=1}^{3} b_i\delta_i^* = 0$ and $\sum_{i=1}^{3} \delta_i^* = 1$ were met for $\delta_i^* = \frac{1}{3}$ where $i \in \{1,2,3\}$, however for this particular problem there does exist an infinite number of combinations provided $0 < \delta_3^* = \delta_1^* < \frac{1}{2}$ and $\delta_2^* = 1 - \delta_3^* - \delta_1^*$ are satisfied. The selection of δ_i^* then leads to the reformulation of the switched affine system according to (27), as shown in (59)

$$
\dot{x}(t) = \begin{bmatrix} \frac{-5}{3} & -3 & 3 \\ 1 & \frac{1}{3} & -3 \\ \frac{2}{3} & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -70 & 4x_1(t) + 9x_2(t) - 35 \\ 0 & -3x_1(t) - 5x_2(t) \\ 0 & -x_1(t) \end{bmatrix} u(x(t)) \tag{59}
$$

where $u(x(t)) = \begin{bmatrix} u_1(x(t)) & u_2(x(t)) \end{bmatrix}^T$. The eigenvalues of $f(x(t))$ were determined to be $\lambda_{1,2} = -1.7294 \pm 0.744i$ and $\lambda_3 = 0.1254$, where i in this instance is to denote imaginary numbers. It is obvious that the redefined system is non-Hurwitz and, as consequence, a controller must be defined to stabilize the system. A linear control law was therefore defined according to (60)

$$
u(x) = \begin{bmatrix} -\frac{1}{105}x_1 + \frac{5}{70}x_2 - \frac{5}{70}x_3 \\ 0 \end{bmatrix} \tag{60}
$$

which obeys the conditions that $u(x(t))$ be continuous and $u(0) = 0$. This feedback control law caused for the system to become as shown below:

$$
\dot{x}(t) = \begin{bmatrix} -1 & -8 & 8 \\ 1 & \frac{1}{3} & -3 \\ \frac{2}{3} & 0 & -2 \end{bmatrix} x(t)
$$

The redefined system with the designed controller is Hurwitz, with eigenvalues at $\lambda_1 = -3.1729$ and $\lambda_{2,3} = -0.2469 \pm 1.6449i$.

The Lyapunov candidate $V = x(t)^T P x(t)$ was chosen and P was solved for in the same manner as was accomplished in Example 1, which used CVX in MATLAB to determine a positive definite symmetric P matrix that provides a positive definite Lyapunov function where it's derivative is negative definite. This yielded the P matrix as described in (61)

$$P = \begin{bmatrix} 1.698 & 2.814 & -5.3653 \\ 2.814 & 39.449 & -40.055 \\ -5.3653 & -40.055 & 54.7603 \end{bmatrix} \tag{61}$$

which yielded the Lyapunov function and it's derivative as shown (62)

$$\begin{aligned} V(x(t)) &= 1.699x_1(t)^2 + 5.628x_1(t)x_2(t) - 10.731x_1(t)x_3(t) + 39.449x_2(t)^2 \\ &\quad -80.109x_2(t)x_3(t) + 54.760x_3(t)^2 \\ \dot{V}(x(t)) &= -4.922x_1(t)^2 + -5.431x_1(t)x_2(t) + 35.3835x_1(t)x_3(t) \\ &\quad -18.724x_2(t)^2 + 27.688x_2(t)x_3(t) - 64.559x_3(t)^2 \end{aligned} \tag{62}$$

Finally, the Lyapunov function was minimized and it was confirmed to be globally positive definite with a single minimum of $V(x) = 0$ at the equilibrium point, and the Lyapunov function's derivative was maximized and it was confirmed to be globally negative definite with a single maximum of $\dot{V}(x) = 0$ at the equilibrium point. Stability of the system with the designed control law was of course known in advance since it is simply a linear system with negative eigenvalues, but the Lyapunov function is required for the design of the switching law, and proper definition of the Lyapunov function is required to be confirmed.

Following the definition of a valid Lyapunov function, the switching laws were defined as 3-dimensional lines in $\mathbb{R}^3$ to accommodate the sliding dynamics. The set of switching surfaces are as defined in (63).

$$\begin{aligned} S_{1,2} &= 375.569x_3 - 196.979x_2 - 118.873x_1 + 131.646x_1x_2 - 87.8842x_1x_3 \\ &\quad -303.97x_2x_3 - 7.43215x_1^2 + 343.84x_2^2 \\ S_{1,3} &= 751.139x_3 - 393.958x_2 - 237.747x_1 \\ S_{2,3} &= 375.569x_3 - 196.979x_2 - 118.873x_1 - 131.646x_1x_2 + 87.8842x_1x_3 \\ &\quad +303.97x_2x_3 + 7.43215x_1^2 - 343.84x_2^2 \end{aligned} \tag{63}$$

The system was again simulated in MATLAB, and a plot of the system's response from the set of initial conditions $x_1(0) = x_2(0) = x_3(0) = 2.0$ with the switching surfaces as defined in (63) is shown in Figure 4.

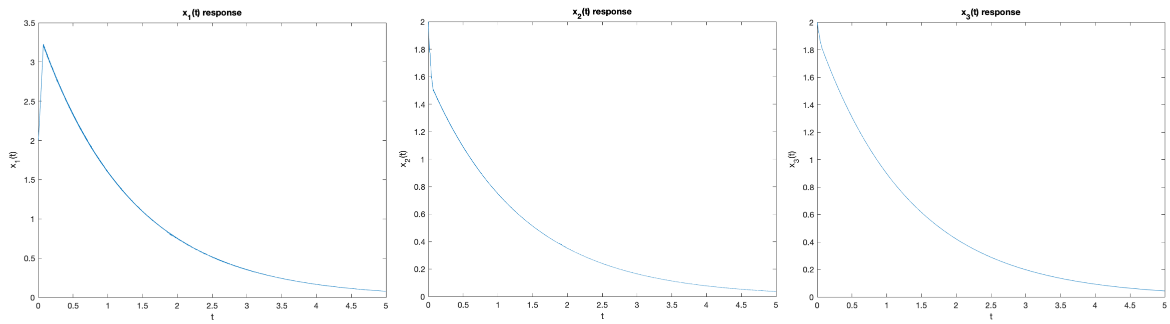


Figure 4: Simulation results for (58) with switching laws per (63)

The plots shown in Figure 4 demonstrate the asymptotic stability of the switched affine system with the defined switching terms from the given initial conditions. However, some observations should be mentioned regarding the solution. Firstly, the state x_1 experiences a significant increase in magnitude by over 50% its initial state. This could prove to be problematic for some systems if there are in fact operational limitations of the system with regards to this state. Secondly, although the states appear to exhibit exponentially decaying curves, they are in fact the act of sliding dynamics along the boundary of modes 1 and 3. A plot of the switching law activity is shown in Figure 5.

Figure 5: System mode switching activity for the simulation presented in Figure 4

As is demonstrated in Figure 5, there is significant switching between modes 2 and 3 initially for a short duration, then between modes 1 and 3 for the remainder of the simulation. This can be extremely problematic for certain systems (i.e. mechanical systems) where switching cannot be executed at such high frequencies. However, for systems that can accommodate high frequency switching (i.e. electrical systems), this solution proves to be an extremely effective means for developing switching laws of switched affine systems for time-efficient convergence to the equilibrium point.

Finally, to demonstrate various responses of the system defined in (58) with the switching laws defined in (63), the system was simulated for 1,330 different initial conditions. In the simulations presented in Figure 6, a combination of initial conditions were applied to the system where each state's initial condition varied from -5 to 5 in steps of magnitude 1, where the equilibrium point (0,0,0) was removed as a candidate initial condition.

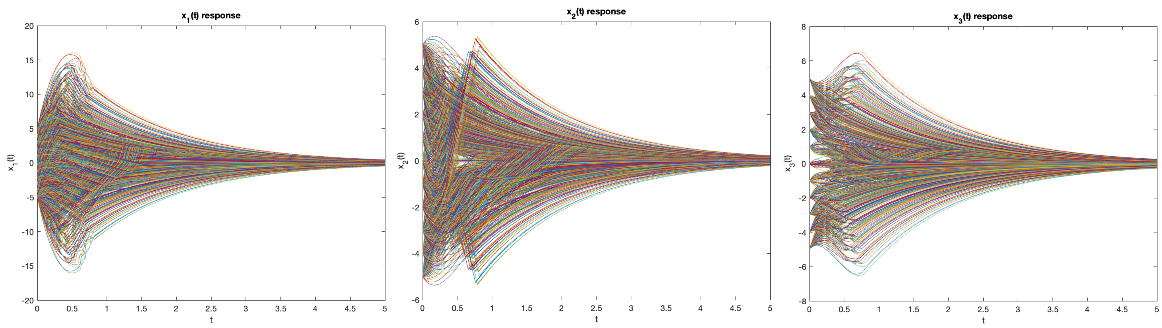


Figure 6: Simulation results for (58) with switching laws per (63)

The results of these 1,330 simulations demonstrate that all 3 states are susceptible to the increase in state magnitude, as well as the high switching frequency. The results also demonstrate the local asymptotic stability of the switched affine system with the defined switching law, where all combinations of initial conditions for state magnitudes between -5 and 5 have converged to the equilibrium point.

APPENDIX I: SET-VALUE MAPPING

Set-value mapping is a mapping that associates sets to points, as opposed to mapping singleton values to points. A time-variant set-value mapping can be defined as shown in (64)

$$F : [0, \infty] \times \mathbb{R}^n \to C(\mathbb{R}^n) \tag{64}$$

where $C(\mathbb{R}^n)$ is the collection of all subsets of $\mathbb{R}^n$. This means that F associates to each point $(t, x) \in [0, \infty] \times \mathbb{R}^n$ the mapped set $F(t, x)$, which is a subset of $\mathbb{R}^n$, or equivalently $F(t, x) \subseteq \mathbb{R}^n$.

Local Boundedness of Set-Value Maps

A set-value map is locally bounded at the point $(t, x) \in [0, \infty] \times \mathbb{R}^n$ if a $\epsilon, \delta \in (0, \infty)$ and an integrable function $m : [t, t + \delta] \to (0, \infty)$ exist such that $||z||_2 \leq m(s)$ for all $z \in F(s, y)$, all $s \in [t, t + \delta]$, and all $y \in B(x, \epsilon)$ [8]. Note that $B(x, \epsilon)$ is an n dimensional ball centred at the state x with some radius $\epsilon > 0$.

In simpler terms, the set-value map is locally bounded at the point $(t, x) \in [0, \infty] \times \mathbb{R}^n$ if for all times within a neighbourhood of t (all $s \in [t, t + \delta]$) and all states within a neighbourhood of x (all $y \in B(x, \epsilon)$), all elements of the set-value map of all regions contained by the ball within the time neighbourhood is bounded by some integrable function $m(s)$.

Upper Semi-Continuity of Set-Value Maps

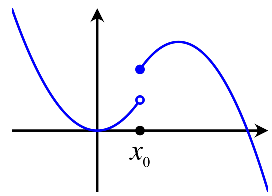
A time-invariant set-value map is upper semi-continuous at $x \in \mathbb{R}^n$ if for all $\epsilon \in (0, \infty)$ there exist some $\delta \in (0, \infty)$ such that (65) is satisfied for all $y \in B(x, \delta)$ [8].

$$F(y) \subset F(x) + B(0, \epsilon) \tag{65}$$

The idea of upper semi-continuous set-value maps is similar to that of functions. The formal definition for functions is that, for a discontinuous function $f(x)$, (66) is satisfied

$$f(x) \leq f(x_0) + \epsilon \quad \forall \quad \epsilon > 0 \tag{66}$$

which describes that the values within a neighbourhood of the discontinuity (x_0) are either close to or less than that of $f(x_0)$. The following figure depicts an upper semi-continuous function.



This idea can be extended to set-value maps, where the set-value map is upper semi-continuous at $x \in \mathbb{R}^n$ if the set-value map for all points within neighbourhood of $x \in \mathbb{R}^n$ (all $y \in B(x, \delta)$) is a subset of the set-value map of the point itself and some small neighbourhood ball ($B(0, \epsilon)$).

APPENDIX II: CARATHEODORY SOLUTIONS

Existence of Solutions for Discontinuous Ordinary Differential Equations

Mathematician Constantin Caratheodory developed a now widely accepted theorem for the existence of solutions for discontinuous ordinary differential equations that meet certain criteria. Let us consider the time-variant ordinary differential equation as described in (67)

$$\dot{x} = X(t, x(t)) \tag{67}$$

where x is the time-variant state of the system, and $X : [0, \infty] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a time-varying discontinuous vector field. The Caratheodory conditions for existence of a (or multiple) solutions of discontinuous ordinary differential equations are as described below:

1. $X(t, x(t))$ is continuous for almost all $t \in [0, \infty]$
2. $X(t, x(t))$ is measurable for all $x \in \mathbb{R}^n$
3. $X(t, x(t))$ is is locally bounded

If these condition are met, then for all $(t_0, x_0) \in [0, \infty] \times \mathbb{R}^n$ there exists at least one Caratheodory solution to the discontinuous ordinary differential equation with initial condition $x(t_0) = x_0$ [8].

Existence of Solutions for Differential Inclusions

A differential inclusion is a generalization of differential equations where at each state the differential inclusion specifies a set of possible changes to the state. This allows for defining a differential equation by an association with a set-valued mapping (see APPENDIX I: SET-VALUE MAPPING). As opposed to equating state velocities to ordinary differential equations (as was show in Section 5.2), differential inclusions define state velocities as an element of a set of possible velocities, as shown in (68):

$$\dot{x}(t) \in F(t, x(t)) \tag{68}$$

From this, we can then say that the point $x_e \in \mathbb{R}^n$ is an equilibrium point of the differential inclusion if $0 \in F(t, x_e)$ for all $t \in [0, \infty]$.

The Caratheodory conditions for existence of a (or multiple) solutions of differential inclusions are as described below:

1. $F(t, x(t))$ is locally bounded
2. $F(t, x(t))$ is non-empty
3. $F(t, x(t))$ is compact (note that if a set is compact it is also bounded)
4. $F(t, x(t))$ is convex
5. $F(t, x(t))$ is upper semicontinuous for each $t \in \mathbb{R}$
6. $F(t, x(t))$ is measureable for each $x \in \mathbb{R}^n$

If these condition are met, then for all $(t_0, x_0) \in [0, \infty] \times \mathbb{R}^n$ there exists at least one Caratheodory solution to the differential inclusion with initial condition $x(t_0) = x_0$ [8].

APPENDIX III: FILIPPOV SOLUTIONS

Filippov Solutions

Caratheodory solutions (see APPENDIX II: CARATHEODORY SOLUTIONS) involve the analysis of vector fields at specific points within the space. However, discontinuities in the vector field across the space can result in a non-existence of Caratheodory solutions. To accommodate these discontinuities, Filippov solutions can be used to analyse vector fields around a point in space, as opposed to at specific points in space (as is done in Caratheodory solutions).

For the differential equation described in (69), it is desirable to associate a set-valued map $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with regards to the vector field around each point in the space.

$$\dot{x} = X(x) \tag{69}$$

To accomplish this, we evaluate the vector field $X(x)$ at all points within the ball $B(x, \epsilon)$ for decreasing $\epsilon > 0$. Furthermore, for solution flexibility and to avoid conflicts of two vector fields at discontinuities, we will remove sets of measure 0 from the ball region. This is because discontinuities of the differential equation occur only at singular points, where there exists different vector fields on either side of each discontinuous point, and so by neglecting the set-valued mapping around those points, it is possible to avoid conflicting vector fields. Furthermore, at singular points the set measure is 0. The set-valued mapping for Filippov solutions can then be formally defined as is shown in (70) [8].

$$F[X](x) \triangleq \cap_{\epsilon>0} \cap_{\mu(S)=0} \overline{conv}\{X(\overline{x}) : \overline{x} \in B(x, \epsilon) \setminus S\} \tag{70}$$

The set-valued map is comprised of the closed convex hull set of all velocities around each state x – where "around" is defined as the ball centred at x with decreasing radius $\epsilon > 0$ – less any singular points where there are conflicting vector fields. It should be noted that the definition for the set of singular points for which there are conflicting vector fields is defined using the measure of the set S where $\mu(S) = 0$, which is known as the Lebesgue measure of the set being 0.

From the Filippov set-valued map, the ordinary differential equations described by vector fields can then be tranlated to differential inclusions as show in (71).

$$\dot{x}(t) = X(x(t)) \Rightarrow \dot{x}(t) \in F[X](x(t)) \tag{71}$$

Filippov Existence

A Filippov solution of a differential equation described in (69) is also a Caratheodory solution to the differential inclusion (70). This means that the existence of a Filippov solution for the described differential equation will exist under the same conditions as for the existence of a Caratheodory solution (see Existence of Solutions for Differential Inclusions). Furthermore, a Filippov solution for the described vector fields is continuous for almost all $t \in [0, t_1]$, where exceptions arise solely at the discontinuities of the system. This is because the Filippov solution allows for discontinuities in the derivative $\frac{dx(t)}{dt}$, but not for discontinuities in the state.

References

[1] L. Hetel and E. Bernuau, “Local stabilization of switched affine systems,” *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 1158–1163, 2015.

[2] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Norwell, MA, USA: Kluwer Academic Publishers, 1988.

[3] A. Juditsky, “Lecture 1: Convex sets,” http://aiweb.techfak.uni-bielefeld.de/content/bworld-robot-control-software/, 2017.

[4] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*. New York, Ny, USA: Springer, 2005.

[5] L. Hetel and E. Fridman, “Robust sampled–data control of switched affine systems,” *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2922–2928, 2013.

[6] M. Grant and S. Boyd, “Cvx: Matlab software for disciplined convex programming, version 2.1,” 2014.

[7] L. Rodrigues, B. Samadi, and M. Moarref, *Piecewise Affine Control: Continuous-Time, Sampled-Data and Networked Systems*, 2019.

[8] J. Cortes, “Discontinuous dynamical systems,” *IEEE Control systems magazine*, vol. 28, no. 3, pp. 36–73, 2008.