# Chapter 4

# Sensitivity Analysis and the Simplex Method

Ragsdale, Cliff T.
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## 4.0 Introduction

In Chapters 2 and 3, we studied how to formulate and solve LP models for a variety of decision problems. However, formulating and solving an LP model does not necessarily mean that the original decision problem has been solved. Several questions often arise about the optimal solution to an LP model. In particular, we might be interested in how sensitive the optimal solution is to changes in various coefficients of the LP model.

Businesses rarely know with certainty what costs will be incurred, or the exact amount of resources that will be consumed or available in a given situation or time period. Thus, management might be skeptical of optimal solutions obtained using models that assume that all relevant factors are known with certainty. Sensitivity analysis can help overcome this skepticism and provide a better picture of how the solution to a problem will change if different factors in the model change. Sensitivity analysis also can help answer several practical managerial questions that might arise about the solution to an LP problem.

## 4.1 The Purpose of Sensitivity Analysis

As noted in Chapter 2, any problem that can be stated in the following form is an LP problem:

$$\begin{array}{ll}
\text{MAX (or MIN):} & c_1X_1 + c_2X_2 + \cdots + c_nX_n \\
\text{Subject to:} & a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \le b_1 \\
& \vdots \\
& a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \ge b_k \\
& \vdots \\
& a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m
\end{array}$$

All the coefficients in this model (the $c_j$, $a_{ij}$, and $b_i$) represent numeric constants. So, when we formulate and solve an LP problem, we implicitly assume that we can specify the exact values for these coefficients. However, in the real world, these coefficients might change from day to day or minute to minute. For example, the price a company charges for its products can change on a daily, weekly, or monthly basis. Similarly, if a skilled machinist calls in sick, a manufacturer might have less capacity to produce items on a given machine than was originally planned.

Realizing that such uncertainties exist, a manager should consider how sensitive an LP model's solution is to changes or estimation errors that might occur in: (1) the objective function coefficients (the $c_j$), (2) the constraint coefficients (the $a_{ij}$), and (3) the RHS values for the constraints (the $b_i$). A manager also might ask a number of "What if?" questions about these values. For example, what if the cost of a product increases by 7%? What if a reduction in setup time allows for additional capacity on a given machine? What if a worker's suggestion results in a product requiring only two hours of labor rather than three? Sensitivity analysis addresses these issues by assessing the sensitivity of the solution to uncertainty or estimation errors in the model coefficients, and also the solution's sensitivity to changes in model coefficients that might occur because of human intervention.

# 4.2 Approaches to Sensitivity Analysis

You can perform sensitivity analysis on an LP model in several ways. If you want to determine the effect of some change in the model, the most direct approach is simply to change the model and re-solve it. This approach is suitable if the model does not take an excessive amount of time to change or solve. In addition, if you are interested in studying the consequences of *simultaneously* changing several coefficients in the model, this might be the only practical approach to sensitivity analysis.

Solver also provides some sensitivity information after solving an LP problem. As mentioned in Chapter 3, one of the benefits of using the simplex method to solve LP problems is its speed—it is considerably faster than the other optimization techniques offered by Solver. However, another advantage of using the simplex method is that it provides more sensitivity analysis information than the other techniques. In particular, the simplex method provides us with information about:

- The range of values the objective function coefficients can assume without changing the optimal solution
- The impact on the optimal objective function value of increases or decreases in the availability of various constrained resources
- The impact on the optimal objective function value of forcing changes in the values of certain decision variables away from their optimal values
- The impact that changes in constraint coefficients will have on the optimal solution to the problem

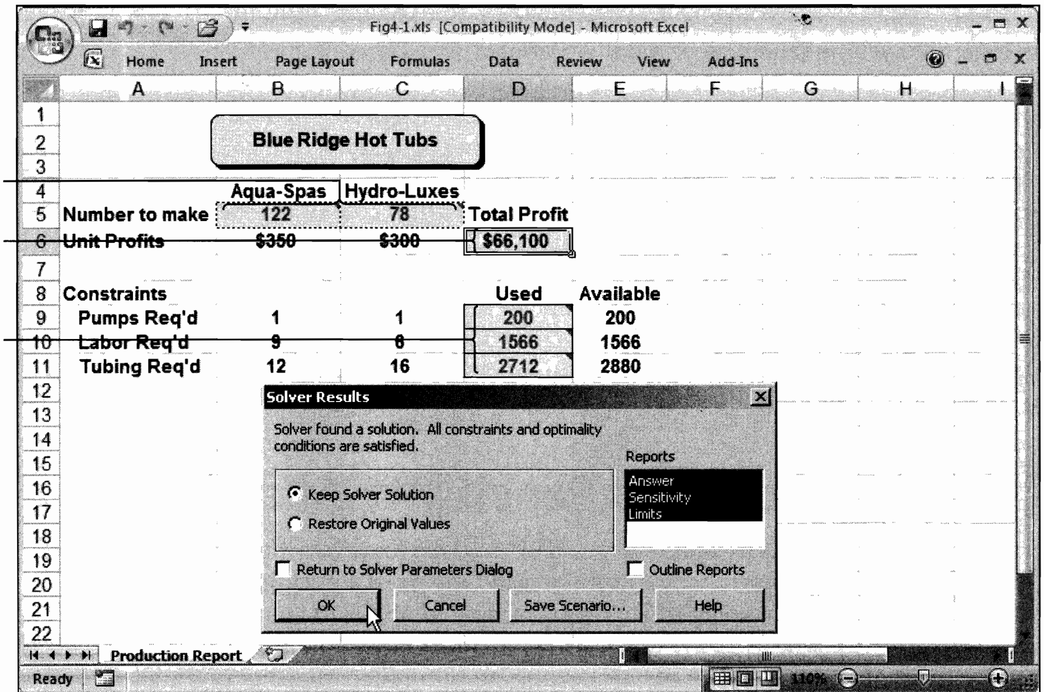
# 4.3 An Example Problem

We will again use the Blue Ridge Hot Tubs problem to illustrate the types of sensitivity analysis information available using Solver. The LP formulation of the problem is repeated here, where $X_1$ represents the number of Aqua-Spas and $X_2$ represents the number of Hydro-Luxes to be produced:

$$
\begin{array}{lrcl}
\text{MAX:} & 350X_1 + 300X_2 & & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 \leq & 200 & \} \text{ pump constraint} \\
 & 9X_1 + 6X_2 \leq & 1{,}566 & \} \text{ labor constraint} \\
 & 12X_1 + 16X_2 \leq & 2{,}880 & \} \text{ tubing constraint} \\
 & X_1, X_2 \geq & 0 & \} \text{ nonnegativity conditions}
\end{array}
$$

**FIGURE 4.1**

*Spreadsheet model for the Blue Ridge Hot Tubs product mix problem*




**Key Cell Formulas**

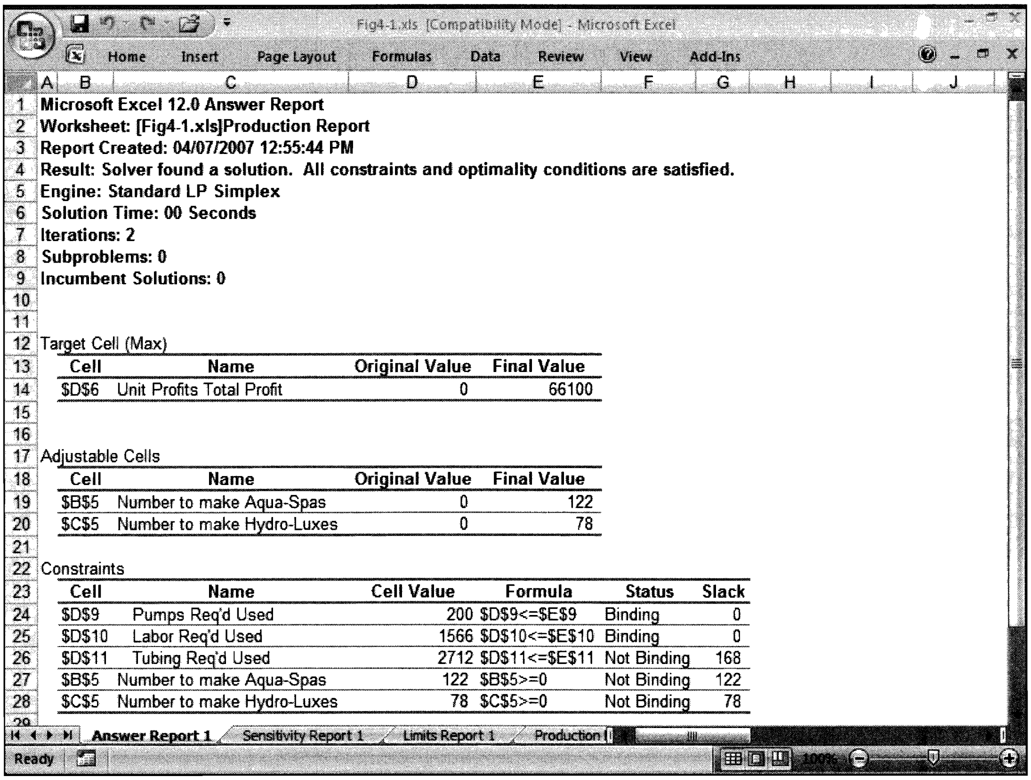
| Cell | Formula | Copied to |
|---|---|---|
| D6 | =SUMPRODUCT(B6:C6,$B$5:$C$5) | D9:D11 |

This model is implemented in the spreadsheet shown in Figure 4.1 (and file Fig4-1.xls on your data disk). (See Chapter 3 for details on the procedure used to create and solve this spreadsheet model.)

After solving the LP model, Solver displays the Solver Results dialog box, shown in Figure 4.1. This dialog box provides three report options: Answer, Sensitivity, and Limits. You can select any of these reports after a model has been solved. To select all three reports, highlight the reports, and then click OK. To access each report, click the appropriate tab at the bottom of the screen.

# 4.4 The Answer Report

Figure 4.2 shows the Answer Report for the Blue Ridge Hot Tubs problem. This report summarizes the solution to the problem, and is fairly self-explanatory. The first section of the report summarizes the original and final (optimal) value of the set cell. The next section summarizes the original and final (optimal) values of the adjustable (or changing) cells representing the decision variables.



Microsoft Excel 12.0 Answer Report
Worksheet: [Fig4-1.xls]Production Report
Report Created: 04/07/2007 12:55:44 PM
Result: Solver found a solution. All constraints and optimality conditions are satisfied.
Engine: Standard LP Simplex
Solution Time: 00 Seconds
Iterations: 2
Subproblems: 0
Incumbent Solutions: 0

Target Cell (Max)

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $D$6 | Unit Profits Total Profit | 0 | 66100 |

Adjustable Cells

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $B$5 | Number to make Aqua-Spas | 0 | 122 |
| $C$5 | Number to make Hydro-Luxes | 0 | 78 |

Constraints

| Cell | Name | Cell Value | Formula | Status | Slack |
|---|---|---|---|---|---|
| $D$9 | Pumps Req'd Used | 200 | $D$9<=$E$9 | Binding | 0 |
| $D$10 | Labor Req'd Used | 1566 | $D$10<=$E$10 | Binding | 0 |
| $D$11 | Tubing Req'd Used | 2712 | $D$11<=$E$11 | Not Binding | 168 |
| $B$5 | Number to make Aqua-Spas | 122 | $B$5>=0 | Not Binding | 122 |
| $C$5 | Number to make Hydro-Luxes | 78 | $C$5>=0 | Not Binding | 78 |



**FIGURE 4.2**

*Answer Report for the hot tub problem*

The final section of this report provides information about the constraints. In particular, the Cell Value column shows the final (optimal) value assumed by each constraint cell. Note that these values correspond to the final value assumed by the LHS formula of each constraint. The Formula column indicates the upper or lower bounds that apply to each constraint cell. The Status column indicates which constraints are binding and which are nonbinding. A constraint is **binding** if it is satisfied as a strict equality in the optimal solution; otherwise, it is **nonbinding**. Notice that the constraints for the number of pumps and amount of labor used are both binding, meaning that *all* the available pumps and labor hours will be used if this solution is implemented. Therefore, these constraints are preventing Blue Ridge Hot Tubs from achieving a higher level of profit.

Finally, the values in the Slack column indicate the difference between the LHS and RHS of each constraint. By definition, binding constraints have zero slack and nonbinding constraints have some positive level of slack. The values in the Slack column indicate that if this solution is implemented, all the available pumps and labor hours will be used, but 168 feet of tubing will be left over. The slack values for the nonnegativity conditions indicate the amounts by which the decision variables exceed their respective lower bounds of zero.

The Answer Report does not provide any information that could not be derived from the solution shown in the spreadsheet model. However, the format of this report gives a convenient summary of the solution that can be incorporated easily into a word-processing document as part of a written report to management.

**Report Headings**

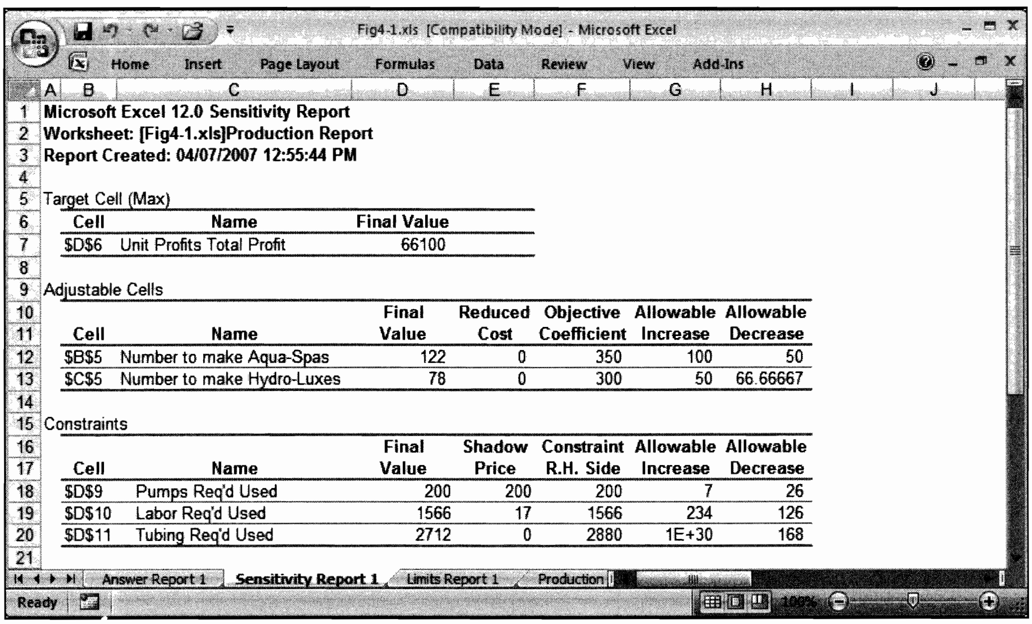
When creating the reports described in this chapter, Solver will try to use various text entries from the original spreadsheet to generate meaningful headings and labels in the reports. Given the various ways in which a model can be implemented, Solver might not always produce meaningful headings. However, you can change any text entry to make the report more meaningful or descriptive.

# 4.5 The Sensitivity Report

Figure 4.3 shows the Sensitivity Report for the Blue Ridge Hot Tubs problem. This report summarizes information about the objective (or target cell), the variable (or adjustable) cells, and constraints for our model. This information is useful in evaluating how sensitive the optimal solution is to changes in various coefficients in the model.

**FIGURE 4.3**

*Sensitivity Report for the hot tub problem*



Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Fig4-1.xls]Production Report
Report Created: 04/07/2007 12:55:44 PM

Target Cell (Max)

| Cell | Name | Final Value |
|---|---|---|
| $D$6 | Unit Profits Total Profit | 66100 |

Adjustable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$5 | Number to make Aqua-Spas | 122 | 0 | 350 | 100 | 50 |
| $C$5 | Number to make Hydro-Luxes | 78 | 0 | 300 | 50 | 66.66667 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $D$9 | Pumps Req'd Used | 200 | 200 | 200 | 7 | 26 |
| $D$10 | Labor Req'd Used | 1566 | 17 | 1566 | 234 | 126 |
| $D$11 | Tubing Req'd Used | 2712 | 0 | 2880 | 1E+30 | 168 |



## 4.5.1 CHANGES IN THE OBJECTIVE FUNCTION COEFFICIENTS

Chapter 2 introduced the level-curve approach to solving a graphical LP problem and showed how to use this approach to solve the Blue Ridge Hot Tubs problem. This graphical solution is repeated in Figure 4.4 (and file Fig4-4.xls on your data disk).

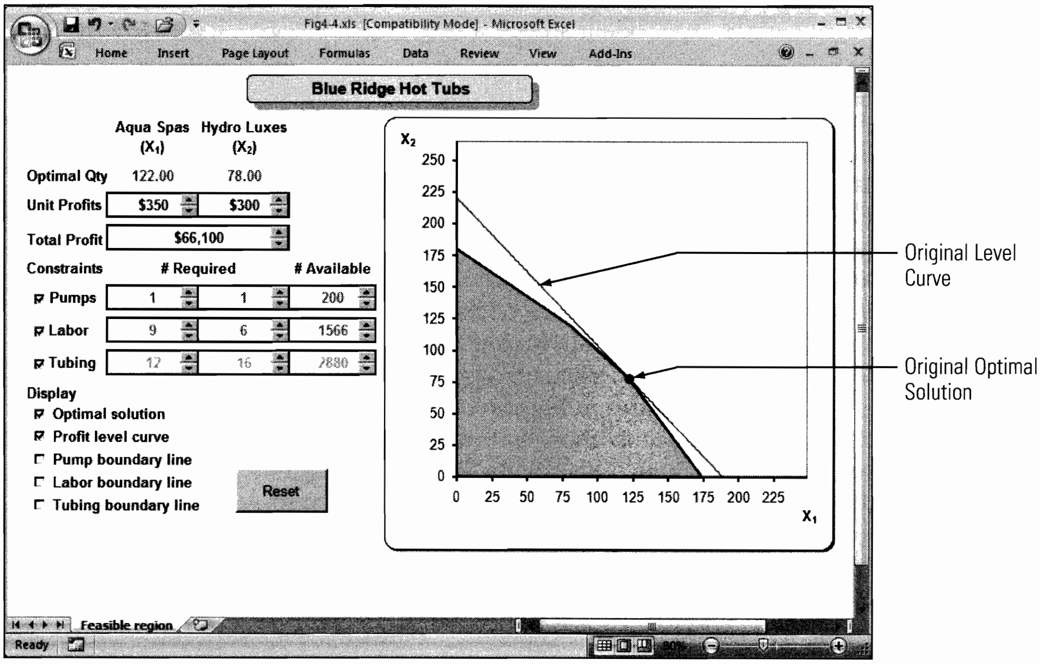



**FIGURE 4.4**

*Graph of original feasible region and optimal solution*

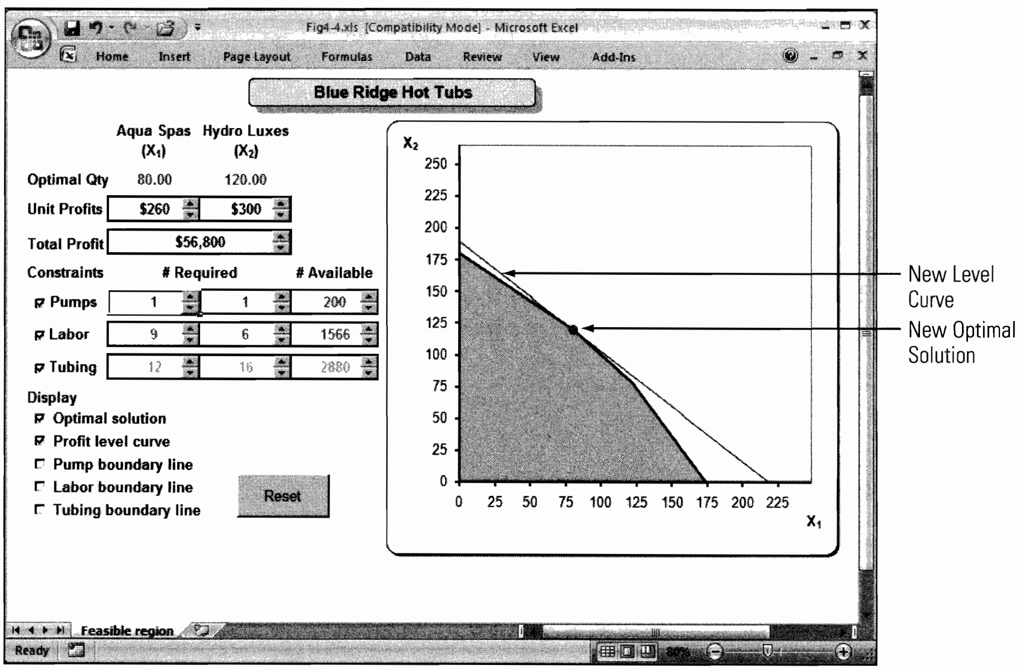



**FIGURE 4.5**

*How a change in an objective function coefficient can change the slope of the level curve and the optimal solution*

The slope of the original level curve in Figure 4.4 is determined by the coefficients in the objective function of the model (the values 350 and 300). In Figure 4.5, we can see that if the slope of the level curve were different, the extreme point represented by $X_1 = 80$, $X_2 = 120$ would be the optimal solution. Of course, the only way to change the level curve for the objective function is to change the coefficients in the objective function. So, if the objective function coefficients are at all uncertain, we might be interested in determining how much these values could change before the optimal solution would change.

For example, if the owner of Blue Ridge Hot Tubs does not have complete control over the costs of producing hot tubs (which is likely because he purchases the fiberglass hot tub shells from another company), the profit figures in the objective function of our LP model might not be the exact profits earned on hot tubs produced in the future. So before the manager decides to produce 122 Aqua-Spas and 78 Hydro-Luxes, he might want to determine how sensitive this solution is to the profit figures in the objective. That is, the manager might want to determine how much the profit figures could change before the optimal solution of $X_1 = 122$, $X_2 = 78$ would change. This information is provided in the Sensitivity Report shown in Figure 4.3.

The original objective function coefficients associated with the variable cells are listed in the Objective Coefficient column in Figure 4.3. The next two columns show the allowable increases and decreases in these values. For example, the objective function value associated with Aqua-Spas (or variable $X_1$) can increase by as much as \$100 or decrease by as much as \$50 without changing the optimal solution, assuming all other coefficients remain constant. (You can verify this by changing the profit coefficient for Aqua-Spas to any value in the range from \$300 to \$450 and re-solving the model.) Similarly, the objective function value associated with Hydro-Luxes (or variable $X_2$) can increase by \$50 or decrease by approximately \$66.67 without changing the optimal values of the decision variables, assuming all other coefficients remain constant. (Again, you can verify this by re-solving the model with different profit values for Hydro-Luxes.)

**Software Note**

When setting up a spreadsheet model for an LP problem for which you intend to generate a sensitivity report, it is a good idea to make sure that the cells corresponding to RHS values of constraints contain constants or formulas that do not involve the decision variables. Thus, any RHS formula related directly or indirectly to the decision variables should be moved algebraically to the left-hand side of the constraint before implementing your model. This will help to reduce problems in interpreting Solver's sensitivity report.

## 4.5.2 A NOTE ABOUT CONSTANCY

The phrase "assuming all other coefficients remain constant" in the previous paragraph reinforces that the allowable increases and decreases shown in the Sensitivity Report apply only if *all* the other coefficients in the LP model do not change. The objective coefficient for Aqua-Spas can assume any value from \$300 to \$450 without changing the optimal solution—*but this is guaranteed to be true only if all the other coefficients in the model remain constant* (*including the objective function coefficient for* $X_2$). Similarly, the objective function coefficient for $X_2$ can assume any value between \$233.33 and \$350 without changing the optimal solution—*but this is guaranteed to be true only if all the other coefficients in the model remain constant* (*including the objective function coefficient for* $X_1$). Later in this chapter, you will see how to determine whether the current solution remains optimal if changes are made in two or more objective coefficients at the same time.

### 4.5.3 ALTERNATE OPTIMAL SOLUTIONS

Sometimes, the allowable increase or allowable decrease for the objective function coefficient for one or more variables will equal zero. In the absence of degeneracy (to be described later), this indicates that alternate optimal solutions exist. Usually you can get Solver to produce an alternate optimal solution (when they exist) by: (1) adding a constraint to your model that holds the objective function at the current optimal value, and then (2) attempting to maximize or minimize the value of one of the decision variables that had an objective function coefficient with an allowable increase or decrease of zero. This approach sometimes involves some "trial and error" in step 2, but should cause Solver to produce an alternate optimal solution to your problem.

### 4.5.4 CHANGES IN THE RHS VALUES

As noted earlier, constraints that have zero slack in the optimal solution to an LP problem are called binding constraints. Binding constraints prevent us from further improving (that is, maximizing or minimizing) the objective function. For example, the Answer Report in Figure 4.2 indicates that the constraints for the number of pumps and hours of labor available are binding, whereas the constraint on the amount of tubing available is nonbinding. This is also evident in Figure 4.3 by comparing the Final Value column with the Constraint R.H. Side column. The values in the Final Value column represent the LHS values of each constraint at the optimal solution. A constraint is binding if its Final Value is equal to its Constraint R.H. Side value.

After solving an LP problem, you might want to determine how much better or worse the solution would be if we had more or less of a given resource. For example, Howie Jones might wonder how much more profit could be earned if additional pumps or labor hours were available. The Shadow Price column in Figure 4.3 provides the answers to such questions.

The **shadow price** for a constraint indicates the amount by which the objective function value changes given a unit *increase* in the RHS value of the constraint, assuming that all other coefficients remain constant. If a shadow price is positive, a unit increase in the RHS value of the associated constraint results in an increase in the optimal objective function value. If a shadow price is negative, a unit increase in the RHS value of the associated constraint results in a decrease in the optimal objective function value. To analyze the effects of decreases in the RHS values, you reverse the sign on the shadow price. That is, the negated shadow price for a constraint indicates the amount by which the optimal objective function value changes given a unit *decrease* in the RHS value of the constraint, assuming that all other coefficients remain constant. The shadow price values apply, provided that the increase or decrease in the RHS value falls within the allowable increase or allowable decrease limits in the Sensitivity Report for each constraint.

For example, Figure 4.3 indicates that the shadow price for the labor constraint is 16.67. Therefore, if the number of available labor hours increased by any amount in the range from 0 to 234 hours, the optimal objective function value changes (increases) by \$16.67 for each additional labor hour. If the number of available labor hours decreased by any amount in the range from 0 to 126 hours, the optimal objective function value changes (decreases) by −\$16.67 for each lost labor hour. A similar interpretation holds for the shadow price for the constraint on the number of pumps. (It is coincidental that the shadow price for the pump constraint (200) is the same as that constraint's RHS and Final Values.)

## 4.5.5 SHADOW PRICES FOR NONBINDING CONSTRAINTS

Now, let's consider the shadow price for the nonbinding tubing constraint. The tubing constraint has a shadow price of zero with an allowable increase of infinity and an allowable decrease of 168. Therefore, if the RHS value for the tubing constraint increases by *any* amount, the objective function value does not change (or changes by zero). This result is not surprising. Because the optimal solution to this problem leaves 168 feet of tubing unused, *additional* tubing will not produce a better solution. Furthermore, because the optimal solution includes 168 feet of unused tubing, we can reduce the RHS value of this constraint by 168 without affecting the optimal solution.

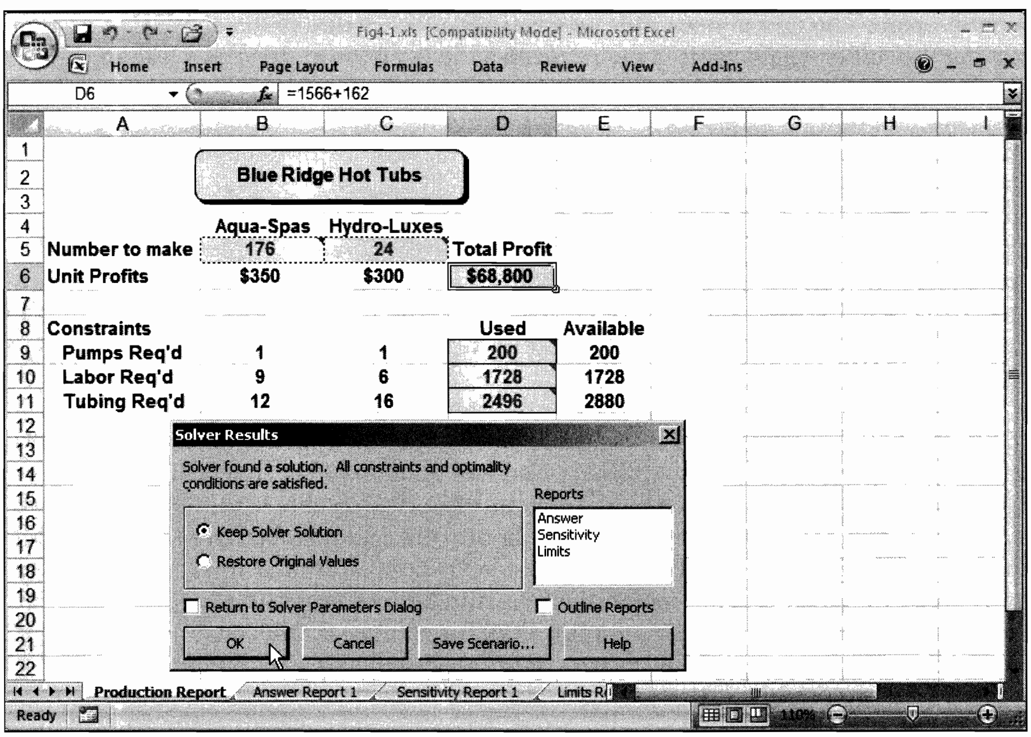
As this example illustrates, the shadow price of a nonbinding constraint is always zero. There is always some amount by which the RHS value of a nonbinding constraint can be changed without affecting the optimal solution.

## 4.5.6 A NOTE ABOUT SHADOW PRICES

One important point needs to be made concerning shadow prices. To illustrate this point, let's suppose that the RHS value of the labor constraint for our example problem increases by 162 hours (from 1,566 to 1,728) due to the addition of new workers. Because this increase is within the allowable increase listed for the labor constraint, you might expect that the optimal objective function value would increase by $16.67 × 162 = $2,700. That is, the new optimal objective function value would be approximately $68,800 ($66,100 + $16.67 × 162 = $68,800). Figure 4.6 shows the re-solved model after increasing the RHS value for the labor constraint by 162 labor hours to 1,728.

**FIGURE 4.6**

*Solution to the revised hot tub problem with 162 additional labor hours*



In Figure 4.6, the new optimal objective function value is $68,800, as expected. But this solution involves producing 176 Aqua-Spas and 24 Hydro-Luxes. That is, the optimal solution to the revised problem is *different* from the solution to the original problem shown in Figure 4.1. This is not surprising, because changing the RHS of a constraint also changes the feasible region for the problem. The effect of increasing the RHS of the labor constraint is shown graphically in Figure 4.7.

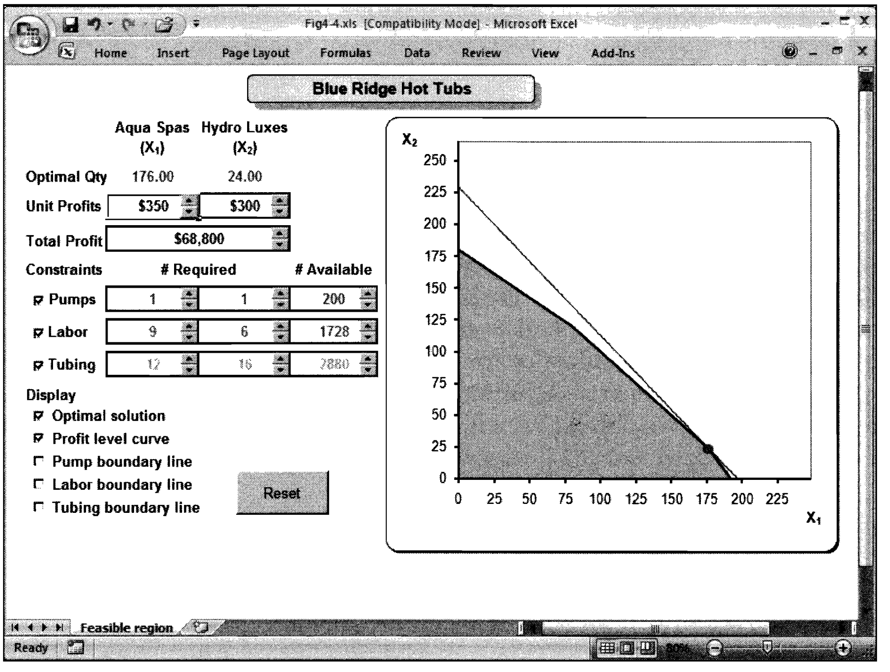



**FIGURE 4.7**

*How a change in the RHS value of the labor constraint changes the feasible region and optimal solution*

So, although shadow prices indicate how the objective function value changes if a given RHS value changes, they *do not* tell you which values the decision variables need to assume to achieve this new objective function value. Determining the new optimal values for the decision variables requires that you make the appropriate changes in the RHS value and re-solve the model.

## Another Interpretation of Shadow Prices

Unfortunately, there is no one universally accepted way of reporting shadow prices for constraints. In some software packages, the signs of the shadow prices do not conform to the convention used by Solver. Regardless of which software package you use, there is another way to look at shadow prices that always should lead to a proper interpretation. The absolute value of the shadow price always indicates the amount by which the objective function will be *improved* if the corresponding constraint is *loosened*. A "less than or equal to" constraint is loosened by *increasing* its RHS value, whereas a "greater than or equal to" constraint is loosened by *decreasing* its RHS value. (The absolute value of the shadow price also can be interpreted as the amount by which the objective will be made *worse* if the corresponding constraint is *tightened*.)

### 4.5.7 SHADOW PRICES AND THE VALUE OF ADDITIONAL RESOURCES

In the previous example, an additional 162 hours of labor allowed us to increase profits by \$2,700. A question then might arise as to how much we should be willing to pay to acquire these additional 162 hours of labor. The answer to this question is, "It depends. . . ."

If labor is a *variable* cost that was subtracted (along with other variable costs) from the selling price of the hot tubs to determine the marginal profits associated with each type of tub, we should be willing to pay up to \$2,700 *above and beyond* what we ordinarily would pay to acquire 162 hours of labor. In this case, notice that both the original and revised profit figures of \$66,100 and \$68,800, respectively, represent the profit earned *after* the normal labor charge has been paid. Therefore, we could pay a premium of up to \$2,700 to acquire the additional 162 hours of labor (or an extra \$16.67 per additional labor hour) and still earn at least as much profit as we would have without the additional 162 hours of labor. Thus, if the normal labor rate is \$12 per hour, we could pay up to \$28.67 per hour to acquire each of the additional 162 hours of labor.

On the other hand, if labor is a sunk cost, which must be paid regardless of how many hot tubs are produced, it would not (or should not) have been subtracted from the selling price of the hot tubs in determining the marginal profit coefficients for each tub produced. In this case, we should be willing to pay a maximum of \$16.67 per hour to acquire each of the additional 162 hours of labor.

### 4.5.8 OTHER USES OF SHADOW PRICES

Because shadow prices represent the marginal values of the resources in an LP problem, they can help us answer several other managerial questions that might arise. For example, suppose that Blue Ridge Hot Tubs is considering introducing a new model of hot tub called the Typhoon-Lagoon. Suppose that each unit of this new model requires 1 pump, 8 hours of labor, and 13 feet of tubing, and can be sold to generate a marginal profit of \$320. Would production of this new model be profitable?

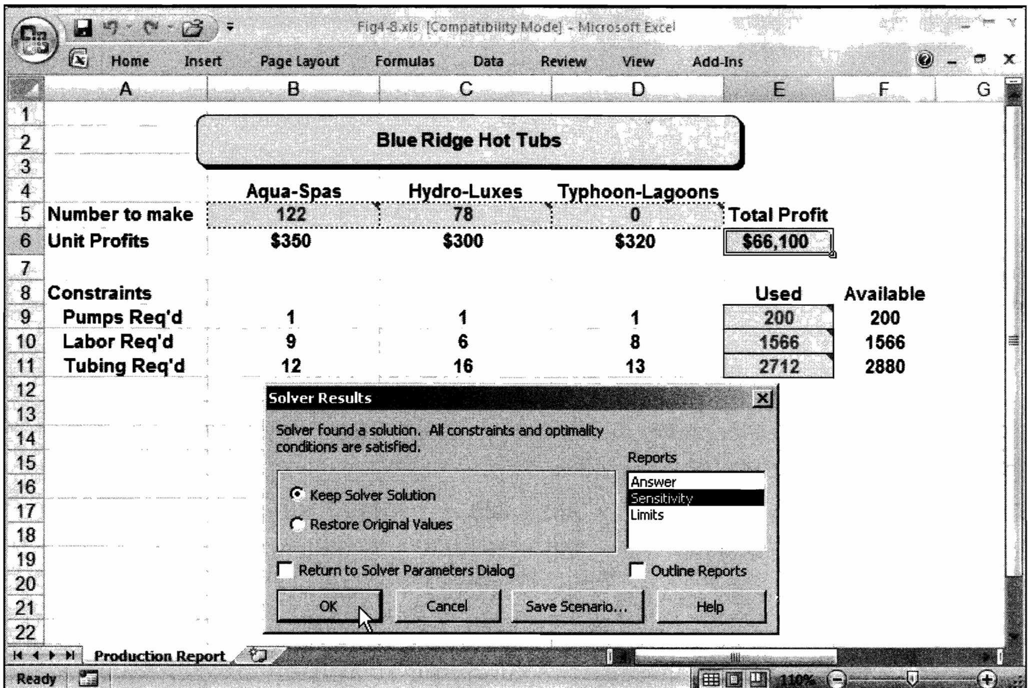
Because Blue Ridge Hot Tubs has limited resources, the production of any Typhoon-Lagoons would consume some of the resources currently devoted to the production of Aqua-Spas and Hydro-Luxes. So, producing Typhoon-Lagoons will reduce the number of pumps, labor hours, and tubing available for producing the other types of hot tubs. The shadow prices in Figure 4.3 indicate that each pump taken away from production of the current products will reduce profits by \$200. Similarly, each labor hour taken away from the production of the current products will reduce profits by \$16.67. The shadow price for the tubing constraint indicates that the supply of tubing can be reduced without adversely affecting profits.

Because each Typhoon-Lagoon requires 1 pump, 8 hours of labor, and 13 feet of tubing, the diversion of resources required to produce one unit of this new model would cause a reduction in profit of $\$200 \times 1 + \$16.67 \times 8 + \$0 \times 13 = \$333.33$. This reduction would be partially offset by the \$320 increase in profit generated by each Typhoon-Lagoon. The net effect of producing each Typhoon-Lagoon would be a \$13.33 reduction in profit ($\$320 - \$333.33 = -\$13.33$). Therefore, the production of Typhoon-Lagoons would not be profitable (although the company might choose to produce a small number of Typhoon-Lagoons to enhance its product line for marketing purposes).

Another way to determine whether or not Typhoon-Lagoons should be produced is to add this alternative to our model and solve the resulting LP problem. The LP model for this revised problem is represented as follows, where $X_1$, $X_2$, and $X_3$ represent the number of Aqua-Spas, Hydro-Luxes, and Typhoon-Lagoons to be produced, respectively:

$$
\begin{array}{llll}
\text{MAX:} & 350X_1 + 300X_2 + 320X_3 & & \}\text{ profit} \\
\text{Subject to:} & 1X_1 + \phantom{1}1X_2 + \phantom{1}1X_3 \leq & 200 & \}\text{ pump constraint} \\
 & 9X_1 + \phantom{1}6X_2 + \phantom{1}8X_3 \leq & 1{,}566 & \}\text{ labor constraint} \\
 & 12X_1 + 16X_2 + 13X_3 \leq & 2{,}880 & \}\text{ tubing constraint} \\
 & X_1, X_2, X_3 \geq & 0 & \}\text{ nonnegativity conditions}
\end{array}
$$

This model is implemented and solved in the spreadsheet, as shown in Figure 4.8. Notice that the optimal solution to this problem involves producing 122 Aqua-Spas ($X_1 = 122$), 78 Hydro-Luxes ($X_2 = 78$), and no Typhoon-Lagoons ($X_3 = 0$). So, as expected, the optimal solution does not involve producing Typhoon-Lagoons. Figure 4.9 shows the Sensitivity Report for our revised model.




**Key Cell Formulas**

| Cell | Formula | Copied to |
|---|---|---|
| E6 | =SUMPRODUCT(B6:D6,$B$5:$D$5) | E9:E11 |

**FIGURE 4.8**

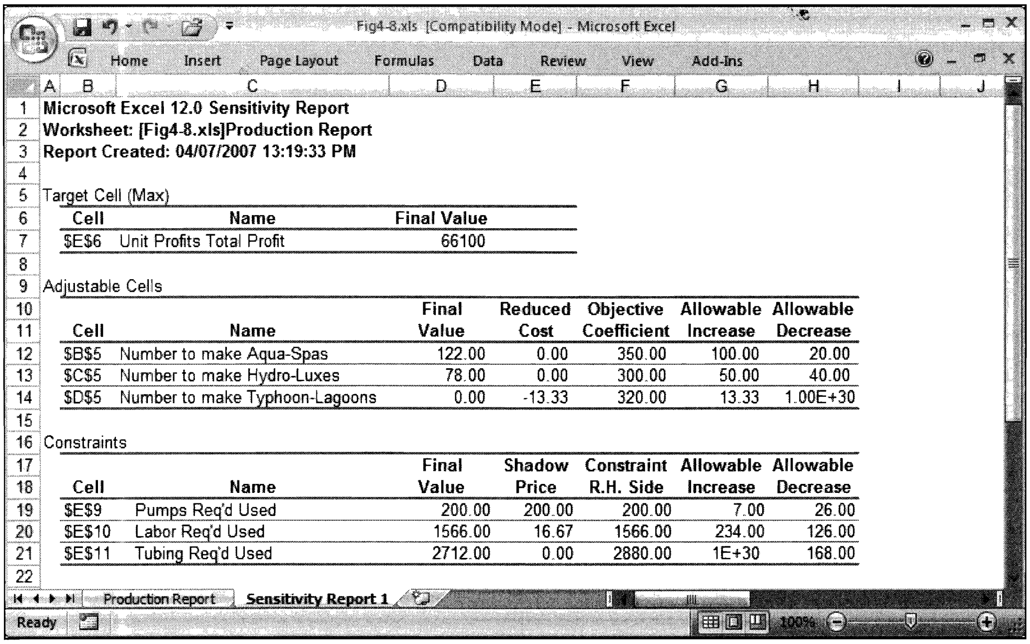
*Spreadsheet model for the revised product mix problem with three hot tub models*

## 4.5.9 THE MEANING OF THE REDUCED COSTS

The Sensitivity Report in Figure 4.9 for our revised model is identical to the sensitivity report for our original model *except* that it includes an additional row in the adjustable cells section. This row reports sensitivity information on the number of Typhoon-Lagoons

**FIGURE 4.9**

*Sensitivity Report for the revised product mix problem with three hot tub models*



Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Fig4-8.xls]Production Report
Report Created: 04/07/2007 13:19:33 PM

Target Cell (Max)

| Cell | Name | Final Value |
|---|---|---|
| $E$6 | Unit Profits Total Profit | 66100 |

Adjustable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$5 | Number to make Aqua-Spas | 122.00 | 0.00 | 350.00 | 100.00 | 20.00 |
| $C$5 | Number to make Hydro-Luxes | 78.00 | 0.00 | 300.00 | 50.00 | 40.00 |
| $D$5 | Number to make Typhoon-Lagoons | 0.00 | -13.33 | 320.00 | 13.33 | 1.00E+30 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $E$9 | Pumps Req'd Used | 200.00 | 200.00 | 200.00 | 7.00 | 26.00 |
| $E$10 | Labor Req'd Used | 1566.00 | 16.67 | 1566.00 | 234.00 | 126.00 |
| $E$11 | Tubing Req'd Used | 2712.00 | 0.00 | 2880.00 | 1E+30 | 168.00 |

to produce. Notice that the Reduced Cost column indicates that the reduced cost value for Typhoon-Lagoons is $-13.33$. This is the same number that we calculated in the previous section when determining whether or not it would be profitable to produce Typhoon-Lagoons.

The reduced cost for each variable is equal to the per-unit amount that the product contributes to profits, minus the per-unit value of the resources it consumes (where the consumed resources are priced at their shadow prices). For example, the reduced cost of each variable in this problem is calculated as:

Reduced cost of Aqua-Spas $= 350 - 200 \times 1 - 16.67 \times 9 - 0 \times 12 = 0$

Reduced cost of Hydro-Luxes $= 300 - 200 \times 1 - 16.67 \times 6 - 0 \times 16 = 0$

Reduced cost of Typhoon-Lagoons $= 320 - 200 \times 1 - 16.67 \times 8 - 0 \times 13 = -13.33$

The allowable increase in the objective function coefficient for Typhoon-Lagoons equals 13.33. This means that the current solution will remain optimal provided that the marginal profit on Typhoon-Lagoons is less than or equal to $\$320 + \$13.33 = \$333.33$ (because this would keep its reduced cost less than or equal to zero). However, if the marginal profit for Typhoon-Lagoons is more than \$333.33, producing this product would be profitable and the optimal solution to the problem would change.

It is interesting to note that the shadow prices (marginal values) of the resources equate exactly with the marginal profits of the products that, at optimality, assume values between their simple lower and upper bounds. This will always be the case. In the optimal solution to an LP problem, the variables that assume values *between* their simple lower and upper bounds always have reduced cost values of zero. (In our example problem, all the variables have implicit simple upper bounds of positive infinity.) The variables with optimal values equal to their simple lower bounds have reduced cost values that are less than or equal to zero for maximization problems, or greater than or equal to zero for minimization problems. Variables with optimal values equal to their simple upper bounds have reduced cost values that are greater than or equal to zero for

| Type of Problem | Optimal Value of Decision Variable | Optimal Value of Reduced Cost |
|---|---|---|
| Maximization | at simple lower bound | $\leq 0$ |
| | between lower and upper bounds | $= 0$ |
| | at simple upper bound | $\geq 0$ |
| Minimization | at simple lower bound | $\geq 0$ |
| | between lower and upper bounds | $= 0$ |
| | at simple upper bound | $\leq 0$ |

**FIGURE 4.10**

*Summary of optimal reduced cost values*

maximization problems, or less than or equal to zero for minimization problems. Figure 4.10 summarizes these relationships.

Generally, at optimality, a variable assumes its largest possible value (or is set equal to its simple upper bound) if this variable helps improve the objective function value. In a maximization problem, the variable's reduced cost must be nonnegative to indicate that if the variable's value increased, the objective value would increase (improve). In a minimization problem, the variable's reduced cost must be nonpositive to indicate that if the variable's value increased, the objective value would decrease (improve).

Similar arguments can be made for the optimal reduced costs of variables at their lower bounds. At optimality, a variable assumes its smallest (lower bound) value if it cannot be used to improve the objective value. In a maximization problem, the variable's reduced cost must be nonpositive to indicate that if the variable's value increased, the objective value would decrease (worsen). In a minimization problem, the variable's reduced cost must be nonnegative to indicate that if the variable's value increased, the objective value would increase (worsen).

## Key Points

Our discussion of Solver's sensitivity report highlights some key points concerning shadow prices and their relationship to reduced costs. These key points are summarized as:

- The shadow prices of resources equate the marginal value of the resources consumed with the marginal benefit of the goods being produced.
- Resources in excess supply have a shadow price (or marginal value) of zero.
- The reduced cost of a product is the difference between its marginal profit and the marginal value of the resources it consumes.
- Products whose marginal profits are less than the marginal value of the goods required for their production will not be produced in an optimal solution.

## 4.5.10 ANALYZING CHANGES IN CONSTRAINT COEFFICIENTS

Given what we know about reduced costs and shadow prices, we can now analyze how changes in some constraint coefficients affect the optimal solution to an LP problem. For example, it is unprofitable for Blue Ridge Hot Tubs to manufacture Typhoon-Lagoons assuming that each unit requires 8 hours of labor. However, what would happen if the product could be produced in only 7 hours? The reduced cost value for Typhoon-Lagoons is calculated as:

$$\$320 - \$200 \times 1 - \$16.67 \times 7 - \$0 \times 13 = \$3.31$$

Because this new reduced cost value is positive, producing Typhoon-Lagoons would be profitable in this scenario and the solution shown in Figure 4.8 no longer would be optimal. We could also reach this conclusion by changing the labor requirement for Typhoon-Lagoons in our spreadsheet model and re-solving the problem. In fact, we have to do this to determine the new optimal solution if each Typhoon-Lagoon requires only 7 hours of labor.

As another example, suppose that we wanted to know the maximum amount of labor that is required to assemble a Typhoon-Lagoon while keeping its production economically justifiable. The production of Typhoon-Lagoons would be profitable provided that the reduced cost for the product is greater than or equal to zero. If $L_3$ represents the amount of labor required to produce a Typhoon-Lagoon, we want to find the maximum value of $L_3$ that keeps the reduced cost for Typhoon-Lagoons greater than or equal to zero. That is, we want to find the maximum value of $L_3$ that satisfies the inequality:

$$\$320 - \$200 \times 1 - \$16.67 \times L_3 - \$0 \times 13 \geq 0$$

If we solve this inequality for $L_3$, we obtain:

$$L_3 \leq \frac{120}{16.67} = 7.20$$

Thus, the production of Typhoon-Lagoons would be economically justified provided that the labor required to produce them does not exceed 7.20 hours per unit. Similar types of questions can be answered using knowledge of the basic relationships between reduced costs, shadow prices, and optimality conditions.

## 4.5.11 SIMULTANEOUS CHANGES IN OBJECTIVE FUNCTION COEFFICIENTS

Earlier, we noted that the values in the Allowable Increase and Allowable Decrease columns in the Sensitivity Report for the objective function coefficients indicate the maximum amounts by which each objective coefficient can change without altering the optimal solution—*assuming that all other coefficients in the model remain constant*. A technique known as The 100% Rule determines whether the current solution remains optimal when more than one objective function coefficient changes. The following two situations could arise when applying this rule:

**Case 1.** All variables whose objective function coefficients change have non-zero reduced costs.

**Case 2.** At least one variable whose objective function coefficient changes has a reduced cost of zero.

In case 1, the current solution remains optimal provided that the objective function coefficient of each changed variable remains within the limits indicated in the Allowable Increase and Allowable Decrease columns of the Sensitivity Report.

Case 2 is a bit more tricky. In case 2, we must perform the following analysis where:

$c_j$ = the original objective function coefficient for variable $X_j$
$\Delta c_j$ = the planned change in $c_j$
$I_j$ = the allowable increase in $c_j$ given in the Sensitivity Report
$D_j$ = the allowable decrease in $c_j$ given in the Sensitivity Report

$$r_j = \begin{cases} \dfrac{\Delta c_j}{I_j}, & \text{if } \Delta c_j \geq 0 \\[2ex] \dfrac{-\Delta c_j}{D_j}, & \text{if } \Delta c_j < 0 \end{cases}$$

Notice that $r_j$ measures the ratio of the planned change in $c_j$ to the maximum allowable change for which the current solution remains optimal. If only one objective function coefficient changed, the current solution remains optimal provided that $r_j \leq 1$ (or, if $r_j$ is expressed as a percentage, it must be less than or equal to 100%). Similarly, if more than one objective function coefficient changes, the current solution will remain optimal provided that $\Sigma r_j \leq 1$. (Note that if $\Sigma r_j > 1$, the current solution might remain optimal, but this is not guaranteed.)

### 4.5.12 A WARNING ABOUT DEGENERACY

The solution to an LP problem sometimes exhibits a mathematical anomaly known as ***degeneracy***. The solution to an LP problem is degenerate if the RHS values of any of the constraints have an allowable increase or allowable decrease of zero. The presence of degeneracy affects our interpretation of the values on the sensitivity report in a number of important ways:

1. When the solution is degenerate, the methods mentioned earlier for detecting alternate optimal solutions cannot be relied upon.
2. When a solution is degenerate, the reduced costs for the variable cells may not be unique. Additionally, in this case, the objective function coefficients for variable cells must change by at least as much as (and possibly more than) their respective reduced costs before the optimal solution would change.
3. When the solution is degenerate, the allowable increases and decreases for the objective function coefficients still hold and, in fact, the coefficients might have to be changed substantially beyond the allowable increase and decrease limits before the optimal solution changes.
4. When the solution is degenerate, the given shadow prices and their ranges still might be interpreted in the usual way but they might not be unique. That is, a different set of shadow prices and ranges also might apply to the problem (even if the optimal solution is unique).

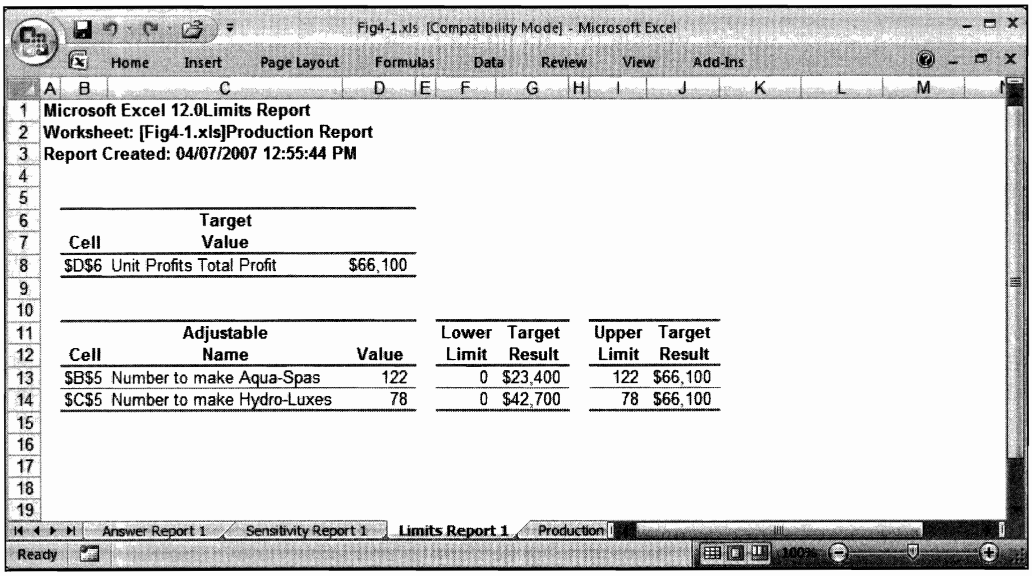
So before interpreting the results on a sensitivity report, you always should check first to see if the solution is degenerate because this has important ramifications for how the numbers on the report should be interpreted. A complete description of the degeneracy anomaly goes beyond the intended scope of this book. However, degeneracy is sometimes caused by having redundant constraints in an LP model. ***Extreme caution*** (and perhaps consultation with an expert in mathematical programming) is in order if important business decisions are being made based on the sensitivity report for a degenerate LP problem.

## 4.6 The Limits Report

The Limits Report for the original Blue Ridge Hot Tubs problem is shown in Figure 4.11. This report lists the optimal value of the set cell. It then summarizes the optimal values for each variable cell and indicates what values the set cell assumes if each variable cell is set to its upper or lower limits. The values in the Lower Limits column indicate the smallest value that each variable cell can assume while the values of all other variable cells remain constant and all the constraints are satisfied. The values in the Upper Limits column indicate the largest value that each variable cell can assume while the values of all other variable cells remain constant and all the constraints are satisfied.

**FIGURE 4.11**

*Limits Report for the original Blue Ridge Hot Tubs problem*




Microsoft Excel 12.0Limits Report
Worksheet: [Fig4-1.xls]Production Report
Report Created: 04/07/2007 12:55:44 PM

| Cell | Target Name | Value |
|---|---|---|
| $D$6 | Unit Profits Total Profit | $66,100 |

| Cell | Adjustable Name | Value | Lower Limit | Target Result | Upper Limit | Target Result |
|---|---|---|---|---|---|---|
| $B$5 | Number to make Aqua-Spas | 122 | 0 | $23,400 | 122 | $66,100 |
| $C$5 | Number to make Hydro-Luxes | 78 | 0 | $42,700 | 78 | $66,100 |

# 4.7 The Sensitivity Assistant Add-in (Optional)

This book comes with an Excel add-in called Sensitivity Assistant that provides two additional tools for performing sensitivity analysis on an *ad hoc* basis: Spider Tables and Solver Tables. A Spider Table summarizes the optimal value for one output cell as individual changes are made to various input cells. A Solver Table summarizes the optimal value of multiple output cells as changes are made to a single input cell. As illustrated in the following sections, these tools can be helpful in developing an understanding of how changes in various model parameters affect the optimal solution to a problem.

## Installing the Sensitivity Assistant Add-In

To use the Sensitivity Assistant add-in, it must first be installed on your computer. To do this:

1. Copy the file Sensitivity.xla from the CD-ROM accompanying this book to the folder on your hard drive containing the file Solver.xla. (On most cases, this is the folder: *d*:\Program Files\Microsoft Office\Office12\Library\Solver.)

*(Continued)*

2. In Excel, click the Office button, Excel Options, Add-Ins, Go; browse to the Sensitivity.xla file and click OK.

This instructs your computer to open the Sensitivity Assistant add-in whenever you start Excel. It also causes the "Sensitivity Assistant . . ." option to be added to the Add-Ins ribbon in Excel. You can deactivate the Sensitivity Assistant add-in at any time by clicking the Office button, Excel Options, Add-Ins, Go command again.

## 4.7.1 CREATING SPIDER TABLES AND PLOTS

Recall that the optimal solution to the original Blue Ridge Hot Tubs problem involves producing 122 Aqua-Spas and 78 Hydro-Luxes for a total profit of $66,100. However, this solution assumes there will be exactly 200 pumps, 1,566 labor hours, and 2,880 feet of tubing available. In reality, pumps and tubing are sometimes defective, and workers sometimes call in sick. So, the owner of the company might wonder how sensitive the total profit is to changes in these resources. Although Solver's Sensitivity Report provides some information about this issue, a Spider Table and Plot is sometimes more helpful in communicating this information to management.

Again, a Spider Table summarizes the optimal value for one output cell as individual changes are made to various input cells. In this case, the output cell of interest is cell D6 representing total profit, and the input cells are E9, E10, and E11 representing the availability of pumps, labor, and tubing. Figure 4.12 (and file Fig4-12.xls on your data disk) shows how to set up a Spider Table for this problem.

The upper-left cell in a Spider Table should contain a formula referring to the output cell you want to track. Thus, in Figure 4.12, cell A14 contains a formula that returns the value of cell D6 representing total profit.

Formula for cell A14: =D6

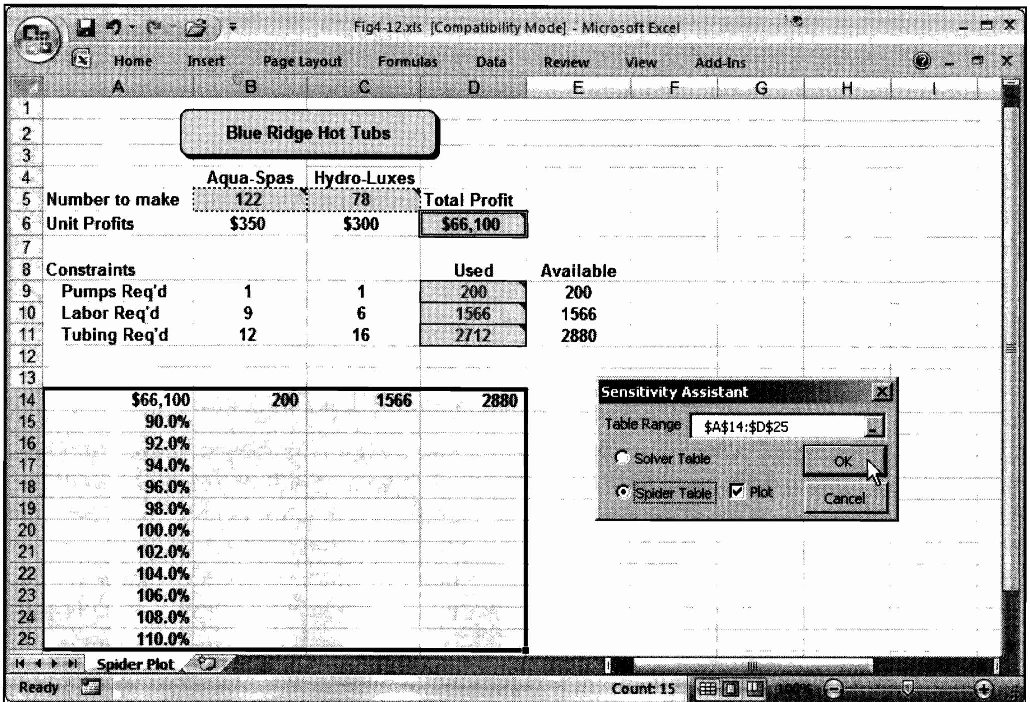
The remaining cells in the first row of a Spider Table should contain formulas referring to the input cells that you want to manipulate. Thus, cells B14, C14, and D14 contain formulas referring, respectively, to cells E9, E10, and E11, which represent the resource availability values that we want to manipulate.

Formula for cell B14: =E9
Formula for cell C14: =E10
Formula for cell D14: =E11

The first cell in each remaining row of a Spider Table should contain a percentage value indicating an amount by which to adjust each of the input cells. Thus, in cells A15 through A25, we entered a series of percentages from 90% to 110%. The Spider Table tool will multiply the value of each of the input cells referenced in cells B14 through D14 by each of the percentages listed in cells A15 through A25, and will keep a record of the corresponding value of the output cell referenced in cell A14. Specifically, the Spider Table tool will first change the value in cell E9 (representing the number of pumps available) to 180 (or 90% of 200), re-solve the problem, and record the optimal profit in cell B15. It will then change the value in cell E9 to 184 (or 92% of 200), re-solve the problem, and record the optimal profit in cell B16. This process continues until all the input cells have been varied from 90% to 110% and the resulting optimal profit values recorded in the table.

**FIGURE 4.12**

*Set up for creating a Spider Table and Plot*




**Key Cell Formulas**

| Cell | Formula | Copied to |
|---|---|---|
| A14 | =D6 | -- |
| B14 | =E9 | -- |
| C14 | =E10 | -- |
| D14 | =E11 | -- |

To complete the Spider Table, we would proceed through the following steps. The resulting table and plot is shown in Figure 4.13.

1. Select the range A14:D25
2. Click Add-Ins, Sensitivity Assistant
3. Select the Spider Table option
4. Click OK

The rationale for the name of this procedure should be readily apparent from the spider-like plot shown in Figure 4.13. The center point in the graph corresponds to the optimal solution to the original model with 100% of the pumps, labor, and tubing available. Each of the points in the graph show the effect on total profit of varying the original resource levels by the indicated percentage.

It is clear from Figure 4.13 that total profit is relatively insensitive to modest decreases or large increases in the availability of tubing. This is consistent with the sensitivity information regarding tubing shown earlier in Figure 4.9. The optimal solution to the original problem involved using all the pumps and all the labor hours, but only 2,712 feet of the 2,880 available feet of tubing. As a result, we could achieve the same level of profit even if the availability of tubing was reduced by 168 feet (or to about 94.2% of its original

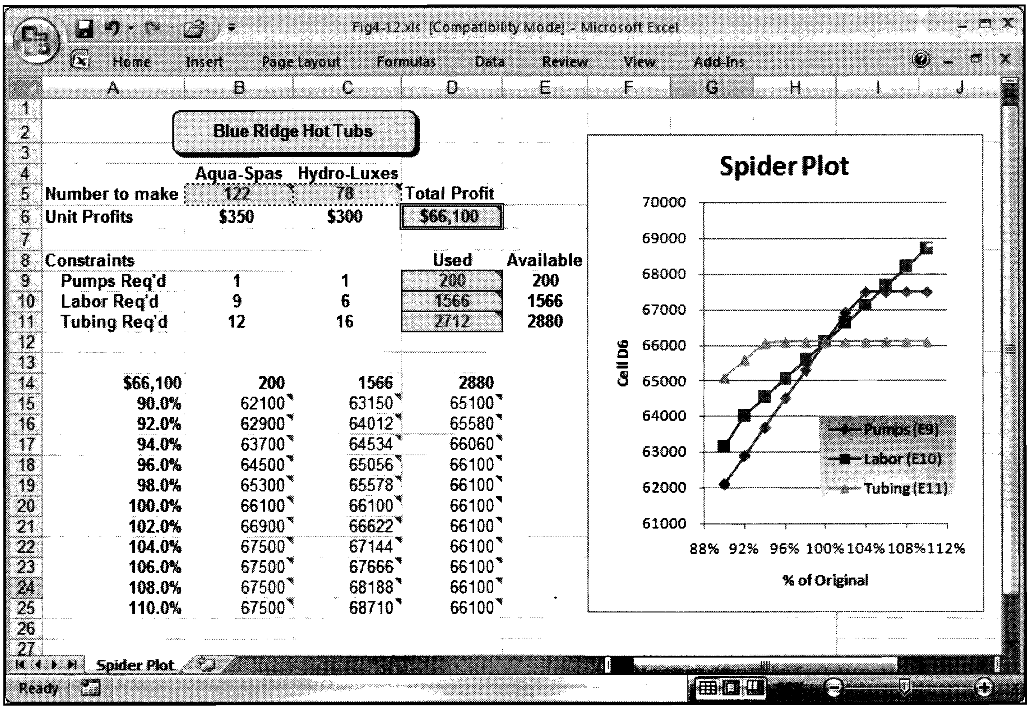



**FIGURE 4.13**

*A Spider Table and Plot showing the relationship between profit and the availability of pumps, labor, and tubing*

value). Similarly, because we are not using all of the available tubing, acquiring more tubing would only increase the surplus and not allow for any improvement in profit. Thus, our Spider Table and Plot suggest that the availability of tubing probably should not be a top concern in this problem. On the other hand, the Spider Plot suggests that changes in the availability of pumps and labor have a more pronounced effect on profit and the optimal solution to the problem.

**Notes on Setting Up a Spider Table**

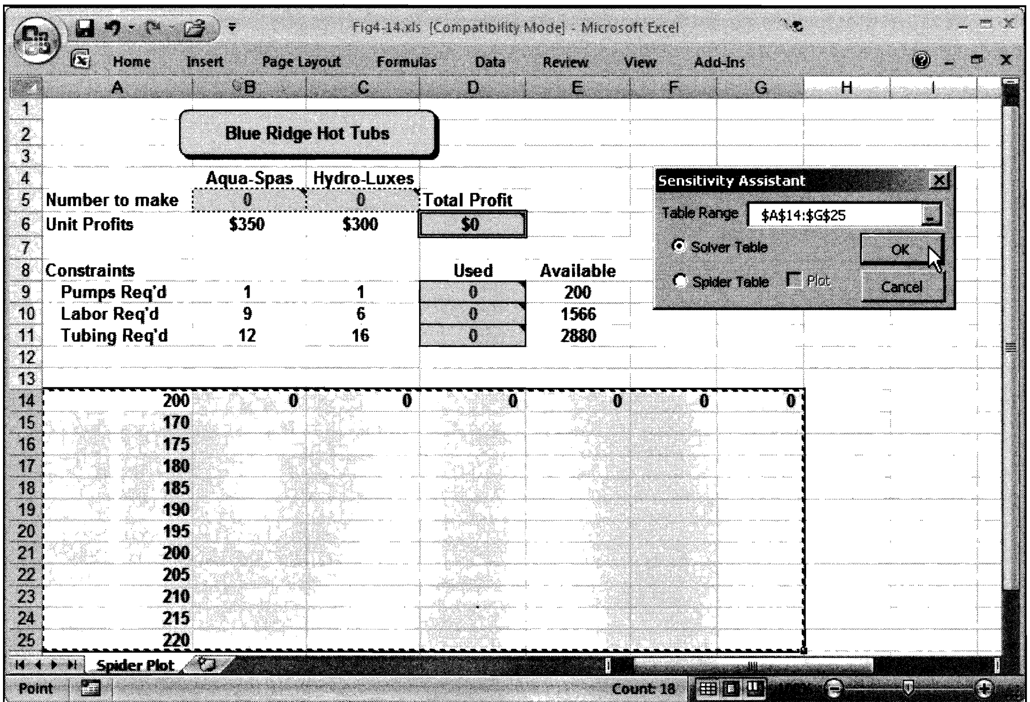
1. The cell in the upper-left corner of the table range should contain a formula returning the value of the output cell you want to track.
2. The remaining cells in the first row of the table range should contain formulas referring to the input cells you want to manipulate.
3. The first cell in each remaining row of the table range should contain a percentage value indicating an amount by which to multiply each of the input cells.

## 4.7.2 CREATING A SOLVER TABLE

The Spider Plot in Figure 4.13 suggests that the total profit earned is most sensitive to changes in the available supply of pumps. We can create a Solver Table to study in greater detail the impact of changes in the available number of pumps. Recall that a Solver Table summarizes the optimal value of multiple output cells as changes are made

**FIGURE 4.14**

*Set up for creating a Solver Table*




**Key Cell Formulas**

| Cell | Formula | Copied to |
|---|---|---|
| A14 | =E9 | -- |
| B14 | =B5 | -- |
| C14 | =C5 | -- |
| D14 | =D6 | -- |
| E14 | =D9 | -- |
| F14 | =D10 | -- |
| G14 | =D11 | -- |

to a single input cell. In this case, the single input we want to change is cell E9, which represents the number of pumps available. We might want to track what happens to several output cells, including the optimal number of Aqua-Spas and Hydro-Luxes (cells B5 and B6), the total profit (cell D6), and the total amount of pumps, labor, and tubing used (cells D9, D10, and D11). Figure 4.14 (and file Fig4-14.xls on your data disk) shows how to set up the Solver Table for this problem.

The upper-left cell in a Solver Table should contain a formula referring to the input cell you want to manipulate. Thus, cell A14 contains a formula referring to cell E9, which represents the available number of pumps.

Formula for cell A14: =E9

The remaining cells in the first row of a Solver Table should contain formulas returning the values of the various output cells you want to track. Thus, cells B14 through G14 each contain a formula referring to one of our output cells.

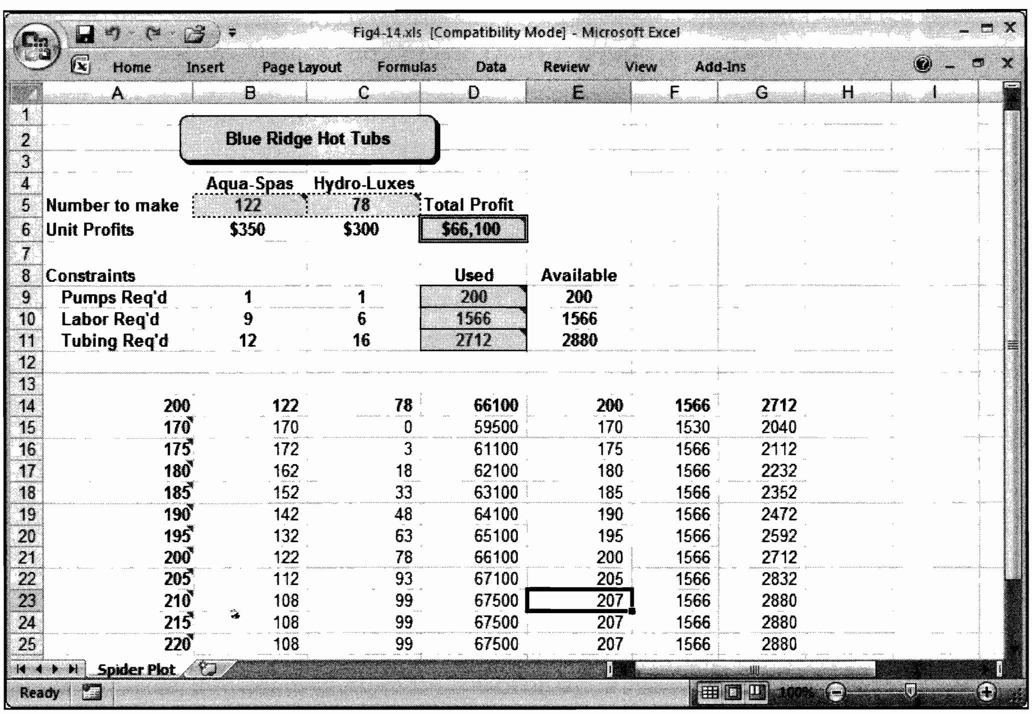
| | |
|---|---|
| Formula for cell B14: | =B5 |
| Formula for cell C14: | =C5 |
| Formula for cell D14: | =D6 |
| Formula for cell E14: | =D9 |
| Formula for cell F14: | =D10 |
| Formula for cell G14 | =D11 |

The first cell in each of the remaining rows of the Solver Table should contain a value to be assumed by the input cell. In this case, we entered values from 170 to 220 (in multiples of five) in cells A15 to A25 as representative values for the input cell (E9) representing the available number of pumps. The Solver Table tool will substitute each of the values in cells A15 through A25 into the input cell (E9), re-solve the problem, and then record the resulting values for the output cells referenced in the first row of the table range.

To complete the Solver Table, we would proceed through the following steps. The resulting table and plot is shown in Figure 4.15.

1. Select the range A14:G25.
2. Click Add-Ins, Sensitivity Assistant.
3. Select the Solver Table option.
4. Click OK.

Several interesting insights emerge from Figure 4.15. First, comparing columns A and E, as the number of available pumps increases from 170 up to 205, they are always all used. With about 175 pumps, we also begin to use all the available labor. However, when the number of available pumps increases to 210 or more, only 207 pumps can be used because we run out of both tubing and labor at that point. This suggests that the




| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 2 | | Blue Ridge Hot Tubs | | | | | |
| 4 | | Aqua-Spas | Hydro-Luxes | | | | |
| 5 | Number to make | 122 | 78 | Total Profit | | | |
| 6 | Unit Profits | $350 | $300 | $66,100 | | | |
| 8 | Constraints | | | Used | Available | | |
| 9 | Pumps Req'd | 1 | 1 | 200 | 200 | | |
| 10 | Labor Req'd | 9 | 6 | 1566 | 1566 | | |
| 11 | Tubing Req'd | 12 | 16 | 2712 | 2880 | | |
| 14 | 200 | 122 | 78 | 66100 | 200 | 1566 | 2712 |
| 15 | 170 | 170 | 0 | 59500 | 170 | 1530 | 2040 |
| 16 | 175 | 172 | 3 | 61100 | 175 | 1566 | 2112 |
| 17 | 180 | 162 | 18 | 62100 | 180 | 1566 | 2232 |
| 18 | 185 | 152 | 33 | 63100 | 185 | 1566 | 2352 |
| 19 | 190 | 142 | 48 | 64100 | 190 | 1566 | 2472 |
| 20 | 195 | 132 | 63 | 65100 | 195 | 1566 | 2592 |
| 21 | 200 | 122 | 78 | 66100 | 200 | 1566 | 2712 |
| 22 | 205 | 112 | 93 | 67100 | 205 | 1566 | 2832 |
| 23 | 210 | 108 | 99 | 67500 | 207 | 1566 | 2880 |
| 24 | 215 | 108 | 99 | 67500 | 207 | 1566 | 2880 |
| 25 | 220 | 108 | 99 | 67500 | 207 | 1566 | 2880 |

**FIGURE 4.15**

*Solver Table showing changes in the optimal solution, profit, and resource usage and the number of pumps changed* company should not be interested in getting more than 7 additional pumps unless it also can increase the amount of tubing and/or labor available.

Also note that the addition or subtraction of 5 pumps from the initial supply of 200 causes the optimal objective function value (column D) to change by $1,000. This suggests that if the company has 200 pumps, the marginal value of each pump is about $200 (i.e., $1000/5 = $200). Of course, this is equivalent to the ***shadow price*** of pumps shown earlier in Figure 4.9.

Finally, it is interesting to note that when the availability of pumps is between 175 and 205, each increase of 5 pumps causes the optimal number of Aqua-Spas to decrease by 10 and the optimal number of Hydro-Luxes to increase by 15. Thus, one advantage of the Solver Table over the Sensitivity Report is that it tells you not only how much the optimal value of the objective function changes as the number of pumps change, but it also can tell you how the optimal solution changes.

**Notes on Setting Up a Solver Table**

1. The cell in the upper-left corner of the table range should contain a formula referring to the input cell you want to manipulate.
2. The remaining cells in the first row of the table range should contain formulas returning the values of the output cells you want to track.
3. The first cell in each remaining row of the table range should contain a value to be assumed by the input cell.

### 4.7.3 COMMENTS

Additional Solver Tables and Spider Tables/Plots could be constructed to analyze every element of the model, including objective function and constraint coefficients. However, these techniques are considered 'computationally expensive' because they require the LP model to be solved repeatedly. For small problems such as Blue Ridge Hot Tubs, this is not really a problem. But as problem size and complexity increases, this approach to sensitivity analysis can become impractical.

## 4.8 The Simplex Method (Optional)

We have mentioned repeatedly that the simplex method is the preferred method for solving LP problems. This section provides an overview of the simplex method and shows how it relates to some of the items that appear on the Answer Report and the Sensitivity Report.

### 4.8.1 CREATING EQUALITY CONSTRAINTS USING SLACK VARIABLES

Because our original formulation of the LP model for the Blue Ridge Hot Tubs problem has only *two* decision variables ($X_1$ and $X_2$), you might be surprised to learn that Solver actually used *five* variables to solve this problem. As you saw in Chapter 2 when we plotted the boundary lines for the constraints in an LP problem, it is easier to work with "equal to" conditions rather than "less than or equal to," or "greater than or equal to" conditions. Similarly, the simplex method requires that *all* constraints in an LP model be expressed as equalities.

To solve an LP problem using the simplex method, Solver temporarily turns all inequality constraints into equality constraints by adding one new variable to each "less than or equal to" constraint and subtracting one new variable from each "greater than or equal to" constraint. The new variables used to create equality constraints are called **slack variables**.

For example, consider the less than or equal to constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \leq b_k$$

Solver can turn this constraint into an equal to constraint by adding the nonnegative slack variable $S_k$ to the LHS of the constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n + S_k = b_k$$

The variable $S_k$ represents the amount by which $a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n$ is less than $b_k$. Now consider the "greater than or equal to" constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \geq b_k$$

Solver can turn this constraint into an "equal to" constraint by subtracting the nonnegative slack variable $S_k$ from the LHS of the constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n - S_k = b_k$$

In this case, the variable $S_k$ represents the amount by which $a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n$ exceeds $b_k$.

To solve the original Blue Ridge Hot Tubs problem using the simplex method, Solver actually solved the following modified problem involving *five* variables:

| | | |
|---|---|---|
| MAX: | $350X_1 + 300X_2$ | } profit |
| Subject to: | $1X_1 + 1X_2 + S_1 = 200$ | } pump constraint |
| | $9X_1 + 6X_2 + S_2 = 1{,}566$ | } labor constraint |
| | $12X_1 + 16X_2 + S_3 = 2{,}880$ | } tubing constraint |
| | $X_1, X_2, S_1, S_2, S_3 \geq 0$ | } nonnegativity conditions |

We will refer to $X_1$ and $X_2$ as the **structural variables** in the model to distinguish them from the slack variables.

Recall that we did not set up slack variables in the spreadsheet or include them in the formulas in the constraint cells. Solver automatically sets up the slack variables it needs to solve a particular problem. The only time Solver even mentions these variables is when it creates an Answer Report like the one shown in Figure 4.2. The values in the Slack column in the Answer Report correspond to the optimal values of the slack variables.

## 4.8.2 BASIC FEASIBLE SOLUTIONS

After all the inequality constraints in an LP problem have been converted into equalities (by adding or subtracting appropriate slack variables), the constraints in the LP model represent a system (or collection) of linear equations. If there are a total of $n$ variables in a system of $m$ equations, one strategy for finding a solution to the system of equations is to select any $m$ variables and try to find values for them that solve the system, assuming that all other variables are set equal to their lower bounds (which usually are zero). This strategy requires more variables than constraints in the system of equations—or that $n \geq m$.

The $m$ variables selected to solve the system of equations in an LP model are sometimes called **basic variables,** whereas the remaining variables are called **nonbasic variables**. If a solution to the system of equations can be obtained using a given set of basic variables (while the nonbasic variables are all set equal to zero), that solution is called a **basic feasible solution.** Every basic feasible solution corresponds to one of the extreme points of the feasible region for the LP problem, and we know that the optimal solution to the LP problem also occurs at an extreme point. So, the challenge in LP is to find the set of basic variables (and their optimal values) that produce the basic feasible solution corresponding to the optimal extreme point of the feasible region.

Because our modified problem involves three constraints and five variables, we could select three basic variables in ten different ways to form possible basic feasible solutions for the problem. Figure 4.16 summarizes the results for these ten options.

The first five solutions in Figure 4.16 are feasible and, therefore, represent basic feasible solutions to this problem. The remaining solutions are infeasible because they violate the nonnegativity conditions. The best feasible alternative shown in Figure 4.16 corresponds to the optimal solution to the problem. In particular, if $X_1$, $X_2$, and $S_3$ are selected as basic variables and $S_1$ and $S_2$ are nonbasic and assigned their lower bound values

**FIGURE 4.16**

*Possible basic feasible solutions for the original Blue Ridge Hot Tubs problem*

| | Basic Variables | Nonbasic Variables | Solution | Objective Value |
|---|---|---|---|---|
| 1 | $S_1, S_2, S_3$ | $X_1, X_2$ | $X_1=0, X_2=0,$ $S_1=200, S_2=1566, S_3=2880$ | 0 |
| 2 | $X_1, S_1, S_3$ | $X_2, S_2$ | $X_1=174, X_2=0,$ $S_1=26, S_2=0, S_3=792$ | 60,900 |
| 3 | $X_1, X_2, S_3$ | $S_1, S_2$ | $X_1=122, X_2=78,$ $S_1=0, S_2=0, S_3=168$ | 66,100 |
| 4 | $X_1, X_2, S_2$ | $S_1, S_3$ | $X_1=80, X_2=120,$ $S_1=0, S_2=126, S_3=0$ | 64,000 |
| 5 | $X_2, S_1, S_2$ | $X_1, S_3$ | $X_1=0, X_2=180,$ $S_1=20, S_2=486, S_3=0$ | 54,000 |
| 6* | $X_1, X_2, S_1$ | $S_2, S_3$ | $X_1=108, X_2=99,$ $S_1=-7, S_2=0, S_3=0$ | 67,500 |
| 7* | $X_1, S_1, S_2$ | $X_2, S_3$ | $X_1=240, X_2=0,$ $S_1=-40, S_2=-594, S_3=0$ | 84,000 |
| 8* | $X_1, S_2, S_3$ | $X_2, S_1$ | $X_1=200, X_2=0,$ $S_1=0, S_2=-234, S_3=480$ | 70,000 |
| 9* | $X_2, S_2, S_3$ | $X_1, S_1$ | $X_1=0, X_2=200,$ $S_1=0, S_2=366, S_3=-320$ | 60,000 |
| 10* | $X_2, S_1, S_3$ | $X_1, S_2$ | $X_1=0, X_2=261,$ $S_1=-61, S_2=0, S_3=-1296$ | 78,300 |

Note: * denotes infeasible solutions

(zero), we try to find values for $X_1$, $X_2$, and $S_3$ that satisfy the following constraints:

$$1X_1 + 1X_2 = 200 \quad \} \text{ pump constraint}$$
$$9X_1 + 6X_2 = 1{,}566 \quad \} \text{ labor constraint}$$
$$12X_1 + 16X_2 + S_3 = 2{,}880 \quad \} \text{ tubing constraint}$$

Notice that $S_1$ and $S_2$ in the modified "equal to" constraints are not included in the above constraint equations because we are assuming that the values of these nonbasic variables are equal to zero (their lower bounds). Using linear algebra, the simplex method determines that the values $X_1 = 122$, $X_2 = 78$, and $S_3 = 168$ satisfy the equations given above. So, a basic feasible solution to this problem is $X_1 = 122$, $X_2 = 78$, $S_1 = 0$, $S_2 = 0$, $S_3 = 168$. As indicated in Figure 4.16, this solution produces an objective function value of $66,100. (Notice that the optimal values for the slack variables $S_1$, $S_2$, and $S_3$ also correspond to the values shown in the Answer Report in Figure 4.2 in the Slack column for constraint cells D9, D10, and D11.) Figure 4.17 shows the relationships between the basic feasible solutions listed in Figure 4.16 and the extreme points of the feasible region for this problem.

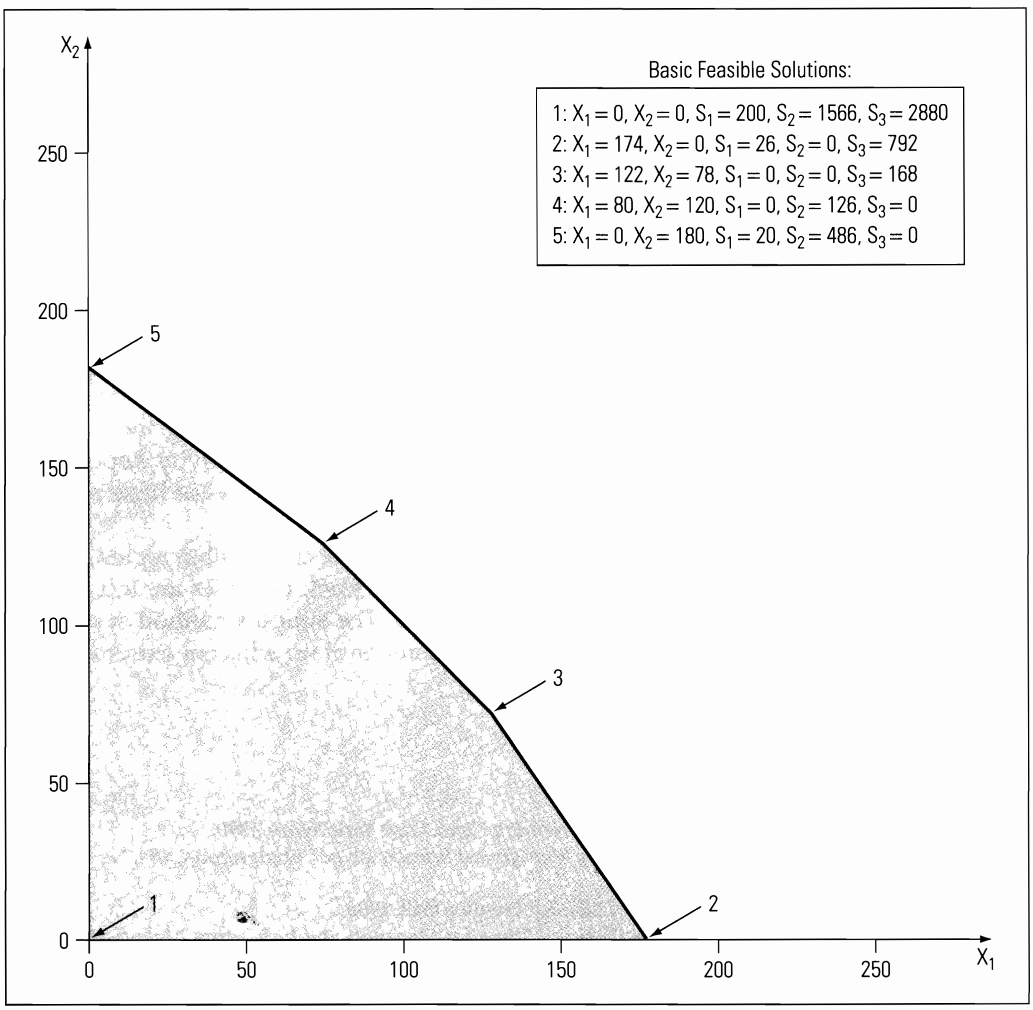



**FIGURE 4.17**

*Illustration of the relationship between basic feasible solutions and extreme points*

### 4.8.3 FINDING THE BEST SOLUTION

The simplex method operates by first identifying any basic feasible solution (or extreme point) for an LP problem, and then moving to an adjacent extreme point, if such a move improves the value of the objective function. When no adjacent extreme point has a better objective function value, the current extreme point is optimal and the simplex method terminates.

The process of moving from one extreme point to an adjacent one is accomplished by switching one of the basic variables with one of the nonbasic variables to create a new basic feasible solution that corresponds to the adjacent extreme point. For example, in Figure 4.17, moving from the first basic feasible solution (point 1) to the second basic feasible solution (point 2) involves making $X_1$ a basic variable and $S_2$ a nonbasic variable. Similarly, we can move from point 2 to point 3 by switching basic variables with nonbasic variables. So, starting at point 1 in Figure 4.17, the simplex method could move to point 2, then to the optimal solution at point 3. Alternatively, the simplex method could move from point 1 through points 5 and 4 to reach the optimal solution at point 3. Thus, although there is no guarantee that the simplex method will take the shortest route to the optimal solution of an LP problem, it will find the optimal solution eventually.

To determine whether switching a basic and nonbasic variable will result in a better solution, the simplex method calculates the reduced cost for each nonbasic variable to determine if the objective function could be improved if any of these variables are substituted for one of the basic variables. (Note that unbounded solutions are detected easily in the simplex method by the existence of a nonbasic variable that could improve the objective value by an infinite amount if it were made basic.) This process continues until no further improvement in the objective function value is possible.

# 4.9 Summary

This chapter described the methods for assessing how sensitive an LP model is to various changes that might occur in the model or its optimal solution. The impact of changes in an LP model can be analyzed easily by re-solving the model. Solver also provides a significant amount of sensitivity information automatically. For LP problems, the maximum amount of sensitivity information is obtained by solving the problem using the simplex method. Before using the information on the Sensitivity Report, you always should check first for the presence of degeneracy because this can have a significant effect on how one should interpret the numbers on this report.

The simplex method considers only the extreme points of the feasible region, and is an efficient way of solving LP problems. In this method, slack variables are introduced first to convert all constraints to "equal to" constraints. The simplex method systematically moves to better and better corner point solutions until no adjacent extreme point provides an improved objective function value.

# 4.10 References

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## THE WORLD OF MANAGEMENT SCIENCE

### *Fuel Management and Allocation Model Helps National Airlines Adapt to Cost and Supply Changes*

Fuel is a major component in the cost structure of an airline. Price and availability of fuel can vary from one air terminal to the next, and it is sometimes advantageous for an aircraft to carry more than the necessary minimum for the next leg of its route. Fuel loaded for the purpose of taking advantage of price or availability at a specific location is said to be tankered. A disadvantage of tankering is that fuel consumption increases when an aircraft is carrying more weight.

The use of LP to determine when and where to fuel aircraft saved National Airlines several million dollars during the first two years of implementation. In particular, National Airlines saw its average fuel costs drop 11.75% during a period when the average fuel cost for all domestic trunk airlines increased by 2.87%.

The objective function in the Fuel Management and Allocation Model consists of fuel costs and increases in operating costs from tankering. The constraints in the model address availability, minimum reserves, and aircraft capacities.

A particularly useful feature of the Fuel Management and Allocation Model is a series of reports that assist management in modifying the fuel-loading plan when sudden changes occur in availability or price. Shadow prices, along with the associated range of applicability, provide information about supply changes. Information about changes in price per gallon comes from the allowable increase and decrease for objective function coefficients.

For example, the availability report might indicate that the optimal quantity to purchase at Los Angeles from Shell West is 2,718,013 gallons; but if its supply decreases and fuel must be purchased from the next most attractive vendor, total cost would increase at the rate of $0.0478 per gallon (the shadow price). This fuel would be replaced by a prior purchase of up to 159,293 gallons from Shell East at New Orleans, tankered to Los Angeles.

The price report shows, for example, that vendor substitutions should be made if the current price of Shell West at Los Angeles, $0.3074, increases to $0.32583 or decreases to $0.27036. The report also indicates what that substitution should be.

Source: Darnell, D. Wayne and Carolyn Loflin, "National Airlines Fuel Management and Allocation Model," *Interfaces*, vol. 7, no. 2, February 1977, pp 1–16.

# Questions and Problems

1. Howie Jones used the following information to calculate the profit coefficients for Aqua-Spas and Hydro-Luxes: pumps cost \$225 each, labor costs \$12 per hour, tubing costs \$2 per foot. In addition to pumps, labor, and tubing, the production of Aqua-Spas and Hydro-Luxes consumes, respectively, \$243 and \$246 per unit in other resources that are not in short supply. Using this information, Howie calculated the marginal profits on Aqua-Spas and Hydro-Luxes as:

| | Aqua-Spas | Hydro-Luxes |
|---|---|---|
| Selling Price | \$950 | \$875 |
| Pump Cost | −\$225 | −\$225 |
| Labor Cost | −\$108 | −\$72 |
| Tubing Cost | −\$24 | −\$32 |
| Other Variable Costs | −\$243 | −\$246 |
| Marginal Profit | \$350 | \$300 |

Howie's accountant reviewed these calculations and thinks Howie made a mistake. For accounting purposes, factory overhead is assigned to products at a rate of \$16 per labor hour. Howie's accountant argues that because Aqua-Spas require nine labor hours, the profit margin on this product should be \$144 less. Similarly, because Hydro-Luxes require six labor hours, the profit margin on this product should be \$96 less. Who is right and why?

2. A variable that assumes an optimal value between its lower and upper bounds has a reduced cost value of zero. Why must this be true? (*Hint:* What if such a variable's reduced cost value is not zero? What does this imply about the value of the objective function?)
3. Implement the following LP problem in a spreadsheet. Use Solver to solve the problem and create a Sensitivity Report. Use this information to answer the following questions:

$$
\begin{aligned}
\text{MAX:} \quad & 4X_1 + 2X_2 \\
\text{Subject to:} \quad & 2X_1 + 4X_2 \leq 20 \\
& 3X_1 + 5X_2 \leq 15 \\
& X_1, X_2 \geq 0
\end{aligned}
$$

   a. What range of values can the objective function coefficient for variable $X_1$ assume without changing the optimal solution?
   b. Is the optimal solution to this problem unique, or are there alternate optimal solutions?
   c. How much does the objective function coefficient for variable $X_2$ have to increase before it enters the optimal solution at a strictly positive level?
   d. What is the optimal objective function value if $X_2$ equals 1?
   e. What is the optimal objective function value if the RHS value for the second constraint changes from 15 to 25?
   f. Is the current solution still optimal if the coefficient for $X_2$ in the second constraint changes from 5 to 1? Explain.

4. Implement the following LP model in a spreadsheet. Use Solver to solve the problem and create a Sensitivity Report. Use this information to answer the following questions:

$$\begin{array}{ll} \text{MAX:} & 2X_1 + 4X_2 \\ \text{Subject to:} & -X_1 + 2X_2 \le 8 \\ & X_1 + 2X_2 \le 12 \\ & X_1 + X_2 \ge 2 \\ & X_1, X_2 \ge 0 \end{array}$$

a. Which of the constraints are binding at the optimal solution?
b. Is the optimal solution to this problem unique, or is there an alternate optimal solution?
c. What is the optimal solution to this problem if the value of the objective function coefficient for variable $X_1$ is zero?
d. How much can the objective function coefficient for variable $X_2$ decrease before changing the optimal solution?
e. Given the objective in this problem, if management could increase the RHS value for any of the constraints for identical costs, which would you choose to increase and why?

5. Implement the following LP model in a spreadsheet. Use Solver to solve the problem and create a Sensitivity Report. Use this information to answer the following questions:

$$\begin{array}{ll} \text{MIN:} & 5X_1 + 3X_2 + 4X_3 \\ \text{Subject to:} & X_1 + X_2 + 2X_3 \ge 2 \\ & 5X_1 + 3X_2 + 2X_3 \ge 1 \\ & X_1, X_2, X_3 \ge 0 \end{array}$$

a. What is the smallest value the objective function coefficient for $X_3$ can assume without changing the optimal solution?
b. What is the optimal objective function value if the objective function coefficient for $X_3$ changes to –1? (*Hint:* The answer to this question is not given in the Sensitivity Report. Consider what the new objective function is relative to the constraints.)
c. What is the optimal objective function value if the RHS value of the first constraint increases to 7?
d. What is the optimal objective function value if the RHS value of the first constraint decreases by 1?
e. Will the current solution remain optimal if the objective function coefficients for $X_1$ and $X_3$ both decrease by 1?

6. The CitruSun Corporation ships frozen orange juice concentrate from processing plants in Eustis and Clermont to distributors in Miami, Orlando, and Tallahassee. Each plant can produce 20 tons of concentrate each week. The company has just received orders of 10 tons from Miami for the coming week, 15 tons for Orlando, and 10 tons for Tallahassee. The cost per ton for supplying each of the distributors from each of the processing plants is shown in the following table.

| | **Miami** | **Orlando** | **Tallahassee** |
|---|---|---|---|
| Eustis | \$260 | \$220 | \$290 |
| Clermont | \$230 | \$240 | \$310 |

The company wants to determine the least costly plan for filling their orders for the coming week.

a. Formulate an LP model for this problem.
b. Implement the model in a spreadsheet and solve it.

c. What is the optimal solution?
d. Is the optimal solution degenerate?
e. Is the optimal solution unique? If not, identify an alternate optimal solution for the problem.
f. How would the solution change if the plant in Clermont is forced to shut for one day resulting in a loss of four tons of production capacity?
g. What would the optimal objective function value be if the processing capacity in Eustis was reduced by five tons?
h. Interpret the reduced cost for shipping from Eustis to Miami.

7. Use Solver to create Answer and Sensitivity Reports for question 15 at the end of Chapter 2 and answer the following questions:
   a. How much excess wiring and testing capacity exists in the optimal solution?
   b. What is the company's total profit if it has 10 additional hours of wiring capacity?
   c. By how much does the profit on alternators need to increase before their production is justified?
   d. Does the optimal solution change if the marginal profit on generators decreases by $50 and the marginal profit on alternators increases by $75?
   e. Suppose the marginal profit on generators decreases by $25. What is the maximum profit that can be earned on alternators without changing the optimal solution?
   f. Suppose the amount of wiring required on alternators is reduced to 1.5 hours. Does this change the optimal solution? Why or why not?
8. Use Solver to create Answer and Sensitivity Reports for question 18 at the end of Chapter 2 and answer the following questions:
   a. If the profit on Razors decreased to $35 would the optimal solution change?
   b. If the profit on Zoomers decreased to $35 would the optimal solution change?
   c. Interpret the shadow price for the supply of polymer.
   d. Why is the shadow price $0 for the constraint limiting the production of pocket bikes to no more than 700 units?
   e. Suppose the company could obtain 300 additional labor hours in production. What would the new optimal level of profit be?
9. Use Solver to create Answer and Sensitivity Reports for question 20 at the end of Chapter 2 and answer the following questions:
   a. How much can the price of watermelons drop before it is no longer optimal to plant any watermelons?
   b. How much does the price of cantaloupes have to increase before it is optimal to grow only cantaloupes?
   c. Suppose the price of watermelons drops by $60 per acre and the price of cantaloupes increases by $50 per acre. Is the current solution still optimal?
   d. Suppose the farmer can lease up to 20 acres of land from a neighboring farm to plant additional crops. How many acres should the farmer lease and what is the maximum amount he should pay to lease each acre?
10. Use Solver to create Answer and Sensitivity Reports for question 21 at the end of Chapter 2 and answer the following questions:
    a. If the profit on doors increased to $700 would the optimal solution change?
    b. If the profit on windows decreased to $200 would the optimal solution change?
    c. Explain the shadow price for the finishing process.
    d. If 20 additional hours of cutting capacity became available how much additional profit could the company earn?

e. Suppose another company wanted to use 15 hours of Sanderson's sanding capacity and was willing to pay $400 per hour to acquire it? Should Sanderson agree to this? How (if at all) would your answer change if the company instead wanted 25 hours of sanding capacity?

11. Create a Sensitivity Report for Electro-Poly's make vs. buy problem in section 3.8 of Chapter 3 and answer the following questions.
   a. Is the solution degenerate?
   b. How much can the cost of making model 1 slip rings increase before it becomes more economical to buy some of them?
   c. Suppose the cost of buying model 2 slip rings decreased by $9 per unit. Would the optimal solution change?
   d. Assume workers in the wiring area normally make $12 per hour and get 50% more when they work overtime. Should Electro-Poly schedule these employees to work overtime to complete this job? If so, how much money would this save?
   e. Assume workers in the harnessing area normally make $12 per hour and get 50% more when they work overtime. Should Electro-Poly schedule these employees to work overtime to complete this job? If so, how much money would this save?
   f. Create a Spider plot that shows the effect of varying each of the wiring and harnessing requirements (in cells B17 thru D18) to 90% of their current levels in 1% increments. If Electro-Poly wanted to invest in training or new technology to reduce one of these values, which one offers the greatest potential for cost savings?
12. Use Solver to create a Sensitivity Report for question 12 at the end of Chapter 3 and answer the following questions:
   a. If the company could get 50 more units of routing capacity, should they do it? If so, how much should they be willing to pay for it?
   b. If the company could get 50 more units of sanding capacity, should they do it? If so, how much should they be willing to pay for it?
   c. Suppose the polishing time on country tables could be reduced from 2.5 to 2 units per table. How much should the company be willing to pay to achieve this improvement in efficiency?
   d. Contemporary tables sell for $450. By how much would the selling price have to decrease before we would no longer be willing to produce contemporary tables? Does this make sense? Explain.
13. Use Solver to create a Sensitivity Report for question 18 at the end of Chapter 3 and answer the following questions:
   a. Which of the constraints in the problem are binding?
   b. If the company was going to eliminate one of its products, which should it be?
   c. If the company could buy 1,000 additional memory chips at the usual cost, should they do it? If so, how much would profits increase?
   d. Suppose the manufacturing costs used in this analysis were estimated hastily and are known to be somewhat imprecise. For which products would you want more precise cost estimates before implementing this solution?
   e. Create a Spider Plot showing the sensitivity of the total profit to the selling price of each product (adjusting the original values by 90%, 92%, . . ., 110%). According to this graph, total profit is most sensitive to which product?
14. Use Solver to create a Sensitivity Report for question 20 at the end of Chapter 3 and answer the following questions:
   a. How much would electric trimmers have to cost for the company to consider purchasing these items rather than making them?

b. If the cost to make gas trimmers increased to $90 per unit, how would the optimal solution change?
c. How much should the company be willing to pay to acquire additional capacity in the assembly area? Explain.
d. How much should the company be willing to pay to acquire additional capacity in the production area? Explain.
e. Prepare a Spider Plot showing the sensitivity of the total cost to changes in costs to make and the costs to buy (adjusting the original values by of 90%, 92%, . . ., 110%). Which of these costs is the total cost most sensitive to?
f. Suppose the hours of production capacity available is uncertain and could vary from 9,500 to 10,500. How does the optimal solution change for every 100-hour change in production capacity within this range?

15. Use Solver to create a Sensitivity Report for question 22 at the end of Chapter 3 and answer the following questions.
    a. Suppose the cost of the first two compounds increases by $1.00 per pound and the cost of the third compound increases by $0.50 per pound. Does the optimal solution change?
    b. How does the solution change if the maximum amount of potassium allowed decreases from 45% to 40%?
    c. How much does the cost of the mix increase if the specifications for CHEMIX change to require at least 31% sulfur? (*Hint:* Remember that the shadow price indicates the impact on the objective function if the RHS value of the associated constraint increases by 1.)
16. Use Solver to create a Sensitivity Report for question 23 at the end of Chapter 3 and answer the following questions.
    a. What is the maximum level profit that can be achieved for this problem?
    b. Are there alternate optimal solutions to this problem? If so, identify the solution that allows the most grade 5 oranges to be used in fruit baskets while still achieving the maximum profit identified in part a.
    c. If Holiday could acquire 1,000 more pounds of grade 4 oranges at a cost of $2.65 per 100 pounds, should they do it? Why?
    d. Create a Spider Table and Plot showing the change in the total profit obtained by changing the required grade of fruit baskets and juice from 90% to 110% in 1% increments. If the department of agriculture wants to increase the required rating of one of these products, which product should the company lobby for?
17. Use Solver to create a Sensitivity Report for question 24 at the end of Chapter 3 and answer the following questions:
    a. Are there alternate optimal solutions to this problem? Explain.
    b. What is the highest possible octane rating for regular gasoline, assuming the company wants to maximize its profits? What is the octane rating for supreme gasoline at this solution?
    c. What is the highest possible octane rating for supreme gasoline, assuming the company wants to maximize its profits? What is the octane rating for regular gasoline at this solution?
    d. Which of the two profit-maximizing solutions identified in parts b and c would you recommend that the company implement? Why?
    e. If the company could buy another 150 barrels of input 2 at a cost of $17 per barrel, should they do it? Why?
18. Use Solver to create a Sensitivity Report for question 28 at the end of Chapter 3 and answer the following questions:
    a. What total profit level is realized if 100 extra hours of labor are available?

b. Assume a marginal labor cost of $11 per hour in determining the unit profits of each of the three products. How much should management pay to acquire 100 additional labor hours?
c. Interpret the reduced cost value for tuners. Why are more tuners not being produced?

19. Use Solver to create a Sensitivity Report for question 29 at the end of Chapter 3 and answer the following questions:
    a. Is the optimal solution unique? How can you tell?
    b. Which location is receiving the fewest cars?
    c. Suppose a particular car at location 1 must be sent to location 3 to meet a customer's request. How much does this increase costs for the company?
    d. Suppose location 6 must have at least eight cars shipped to it. What impact does this have on the optimal objective function value?
20. Refer to the previous question. Suppose location 1 has 15 cars available rather than 16. Create a Sensitivity Report for this problem and answer the following questions:
    a. Is the optimal solution unique? How can you tell?
    b. According to the Sensitivity Report, by how much should the total cost increase if we force a car to be shipped from location 1 to location 3?
    c. Add a constraint to the model to force one car to be shipped from location 1 to location 3. By how much did the total cost increase?
21. Use Solver to create a Sensitivity Report for question 30 at the end of chapter 3 and answer the following questions:
    a. Is the solution unique?
    b. If Sentry wants to increase their production capacity to meet more of the demand for their product, which plant should they use? Explain.
    c. If the cost of shipping from Phoenix to Tacoma increased to $1.98 per unit, would the solution change? Explain.
    d. Could the company make more money if they relaxed the restriction that each distributor must receive at least 80% of the predicted demand? Explain.
    e. How much extra should the company charge the distributor in Tacoma if this distributor insisted on receiving 8,500 units?
22. Use Solver to create a Sensitivity Report for question 31 at the end of Chapter 3 and answer the following questions.
    a. Is the solution degenerate?
    b. Is the solution unique?
    c. How much should the recycler be willing to pay to acquire more cardboard?
    d. If the recycler could buy 50 more tons of newspaper at a cost of $18 per ton, should they do it? Why or why not?
    e. What is the recycler's marginal cost of producing each of the three different types of pulp?
    f. By how much would the cost of converting white office paper into newsprint have to drop before it would become economical to use white office paper for this purpose?
    g. By how much would the yield of newsprint pulp per ton of cardboard have to increase before it would become economical to use cardboard for this purpose?
23. Use Solver to create a Sensitivity Report for question 35 at the end of Chapter 3 and answer the following questions.
    a. Is the solution degenerate?
    b. Is the solution unique?
    c. How much can the profit per ton on commodity 1 decrease before the optimal solution would change?

d. Create a Spider Table and Plot showing the change in the total profit obtained by changing the profit per ton on each commodity from 95% to 105% in 1% increments. If the shipping company wanted to increase the price of transporting one of the commodities, which one would have the greatest influence on total profits?

24. Use Solver to create a Sensitivity Report for question 36 at the end of Chapter 3 and answer the following questions.
    a. Is the solution degenerate?
    b. Is the solution unique?
    c. Use a Solver Table to determine the maximum price the Pelletier Corporation should be willing to pay for a two-month lease.
    d. Suppose the company is not certain that it will need exactly 20,000 sq.ft. in month 3 and believes that the actual amount needed may be as low as 15,000 sq.ft. or as high as 25,000 sq. ft. Use a Solver Table to determine if this would have any effect on the leasing arrangements the company selects in months 1 and 2.
25. Refer to question 45 in Chapter 3 and create a Solver table to answer the following questions.
    a. Suppose the gas producer needs an extra 50,000 cf of storage capacity for the next 10 days and wants to buy this capacity from the gas trading firm. What is the least amount of money the gas trading company should demand to provide this capacity?
    b. How much should the gas trading company be willing to pay to increase their available storage capacity by 500,000 cf?
26. Consider the following LP problem:

$$\begin{array}{ll} \text{MAX:} & 4X_1 + 2X_2 \\ \text{Subject to:} & 2X_1 + 4X_2 \leq 20 \\ & 3X_1 + 5X_2 \leq 15 \\ & X_1, X_2 \geq 0 \end{array}$$

    a. Use slack variables to rewrite this problem so that all its constraints are "equal to" constraints.
    b. Identify the different sets of basic variables that might be used to obtain a solution to the problem.
    c. Of the possible sets of basic variables, which lead to feasible solutions and what are the values for all the variables at each of these solutions?
    d. Graph the feasible region for this problem and indicate which basic feasible solution corresponds to each of the extreme points of the feasible region.
    e. What is the value of the objective function at each of the basic feasible solutions?
    f. What is the optimal solution to the problem?
    g. Which constraints are binding at the optimal solution?
27. Consider the following LP problem:

$$\begin{array}{ll} \text{MAX:} & 2X_1 + 4X_2 \\ \text{Subject to:} & -X_1 + 2X_2 \leq 8 \\ & X_1 + 2X_2 \leq 12 \\ & X_1 + X_2 \geq 2 \\ & X_1, X_2 \geq 0 \end{array}$$

    a. Use slack variables to rewrite this problem so that all its constraints are "equal to" constraints.
    b. Identify the different sets of basic variables that might be used to obtain a solution to the problem.

c. Of the possible sets of basic variables, which lead to feasible solutions and what are the values for all the variables at each of these solutions?
d. Graph the feasible region for this problem and indicate which basic feasible solution corresponds to each of the extreme points of the feasible region.
e. What is the value of the objective function at each of the basic feasible solutions?
f. What is the optimal solution to the problem?
g. Which constraints are binding at the optimal solution?

28. Consider the following LP problem:

$$\begin{aligned} \text{MIN:} \quad & 5X_1 + 3X_2 + 4X_3 \\ \text{Subject to:} \quad & X_1 + X_2 + 2X_3 \geq 2 \\ & 5X_1 + 3X_2 + 2X_3 \geq 1 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

a. Use slack variables to rewrite this problem so that all its constraints are "equal to" constraints.
b. Identify the different sets of basic variables that might be used to obtain a solution to the problem.
c. Of the possible sets of basic variables, which lead to feasible solutions and what are the values for all the variables at each of these solutions?
d. What is the value of the objective function at each of the basic feasible solutions?
e. What is the optimal solution to the problem?
f. Which constraints are binding at the optimal solution?

29. Consider the following constraint, where S is a slack variable:

$$2X_1 + 4X_2 + S = 16$$

a. What was the original constraint before the slack variable was included?
b. What value of S is associated with each of the following points:

i) $X_1 = 2, X_2 = 2$
ii) $X_1 = 8, X_2 = 0$
iii) $X_1 = 1, X_2 = 3$
iv) $X_1 = 4, X_2 = 1$

30. Consider the following constraint, where S is a slack variable:

$$3X_1 + 4X_2 - S = 12$$

a. What was the original constraint before the slack variable was included?
b. What value of S is associated with each of the following points:

i) $X_1 = 5, X_2 = 0$
ii) $X_1 = 2, X_2 = 2$
iii) $X_1 = 7, X_2 = 1$
iv) $X_1 = 4, X_2 = 0$

# A Nut Case

**CASE 4.1**

The Molokai Nut Company (MNC) makes four different products from macadamia nuts grown in the Hawaiian Islands: chocolate-coated whole nuts (Whole), chocolate-coated nut clusters (Cluster), chocolate-coated nut crunch bars (Crunch), and plain roasted nuts (Roasted). The company is barely able to keep up with the increasing demand for these products. However, increasing raw material prices and foreign competition are forcing MNC to watch its margins to ensure it is operating in the most efficient manner possible. To meet marketing demands for the coming week, MNC needs to produce at least 1,000 pounds of the Whole product, between 400 and 500 pounds of the Cluster product, no more than 150 pounds of the Crunch product, and no more than 200 pounds of Roasted product.

Each pound of the Whole, Cluster, Crunch, and Roasted product contains, respectively, 60%, 40%, 20%, and 100% macadamia nuts with the remaining weight made up of chocolate coating. The company has 1100 pounds of nuts and 800 pounds of chocolate available for use in the next week. The various products are made using four different machines that hull the nuts, roast the nuts, coat the nuts in chocolate (if needed), and package the products. The following table summarizes the time required by each product on each machine. Each machine has 60 hours of time available in the coming week.

| | Minutes Required per Pound | | | |
|---|---|---|---|---|
| **Machine** | **Whole** | **Cluster** | **Crunch** | **Roasted** |
| Hulling | 1.00 | 1.00 | 1.00 | 1.00 |
| Roasting | 2.00 | 1.50 | 1.00 | 1.75 |
| Coating | 1.00 | 0.70 | 0.20 | 0.00 |
| Packaging | 2.50 | 1.60 | 1.25 | 1.00 |

The controller recently presented management with the following financial summary of MNC's average weekly operations over the past quarter. From this report, the controller is arguing that the company should cease producing its Cluster and Crunch products.

| | Product | | | | |
|---|---|---|---|---|---|
| | **Whole** | **Cluster** | **Crunch** | **Roasted** | **Total** |
| Sales Revenue | $5,304 | $1,800 | $510 | $925 | $8,539 |
| Variable Costs | | | | | |
| Direct materials | $1,331 | $560 | $144 | $320 | $2,355 |
| Direct labor | $1,092 | $400 | $96 | $130 | $1,718 |
| Manufacturing overhead | $333 | $140 | $36 | $90 | $599 |
| Selling & Administrative | $540 | $180 | $62 | $120 | $902 |
| Allocated Fixed Costs | | | | | |
| Manufacturing overhead | $688 | $331 | $99 | $132 | $1,250 |
| Selling & Administrative | $578 | $278 | $83 | $111 | $1,050 |
| Net Profit | $742 | −$88 | −$11 | $22 | $665 |
| Units Sold | 1040 | 500 | 150 | 200 | 1890 |
| Net Profit Per Unit | $0.71 | −$0.18 | −$0.07 | $0.11 | $0.35 |

a. Do you agree with the controller's recommendation? Why or why not?
b. Formulate an LP model for this problem.
c. Create a spreadsheet model for this problem and solve it using Solver.
d. What is the optimal solution?
e. Create a sensitivity report for this solution and answer the following questions.
f. Is the solution degenerate?
g. Is the solution unique?

h. If MNC wanted to decrease the production on any product, which one would you recommend and why?
i. If MNC wanted to increase the production of any product, which one would you recommend and why?
j. Which resources are preventing MNS from making more money? If they could acquire more of this resource how much should they acquire & how much should they be willing to pay to acquire it?
k. How much should MNC be willing to pay to acquire more chocolate?
l. If the marketing department wanted to decrease the price of the Whole product by $0.25, would the optimal solution change?
m. Create a Spider plot showing the impact on net profit of changing each product's required time in the roasting process from between 70% to 130% of their original values in 5% increments. Interpret the information in the resulting chart.
n. Create a Spider plot showing the impact on net profit of changing the availability of nuts and chocolate from between 70% to 100% of their original values in 5% increments. Interpret the information in the resulting chart.

## Parket Sisters

**CASE 4.2**

(Contributed by Jack Yurkiewicz, Lubin School of Business, Pace University, New York.)

Computers and word processors notwithstanding, the art of writing by hand recently entered a boom era. People are buying fountain pens again, and mechanical pencils are becoming more popular than ever. Joe Script, the president and CEO of Parket Sisters, a small but growing pen and pencil manufacturer, wants to establish a better foothold in the market. The writing market is divided into two main sectors. One, dominated by Mont Blanc, Cross, Parker Brothers, Waterman, Schaffer, and a few others, caters to people who want writing instruments. The product lines from these companies consist of pens and pencils of elaborate design, lifetime warranty, and high price. At the other end of the market are manufacturers like BIC, Pentel, and many companies from the far east, offering good quality items, low price, few trims, and limited diversity. These pens and pencils are meant to be used for a limited time and disposed of when the ink in a ballpoint pen runs out, or when the lead in a mechanical pencil won't retract or extend. In short, these items are not meant for repair.

Joe thinks that there must be a middle ground, and that is where he wants to position his company. Parket Sisters makes high-quality items, with limited trim and diversity, but also offers lifetime warranties. Furthermore, its pens and pencils are ergonomically efficient. Joe knows that some people want the status of the Mont Blanc Meisterstuck pen, for example, but he has never met a person who said that writing with such a pen is enjoyable. The pen is too large and clumsy for smooth writing. Parket Sisters' products, on the other hand, have a reputation for working well, are easy to hold and use, and cause limited "writer's fatigue."

Parket Sisters makes only three items—a ballpoint pen, a mechanical pencil, and a fountain pen. All are available in just one color, black, and are sold mostly in specialty stores and from better catalog companies. The per-unit profit of the items is $3.00 for the ballpoint pen, $3.00 for the mechanical pencil, and $5.00 for the fountain pen. These values take into account labor, the cost of materials, packing, quality control, and so on.

The company is trying to plan its production mix for each week. Joe believes that the company can sell any number of pens and pencils it produces, but production is currently limited by the available resources. Because of a recent strike and certain cash-flow problems, the suppliers of these resources are selling them to Parket Sisters in limited amounts. In particular, Joe can count on getting at most 1,000 ounces of plastic, 1,200 ounces of chrome, and 2,000 ounces of stainless steel each week from his suppliers, and these figures are not likely to change in the near future. Because of Joe's excellent reputation, the suppliers will sell Joe any amount (up to his limit) of the resources he needs when he requires them. That is, the suppliers do not require Joe to buy some fixed quantities of resources in advance of his production of pens and pencils; therefore, these resources can be considered variable costs rather than fixed costs for the pens and pencils.

Each ballpoint pen requires 1.2 ounces of plastic, 0.8 ounces of chrome, and 2 ounces of stainless steel. Each mechanical pencil requires 1.7 ounces of plastic, no chrome, and 3 ounces of stainless steel. Each fountain pen requires 1.2 ounces of plastic, 2.3 ounces of chrome, and 4.5 ounces of stainless steel. Joe believes LP could help him decide what his weekly product mix should consist of.

Getting his notes and notebooks, Joe grapples with the LP formulation. In addition to the constraints of the available resources, he recognizes that the model should include many other constraints (such as labor time availability and materials for packing). However, Joe wants to keep his model simple. He knows that eventually he'll have to take other constraints into account, but as a first-pass model, he'll restrict the constraints to just the three resources: plastic, chrome, and stainless steel.

With only these three constraints, Joe can formulate the problem easily as:

$$
\begin{array}{ll}
\text{MAX:} & 3.0X_1 + 3.0X_2 + 5.0X_3 \\
\text{Subject to:} & 1.2X_1 + 1.7X_2 + 1.2X_3 \le 1{,}000 \\
 & 0.8X_1 + \phantom{1.}0X_2 + 2.3X_3 \le 1{,}200 \\
 & 2.0X_1 + 3.0X_2 + 4.5X_3 \le 2{,}000 \\
 & \qquad\qquad X_1, X_2, X_3 \ge 0
\end{array}
$$

where:

$X_1$ = the number of ballpoint pens
$X_2$ = the number of mechanical pencils
$X_3$ = the number of fountain pens

Joe's knowledge of Excel and the Solver feature is limited, so he asks you to enter and solve the problem for him, then answer the following questions. (Assume that each question is independent unless otherwise stated.)

a. What should the weekly product mix consist of, and what is the weekly net profit?
b. Is the optimal solution to question 1 degenerate? Explain your response.
c. Is the optimal solution from question 1 unique, or are there alternate answers to this question? Explain your response.
d. What is the marginal value of one more unit of chrome? Of plastic?
e. A local distributor has offered to sell Parket Sisters an additional 500 ounces of stainless steel for $0.60 per ounce more than it ordinarily pays. Should the company buy the steel at this price? Explain your response.
f. If Parket Sisters buys the additional 500 ounces of stainless steel noted in question 5, what is the new optimal product mix and what is the new optimal profit? Explain your response.
g. Suppose that the distributor offers to sell Parket Sisters some additional plastic at a price of only $1.00 over its usual cost of $5.00 per ounce. However, the distributor

will sell the plastic only in lot sizes of 500 ounces. Should Parket Sisters buy one such lot? Explain your response.

h. The distributor is willing to sell the plastic in lots of just 100 ounces instead of the usual 500-ounce lots, still at $1.00 over Parket Sisters' cost of $5.00 per ounce. How many lots (if any) should Parket Sisters buy? What is the optimal product mix if the company buys these lots, and what is the optimal profit?
i. Parket Sisters has an opportunity to sell some of its plastic for $6.50 per ounce to another company. The other company (which does not produce pens and pencils and, therefore, is not a competitor) wants to buy 300 ounces of plastic from Parket Sisters. Should Parket Sisters sell the plastic to the other company? What happens to Parket Sisters' product mix and overall profit if it does sell the plastic? Be as specific as possible.
j. The chrome supplier might have to fulfill an emergency order, and would be able to send only 1,000 ounces of chrome this week instead of the usual 1,200 ounces. If Parket Sisters receives only 1,000 ounces of chrome, what is the optimal product mix and optimal profit? Be as specific as possible.
k. The R&D department at Parket Sisters has been redesigning the mechanical pencil to make it more profitable. The new design requires 1.1 ounces of plastic, 2.0 ounces of chrome, and 2.0 ounces of stainless steel. If the company can sell one of these pencils at a net profit of $3.00, should it approve the new design? Explain your response.
l. If the per-unit profit on ballpoint pens decreases to $2.50, what is the optimal product mix and what is the company's total profit?
m. The marketing department suggested introducing a new felt tip pen that requires 1.8 ounces of plastic, 0.5 ounces of chrome, and 1.3 ounces of stainless steel. What profit must this product generate to make it worthwhile to produce?
n. What must the minimum per-unit profit of mechanical pencils be to make them worthwhile to produce?
o. Management believes that the company should produce at least 20 mechanical pencils per week to round out its product line. What effect would this have on overall profit? Give a numerical answer.
p. If the profit on a fountain pen is $6.75 instead of $5.00, what is the optimal product mix and optimal profit?

# Kamm Industries

**CASE 4.3**

If your home or office is carpeted, there's a good chance that carpet came from Dalton, Georgia—also known as the "Carpet Capital of the World." Manufacturers in the Dalton area produce more than 70 percent of the total output of the $9 billion world-wide carpet industry. Competition in this industry is intense, which forces producers to strive for maximum efficiency and economies of scale. It also forces producers to continually evaluate investments in new technology.

Kamm Industries is one of the leading carpet producers in the Dalton area. Its owner, Geoff Kamm, has asked for your assistance in planning the production schedule for the next quarter (13 weeks). The company has orders for 15 different types of carpets that the company can produce on either of two types of looms: Dobbie looms and Pantera looms. Pantera looms produce standard tufted carpeting. Dobbie looms also can produce standard tufted carpeting, but they also allow the incorporation of designs (such as flowers or corporate logos) into the carpeting. The following table summarizes the orders for each type of carpet that must be produced in the coming quarter along with their production rates and costs on each type of loom, and the cost of subcontracting each order. Note that the first 4 orders involve special production requirements that can be achieved only on a Dobbie loom or via subcontracting. Assume that any portion of an order may be subcontracted.

| | Demand | Dobbie | | Pantera | | Subcontract |
|---|---|---|---|---|---|---|
| Carpet | Yds | Yd/Hr | Cost/Yd | Yd/Hr | Cost/Yd | Cost/Yd |
| 1 | 14,000 | 4.510 | $2.66 | na | na | $2.77 |
| 2 | 52,000 | 4.796 | 2.55 | na | na | 2.73 |
| 3 | 44,000 | 4.629 | 2.64 | na | na | 2.85 |
| 4 | 20,000 | 4.256 | 2.56 | na | na | 2.73 |
| 5 | 77,500 | 5.145 | 1.61 | 5.428 | $1.60 | 1.76 |
| 6 | 109,500 | 3.806 | 1.62 | 3.935 | 1.61 | 1.76 |
| 7 | 120,000 | 4.168 | 1.64 | 4.316 | 1.61 | 1.76 |
| 8 | 60,000 | 5.251 | 1.48 | 5.356 | 1.47 | 1.59 |
| 9 | 7,500 | 5.223 | 1.50 | 5.277 | 1.50 | 1.71 |
| 10 | 69,500 | 5.216 | 1.44 | 5.419 | 1.42 | 1.63 |
| 11 | 68,500 | 3.744 | 1.64 | 3.835 | 1.64 | 1.80 |
| 12 | 83,000 | 4.157 | 1.57 | 4.291 | 1.56 | 1.78 |
| 13 | 10,000 | 4.422 | 1.49 | 4.558 | 1.48 | 1.63 |
| 14 | 381,000 | 5.281 | 1.31 | 5.353 | 1.30 | 1.44 |
| 15 | 64,000 | 4.222 | 1.51 | 4.288 | 1.50 | 1.69 |

Kamm currently owns and operates 15 Dobbie looms and 80 Pantera looms. To maximize efficiency and keep pace with demand, the company operates 24 hours a day, 7 days a week. Each machine is down for routine maintenance for approximately 2 hours per week. Create a spreadsheet model for this problem that can be used to determine the optimal production plan and answer the following questions.

a. What is the optimal production plan and associated cost?
b. Is the solution degenerate?
c. Is the solution unique?
d. What would happen to the total cost if one of the Dobbie machines broke and could not be used at all during the quarter?
e. What would happen to the total cost if an additional Dobbie machine was purchased and available for the quarter?
f. What would happen to the total cost if one of the Pantera machines broke and could not be used at all during the quarter?
g. What would happen to the total cost if an additional Pantera machine was purchased and available for the quarter?
h. Explain the shadow prices and the values in the "Allowable Increase" column of the Sensitivity Report for the products that are being outsourced.
i. How much money does it cost to produce carpet order 2? How much would the total cost decrease if that order were eliminated? Explain.
j. If the carpets in orders 5 through 15 all sell for the same amount, which type of carpet should Kamm encourage its sales force to sell more of? Why?
k. If the cost of buying the carpet in order 1 increased to $2.80 per yard, would the optimal solution change? Why?
l. If the cost of buying the carpet in order 15 decreased to $1.65 per yard, would the optimal solution change? Why?