Sensitivity Analysis

Lecture 9

Chapter 3: Optimization

Institute of Engineering
Asst. Prof. Anita Prajapati, Ph.D.

30 June 2023

Chapter 4

Sensitivity Analysis and the Simplex Method

Introduction

- When solving an LP model we assume that all relevant factors are known with certainty.
- Such certainty rarely exists.
- Sensitivity analysis helps answer questions about how sensitive the optimal solution is to changes in various coefficients in an LP model.

General Form of a Linear Programming (LP) Problem

MAX (or MIN): $c_1X_1 + c_2X_2 + \ldots + c_nX_n$

Subject to: $a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n <= b_1$

$$\vdots$$

$$a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n <= b_k$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$$

- How sensitive is a solution to changes in the c_i, a_{ij}, and b_i?

Approaches to Sensitivity Analysis

- Change the data and re-solve the model!
  - Sometimes this is the only practical approach.

- Solver also produces sensitivity reports that can answer questions about:
  - amounts objective function coefficients can change without changing the solution.
  - the impact on the optimal objective function value due to changes in various constrained resources.
  - the impact on the optimal objective function value due to forced changes in certain decision variables.
  - the impact on the optimal solution due to changes in constraint coefficients

Software Note

- When solving LP problems, be sure to select the "Assume Linear Model" option in the Solver Options dialog box as this allows Solver to provide more sensitivity information than it could otherwise do.

Once Again, We'll Use The Blue Ridge Hot Tubs Example...

MAX:	$350X_1 + 300X_2$	} profit
S.T.:	$1X_1 + 1X_2 <= 200$	} pumps
	$9X_1 + 6X_2 <= 1566$	} labor
	$12X_1 + 16X_2 <= 2880$	} tubing
	$X_1, X_2 >= 0$	} non-negativity

The Answer Report

See file Fig4-1.xls

Target Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	Unit Profits Total Profit	0	66100

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$5	Number to make Aqua-Spas	0	122
$C$5	Number to make Hydro-Luxes	0	78

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$9	Pumps Req'd Used	200	$D$9<=$E$9	Binding	0
$D$10	Labor Req'd Used	1566	$D$10<=$E$10	Binding	0
$D$11	Tubing Req'd Used	2712	$D$11<=$E$11	Not Binding	168
$B$5	Number to make Aqua-Spas	122	$B$5>=0	Not Binding	122
$C$5	Number to make Hydro-Luxes	78	$C$5>=0	Not Binding	78

Answer Report 1 / Sensitivity Report 1 / Limits Report 1 / Production

The Sensitivity Report

See file Fig4-1.xls

Target Cell (Max)

Cell	Name	Final Value
$D$6	Unit Profits Total Profit	66100

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to make Aqua-Spas	122	0	350	100	50
$C$5	Number to make Hydro-Luxes	78	0	300	50	66.66667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	Pumps Req'd Used	200	200	200	7	26
$D$10	Labor Req'd Used	1566	17	1566	234	126
$D$11	Tubing Req'd Used	2712	0	2880	1E+30	168

Answer Report 1 | Sensitivity Report 1 | Limits Report 1 | Production

Changes in Objective Function Coefficients

- Values in the "Allowable Increase" and "Allowable Decrease" columns for the Changing Cells indicate the amounts by which an objective function coefficient can change without changing the optimal solution, *assuming all other coefficients remain constant*.

Changes in Constraint RHS Values

- The *shadow price* of a constraint indicates the amount by which the objective function value changes given a unit *increase* in the RHS value of the constraint, *assuming all other coefficients remain constant*.

- Shadow prices hold only within RHS changes falling within the values in “Allowable Increase” and “Allowable Decrease” columns.

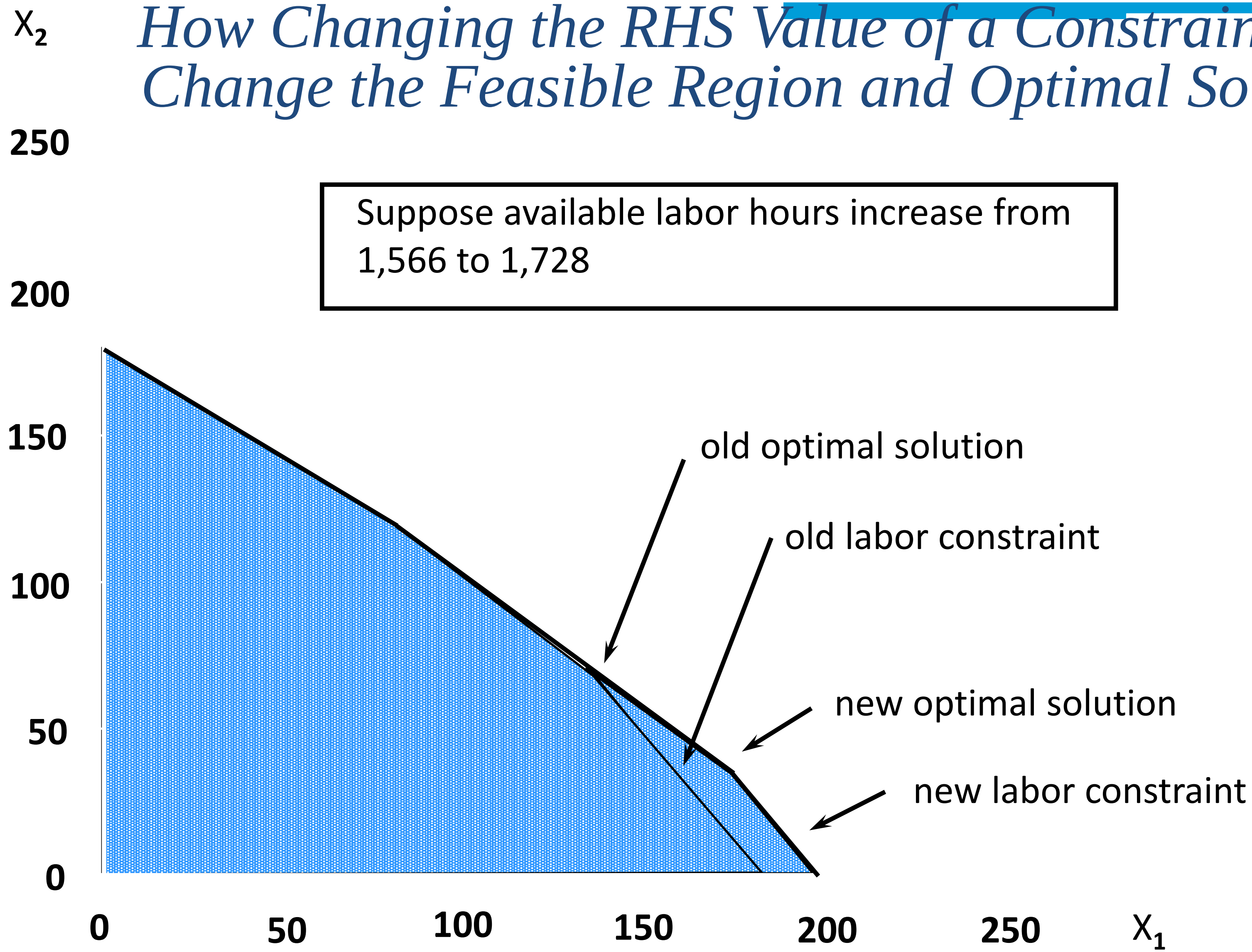
- Shadow prices for nonbinding constraints are always zero.

Comments About Changes in Constraint RHS Values

- Shadow prices only indicate the changes that occur in the objective function value as RHS values change.
- Changing a RHS value for a binding constraint also changes the feasible region and the optimal solution (see graph on following slide).
- To find the optimal solution after changing a binding RHS value, you must re-solve the problem.

How Changing the RHS Value of a Constraint Can Change the Feasible Region and Optimal Solution

Suppose available labor hours increase from 1,566 to 1,728



Other Uses of Shadow Prices

- Suppose a new Hot Tub (the Typhoon-Lagoon) is being considered. It generates a marginal profit of $320 and requires:
  1 pump (shadow price = $200)
  8 hours of labor (shadow price = $16.67)
  13 feet of tubing (shadow price = $0)

Q: Would it be profitable to produce any?

A: $320 - $200*1 - $16.67*8 - $0*13 = -$13.33 = *No!*

The Meaning of Reduced Costs

- The Reduced Cost for each product equals its per-unit marginal profit minus the per-unit value of the resources it consumes (priced at their shadow prices).

Type of Problem	Optimal Value of Decision Variable	Optimal Value of Reduced Cost
Maximization	at simple lower bound	<=0
	between lower & upper bounds	=0
	at simple upper bound	>=0
Minimization	at simple lower bound	>=0
	between lower & upper bounds	=0
	at simple upper bound	<=0

$$\text{Reduced cost of Aqua-Spas} = 350 - 200 \times 1 - 16.67 \times 9 - 0 \times 12 = 0$$

$$\text{Reduced cost of Hydro-Luxes} = 300 - 200 \times 1 - 16.67 \times 6 - 0 \times 16 = 0$$

$$\text{Reduced cost of Typhoon-Lagoons} = 320 - 200 \times 1 - 16.67 \times 8 - 0 \times 13 = -13.33$$

Key Points

- The shadow prices of resources equate the marginal value of the resources consumed with the marginal benefit of the goods being produced.
- Resources in excess supply have a shadow price (or marginal value) of zero.
- The reduced cost of a product is the difference between its marginal profit and the marginal value of the resources it consumes.
- Products whose marginal profits are less than the marginal value of the goods required for their production will not be produced in an optimal solution.

Analyzing Changes in Constraint Coefficients

Q: Suppose a Typhoon-Lagoon required only 7 labor hours rather than 8. Is it now profitable to produce any?

A: $320 - $200*1 - $16.67*7 - $0*13 = $3.31 = *Yes!*

Q: What is the maximum amount of labor Typhoon-Lagoons could require and still be profitable?

A: We need $320 - $200*1 - $16.67*$L_3$ - $0*13 >=0

The above is true if L_3 <= $120/$16.67 = $7.20

The Limits Report

See file Fig4-1.xls

Cell	Target Name	Value
$D$6	Unit Profits Total Profit	$66,100

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
$B$5	Number to make Aqua-Spas	122	0	$23,400	122	$66,100
$C$5	Number to make Hydro-Luxes	78	0	$42,700	78	$66,100

Answer Report 1 / Sensitivity Report 1 / Limits Report 1 / Production

Simplex Method Summary

- The simplex method operates by first identifying any basic feasible solution (or extreme point) for an LP problem, then moving to an adjacent extreme point, if such a move improves the value of the objective function.

- When no adjacent extreme point has a better objective function value, the current extreme point is optimal and the simplex method terminates.

- The process of moving from one extreme point to an adjacent one is accomplished by switching one of the basic variables with one of the non-basic variables to create a new basic feasible solution that corresponds to the adjacent extreme point.

Assignment 4

Question no.: 5, 11

Due on 14^{th} July

End of Chapter 4