Network Modelling

Lecture 10

Chapter 3: Optimization

Institute of Engineering
Asst. Prof. Anita Prajapati, Ph.D.

5 July 2023

Chapter 5

Network Modeling

Introduction

- A number of business problems can be represented graphically as networks.

- This chapter focuses on several types of network flow problems:
  - Transshipment Problems
  - Shortest Path Problems
  - Maximal Flow Problems
  - Transportation/Assignment Problems
  - Generalized Network Flow Problems

- We also consider a different type of network problem called the Minimum Spanning Tree Problem

Characteristics of Network Flow Problems

- Network flow problems can be represented as a collection of nodes connected by arcs.

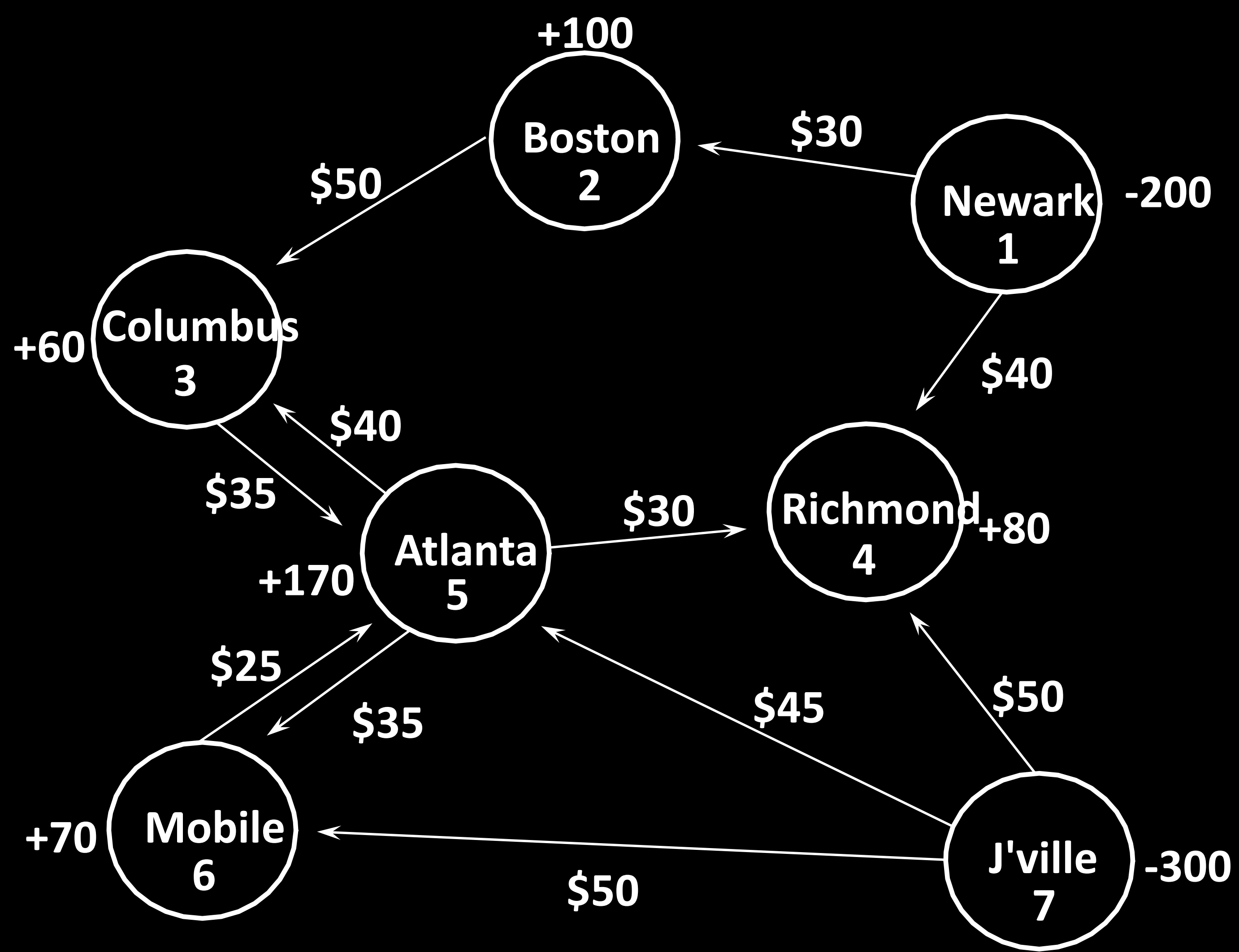
- There are three types of nodes:
  - Supply
  - Demand
  - Transshipment

- We'll use negative numbers to represent supplies and positive numbers to represent demand.

Transshipment Problem(Characteristics)

- Transshipment nodes can both send to and receive from other nodes in the network.
- Positive numbers represent the demand at a given node.
- Negative numbers represent the supply available at a node

A Transshipment Problem: The Bavarian Motor Company



Transhipment Example

The Bavarian Motor Company (BMC) manufactures expensive luxury cars in Hamburg, Germany, and exports cars to sell in the United States. The exported cars are shipped from Hamburg to ports in Newark, New Jersey, and Jacksonville, Florida. From these ports, the cars are transported by rail or truck to distributors located in Boston, Massachusetts; Columbus, Ohio; Atlanta, Georgia; Richmond, Virginia; and Mobile, Alabama. Figure 5.1 shows the possible shipping routes available to the company along with the transportation cost for shipping each car along the indicated path.

Currently, 200 cars are available at the port in Newark and 300 are available in Jacksonville. The numbers of cars needed by the distributors in Boston, Columbus, Atlanta, Richmond, and Mobile are 100, 60, 170, 80, and 70, respectively. BMC wants to determine the least costly way of transporting cars from the ports in Newark and Jacksonville to the cities where they are needed.

Defining the Decision Variables

For each arc in a network flow model
we define a decision variable as:

X_{ij} = the amount being shipped (or flowing) <u>from</u> node i <u>to</u> node j

For example,

X_{12} = the number of cars shipped *from* node 1 (Newark) *to* node 2 (Boston)

X_{56} = the number of cars shipped *from* node 5 (Atlanta) *to* node 6 (Mobile)

Note: The number of arcs determine the number of variables in a network flow problem!

The network in Figure contains 11 arcs. Therefore, the LP formulation of this model requires 11 decision variables

X_{12} = the number of cars shipped *from* node 1 (Newark) *to* node 2 (Boston)

X_{14} = the number of cars shipped *from* node 1 (Newark) *to* node 4 (Richmond)

X_{23} = the number of cars shipped *from* node 2 (Boston) *to* node 3 (Columbus)

X_{35} = the number of cars shipped *from* node 3 (Columbus) *to* node 5 (Atlanta)

X_{53} = the number of cars shipped *from* node 5 (Atlanta) *to* node 3 (Columbus)

X_{54} = the number of cars shipped *from* node 5 (Atlanta) *to* node 4 (Richmond)

X_{56} = the number of cars shipped *from* node 5 (Atlanta) *to* node 6 (Mobile)

X_{65} = the number of cars shipped *from* node 6 (Mobile) *to* node 5 (Atlanta)

X_{74} = the number of cars shipped *from* node 7 (Jacksonville) *to* node 4 (Richmond)

X_{75} = the number of cars shipped *from* node 7 (Jacksonville) *to* node 5 (Atlanta)

X_{76} = the number of cars shipped *from* node 7 (Jacksonville) *to* node 6 (Mobile)

Defining the Objective Function

Minimize total shipping costs.

$$
\begin{aligned}
\text{MIN:}\quad & 30X_{12} + 40X_{14} + 50X_{23} + 35X_{35} \\
& +40X_{53} + 30X_{54} + 35X_{56} + 25X_{65} \\
& + 50X_{74} + 45X_{75} + 50X_{76}
\end{aligned}
$$

Constraints for Network Flow Problems: The Balance-of-Flow Rules

For Minimum Cost Network Flow Problems Where:	Apply This Balance-of-Flow Rule At Each Node:
Total Supply > Total Demand	Inflow-Outflow >= Supply or Demand
Total Supply < Total Demand	Inflow-Outflow <=Supply or Demand
Total Supply = Total Demand	Inflow-Outflow = Supply or Demand

Defining the Constraints

- In the BMC problem:

  Total Supply = 500 cars

  Total Demand = 480 cars

- So for each node we need a constraint of the form:

  Inflow - Outflow >= Supply or Demand

- Constraint for node 1:

  $-X_{12} - X_{14} >= -200$ **(there is no inflow for node 1!)**

- This is equivalent to:
  - $+X_{12} + X_{14} <= 200$

Defining the Constraints

Flow constraints

$-X_{12} - X_{14} >= -200$ } node 1

$+X_{12} - X_{23} >= +100$ } node 2

$+X_{23} + X_{53} - X_{35} >= +60$ } node 3

$+ X_{14} + X_{54} + X_{74} >= +80$ } node 4

$+ X_{35} + X_{65} + X_{75} - X_{53} - X_{54} - X_{56} >= +170$ } node 5

$+ X_{56} + X_{76} - X_{65} >= +70$ } node 6

$-X_{74} - X_{75} - X_{76} >= -300$ } node 7

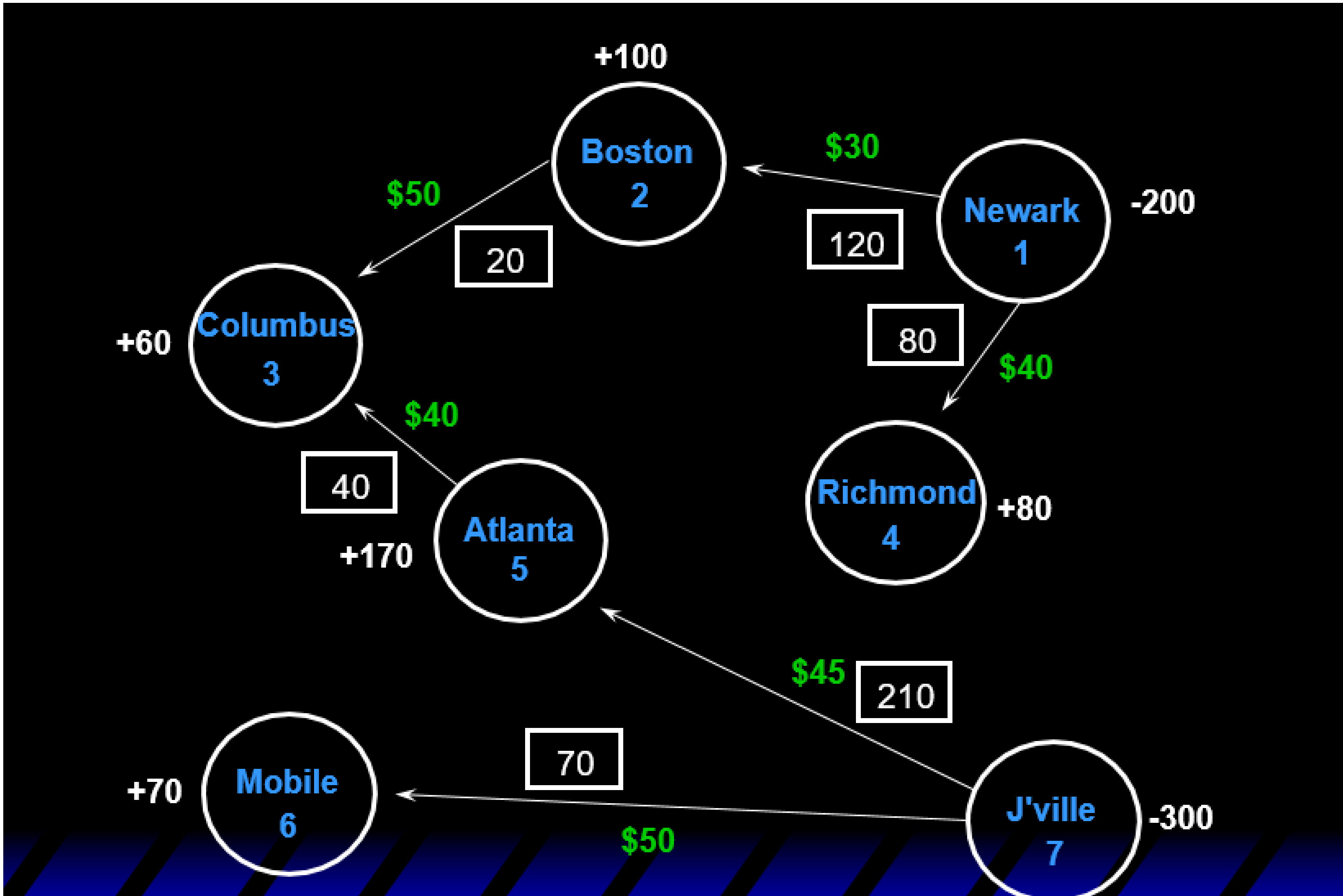
Non-negativity conditions

$X_{ij} >= 0$ for all *ij*

Implementing the Model

See file Fig5-2.xls

Optimal Solution to the BMC Problem



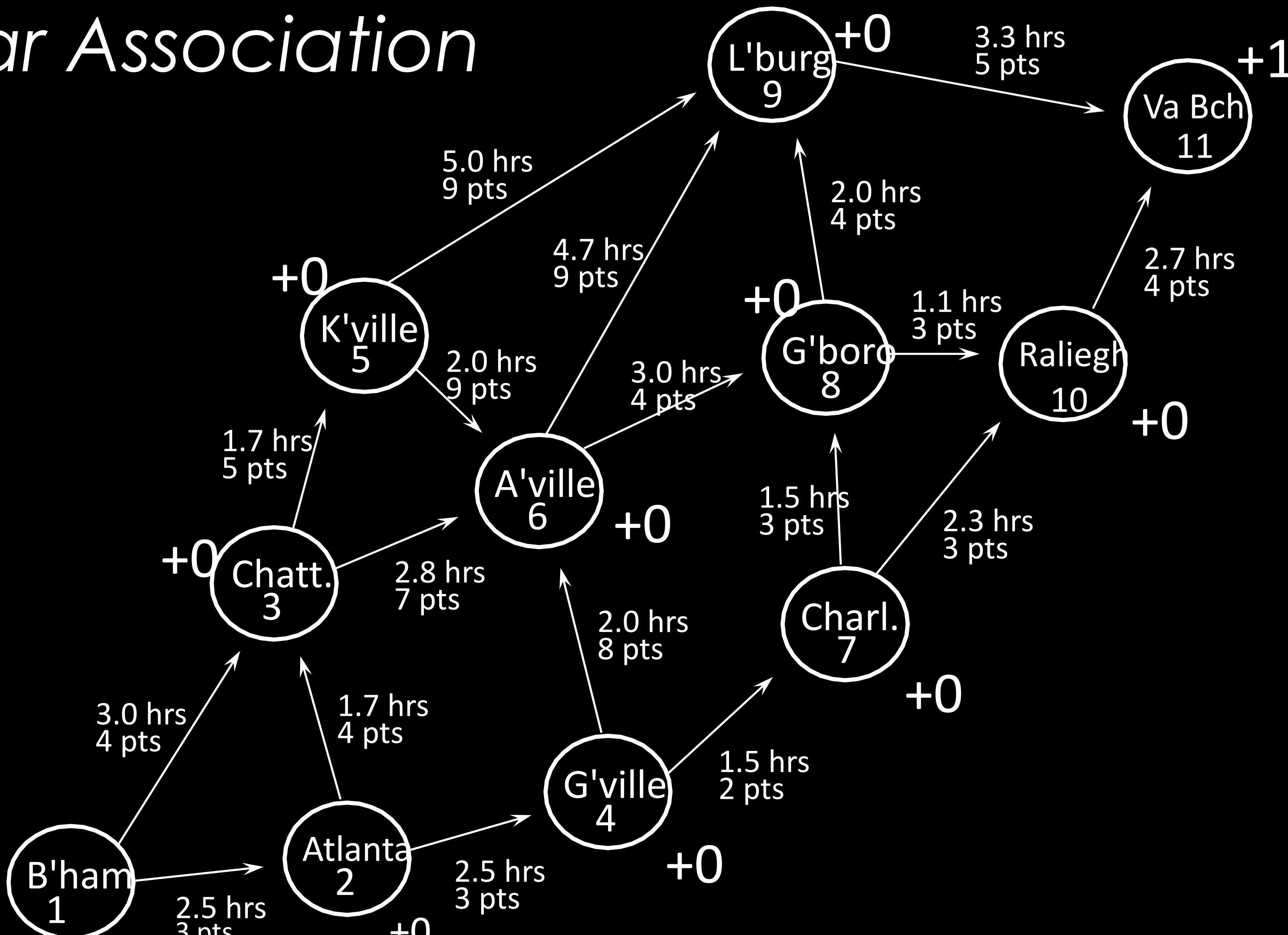
The Shortest Path Problem

- Many decision problems boil down to determining the shortest (or least costly) route or path through a network.

    Ex. Emergency Vehicle Routing

- This is a special case of a transshipment problem where:
    - there is a supply node with a supply of -1
    - there is a demand node with a demand of +1
    - all other nodes have supply/demand of +0

The American Car Association



The American Car Association (ACA) provides a variety of travel-related services to its members, including information on vacation destinations, discount hotel reservations, emergency road assistance, and travel route planning. This last service, travel route planning, is one of its most popular services. When members of the ACA are planning to take a driving trip, they call the organization's toll-free 800 number and indicate what cities they will be traveling from and to. The ACA then determines an optimal route for traveling between these cities. The ACA's computer databases of major highways and interstates are kept up-to-date with information on construction delays and detours and estimated travel times along various segments of roadways.

Members of the ACA often have different objectives in planning driving trips. Some are interested in identifying routes that minimize travel times. Others, with more leisure time on their hands, want to identify the most scenic route to their desired destination. The ACA wants to develop an automated system for identifying an optimal travel plan for its members.

Solving the Problem

- There are two possible objectives for this problem
  - Finding the quickest route (minimizing travel time)
  - Finding the most scenic route (maximizing the scenic rating points)

- See file Fig5-7.xls

Select Route?	From		To		Driving Time	Scenic Rating
1.0	1	Birmingham	2	Atlanta	2.5	3
0.0	1	Birmingham	3	Chattanooga	3.0	4
0.0	2	Atlanta	3	Chattanooga	1.7	4
1.0	2	Atlanta	4	Greenville	2.5	3
0.0	3	Chattanooga	5	Knoxville	1.7	5
0.0	3	Chattanooga	6	Asheville	2.8	7
0.0	4	Greenville	6	Asheville	2.0	8
1.0	4	Greenville	7	Charlotte	1.5	2
0.0	5	Knoxville	6	Asheville	2.0	9
0.0	5	Knoxville	9	Lynchburg	5.0	9
0.0	6	Asheville	8	Greensboro	3.0	4
0.0	6	Asheville	9	Lynchburg	4.7	9
0.0	7	Charlotte	8	Greensboro	1.5	3
1.0	7	Charlotte	10	Raleigh	2.3	3
0.0	8	Greensboro	9	Lynchburg	2.0	4
0.0	8	Greensboro	10	Raleigh	1.1	3
0.0	9	Lynchburg	11	Virginia Beach	3.3	5
1.0	10	Raleigh	11	Virginia Beach	2.7	4
				Total	11.5	15

Nodes		Net Flow	Supply/Deman
1	Birmingham	-1	-1
2	Atlanta	0	0
3	Chattanooga	0	0
4	Greenville	0	0
5	Knoxville	0	0
6	Asheville	0	0
7	Charlotte	0	0
8	Greensboro	0	0
9	Lynchburg	0	0
10	Raleigh	0	0
11	Virginia Beach	1	1

Minimum Driving Time

Select Route?	From		To		Driving Time	Scenic Rating
1.0	1	Birmingham	2	Atlanta	2.5	3
0.0	1	Birmingham	3	Chattanooga	3.0	4
1.0	2	Atlanta	3	Chattanooga	1.7	4
0.0	2	Atlanta	4	Greenville	2.5	3
1.0	3	Chattanooga	5	Knoxville	1.7	5
0.0	3	Chattanooga	6	Asheville	2.8	7
0.0	4	Greenville	6	Asheville	2.0	8
0.0	4	Greenville	7	Charlotte	1.5	2
1.0	5	Knoxville	6	Asheville	2.0	9
0.0	5	Knoxville	9	Lynchburg	5.0	9
0.0	6	Asheville	8	Greensboro	3.0	4
1.0	6	Asheville	9	Lynchburg	4.7	9
0.0	7	Charlotte	8	Greensboro	1.5	3
0.0	7	Charlotte	10	Raleigh	2.3	3
0.0	8	Greensboro	9	Lynchburg	2.0	4
0.0	8	Greensboro	10	Raleigh	1.1	3
1.0	9	Lynchburg	11	Virginia Beach	3.3	5
0.0	10	Raleigh	11	Virginia Beach	2.7	4
				Total	15.9	35

Nodes		Net Flow	Supply/Demand
1	Birmingham	-1	-1
2	Atlanta	0	0
3	Chattanooga	0	0
4	Greenville	0	0
5	Knoxville	0	0
6	Asheville	0	0
7	Charlotte	0	0
8	Greensboro	0	0
9	Lynchburg	0	0
10	Raleigh	0	0
11	Virginia Beach	1	1

Maximize scenic rating

Transportation & Assignment Problems

Tropicsun is a leading grower and distributor of fresh citrus products with three large citrus groves scattered around central Florida in the cities of Mt. Dora, Eustis, and Clermont. Tropicsun currently has 275,000 bushels of citrus at the grove in Mt. Dora, 400,000 bushels at the grove in Eustis, and 300,000 bushels at the grove in Clermont. Tropicsun has citrus processing plants in Ocala, Orlando, and Leesburg with processing capacities to handle 200,000, 600,000, and 225,000 bushels, respectively. Tropicsun contracts with a local trucking company to transport its fruit from the groves to the processing plants. The trucking company charges a flat rate for every mile that each bushel of fruit must be transported. Each mile a bushel of fruit travels is known as a bushel-mile. The following table summarizes the distances (in miles) between the groves and processing plants:

	Distances (in miles) Between Groves and Plants		
Grove	**Ocala**	**Orlando**	**Leesburg**
Mt. Dora	21	50	40
Eustis	35	30	22
Clermont	55	20	25

Tropicsun wants to determine how many bushels to ship from each grove to each processing plant to minimize the total number of bushel-miles the fruit must be shipped.

Specifically, the nine decision variables are:

X_{14} = number of bushels to ship from Mt. Dora (node 1) to Ocala (node 4)

X_{15} = number of bushels to ship from Mt. Dora (node 1) to Orlando (node 5)

X_{16} = number of bushels to ship from Mt. Dora (node 1) to Leesburg (node 6)

X_{24} = number of bushels to ship from Eustis (node 2) to Ocala (node 4)

X_{25} = number of bushels to ship from Eustis (node 2) to Orlando (node 5)

X_{26} = number of bushels to ship from Eustis (node 2) to Leesburg (node 6)

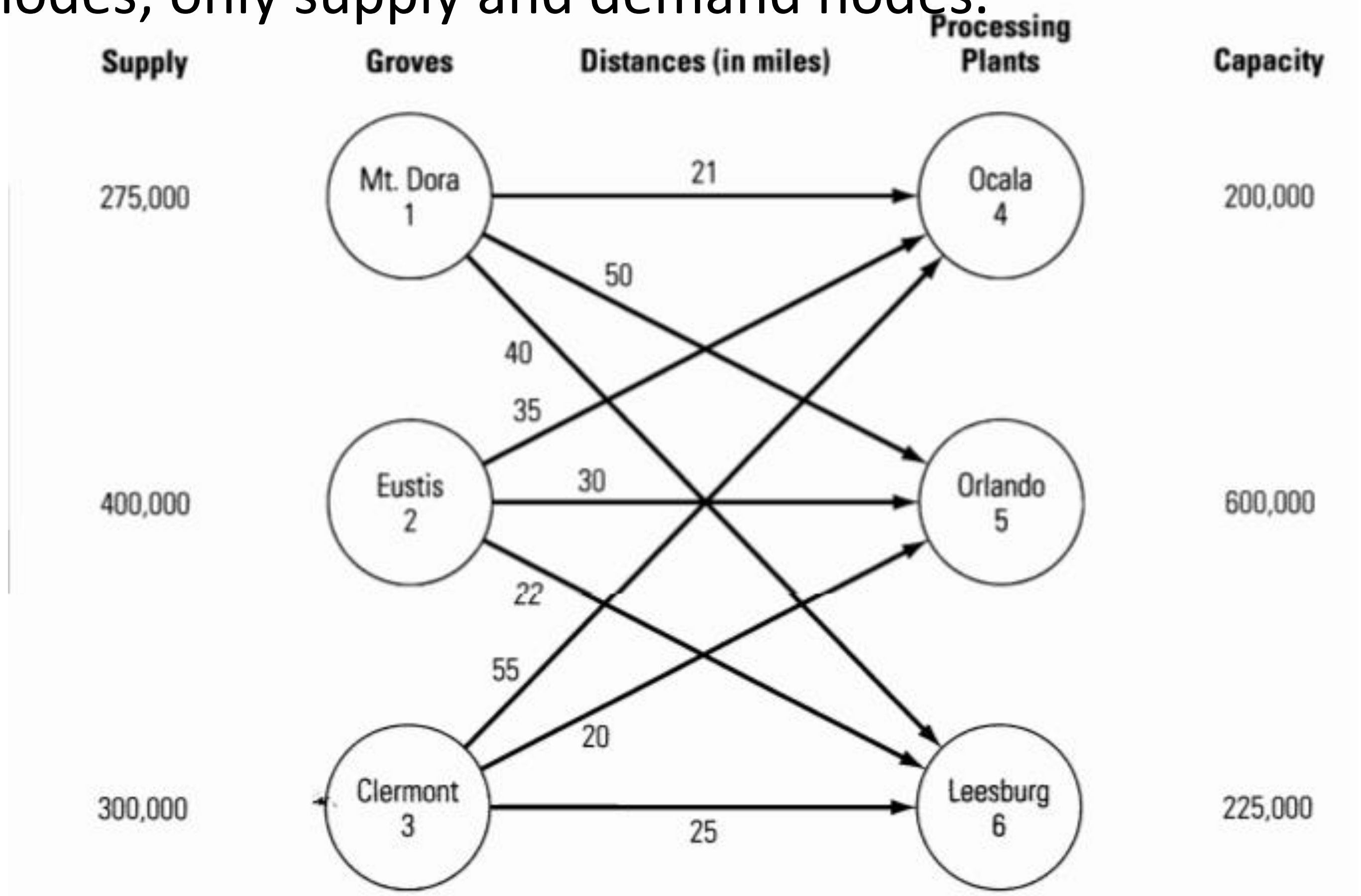
X_{34} = number of bushels to ship from Clermont (node 3) to Ocala (node 4)

X_{35} = number of bushels to ship from Clermont (node 3) to Orlando (node 5)

X_{36} = number of bushels to ship from Clermont (node 3) to Leesburg (node 6)

Graphical Representation

- Some network flow problems don't have trans-shipment nodes; only supply and demand nodes.



- These problems are implemented more effectively using the technique described in Chapter 3.

Objective Function

The goal in this problem is to determine how many bushels to ship from each grove to each processing plant while minimizing the total distance (or total number of bushel miles) the fruit must travel

$$\text{MIN:} \quad 21X_{14} + 50X_{15} + 40X_{16} + 35X_{24} + 30X_{25} + 22X_{26} + 55X_{34} + 20X_{35} + 25X_{36}$$

Two physical constraint

First, there is a limit on the amount of fruit that can be shipped to each processing plant. Tropicsun can ship no more than 200,000, 600,000, and 225,000 bushels to Ocala, Orlando, and Leesburg, respectively. So,

$$X_{14} + X_{24} + X_{34} \le 200{,}000 \quad \} \text{ capacity restriction for Ocala}$$
$$X_{15} + X_{25} + X_{35} \le 600{,}000 \quad \} \text{ capacity restriction for Orlando}$$
$$X_{16} + X_{26} + X_{36} \le 225{,}000 \quad \} \text{ capacity restriction for Leesburg}$$

The second set of constraints ensures that the supply of fruit at each grove is shipped to a processing plant. That is, all of the 275,000, 400,000, and 300,000 bushels at Mt. Dora, Eustis, and Clermont, respectively, must be processed somewhere. So,

$$X_{14} + X_{15} + X_{16} = 275{,}000 \quad \} \text{ supply available at Mt. Dora}$$
$$X_{24} + X_{25} + X_{26} = 400{,}000 \quad \} \text{ supply available at Eustis}$$
$$X_{34} + X_{35} + X_{36} = 300{,}000 \quad \} \text{ supply available at Clermont}$$

Model

MIN: $21X_{14} + 50X_{15} + 40X_{16} +$
$35X_{24} + 30X_{25} + 22X_{26} +$
$55X_{34} + 20X_{35} + 25X_{36}$ } total distance fruit is shipped (in bushel-miles)

Subject to:

$X_{14} + X_{24} + X_{34} \leq 200{,}000$ } capacity restriction for Ocala
$X_{15} + X_{25} + X_{35} \leq 600{,}000$ } capacity restriction for Orlando
$X_{16} + X_{26} + X_{36} \leq 225{,}000$ } capacity restriction for Leesburg
$X_{14} + X_{15} + X_{16} = 275{,}000$ } supply available at Mt. Dora
$X_{24} + X_{25} + X_{26} = 400{,}000$ } supply available at Eustis
$X_{34} + X_{35} + X_{36} = 300{,}000$ } supply available at Clermont
$X_{ij} \geq 0$, for all i and j } nonnegativity conditions

Generalized Network Flow Problems

- In some problems, a gain or loss occurs in flows over arcs. Examples
  - Oil or gas shipped through a leaky pipeline
  - Imperfections in raw materials entering a production process
  - Spoilage of food items during transit
  - Theft during transit
  - Interest or dividends on investments

- These problems require some modeling changes.

Nancy Grant is the owner of Coal Bank Hollow Recycling, a company that specializes in collecting and recycling paper products. Nancy's company uses two different recycling processes to convert newspaper, mixed paper, white office paper, and cardboard into paper pulp. The amount of paper pulp extracted from the recyclable materials and the cost of extracting the pulp differs depending on which recycling process is used. The following table summarizes the recycling processes:

	Recycling Process 1		Recycling Process 2	
Material	**Cost per ton**	**Yield**	**Cost per Ton**	**Yield**
Newspaper	$13	90%	$12	85%
Mixed Paper	$11	80%	$13	85%
White Office Paper	$9	95%	$10	90%
Cardboard	$13	75%	$14	85%

For instance, every ton of newspaper subjected to recycling process 1 costs $13 and yields 0.9 tons of paper pulp. The paper pulp produced by the two different recycling processes goes through other operations to be transformed into pulp for newsprint, packaging paper, or print stock quality paper. The yields associated with transforming the recycled pulp into pulp for the final products are summarized in the following table:

	Newsprint Pulp		Packaging Paper Pulp		Print Stock Pulp	
Pulp Source	**Cost per Ton**	**Yield**	**Cost per Ton**	**Yield**	**Cost per Ton**	**Yield**
Recycling Process 1	$5	95%	$6	90%	$8	90%
Recycling Process 2	$6	90%	$8	95%	$7	95%

For instance, a ton of pulp exiting recycling process 2 can be transformed into 0.95 tons of packaging paper at a cost of $8.

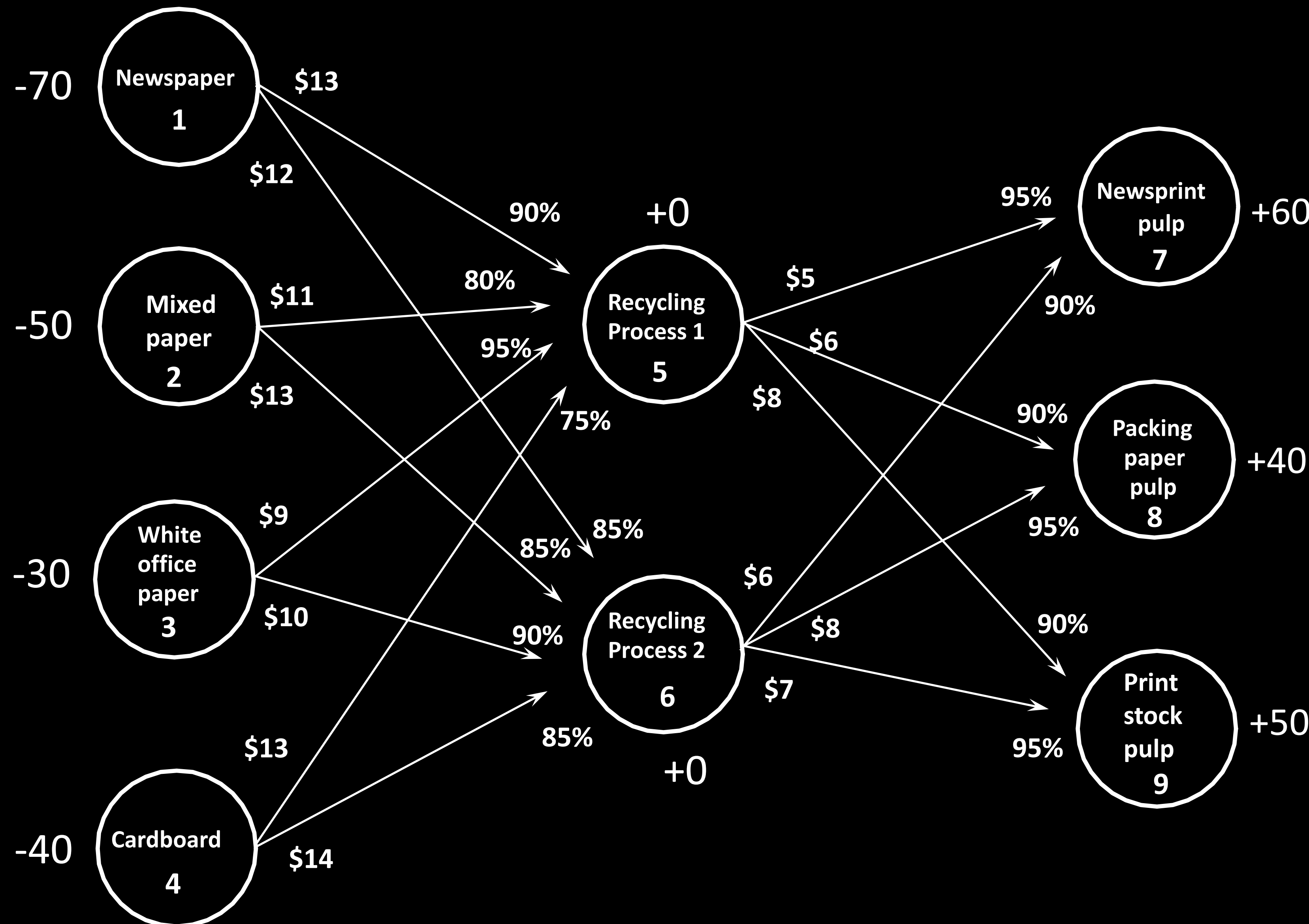
Nancy currently has 70 tons of newspaper, 50 tons of mixed paper, 30 tons of white office paper, and 40 tons of cardboard. She wants to determine the most efficient way of converting these materials into 60 tons of newsprint pulp, 40 tons of packaging paper pulp, and 50 tons of print stock pulp.

Coal Bank Hollow Recycling

	Recycling Process 1		Recycling Process 2		
Material	**Cost**	**Yield**	**Cost**	**Yield**	**Supply**
Newspaper	$13	90%	$12	85%	70 tons
Mixed Paper	$11	80%	$13	85%	50 tons
White Office Paper	$9	95%	$10	90%	30 tons
Cardboard	$13	75%	$14	85%	40 tons

	Newsprint		Packaging Paper		Print Stock	
Pulp Source	**Cost**	**Yield**	**Cost**	**Yield**	**Cost**	**Yield**
Recycling Process 1	$5	95%	$6	90%	$8	90%
Recycling Process 2	$6	90%	$8	95%	$7	95%
Demand	60 tons		40 tons		50 tons	

Network for Recycling Problem



Defining the Objective Function

Minimize total cost.

$$\begin{aligned}\text{MIN: } & 13X_{15} + 12X_{16} + 11X_{25} + 13X_{26} \\ & + 9X_{35} + 10X_{36} + 13X_{45} + 14X_{46} + 5X_{57} \\ & + 6X_{58} + 8X_{59} + 6X_{67} + 8X_{68} + 7X_{69}\end{aligned}$$

Defining the Constraints

Raw Materials

$-X_{15} - X_{16} >= -70$ } node 1
$-X_{25} - X_{26} >= -50$ } node 2
$-X_{35} - X_{36} >= -30$ } node 3
$-X_{45} - X_{46} >= -40$ } node 4

Recycling Processes

$+0.9X_{15} + 0.8X_{25} + 0.95X_{35} + 0.75X_{45} - X_{57} - X_{58} - X_{59} >= 0$ } node 5
$+0.85X_{16} + 0.85X_{26} + 0.9X_{36} + 0.85X_{46} - X_{67} - X_{68} - X_{69} >= 0$ } node 6

Paper Pulp

$+0.95X_{57} + 0.90X_{67} >= 60$ } node 7
$+0.90X_{57} + 0.95X_{67} >= 40$ } node 8
$+0.90X_{57} + 0.95X_{67} >= 50$ } node 9

Implementing the Model

See file Fig5-17.xls

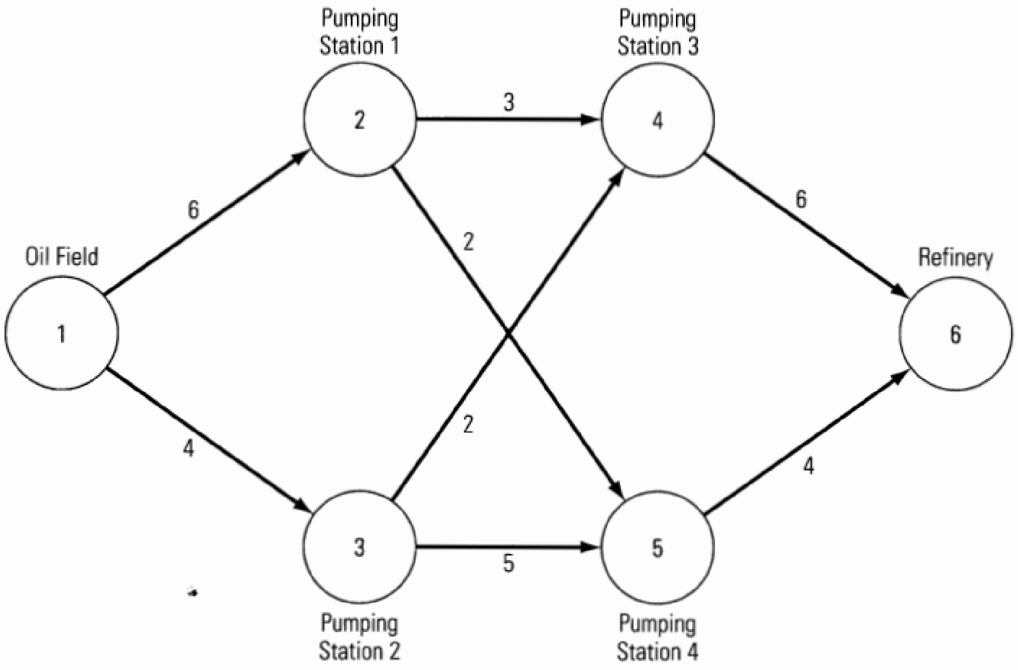
Flow	From Node		Yield	Flow	Into Node		Cost
43.4	1	Newspaper	0.90	39.1	5	Process 1	$13
26.6	1	Newspaper	0.85	22.6	6	Process 2	$12
50.0	2	Mixed Paper	0.80	40.0	5	Process 1	$11
0.0	2	Mixed Paper	0.85	0.0	6	Process 2	$13
30.0	3	White Office	0.95	28.5	5	Process 1	$9
0.0	3	White Office	0.90	0.0	6	Process 2	$10
0.0	4	Cardboard	0.75	0.0	5	Process 1	$13
35.4	4	Cardboard	0.85	30.1	6	Process 2	$14
63.2	5	Process 1	0.95	60.0	7	Newsprint	$5
44.4	5	Process 1	0.90	40.0	8	Packaging	$6
0.0	5	Process 1	0.90	0.0	9	Print Stock	$8
0.0	6	Process 2	0.90	0.0	7	Newsprint	$6
0.0	6	Process 2	0.95	0.0	8	Packaging	$8
52.6	6	Process 2	0.95	50.0	9	Print Stock	$7
						Total Cost	$3,149

Node		Net Flow	Supply/Demand
1	Newspaper	-70.0	-70
2	Mixed Paper	-50.0	-50
3	White Office	-30.0	-30
4	Cardboard	-35.4	-40
5	Process 1	0.0	0
6	Process 2	0.0	0
7	Newsprint	60.0	60
8	Packaging	40.0	40
9	Print Stock	50.0	50

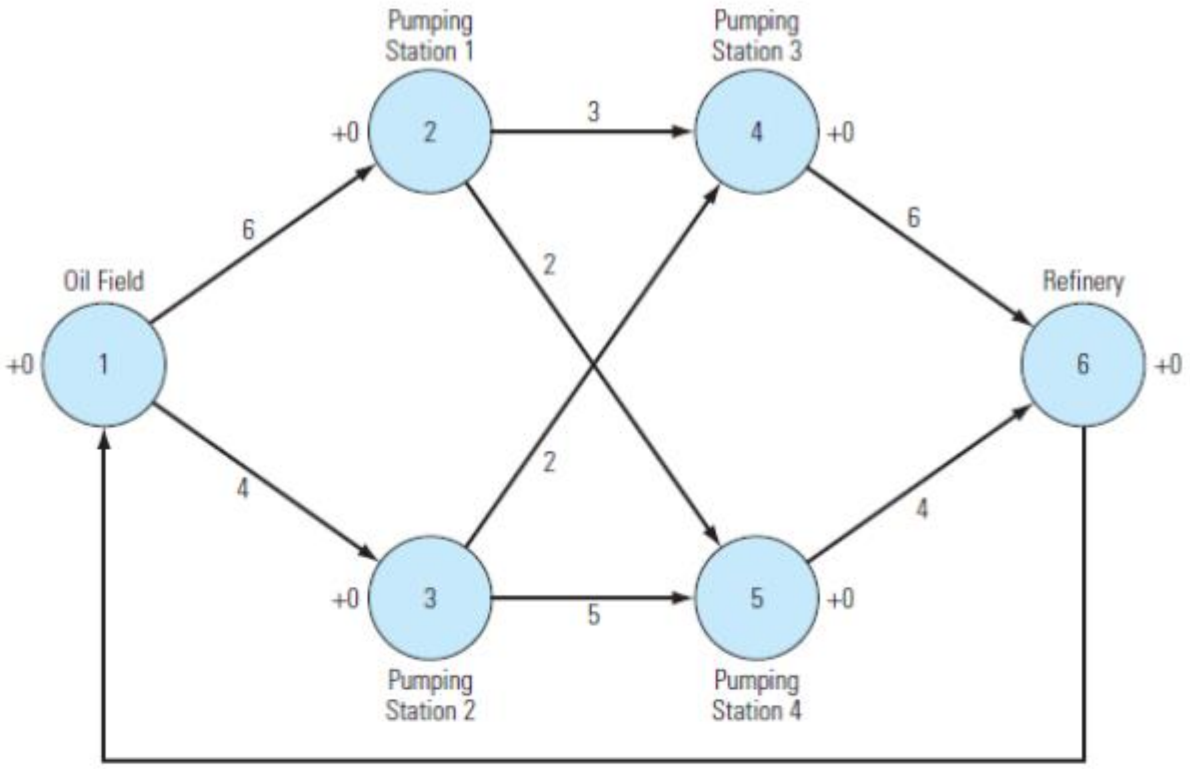
The Maximal Flow Problem

- In some network problems, the objective is to determine the maximum amount of flow that can occur through a network.
- The arcs in these problems have upper and lower flow limits.
- Examples
  - How much water can flow through a network of pipes?
  - How many cars can travel through a network of streets?

The Northwest Petroleum Company operates an oil field and refinery in Alaska. The crude obtained from the oil field is pumped through the network of pumping sub-stations shown in Figure 5.22 to the company's refinery located 500 miles from the oil field. The amount of oil that can flow through each of the pipelines, represented by the arcs in the network, varies due to differing pipe diameters. The numbers next to the arcs in the network indicate the maximum amount of oil that can flow through the various pipelines (measured in thousands of barrels per hour). The company wants to determine the maximum number of barrels per hour that can flow from the oil field to the refinery.



Maximal Flow Problem (Adding Return Arc)



Formulation of the Max Flow Problem

MAX: X_{61}

Subject to:

$$
\begin{aligned}
+X_{61} - X_{12} - X_{13} &= 0 \\
+X_{12} - X_{24} - X_{25} &= 0 \\
+X_{13} - X_{34} - X_{35} &= 0 \\
+X_{24} + X_{34} - X_{46} &= 0 \\
+X_{25} + X_{35} - X_{56} &= 0 \\
+X_{46} + X_{56} - X_{61} &= 0
\end{aligned}
$$

with the following bounds on the decision variables:

$$
\begin{array}{lll}
0 <= X_{12} <= 6 & 0 <= X_{25} <= 2 & 0 <= X_{46} <= 6 \\
0 <= X_{13} <= 4 & 0 <= X_{34} <= 2 & 0 <= X_{56} <= 4 \\
0 <= X_{24} <= 3 & 0 <= X_{35} <= 5 & 0 <= X_{61} <= \text{inf}
\end{array}
$$

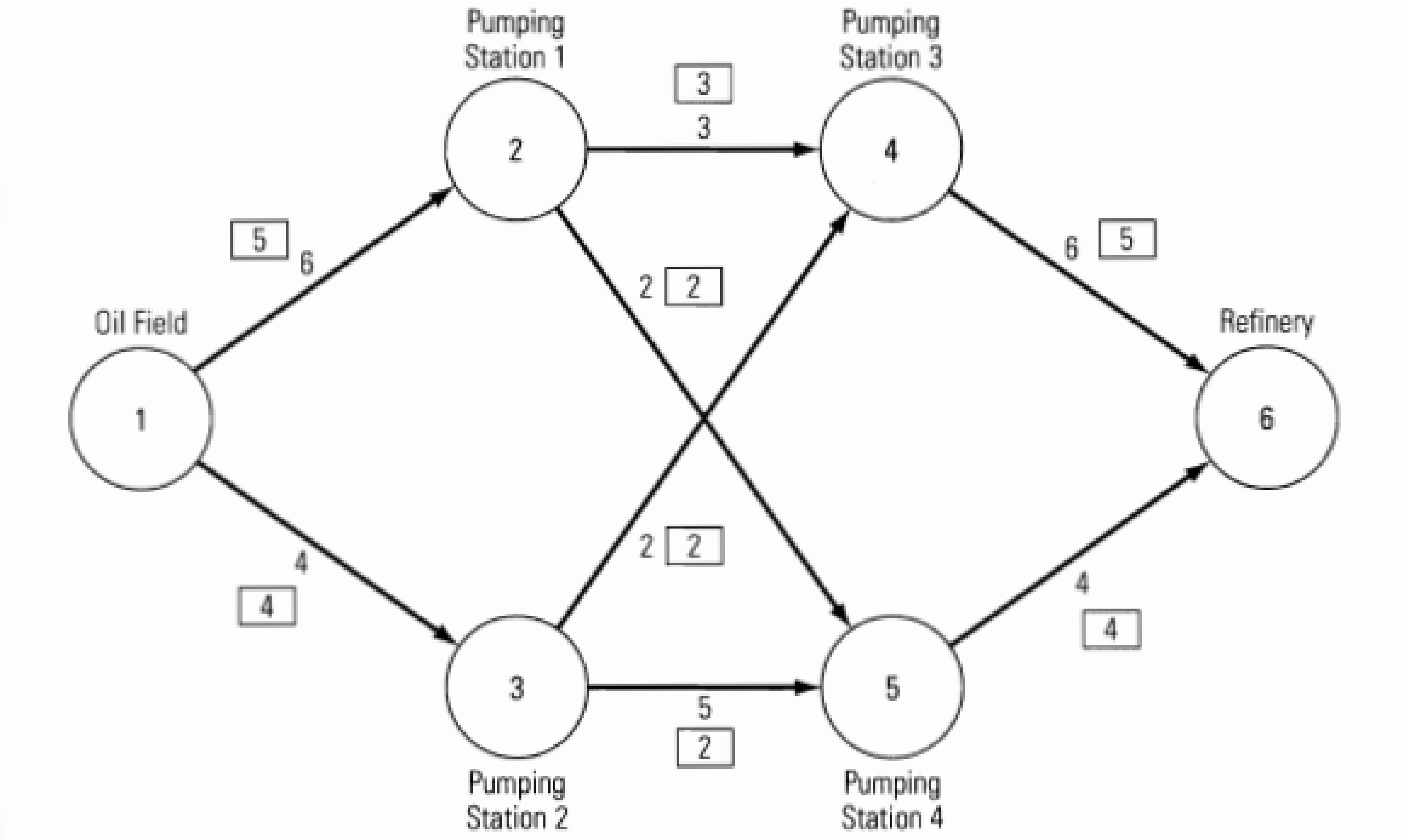
Implementing the Model

See file Fig5-24.xls

Units	-- Arcs --				Upper
of Flow	From		To		Bound
5.0	1	Oil Field	2	Station 1	6
4.0	1	Oil Field	3	Station 2	4
3.0	2	Station 1	4	Station 3	3
2.0	2	Station 1	5	Station 4	2
2.0	3	Station 2	4	Station 3	2
2.0	3	Station 2	5	Station 4	5
5.0	4	Station 3	6	Refinery	6
4.0	5	Station 4	6	Refinery	4
9.0	6	Refinery	1	Oil Field	9999
9.0	Maximal Flow				

Nodes		Net Flow	Supply/Demand
1	Oil Field	0	0
2	Station 1	0	0
3	Station 2	0	0
4	Station 3	0	0
5	Station 4	0	0
6	Refinery	0	0

Optimal Solution



The Minimal Spanning Tree Problem

- For a network with n nodes, a *spanning tree* is a set of n-1 arcs that connects all the nodes and contains no loops.

- The minimal spanning tree problem involves determining the set of arcs that connects all the nodes at minimum cost.

Assignment 5

Question no: 9, 11, 14, 16, 17, 18, 21, 23, 27, 30

Due on 14 July

End of Chapter 5