Introduction to Computer Programming

Week 8.2: Symbolic computation with SymPy

Bristol

What is symbolic computation?

Symbolic computation is about performing exact mathematical operations that go beyond the basics like addition and multiplication (think differentiation and integration)

The idea is to introduce a new type of variable, called a **symbol**, that behaves like an algebraic variable.

Symbols can be operated on without having a precise value assigned to them.

This is different from the variable types we have seen so far (e.g. int, float), which need a value assigned to them when they are created

```
x = 1.33
v = [1, 2, 3, 4]
```

SymPy

SymPy is a Python library for carrying out exact computations using symbolic computing.

Features of SymPy include:

- Solving algebraic equations (linear equations, polynomials, nonlinear equations)
- Differentiating and integrating functions
- Solving linear algebra problems (linear systems, determinants, eigenvalue problems)
- Solving differential equations
- And much more: see https://www.sympy.org/en/index.html for more info!

Getting started

To get started, let's load the SymPy library into Python:

In [1]:
```
from sympy import *
```

Some special variables

SymPy has exact representations of useful mathematical quantities

- `pi` represents π
- `E` represents Euler's number e
- `oo` (two o's) represents infinity ∞
- `I` represents the complex number $i = \sqrt{-1}$

Displaying maths

The `pprint` and `display` functions can be used to print and display mathematical expressions in Sypder (and also in Jupyter notebooks)

The `display` function works better, but it requires additional software (LaTeX) to be installed -- not an issue for the lab computers

Note: the `display` function uses black font, which is not very visible in the default Spyder scheme. The Spyder colours can be changed by going to:

Tools -> Preferences -> Appearance -> Syntax highlighting scheme

Example: print π^2 using `pprint`, `display`, and using a Jupyter notebook

In [2]:
```
# printing pi**2 using pprint
pprint(pi**2)
```

```
 2
π
```

In [3]:
```
# printing pi with the display function
display(pi**2)
```

π^2

In [4]:
```
# printing with Jupyter notebook without display or pprint
pi**2
```

Out[4]: π^2

Examples: Let's explore using some of these special variables

In [5]:
```
sin(pi)
```

Out[5]: 0

In [6]:
```
log(E)
```

Out[6]: 1

In [7]:
```
1 / oo
```

Out[7]: 0

In [8]:
```
I**2
```

Out[8]: -1

Defining variables as symbols

In order to make use of the capabilities of SymPy, we need to define variables as symbols.

This is done using the `symbols` function

In [9]:
```
x = symbols('x')
```

This creates a variable x of type symbol.

Even though we haven't assigned a value to x, we can still perform operations on it and use it to define new variables

In [10]:
```
y = x + 1
display(y)
```

$x + 1$

This is the power of symbolic computing

Defining mathematical functions

Once we define a symbol, we can use it to create mathematical functions.

Example: Define the function $y(x) = \sqrt{x}$

In [11]:
```
y = sqrt(x)
```

Values of x can be substituted into y using the `subs` method

Example: Substitute $x = 4$ into $y(x) = \sqrt{x}$

In [12]:
```
y.subs(x, 4)
```

Out[12]: 2

Let's see what happens when we substitute $x = 8$ into the function $y = \sqrt{x}$

In [13]:
```
y.subs(x, 8)
```

Out[13]: $2\sqrt{2}$

The number $2\sqrt{2}$ is represented exactly as a symbol rather than being approximated by a float

The `evalf` method evaluates a symbolic expression as a floating point number

In [10]:
```
y_at_8 = y.subs(x, 8)
y_at_8.evalf()
```

Out[10]: 2.82842712474619

There are some other ways we can do this. The simplest is to substitute $x = 8.0$ into y, which automatically triggers the floating-point evaluation

In [11]:
```
y.subs(x, 8.0)
```

Out[11]: 2.82842712474619

The substitution and evaluation can be done at the same time using dictionaries (helpful when substituting multiple values)

In [12]:
```
y.evalf(subs = {x:8})
```

Out[12]: 2.82842712474619

Exercise:

Define the function

$$y(t) = \frac{t^2}{1 + 2t}.$$

Then evaluate $y(2)$ exactly and find a floating-point approximation to this value

Solution:

In [22]:
```
# create the symbol t and define the function y
t = symbols('t')
y = t**2 / (1 + 2*t)

# computing y(2) exactly and displaying the value
y_at_2 = y.subs(t,2)
display(y_at_2)

# finding a floating-point approx to y(2) and displaying the value
display(y_at_2.evalf())
```

$\frac{4}{5}$

0.8

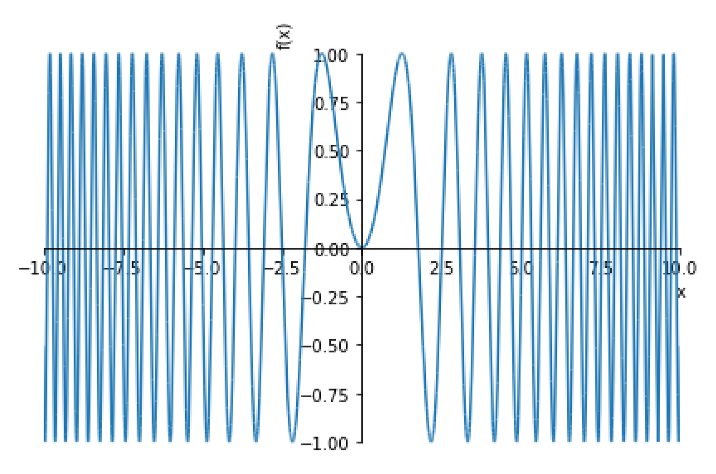
Plotting functions

SymPy enables plotting of functions using the matplotlib package, which will be covered in more detail next week.

Functions can be plotted using the `plot` function

Example: Plot the function $y(x) = \sin(x^2)$

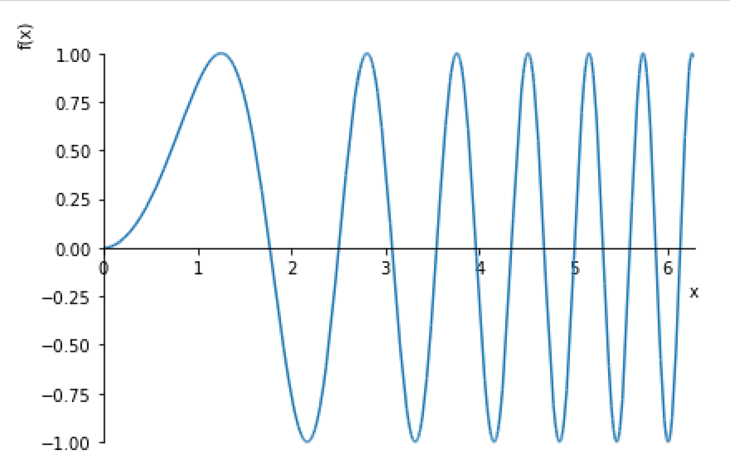
In [3]:
```
y = sin(x**2)
plot(y)
```



Out[3]: `<sympy.plotting.plot.Plot at 0x7fb9f59c3490>`

It is possible to specify plotting range by passing a tuple to the `plot` function

In [5]:
```
plot(y, (x, 0, 2 * pi))
```



Out[5]: `<sympy.plotting.plot.Plot at 0x7fb9e7683b80>`

Differentiating functions

The `diff` function enables functions to be differentiated an arbitrary number of times

Example: Compute y' and y''' when $y = \sqrt{x}$

In [13]:
```
y = sqrt(x)
diff(y,x)
```

Out[13]: $\frac{1}{2\sqrt{x}}$

In [14]:
```
diff(y, x, 3)
```

Out[14]: $\frac{3}{8x^{\frac{5}{2}}}$

Exercise:

Plot the function $y(x) = \cos(4\pi x)e^{-3x}$ from $x = 0$ to $x = 2$. Compute $y'''(1)$.

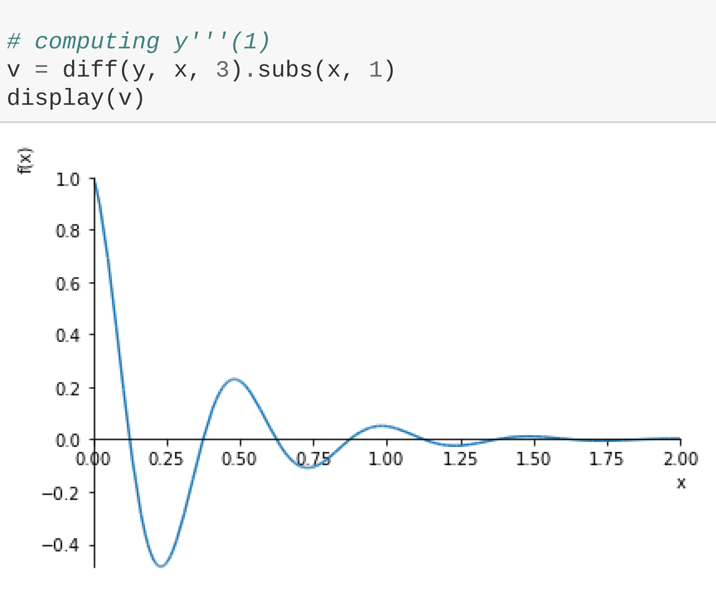
Hint: use the `cos` and `exp` functions for the cosine and exponential

Solution:

In [27]:
```
y = cos(4 * pi * x) * exp(-3 * x)

# plotting
plot(y, (x, 0, 2))

# computing y'''(1)
v = diff(y, x, 3).subs(x, 1)
display(v)
```



$\frac{-27 + 144\pi^2}{e^3}$

Integrating functions

The `integrate` function computes the indefinite integral of a function (if it exists)

Example: Compute the indefinite integral of $y = x$

In [4]:
```
y = x
integrate(y, x)
```

Out[4]: $\frac{x^2}{2}$

Warning: SymPy does not add the constant of integration to indefinite integrals!

The `integrate` function can also handle definite integrals.

Example: Compute

$$\int_0^1 x^2 \, dx$$

In [5]:
```
integrate(x**2, (x, 0, 1))
```

Out[5]: $\frac{1}{3}$

Example: Compute the following integral exactly and approximately

$$\int_0^\infty \frac{\sin x}{x} \, dx$$

In [8]:
```
# compute the exact value
integrate(sin(x) / x, (x, 0, oo))
```

Out[8]: $\frac{\pi}{2}$

In [6]:
```
# compute a floating-point approximation to the integral
integrate(sin(x) / x, (x, 0, oo)).evalf()
```

Out[6]: 1.5707963267949

Solving an equation

The `solve` function solves algebraic equations of the form $F(x) = 0$

Example: Solve $x^3 = ax$ where a is a parameter

Solution: First we write this as $F(x) = x^3 - ax = 0$

In [30]:
```
# define a and x as symbols
a, x = symbols('a x')

# define the function and solve
F = x**3 - a * x
solve(F, x)
```

Out[30]: `[0, -sqrt(a), sqrt(a)]`

Summary

SciPy performs **exact** mathematical calculations using **symbolic computing**

- `symbols` is used to define algebraic variables
- `subs` and `evalf` are for substituting values and computing floating-point approximations
- `diff` and `integrate` compute derivatives and integrals
- `solve` solves algebraic equations

This is very useful for calculus homework!

For more details of SymPy, see https://www.sympy.org/en/index.html