High Order Limit State Functions in the Response Surface Method for Structural Reliability Analysis

Henri P. Gavin *, Siu Chung Yau

Department of Civil and Environmental Engineering, Duke University, Durham NC 27708-0287

Abstract

The stochastic response surface method (SRSM) is a technique for the reliability analysis of complex systems with low failure probabilities, for which Monte Carlo simulation (MCS) is too computationally intensive and for which approximate methods are inaccurate. Typically, the SRSM approximates a limit state function with a multidimensional quadratic polynomial by fitting the polynomial to a number of sampling points from the limit state function. This method can give biased approximations of the failure probability for cases in which the quadratic response surface can not conform to the true limit state function's nonlinearities. In contrast to recently proposed algorithms which focus on the positions of sample points to improve the accuracy of the quadratic SRSM, this paper describes the use of higher order polynomials in order to approximate the true limit state more accurately. The use of higher order polynomials has received relatively little attention to date because of problems associated with ill-conditioned systems of equations and an approximated limit state which is very inaccurate outside the domain of the sample points. To address these problems, an algorithm using orthogonal polynomials is proposed to determine the necessary polynomial orders. Four numerical examples compare the proposed algorithm with the conventional quadratic polynomial SRSM and a detailed MCS. `doi:10.1016/j.strusafe.2006.10.003`

Key words: Chebyshev polynomial, Failure probability, Monte Carlo, Statistical test, Structural reliability, Response surface

* Department of Civil and Environmental Engineering, Duke Univ.
URL: `http://www.duke.edu/~hpgavin/` (Henri P. Gavin).

1 Introduction

In system reliability analysis, Monte Carlo Simulation (MCS) [17] is the only known technique to accurately estimate the probability of failure, P_f, regardless of the complexity of the system or the limit state. MCS becomes computationally intensive for the reliability analysis of complex systems with low failure probabilities and MCS can be infeasible when the analysis requires a large number of computationally intensive simulations. In such cases, the first order reliability method (FORM) [22] and the second order reliability method (SORM) are also difficult to apply since the true limit state usually cannot be easily expressed explicitly.

The stochastic response surface method (SRSM) [10] is a recently-developed technique which can provide an efficient and accurate estimation of structural reliability regardless of the complexity of the failure process. The SRSM approximates the true limit state function using simple and explicit mathematical functions (typically quadratic polynomials) of the random variables involved in the limit state function. By fitting the response surface to a number of designated sample points of the true limit state, an approximated limit state function is constructed. As the approximated limit state function is explicit, FORM or SORM can be applied to estimate the probability of failure directly. Alternatively, MCS can be used efficiently since the evaluation of the response surface function requires very little computational effort.

Recent studies have investigated the limitations of the SRSM and have shown that the method fails to estimate the probability of failure accurately in some problems with highly nonlinear limit state functions and in some problems with low probabilities of failure [4,14,21]. Most of the proposed improvements to the SRSM have been concerned with the relocation of the sample points to positions close to the true limit state or design point, so that a quadratic polynomial can better approximate the true limit state function in that region [2,4,12,16,21]. These studies have shown that if a second order polynomial is used, methods of experiment design (simply working on the locations of sample points) do not neccessarily improve results in cases where the a quadratic form can not accurately fit the limit state function over a broad domain. If the true limit state function is highly nonlinear, the accuracy of the approximation depends very much upon the location and distribution of the sample points [19]. Therefore, for highly nonlinear limit states, experimental design alone cannot solve the problem.

In this study, a different approach is proposed, suggesting the use of higher order polynomials, in order to approximate the true limit state over a broad region of the space of random parameters. The use of higher order polynomials has received little attention because doing so can require an excessively large number of sample points. It can also result in ill-conditioned systems of equations, and huge differences between the approximated and true limit state functions outside the domain of the sample points [8,19,21]. This study briefly discusses how to determine polynomial orders so as to avoid unnecessary higher order terms and associated ill-conditioning. The method involves the use of Chebyshev polynomials, a statistical analysis of the Chebyshev polynomial coefficients, and a statistical analysis of the high-order response surface.

2 Background, Review and Motivation

2.1 Introduction to SRSM

In a large and complex structural system, the true limit state is usually implicit and expressed in terms of a set of n random variables, such as material properties, dimensions, and load densities. Let $\mathbf{X}$ denote the $1 \times n$ vector containing these variables and let $g(\mathbf{X})$ be the true *limit state function*. The function $g(\mathbf{X}) = 0$ defines the true *limit state*, and $g(\mathbf{X}) < 0$ indicates a failure condition of the system. The failure of probability is given by $P_f = \mathrm{Prob}[g(\mathbf{X}) < 0]$.

In the stochastic response surface method (SRSM), the true limit state function, $g(\mathbf{X})$, is approximated by a simple and explicit mathematical expression $\tilde{g}(\mathbf{X})$, which is typically a k-th order polynomial, with undetermined coefficients. The value of the true limit state function is evaluated at a number of samples of $\mathbf{X}$, to determine the unknown coefficients such that the error of approximation at the samples of $\mathbf{X}$ is minimized.

2.2 The Second Order SRSM

The selection of the form of the approximated limit state function, $\tilde{g}(\mathbf{X})$, i.e. the response surface, ideally should be based on the shape and the nonlinearity of the true limit state function, $g(\mathbf{X})$. Since $g(\mathbf{X})$ is usually unknown, there has been a tendency to develop a generic form for a response surface which can be applied across a wide range of strutucal reliability problems. The most common form is the quadratic polynomial [10],

$$\tilde{g}(\mathbf{X}) = a + \sum_{i=1}^{n} b_i X_i + \sum_{i=1}^{n} c_i X_i^2 \tag{1}$$

Higher order polynomials are usually not used because if $\tilde{g}(\mathbf{X})$ is of a much

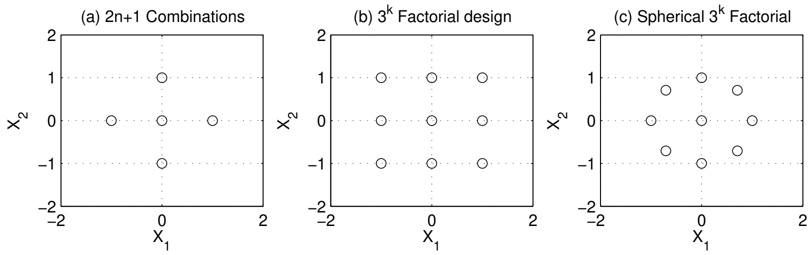

Fig. 1. Different sample methods ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $h = 1$)

higher degree than $g(\mathbf{X})$, ill-conditioned systems of equations may be encountered [19,17,21] and there may be huge differences between $\tilde{g}(\mathbf{X})$ and $g(\mathbf{X})$ outside the domain of sample points [8]. Comparatively, though the use of the quadratic polynomial has no physical justification, it provides a simple and readily-available smoothing surface which allows easy identification of a single design point [19]. This is particularly useful for the application of FORM and SORM on the approximated limit state and the iterative scheme of response surface approximation which locates the design point on the approximated limit state [2].

In equation (1), a, b_i and c_i, are the $2n + 1$ unknown coefficients. The values of the coefficients can be determined via singular value decomposition using a set of sample points from the true limit state function, $g(\mathbf{X})$. The number of sample points must be larger or at least equal to the number of coefficients. Among various sampling methods, a common experiment-design approach is to evaluate $g(\mathbf{X})$ at $2n+1$ combinations of μ_i and $\mu_i \pm h\sigma_i$, as shown in Figure 1a, where μ_i and σ_i are the mean and standard deviation of X_i, and h is an arbitrary factor. In this case, the number of sample points is just sufficient for the determination of the coefficients, thereby minimizing the number of sample points and the number of evaluations of the limit state function.

In order to capture the nonlinearity of the true limit state more precisely, mixed terms are sometimes included [15] into the quadratic polynomial $\tilde{g}(\mathbf{X})$:

$$\tilde{g}(\mathbf{X}) = a + \sum_{i=1}^{n} b_i X_i + \sum_{i=1}^{n} c_i X_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij} X_i X_j \qquad (2)$$

where the number of coefficients is now $1+2n+\frac{n(n-1)}{2}$. In this case, 3^k factorial design is a common sampling approach. In 3^k factorial design, sample points are chosen at all possible combinations of the mean values μ_i and $\mu_i \pm h\sigma_i$, as shown in Figure 1b. In 3^k factorial sampling, the number of sample points can be excessively large if the number of random variables, n, is large. An alternate sampling strategy, 'spherical' 3^k factorial design, can also be applied, where every sample point except the center point has the same distance to the origin, as shown in Figure 1c.

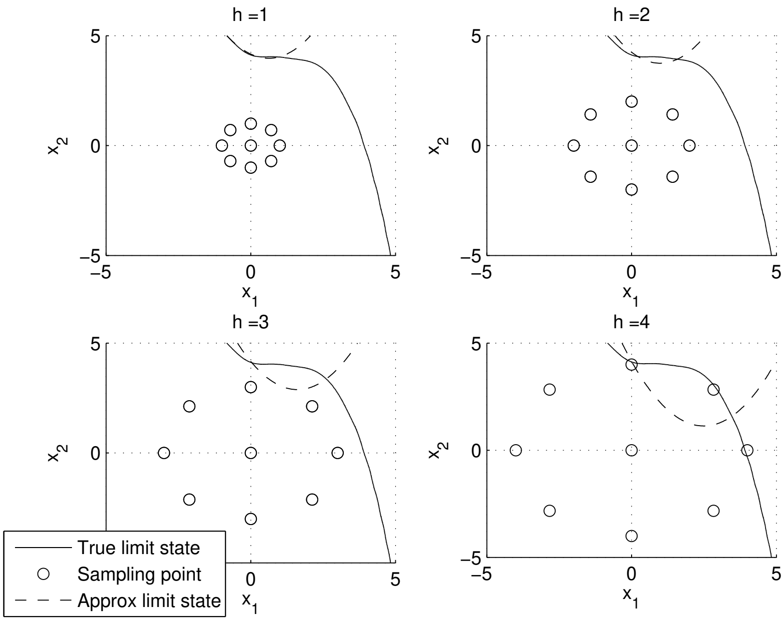

Fig. 2. Influence of parameter h in the quadratic approximation of the limit state

The two basic forms of the SRSM mentioned above can approximate the true limit state function, $g(\mathbf{X})$, accurately for rougly linear and quadratic limit states. But when the shape of the true limit state is not close to linear or quadratic, the parameter h, which controls the size of tbe sampling domain, plays a significant role in the accuracy of the 2$^{\text{nd}}$ order SRSM approximation [11]. For example, consider the approximation of

$$g(\mathbf{X}) = -0.16(X_1 - 1)^3 - X_2 + 4 - 0.04\cos(X_1 X_2) \tag{3}$$

using equation (2). As shown in Figure 2, as long as the quadratic polynomial cannot conform to the true limit state, the 2$^{\text{nd}}$ order response surfaces depend critically on the value selected for h. Since the form of the true limit state function and the failure probability are usually unknown before the approximation and vary from problem to problem, there is no typical value of h which can be applied across a wide range of reliability problems.

2.3 Improvements on the Second Order SRSM

For structures with high reliability and highly nonlinear limit states, the approximation using the sampling methods in Figure 1 may be inaccurate because the sample points, which are located around the mean values, can be far from the true limit state, $g(\mathbf{X}) = 0$. Some recent developments suggest algoritms of repeated response surface approximations to shift the sample points closer to the true limit state, or the design point [4,16,21].

Bucher and Bourgund [4] proposed an algorithm to locate a new center of sample points $\mathbf{x}_m$, which is closer to the true limit state than the mean value vector $\boldsymbol{\mu}$, by adaptive linear interpolation:

$$\mathbf{x}_m = \boldsymbol{\mu} - g(\boldsymbol{\mu})\frac{\boldsymbol{\mu} - \mathbf{x}_D}{g(\boldsymbol{\mu}) - g(\mathbf{x}_D)} \tag{4}$$

where $\mathbf{x}_D$ is the approximated design point, defined as a point lying on the approximated limit state that is closest to $\boldsymbol{\mu}$. For standardized random variables, $\boldsymbol{\mu}$=0 After an approximation of the true limit state function, equation (4) is applied. A new set of sample points, which are combinations of x_{mi} and $x_{mi} \pm h\sigma_i$, are used to construct a second approximated limit state function, and the reliability is estimated based on this new approximated limit state. A drawback of the method is a doubling of the computational effort. The procedure also assumes a single design point and fails to incoporate mulitple design points.

To approximate the true limit state function more accurately around the single design point, a sequence of repeated response surface approximations can be persued [3]. Rajashekhar and Ellingwood [21] suggested that an iteration of the procedure proposed by Bucher and Bourgund and that a reduction of h in the later iterations can improve the approximation. Liu and Moses [16] proposed the iteration of the procedure until a convergence criterion is satisfied. Gupta and Manohar [12] recognized the difficulty in the approximation of limit states with multiple design points. They proposed a method which involves the use of a higher order polynomial and Bucher and Bourgund's algorithm to incorporate the multiple design points in the approximation. Their sample strategy is complicated and requires considerably many sample points.

Most of the recent improvements of the second order SRSM require a significant increase in the computational effort, but they may not produce accurate approximations in highly nonlinear limit state functions or limit states with mulitple design point. For example, consider the application of iterative Bucher and Bourgund procedure at constant h on the limit state funciton in equation 3, as illustrated in Figure 3. The process starts with sample points centered at the origin. In each figure a 2^{nd} order response surface approximation (solid line) is determined from a set of sample points (circles) to locate a design point, $\mathbf{x}_D$ (star), which is the point on the approximation closest to the origin. The previous response surface(s) are shown as dashed lines, and the true limit state function is shown as a dotted line. The next response surface is determined from a set of sample points centered at $\mathbf{x}_m$. The upper row of the figure corresponds to h=1 and the lower row of the figure corresponds to h=2. As shown in the figure, though the quadratic polynomial could accurately fit the local nonlinearity around the region of the sample points, the approximation outside the domain can be so inaccurate that it affects the estimate the

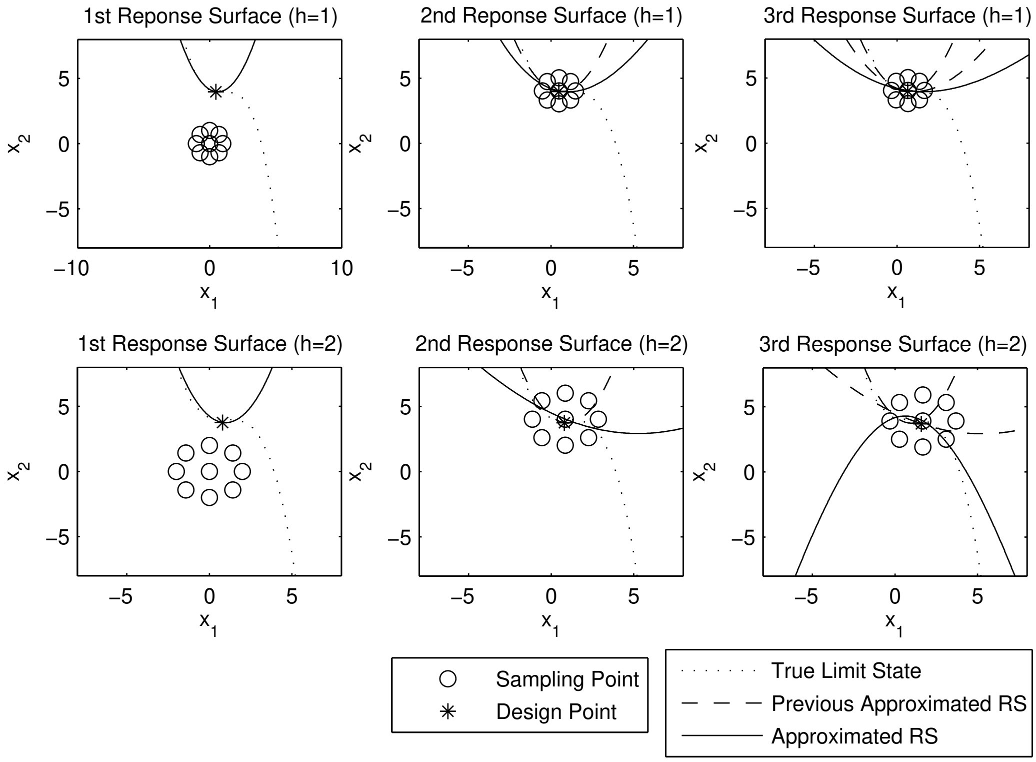

Fig. 3. Iterative Response Surface Approximation

probability of failure significantly.

2.4 Higher Order Approximated Limit State Functions

A polynomial of a fixed degree, e.g. a quadratic polynomial, is unable to approximate the limit states accurately for a wide range of structural reliability problems, since the shape of the limit state varies for each problem. Therefore it is preferable to use a more flexible polynomial which has a form suited to the nonlinearity of the true limit state. A flexible polynomial not only produces more accurate results, but is also more robust to the variations in the positions of sample points. This paper proposes the use of statistical analyses of trial response surfaces, in order to determine the appropriate order of an approximated limit state at small expense of computational effort. This avoids any excessively high order terms, which can involve ill-conditioned systems of equations. Compared to the recursive response surface approximation, which involves a small domain of sample points in each iteration, the objective of this paper is to provide an accurate estimation of the failure probability by sampling a boader domain size only once. After the approximation, the estimated probability of failure can serve as an indication of the domain size of sample points, given that the approximated limit state can conform well to the true limit state.

3 The High Order Stochastic Response Surface Method, HO-SRSM

The method proposed in this study approximates the true limit state function with an approximate polynomial response surface of arbitrary order,

$$\tilde{g}(\mathbf{X}) = a + \sum_{i=1}^{n} \sum_{j=1}^{k_i} b_{ij} X_i^j + \sum_{q=1}^{m} c_q \prod_{i=1}^{n} X_i^{p_{iq}} \ , \tag{5}$$

where the coefficients b_{ij} correspond to terms involving only one random variable, and the coefficients c_q correspond to mixed terms, involving the product of two or more random variables. The polynomial order, k_i, the total number of mixed terms, m, and the order of a random variable in a mixed term, p_{iq}, are determined in the algorithm below.

The algorithm of the proposed method, called the High Order Stochastic Response Surface Method (HO-SRSM), has four stages. First, orders for the response surface are identified. Second, the number and types of mixed terms are determined. These first two stages result in the formulation of the higher order polynomial to be used for the response surface. After the formulation of the higher order polynomial is completed, the coefficients of the higher order response surface polynomial are estimated in the third stage, using singular value decomposition. Fourth, MCS is carried out on the response surface to determine the probability of failure, P_f.

3.1 Polynomial Orders

In the first stage of the method, the mixed terms are neglected and the polynomial orders, k_i, are determined by statistically and numerically testing the significance of polynomial coefficients. The statistical test for these coefficients requires that the coefficients be statistically uncorrelated. The numerical test for the coefficients requires that all local maxima and minima of all polynomial basis functions have the same magnitude. These two conditions are satisfied by fitting the limit state function in a basis of Chebyshev polynomials. The orthogonality of the Chebyshev polynomials guarantees uncorrelated coefficients, and Chebyshev polynomials evaluated over the domain $[-1, 1]$ are bounded to within the range of $[-1, 1]$, as shown in Figure 4. A Chebyshev

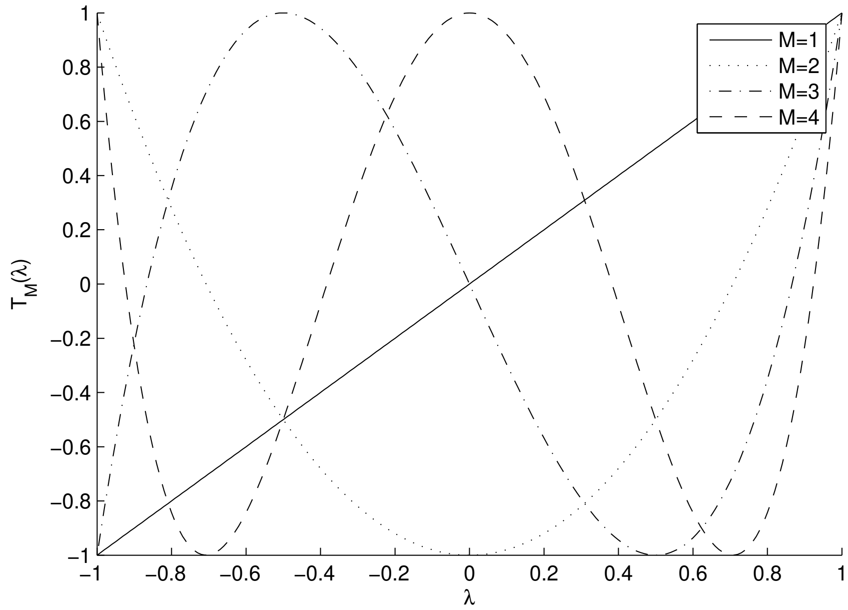

Fig. 4. Chebyshev polynomials of different orders

polynomial [20] of degree M in λ is given by

$$T_M(\lambda) = \cos(M \arccos \lambda) \ , \tag{6}$$

where $\min(T_M(\lambda)) = -1$, and $\max(T_M(\lambda)) = 1$, for all λ such that $-1 \leq \lambda \leq 1$.

The polynomial $T_M(\lambda)$ has M roots in the interval $[-1, 1]$ at

$$\lambda = \cos\left(\frac{\pi(m - \frac{1}{2})}{M}\right) \qquad \text{where} \qquad m = 1, \ldots, M \tag{7}$$

The discrete orthogonality relation for Chebyshev polynomials is given by:

$$\sum_{m=1}^{M} T_i(\lambda_m) T_j(\lambda_m) = \begin{cases} 0 & : \ i \neq j \\ M/2 & : \ i = j \neq 0 \\ M & : \ i = j = 0 \end{cases} \tag{8}$$

where $\lambda_m (m = 1, \ldots, M)$ are the M roots of $T_M(\lambda)$ given by equation (7) and M must be greater than or equal to the larger of i and j. If the range of the desired sample points is other than the interval $[-1, 1]$, the sample points must be interpolated to the interval $[-1, 1]$.

The orders of the variables k_i in equation (5) are estimated one-by-one along dimension X_i using one-dimensional Chebyshev polynomials,

$$\hat{g}_i(X_i) = d_0 T_0(\lambda_i) + d_1 T_1(\lambda_i) + d_2 T_2(\lambda_i) + \ldots + d_{k_i} T_{k_i}(\lambda_i), \tag{9}$$

where λ_i is the interpolated values of X_i from interval $[\mu_i - h_{\text{ord}}\sigma_i, \mu_i + h_{\text{ord}}\sigma_i]$ to $[-1, 1]$, i.e.

$$X_i = \mu_i + h_{\text{ord}}\lambda\sigma_i \tag{10}$$

Parameter h_{ord} is the domain of the sampling points used to determine the polynomial degree of the approximation. Ideally the zeros of the Chebyshev polynomial should be interpolated to the interval about the design point of the true limit state, instead of μ_i, but the location of the true design point is unknown at this stage of the algorithm. In the estimation of the order k_i for a random variable X_i, all other variables, X_1,. . .,X_{i-1},X_{i+1},. . .,X_n, are set to their mean values. $(\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ...\mu_n)$. This approximated limit state function serves only to determine the appropriate polynomial order k_i in equation (5), and is not used to estimate the structural reliability. One-dimensional approximation, instead of a multi-dimensional approximation, is used because it is much more computationally efficient, especially in cases involving a large number of random variables.

The Chebyshev polynomial coefficients, d_j, are determined by the least squares method.

$$\mathbf{d} = [\mathbf{T}^T\mathbf{T}]^{-1}\mathbf{T}^T\mathbf{g}_i(\mathbf{x}_i) \tag{11}$$

where the $\mathbf{T}_{jk} = T_j(\lambda_k)$, λ_k is the k-th root of $T_K(\lambda)$, and $\mathbf{g}_i(\mathbf{x}_i)$ is a vector of the values of true limit state function evaluated with discrete values of random variable X_i set to

$$x_{ik} = \mu_i + h_{\text{ord}}\lambda_k\sigma_i \qquad \text{where} \qquad k = 1, \ldots, K \ , \tag{12}$$

and with all other elements of $\mathbf{X}$ set to their mean values.

The coefficient covariance matrix,

$$V_d = [\mathbf{T}^T\mathbf{T}]^{-1} \sum_{k=1}^{K} (\hat{g}_i(x_{ik}) - g_i(x_{ik}))^2 / (K - k_i) \ , \tag{13}$$

is diagonal, due to the discrete orthogonality relationship of the Chebyshev polynomials, given in equation (8). In fact, every diagonal term of V_d, except the first, is

$$\sigma_d^2 = \frac{K}{2} \sum_{k=1}^{K} (\hat{g}_i(x_{ik}) - g_i(x_{ik}))^2 / (K - k_i) \ . \tag{14}$$

Because all Chebyshev polynomials are bounded to within $[-1, 1]$ the contribution of T_j to $\hat{g}_i$ is related only to d_j.

The variance of the coefficients, σ_d^2, is used to reveal the statistical significance of each of the terms in equation (9), indicating which terms could be neglected without significant influence. The test of statistical significance of an individual term d_j in equation (9) involves the test of the null hypothesis, H_0: the true coefficient of the term is 0. The test is performed by calculating values of the t-statistics [6],

$$t_j = \frac{d_j}{\sigma_d} \ . \tag{15}$$

Using a two-sided test and 90% confidence intervals, if the absolute value of t_j is smaller than the value of $t_{0.05} = 3.499$, the null hypothesis can not be rejected and the $T_j(\lambda_i)$ term is determined to be statistically insignificant. Also if d_j is less than one percent of the other coefficients, in magnitude, then $T_j(\lambda_i)$ is determined to be numerically insignificant, because $T_j(\lambda_i)$ is bounded within [-1,1] for all i and j.

The algorithm starts with $k_i = 3$, the approximation first determines whether a second order is sufficient for X_i. If $T_3(\lambda_i)$ is determined to be statistically or numerically insignificant, it is concluded that the response surface should be second order in X_i, i.e. k_i=2. On the other hand, if $T_3(\lambda_i)$ is significant, another approximation including a $T_4(\lambda_i)$ term is carried out and the significance of the forth order term is tested. These iterations continue until the highest order term is determined to be insignificant or the order is estimated to be at most five. It is presumed that it is very rare to exceed the fifth order. By removing the insignificant high order terms, the orders $k_1, \ldots, k_n$ can be kept small yet sufficient to provide an accurate approximation of the limit state.

3.2 Mixed Terms

After the orders, k_i, of the polynomials in X_i, $i = 1, \cdots, n$, are determined, the formulation of the mixed terms can be determined. As a rule, the number of mixed terms should be kept as small as possible. In general, a mixed term can be expressed as $X_1^{p_1} X_2^{p_2} \ldots X_n^{p_n}$. There are two criteria for a valid mixed term: (1) the power of a variable in a mixed term should not be larger than

Table 1
Valid mixed term powers in a limit state function where $k_1 = 2$, $k_2 = 3$, $k_3 = 3$

q	p_{1q}	p_{2q}	p_{3q}
1	0	1	1
2	1	0	1
3	1	1	0
4	0	1	2
5	0	2	1
6	1	0	2
7	2	0	1
8	1	2	0
9	2	1	0

the estimated order of the variable alone, i.e., $p_i \le k_i$ and (2) the total order of the mixed term, $\sum_i p_i$, should not be larger than the highest order term, i.e., $\sum_i p_i \le \max(k_i)$. For example, in a limit state function of three variables, suppose it is found in stage 1 that k_1=2, k_2=3, k_3=3, then the nine valid mixed terms, $c_q \prod_{i=1}^{n} X_i^{p_{iq}}$, (and q=1, . . . , 9), have the powers listed in Table 1, where q is the mixed term with coefficient c_q.

For large and complex problems, it may be worthwhile to examine and compare the results of several reasonable values of h_{ord} after the number of mixed terms are calculated, since the computational cost for the determination of the order is relatively small compared to the approximation of the true limit state. It may be desirable to use a value for h_{ord} resulting in a small number of coefficients in order to reduce the computational cost, as there is no general guideline to decide which h_{ord} should be used before P_f is estimated.

3.3 Response Surface Approximation

Once the response surface has been formulated, the coefficients are estimated via singular value decomposition using sample points from the true limit state function. In this study, the number of sample points is chosen to be twice the number of coefficients. This number is more than sufficient and was found to be adequate in the examples considered here. A "full factorial design" has P sample points where

$$P = \prod_{i=1}^{n} (k_i + 1) \ . \tag{16}$$

In problems with $n > 3$, the value of P, even 3^n, is much larger than the number of coefficients. Thus uniformly distributed random sample points are taken within the domain $[\boldsymbol{\mu} + h_{\text{reg}}\boldsymbol{\sigma}, \boldsymbol{\mu} - h_{\text{reg}}\boldsymbol{\sigma}]$, where $\boldsymbol{\mu}$ is a vector of the mean values of $\mathbf{X}$ and $\boldsymbol{\sigma}$ is a vector containing the standard deviation of $\mathbf{X}$. Analogous to the parameter h in regular 2$^{\text{nd}}$ order SRSM, the parameter h_{reg} here indicates the size of the domain of the sample points used in the regression for the polynomial coefficients. The true limit state function is evaluated at the uniformly-distributed sample points within the domain $[\boldsymbol{\mu} + h_{\text{reg}}\boldsymbol{\sigma}, \boldsymbol{\mu} - h_{\text{reg}}\boldsymbol{\sigma}]$. The use of randomly distributed sample points may increase the randomness of the results, but the number of sample points are believed to be large enough to diminish the effect of their randomness, while still being much smaller than P or a 3^n design. If the true limit state function is more accurately approximated by a higher-order limit state function, then the regression coefficients of the higher order aproximation will be less sensitive to h_{reg} than those of a 2$^{\text{nd}}$ order approximation, because the 2$^{\text{nd}}$ order approximation will be accurate only with smaller domains.

3.4 Monte Carlo Simulation

In the fourth stage, a full scale MCS on the approximated limit state is carried out to determine the probability of failure, P_f.

Recall that in the proposed method, there are two input parameters h_{ord} and h_{reg}, while the regular quadratic SRSM has one domain size parameter. Parameter h_{ord} is the size of the domain of the sampling points used to determine the polynomial degree of the response surface, and h_{reg} is the size of domain of the sampling points used to determine the coefficients of the response surface.

4 Numerical Examples

The performance of the proposed method is illustrated by four numerical examples. The first two examples have explicit hypothetical limit state functions and the other two are realistic structural reliability problems. In each example, the probability of failure is estimated by the following three methods:

(1) Direct full scale Monte Carlo simulation (MCS).
(2) Stochastic response surface method which approximates the true limit state with a quadratic polynomial including the mixed terms (regular 2$^{\text{nd}}$ order SRSM). The spherical 3^k factorial design is used to locate the positions of sample points. The effect of the parameter h_{reg} on the estimation result is investigated for a set of values of h_{reg}. In the first example, the

procedure proposed by Bucher and Bourgund will be applied for further comparison.

(3) High order stochastic response surface method (HO-SRSM). In each example, a range of values of parameter h_{ord} are used to identify appropriate polynomial orders. Then the limit state is approximated for that polynomial order. Again, different values of h_{reg} will be used to investigate its influence on the estimated failure probabilities.

In addition, the values of adjusted R^2, indicating the accuracy of the approximation, are calculated from the statistical data of each approximated limit state [6]. The definition of adjusted R^2 is given by

$$\text{Adjusted}R^2 = \frac{(P-1)R^2 - N}{P-(N+1)} \tag{17}$$

where P is the total number of sample points, N is the total number of coefficients, and

$$R^2 = 1 - \frac{\sum_{i=1}^{P}\left(\tilde{g}(x_i) - g(x_i)\right)^2}{\sum_{i=1}^{P}\left(g(x_i) - \bar{g}\right)^2} \tag{18}$$

The mean value of the limit state function, $\bar{g}$, is given by

$$\bar{g} = \frac{1}{P}\sum_{i=1}^{P} g(x_i) \tag{19}$$

The adjusted R^2 is bounded by the interval [0,1]. If its value is large, the approximated limit state is close to the true limit state at the sample points.

The general-purpose computer code for the HO-SRSM, and examples are available at http://www.duke.edu/~hpgavin/HOSRSM/.

4.1 Example 1 - Hypothetical limit state with 2 variables

A hypothetical limit state with two independent standard normal variables is considered.

$$g(\mathbf{X}) = -0.16(X_1-1)^3 - X_2 + 4 - 0.04\cos(X_1X_2) \tag{20}$$

The limit state, $g(\mathbf{X}) = 0$, has the shape shown in top-left figure in Figure 5. The limit state function is roughly cubic in X_1 and quadratic in X_2. The cosine term represents the small effects of higher-order terms. The true value

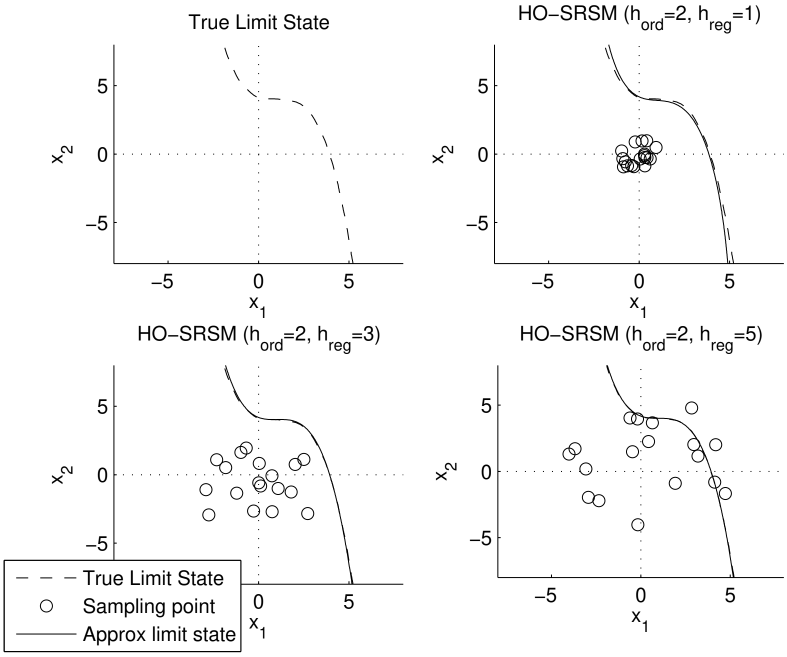

Fig. 5. True limit state and approximated higher-order limit states in Example 1

Table 2
Determination of polynomial orders for Example 1

h_{ord}	k_1	k_2
1.0	2	2
2.0	3	2
3.0	3	2
4.0	3	2
5.0	3	2
6.0	3	2
7.0	3	2

of $P_f = 0.000095$ is obtained by a direct full scale MCS of 10,000,000 sample size.

Table 2 suggests that the parameter h_{ord} in the HO-SRSM has almost no influence in the estimation of the polynomial orders in this problem, since all values of h_{ord} larger than 1.0 generate exactly the same results. The failure to detect the third order at $h_{\mathrm{ord}} = 1$ is reasonable because over a small sample area, the limit state does not demonstrate sufficient nonlinearity. The polynomial orders given by h_{ord} from 2 to 7, i.e., $k_1 = 3$ and $k_2 = 2$, are used to approximate the true limit state function via the HO-SRSM.

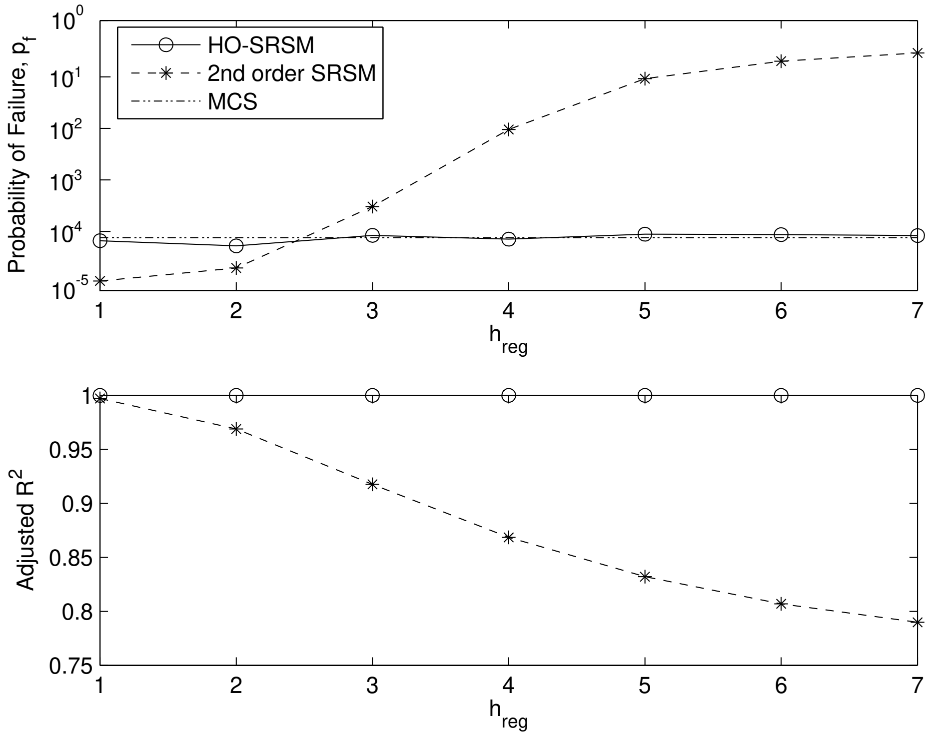

Fig. 6. Estimation results of Example 1

The shapes of the approximated limit states at different h_{reg} in the regular 2^{nd} order SRSM and HO-SRSM are shown in Figures 2 and 5 respectively. The estimation of P_f by the HO-SRSM is consistent over a large range of h_{reg}, as compared to the regular 2^{nd} order SRSM, as shown in Figure 6. The P_f of the HO-SRSM ranges from 0.000067 (-29%) (at h_{reg}=2) to 0.00011 (+16%) (at h_{reg}=5), while that of the 2^{nd} order SRSM ranges from 0.000015 (-84%) (at h_{reg}=1) to 0.25 (+2,600,000%) (at h_{reg}=6). The estimation results of the 2^{nd} order SRSM are very heavily dependent on h_{reg}. Comparatively, there is no obvious trend over values of h_{reg} in the HO-SRSM. All the values of P_f mentioned above are obtained by running a MCS of 1,000,000 sample size on the response surface.

The lower plot of Figure 6 suggests the reasons for the large errors in the regular 2^{nd} order SRSM. For the 2^{nd} order SRSM, at h_{reg}=1 and at h_{reg}=2, the values of R^2 are large, implying the approximation is fairly accurate within the domain and the inaccurate P_f estimation is due to a lack-of-fit of the approximated limit state outside the small domain. While h_{reg} increases, the value of the adjusted R^2 decreases because as the sample domain becomes larger, the quadratic polynomial increasingly fails to conform accurately to the shape of the true limit state within the domain. On the other hand, the HO-SRSM method is able to capture the high nonlinearity regardless of the size of the domain.

Table 3
Iterative Bucher and Bourgund procedure for Example 1

h_{reg}	Iteration	Adjusted R^2	P_f
1.0	0	0.9970	0.000015
1.0	1	0.9894	0.000015
1.0	2	0.9906	0.000014
2.0	0	0.9689	0.000026
2.0	1	0.9510	0.000018
2.0	2	0.9578	0.003095

An iterative scheme of the procedure proposed by Bucher and Bougund is carried out at h_{reg}=1 and 2. Small sampling areas are chosen since it is expected that the samples are taken near the design point. The procedure proposed by Bucher and Bourgund does not improve the accuracy of the estimation of P_f in this example. As shown in Table 3, the approximated limit state labeled '2nd Response Surface' and '3rd Response Surface' do not approximate the true limit state better than the one labeled '1st Response Surface' as illustrated in Figure 3. At the second iteration at h_{reg}=2, the estimated value of P_f becomes exceptionally large. The values of adjusted R^2 are consistent and close to 1 throughout the iteration process, implying the procedure may have led to an accurately approximated limit state around an inaccurate design point in this case.

4.2 Example 2 - Hypothetical limit state with 3 variables

A second hypothetical example, with three independent standard normal variables, is considered:

$$g(\mathbf{X}) = -0.32(X_1 - 1)^2 X_2^2 - X_2 + X_3^3 - 0.2\sin(X_1 X_3) \tag{21}$$

A sine function is included to prevent the hypothetical limit state from being perfectly fit by a third order polynomial. The value of $P_f = 0.037216$ is obtained by a direct full scale MCS of 10,000,000 sample size.

As in Example 1, parameter h_{ord} has no influence on the estimation of the orders of the polynomials, as shown in Table 4. All values of h_{ord} give exactly the same results.

The failure probability estimation results of the HO-SRSM again are consistent over different values of parameter h_{reg}, while the estimated P_f in the regular 2^{nd} order SRSM is heavily dependent on h_{reg}. In the proposed method, P_f

Table 4
Determination of polynomial orders for Example 2

h_{ord}	k_1	k_2	k_3
1.0	2	2	3
2.0	2	2	3
3.0	2	2	3
4.0	2	2	3
5.0	2	2	3
6.0	2	2	3
7.0	2	2	3

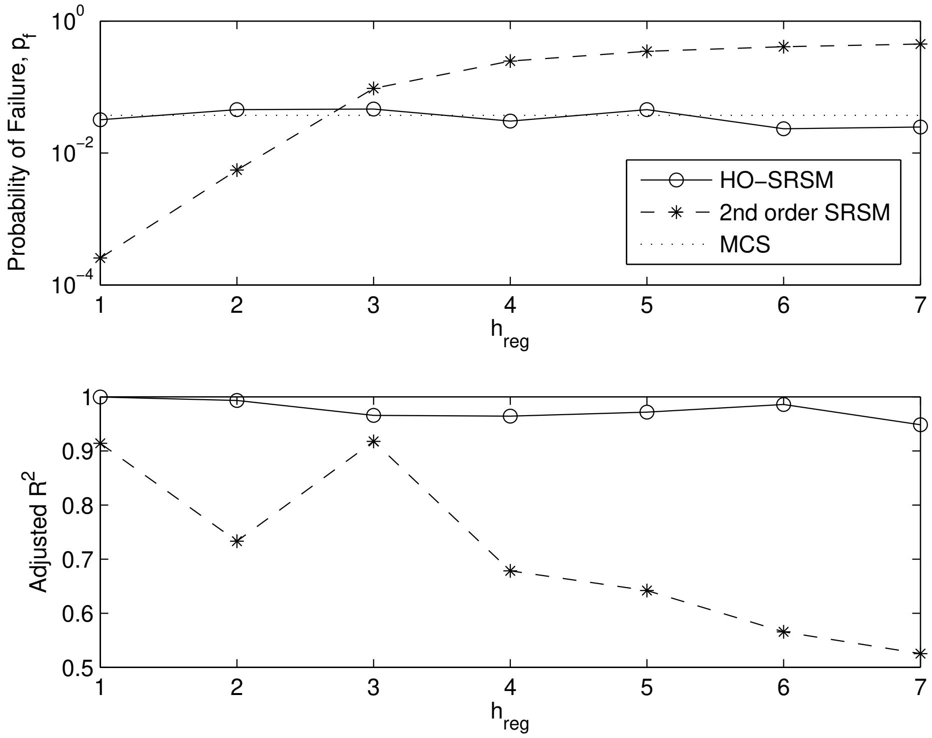

Fig. 7. Estimation results of Example 2

ranges from 0.023212(-37.6%) (at h_{reg}=6) to 0.046269 (+24.3%) (at h_{reg}=3), and in the regular 2$^{\text{nd}}$ order SRSM, P_f increases from 0.000255(-99%) to 0.446921(+1100%) from h_{reg}=1 to h_{reg}=6. All of the estimated P_f values are obtained by running a MCS of 1,000,000 sample size on the approximated limit state function. The lower plot in Figure 7 suggests that the accuracy of the regular 2$^{\text{nd}}$ order SRSM decreases much more significantly than does the HO-SRSM as h_{reg} increases.

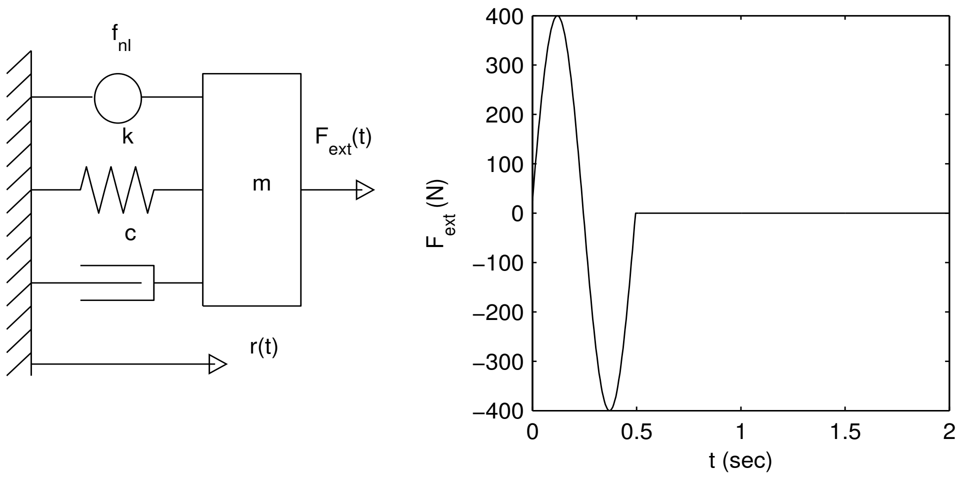

Fig. 8. Example 3 - A simple dynamic problem with an hysteretic force

4.3 Example 3 - Dynamic Problem of One Degree of Freedom

This example examines a simple dynamic problem of a hysteretic [23] one degree of freedom system excited by a full sine-wave force impulse. The equation of motion is given by:

$$\begin{aligned} &m\,\ddot{r}(t) + c\dot{r}(t) + kr(t) + f_{\mathrm{nl}}(t) = f_{\mathrm{ext}}(t) \\ &\dot{z} \;= \left(1 - \frac{1}{2}z^2(\mathrm{sign}(\dot{r}z) + 1)\right)\frac{\dot{r}}{r_y} \\ &f_{\mathrm{nl}} = f_y z \end{aligned} \tag{22}$$

The description, mean and coefficient of variation of each random variable is listed in Table 5. All variables are lognormal and are assumed to be independent. The limit state function is defined as:

$$g(\mathbf{X}) = 0.04 - \max_t |\ddot{r}(t)| \tag{23}$$

where $\ddot{r}(t)$ is the acceleration of the mass. This limit state means that the system fails if the maximum acceleration over time exceeds 0.04 m/s^2. The value of P_f generated by the direct MCS of 100,000 sample size is 0.00529.

Table 6 shows the polynomial orders for different values of h_{ord}. Only the orders in the force magnitude F_p and force period T_p differ at different values of h_{ord}, while the other five variables have orders of two. F_p increases from order 2 to 5 and T_p fluctuates between 2 and 5 at as h_{ord} increases. This suggests an important finding, that the excitation force and period in problems with a hysteretic restoring force have a significant nonlinear effect on the peak acceleration of the mass. The fluctuating order of T_p may be because

Table 5
Statistical properties of random variables for Example 3

Variable	Description	Units	Mean value	C.O.V.
m	Mass	kg	1,000	0.05
k	Stiffness	N/m	4,000	0.10
c	Damping Coefficient	N/m/s	250	0.20
f_y	Yield Force	N	400	0.10
r_y	Yield Displacement	m	0.01	0.15
F_p	Force Amplitude	N	400	0.20
T_p	Force Period	s	0.5	0.50

its effect on the limit state cannot simply be described by integer powers. In the estimation of polynomial orders, the required number of evaluations of the true limit state varies for different values of h_{ord}. At h_{ord}=1, there are only 24 evaluations and at h_{ord}=5 there are 39. Compared to the hundreds of evaluations in the process of limit state approximation, the computational effort spent in the order estimation is modest.

The estimations of the failure probability are shown Figure 9. In the regular 2$^{\text{nd}}$ order SRSM, the estimated result, which is based on 36 coefficients and 2187 sample points, is positively correlated with the parameter h_{reg}. It increases from 0.00199 (-62%) to 0.00962 (82%) from h_{reg}=1 to h_{reg}=6. In the HO-SRSM, in order to show the effect of parameter h_{ord}, two approximate limit states are constructed using the values of h_{ord}=3.0 and h_{ord}=5.0. The estimated P_f for h_{ord}=3.0 ranges from 0.00366(-31%) (h_{reg}=2) to 0.00562 (+6%) (h_{reg}=6). For h_{ord}=5.0, P_f has estimated values from 0.00332 (-37%) (h_{reg}=6) to 0.00600 (+13%) (h_{reg}=2). Both cases of h_{ord}=3.0 and 5.0 involve 288 coefficients and 576 sample points. The two estimation results do not show any correlation with the parameter h_{reg} and are more accurate than the 2$^{\text{nd}}$ order SRSM, in general. The lower plot shows that the values of adjusted R^2 are about the same for both h_{ord}, and they are significantly higher than those of the 2$^{\text{nd}}$ order SRSM.

To roughly indicate if the value of h_{ord} is appropriate and at what value of h_{reg} the estimation result is accurate, the standard normal cumulative distribution function (cdf) may be used. After P_f is estimated, it can be compared with the normal cdf at the negative value of h_{ord}, $\Phi(-h_{\text{ord}})$. If the estimated P_f is larger than $\Phi(-h_{\text{ord}})$ by one or two order of magnitude, the value of h_{ord} may be too large. An excessively-large value of h_{ord} may result in inefficient computation because large h_{ord} values give higher estimated orders and hence a larger number of coefficients and sampling points. For this problem, the estimated P_f is about 0.005. The standard normal cdf at $-h_{\text{ord}}$ =-3.0 has

Table 6
Determination of order of variables for Example 3

h_{ord}	m	k	c	f_y	r_y	F_p	T_p
1.0	2	2	2	2	2	2	2
2.0	2	2	2	2	2	2	3
3.0	2	2	2	2	2	3	4
4.0	2	2	2	2	2	3	5
5.0	2	2	2	2	2	4	3
6.0	2	2	2	2	2	4	5

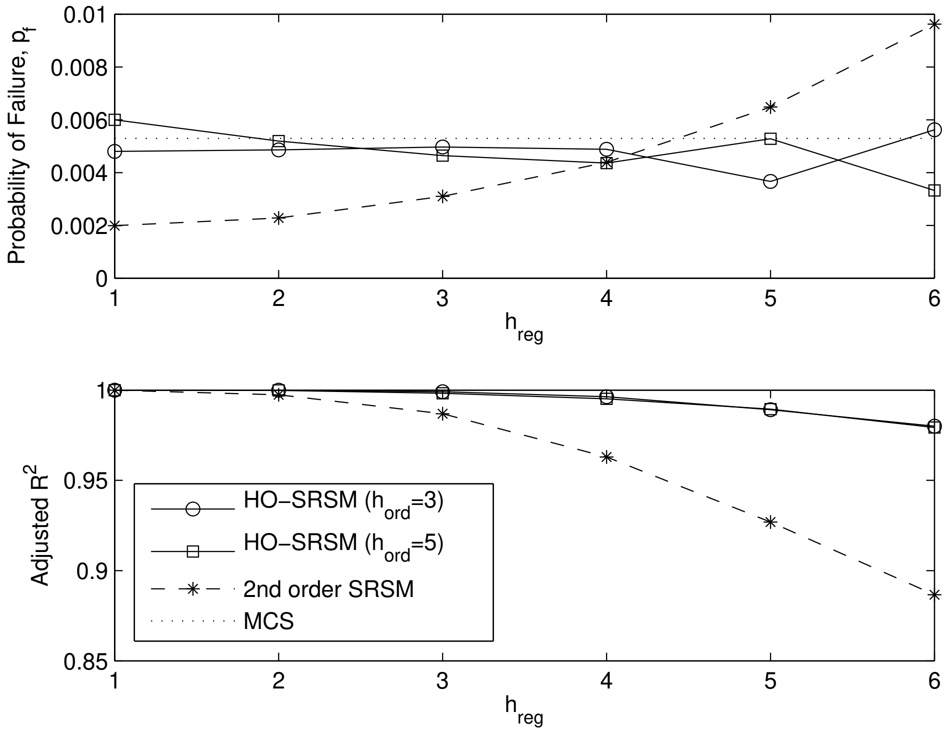

Fig. 9. Estimation results of Example 3

a value of 0.001350 and has a value of 2.867×10^{-7} at $-h$ord=-5.0. Since 2.867×10^{-7} is largely different from the actual P_f of 0.005, it can be suggested that at h_{ord}=5, the sampling domain is too large. In this specific problem, using h_{ord}=3 does not save any computational time compared to h_{ord}=5. But in general a smaller value of h_{ord} would result in smaller estimated order and less required computation. Nonetheless, compared to the regular 2$^{\text{nd}}$ order SRSM, both values of h_{reg} provide an accurate estimation of P_f. Similarly, a reasonable value of h_{reg} could be obtained by this comparison. In fact, if both h_{ord} and h_{reg} are equal to 3.0, the result is 0.00497 (-6%), which is sufficiently accurate. This approach may be inaccurate for the determination of h_{reg} for the regular 2nd order SRSM because the shape of the approximated limit state may be different from the true limit state.

4.4 Example 4 - Dynamic Problem with Six Degrees of Freedom

Figure 10 shows the problem of a four-story building excited by a single period sinusoidal impulse of ground acceleration. The building contains isolated equipment on the second floor. The motion of the lowest floor is resisted by a nonlinear hysteresis force [23] due to the building's base isolation bearings and an additional stiffness force, if its displacement exceeds d_c. Each floor has a mass of m_f and between floors the stiffness and damping coefficient are k_f and c_f respectively [13]. For simplicity, and to decrease the number of variables, m_f, k_f and c_f are assumed to be the same for each floor. On the second floor there are two isolated masses, representing isolated, shock-sensitive equipment. The larger mass is connected to the floor by a relatively flexible spring, k_1, and a damper, c_1, representing the isolation system. The smaller mass is connected to the larger mass by a relatively stiff spring, k_2 and damper, c_2, representing the equipment itself. There are six degrees of freedom, four at the floors and two at the equipment mass blocks. The statistical parameters of the basic random variables are listed in Table 7. All variables are assumed to be lognormal and independent.

The limit state function is defined by

$$\begin{aligned} g(\mathbf{X}) = {} & 12.5\,(0.04 - \max_t |r_{f_i}(t) - r_{f_{i-1}}(t)|)_{i=2,3,4} \\ & + \ (0.5 - \max_t |\ddot{z}_g(t) + \ddot{r}_{m_2}(t)|) \\ & + \ 2(0.25 - \max_t |r_{f_2}(t) - r_{m_1}(t)|) \end{aligned} \tag{24}$$

where r_{f_i} refers to the displacement of i-th floor and $r_{f_i}(t) - r_{f_{i-1}}(t)$ is the inter-story displacement of two consecutive floors. The accelerations $\ddot{z}_g$ and $\ddot{r}_{m_2}$ are of the ground and the smaller mass block respectively. The displacement r_{m_1} is of the larger mass block, and represents the displacement of the equipment isolation system. The limit state function in equation (24) is the sum of three expressions of failure modes. The first term describes the damage to the structural system due to excessive deformation. The second term represents the damage to equipment caused by excessive acceleration. The last term represents the damage of the isolation system. They are multiplied by weighing factors, which emphasize the three faulure modes equally. Equation (24) means that it is desirable that (1) none of the inter-story displacements exceeds 0.04 m, (2) the peak acceleration of the smaller mass block (the equipment) is less than 0.5 m/s^2, and (3) the displacement across the equipment isolation system is less than 0.25 m. Failing one or two of the conditions does

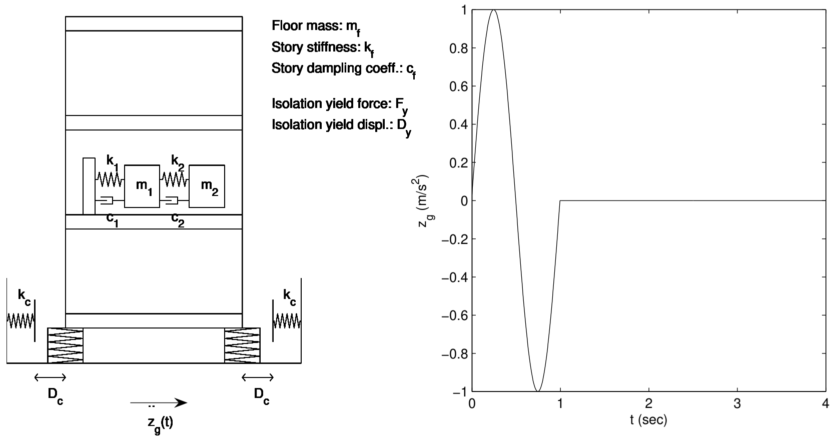

Fig. 10. Example 4 - Base isolated structure with an equipment isolation system on the 2^{nd} floor, and including the effects of isolation displacement limits

not necessarily lead to a failure in the limit state function, e.g. $g(\mathbf{X} < 0)$, but will decrease the value of the limit state function. In these simulations, the system fails mainly because of the large acceleration of the smaller mass. The estimation result of P_f by direct full scale MCS of 100,000 sample size is 0.19599.

Table 8 shows the polynomial orders of the random variables. The polynomial orders for m_f, k_f, c_f and k_c remain at 2 for all values of h_{ord}. The other four variables d_y, f_y, T and A remain at orders of 2 or 3 from h_{ord}=1.0 to 3.0 and then fluctuate about 4 and 5 at higher values of h_{ord}. This result shows that the parameters of the hysteresis force as well as the excitation period and the amplitude of an earthquake have a complicated effect on the limit state function. To illustrate the effect of h_{ord}, results corresponding to h_{ord}=2 and 6 are used to approximate the true limit state. To determine which h_{ord} estimates the shape of the limit state more accurately, the estimated values of P_f are compared to the normal cdf after the approximation.

The upper plot of Figure 11 shows that P_f is about 20 percent in this problem. The HO-SRSM has better estimations of P_f at h_{reg}=1 and 2. At $h_{\mathrm{reg}} = 1$, the estimated values of P_f for the HO-SRSM are extremely accurate, being 0.19325(-1.4%) and 0.19512(-0.4%) for h_{ord}=2.0 and h_{ord}=6.0 respectively. For the 2^{nd} order SRSM it is 0.20374(+4.0%). At h_{reg}=2, the estimated values of P_f are almost the same, being equal to 0.19355(-1.2%) and 0.19643(-0.2%) for h_{ord}=2.0 and h_{ord}=6.0 in the HO-SRSM, and 0.20051(+2.3%) for the 2^{nd} order SRSM. From h_{reg}=3 to 6, the estimated P_f in the 2^{nd} order SRSM increases gradually from 0.19999(-0.2%) to 0.32196(+64%). The proposed method underestimates the value of P_f at values of h_{reg} larger than 2. For h_{ord}=2.0, the smallest estimated value is 0.10035(-49%)(at h_{reg}=6); it is 0.08554(-56%) (at h_{reg}=4) for h_{ord}=6.0. Regarding the compuatational effort, the number of sample points is 6561 in the 2^{nd} order SRSM, 316 in the HO-SRSM of h_{ord} =2,

Table 7
Statistical properties of random variables for Example 4

Variable	Description	Units	Mean value	C.O.V.
m_f	Floor Mass	kg	6,000	0.10
k_f	Floor Stiffness	N/m	30,000,000	0.10
c_f	Floor Damping Coefficient	N/m/s	60,000	0.20
d_y	Isolation Yield Displacement	m	0.05	0.20
f_y	Isolation Yield Force	N	20,000	0.20
d_c	Isolation Contact Displacement	m	0.5	0
k_c	Isolation Contact Stiffness	N/m	30,000,000	0.30
m_1	Mass of Block 1	kg	500	0
m_2	Mass of Block 2	kg	100	0
k_1	Stiffness of Spring 1	N/m	2,500	0
k_2	Stiffness of Spring 2	N/m	100,000	0
c_1	Damping Coefficient of Damper 1	N/m/s	350	0
c_2	Damping Coefficient of Damper 2	N/m/s	200	0
T	Pulse Excitation Period	s	1.0	0.20
A	Pulse Amplitude	m/s/s	1.0	0.50

Table 8
Determination of polynomial orders, k_i, for variables in Example 4

h_{ord}	m_f	k_f	c_f	d_y	f_y	k_c	T	A
1.0	2	2	2	2	2	2	2	2
2.0	2	2	2	2	2	2	2	3
3.0	2	2	2	2	2	2	2	3
4.0	2	2	2	5	4	2	2	5
5.0	2	2	2	4	5	2	5	5
6.0	2	2	2	2	5	2	5	3

and 2106 in the case of h_{ord}=6. The number of coefficients are 45, 158 and 1053, respectively.

The results of the HO-SRSM are disappointing at larger h_{reg} values. After P_f is estimated, it should nevertheless be noted that, in this case, the values of both h_{reg} and h_{ord} should be small. Checking against the values of $\Phi(-h_{\text{ord}})$, as mentioned in Example 3, reveals that the estimation result should be sufficiently accurate for h_{ord}=1 and h_{reg}=1. Referring to Table 8, at h_{reg}=1, all

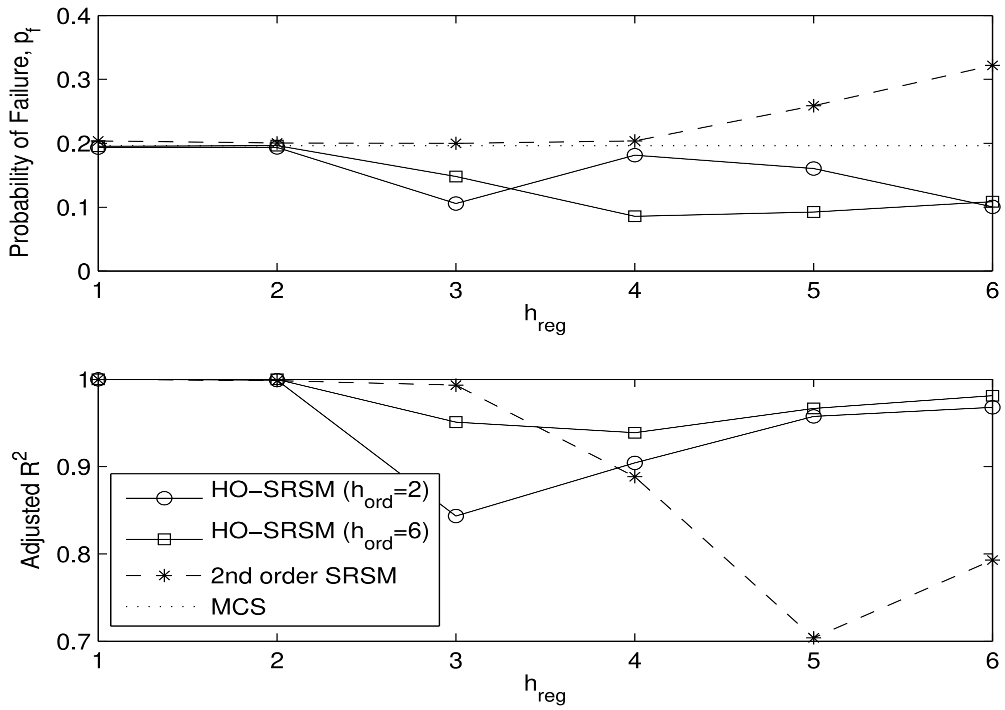

Fig. 11. Estimation results of Example 4

variables have 2$^{\text{nd}}$ order. This explains why the 2$^{\text{nd}}$ order SRSM performs satisfactorily in this method. This example shows that in problems where the 2$^{\text{nd}}$ order SRSM is sufficient, the proposed method is still useful in the confirmation of the accuracy of the results. It is suggested that several values of h_{ord} should be investigated before the approximation of the true limit state. After the estimation of the order of variables and the number of coefficients, the result of h_{ord} which yields the smallest number of coefficients is used to approximate the limit state function in order to save computational effort. Then the estimated value of P_f is compared to $\Phi(-h_{\text{ord}})$ to determine the sufficient value of h_{ord}. Implementation of this approach in this problem would provide as accurate an estimation of P_f as the 2$^{\text{nd}}$ order SRSM, but in a more efficient way, since the 3^k factorial design involves a very large number of sample points.

5 Conclusions

This study proposes the use of higher order polynomials in the stochastic response surface method, and an algorithm to determine the polynomial orders using a statistical analysis of polynomial coefficients. The numerical examples have shown that the proposed method not only provides more accurate results of the probability of failure than the 2^{nd} order SRSM in highly nonlinear problems, but also allows an indication of the accuracy of the estimated failure probability. Moreover, unlike the 2^{nd} order SRSM, the failure probabilities computed using the proposed method do not show any significant correlation with the size of the domain of the sample points. The method proposed in this paper checks the accuracy of the response surface using a goodness-of-fit criteria and checks the failure probability by a comparison to the size of the domain of sample points.

Future work could implement better experimental design into the method, since the method currently uses random sampling points. For example, the proposed method could be improved by shifting the location of the sample points closer to the limit state, as is achieved by Bucher and Bourgund, in improving the 2^{nd} order SRSM. Implementation of such a concept in the proposed method is, however, more complicated, since it may not be easy to find a single design point if the limit state function has a highly nonlinear shape. Also, estimation results may become more inaccurate if sample points are concentrated at one of the multiple design points. To further improve the proposed method, relocation of the sample points to incorporate the effect of multiple design points, at small expense of computational time, should be investigated. The fact that random sampling gives good results in our examples, however, shows that the location of sample points may not critically important, as long as appropiate polynomial orders and appropriate cross-terms are utilized and as long as the sample points lie in the domain of interest. The criteria in the determination of the significance of polynomial coefficients also requires further investigation.

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