State Space Models

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Master Project Report
Barcelona Graduate School of Economics
Master Degree in Data Science
2017

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1 Introduction

2 State Space Models

State space models consist of two series of data:

1. A series of latent **states** $\{x_t\}$ forming a Markov chain. Thus, x_t is independent of all past states but x_{t-1}.
2. A series of **observations** $\{y_t\}$ where y_t only depends on x_t.

2.1 Local Level Model

The arguably simplest state space model is the (univariate) local level. It has the following form:

$$\begin{aligned} \text{observation:} \quad & y_t = x_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \\ \text{state:} \quad & x_{t+1} = x_t + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta^2) \end{aligned}$$

with some initial state $x_1 \sim N(a_1, P_1)$.

3 Filtering

The object of filtering is to update our knowledge of the system each time a new observation y_t is brought in.

3.1 Kalman Filter

3.1.1 Likelihood evaluation

$$\log L(Y_n) = -\frac{nd}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}(\log|F_t| + v_t^T F_t^{-1} v_t)$$

3.2 Particle Filter

3.3 Importance Sampling Particle Filter

4 Illustration

4.1 Trivariate Local Level Model

4.1.1 The Model

Consider a time series of length T with each observation $\boldsymbol{y}_t = [y_{1t}, y_{2t}, y_{3t}]^T$ and each state $\boldsymbol{x}_t = [x_{1t}, x_{2t}, x_{3t}]^T$ being described by a 3-dimensional vector.

$$\begin{aligned} \text{observation:} \quad \boldsymbol{y}_t &= \boldsymbol{x}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \sigma_\epsilon^2 I_3) \\ \text{state:} \quad \boldsymbol{x}_{t+1} &= \boldsymbol{x}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \Sigma_\eta) \end{aligned}$$

with initial state $\boldsymbol{x}_1 \sim N(\boldsymbol{a}_1, P_1)$ and where we restrict the covariance matrix of the state disturbances, Σ_η, to the form

$$\Sigma_\eta = \begin{bmatrix} \sigma_{\eta 1}^2 & \rho\sigma_{\eta 1}\sigma_{\eta 2} & \rho\sigma_{\eta 1}\sigma_{\eta 3} \\ \rho\sigma_{\eta 1}\sigma_{\eta 2} & \sigma_{\eta 2}^2 & \rho\sigma_{\eta 2}\sigma_{\eta 3} \\ \rho\sigma_{\eta 1}\sigma_{\eta 3} & \rho\sigma_{\eta 2}\sigma_{\eta 3} & \sigma_{\eta 3}^2 \end{bmatrix}$$

Thus, Σ_η can be described by $\sigma_{\eta 1}^2$, $\sigma_{\eta 2}^2$, $\sigma_{\eta 3}^2 > 0$ and $\rho \in [0, 1]$. Furthermore, we assume for simplicity that the observation noise has the same variance in each dimension $\sigma_\epsilon^2 > 0$. Therefore, the model is fully specified by the following vector of parameters:

$$\boldsymbol{\theta} = [\rho, \sigma_{\eta 1}^2, \sigma_{\eta 2}^2, \sigma_{\eta 3}^2, \sigma_\epsilon^2]^T$$

The initial state parameters $\boldsymbol{a}_1$ and P_1 are assumed to be known.

4.1.2 Realization

Figure 4.1 plots the states and observations for a realization of the trivariate local level model with length $T = 100$. The model parameters are

$$\boldsymbol{\theta} = [\rho = 0.7, \sigma_{\eta 1}^2 = 4.2, \sigma_{\eta 2}^2 = 2.8, \sigma_{\eta 3}^2 = 0.9, \sigma_\epsilon^2 = 1.0]^T$$

The initial daily and intra-daily state components where drawn from a standard normal.

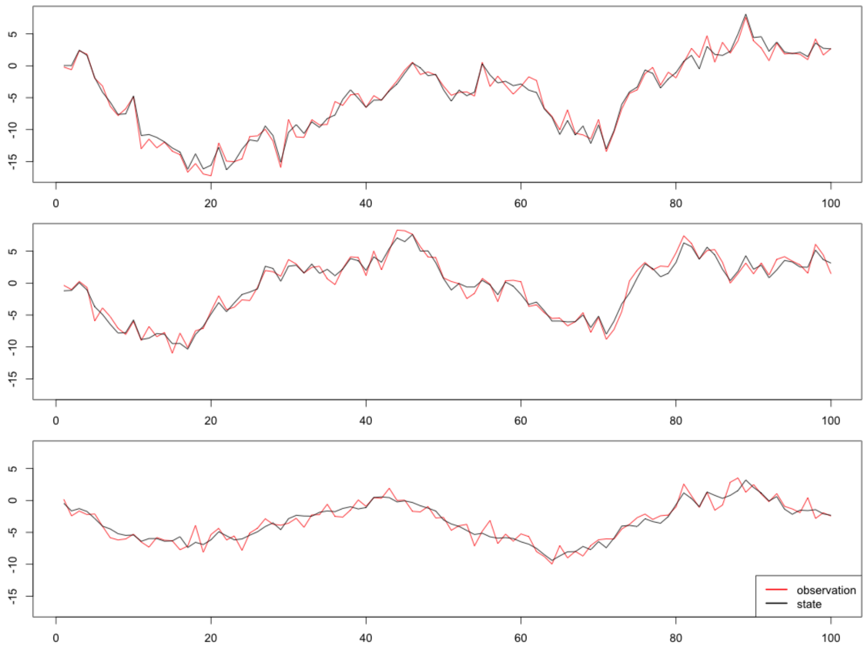

Figure 4.1: Realization of the model with $T = 100$

4.2 Hierarchical Dynamic Poisson Model

Explain the main idea and potential use cases.

4.2.1 The Model

Consider a time series over M days, each consisting of N intra-daily observations. Let m denote the day and n be the intraday index.

$$\begin{aligned} \text{observation:} \quad y_{mn} &= \text{Poisson}(\lambda_{mn}) \\ \text{state:} \quad \log\lambda_{mn} &= \log\lambda_m^{(D)} + \log\lambda_{mn}^{(I)} + \log\lambda_n^{(P)} \end{aligned}$$

where the state consists of a daily, an intra-daily, and a periodic component:

$$\begin{aligned} \text{daily component:} \quad & \log\lambda_{m+1}^{(D)} = \phi_0^{(D)} + \phi_1^{(D)}\log\lambda_m^{(D)} + \eta_m^{(D)} && \eta_t \sim N(0, \sigma_{(D)}^2) \\ \text{intra-daily component:} \quad & \log\lambda_{mn+1}^{(I)} = \phi_1^{(I)}\log\lambda_{mn}^{(I)} + \eta_{mn}^{(I)} && \eta_{mn} \sim N(0, \sigma_{(I)}^2) \\ \text{periodic component:} \quad & \log\lambda_n^{(P)} = \phi_1^{(P)}\sin(\pi(n-1)/M) \end{aligned}$$

The initial daily and intra-daily component is drawn from a normal with mean a_1 and covariance P_1:

$$\log \lambda_1^{(D)}, \log \lambda_1^{(I)} \sim N(a_1, P_1)$$

Note that both the daily and intra-daily component constitute an AR(1) model, with the mean of the intra-daily component $\phi_0^{(I)}$ set to 0. The model is fully specified by the following vector of parameters:

$$\boldsymbol{\theta} = [\phi_0^{(D)}, \phi_1^{(D)}, \sigma^2_{(D)}, \phi_1^{(I)}, \sigma^2_{(I)}, \phi_1^{(P)}]^T$$

Again, the initial state parameters a_1 and P_1 are assumed to be known.

4.2.2 Realization

Figure 4.2 plots the states and observations for a realization of the hierarchical dyanmic Poisson model over $N = 5$ days with $M = 20$ intra-daily observations. The model parameters are

$$\boldsymbol{\theta} = [\phi_0^{(D)} = 0.7, \phi_1^{(D)} = 0.6, \sigma^2_{(D)} = 0.6, \phi_1^{(I)} = 0.3, \sigma^2_{(I)} = 0.2, \phi_1^{(P)} = 0.8]^T$$

The initial daily and intra-daily state components where drawn from a standard normal.

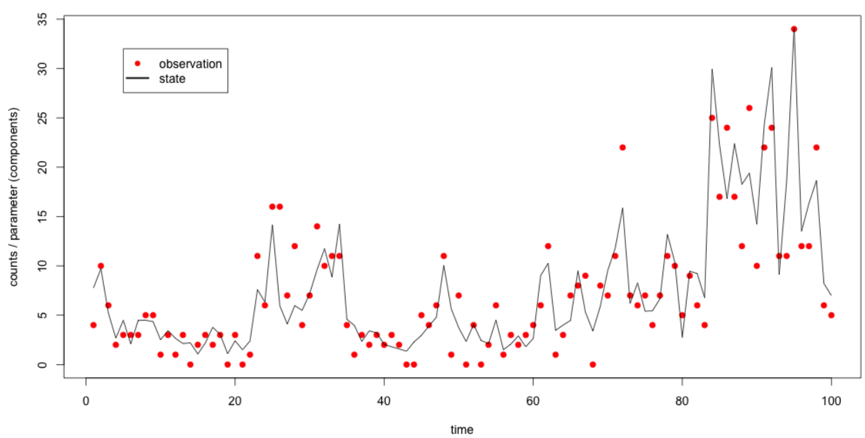

Figure 4.2: Realization of the model with $N = 5$ and $M = 20$

4.2.3 Densities

State transition and prediction density and how they are used in the particle filter

4.2.4 Maximum Likelihood Estimation

Show log-likelihood plots

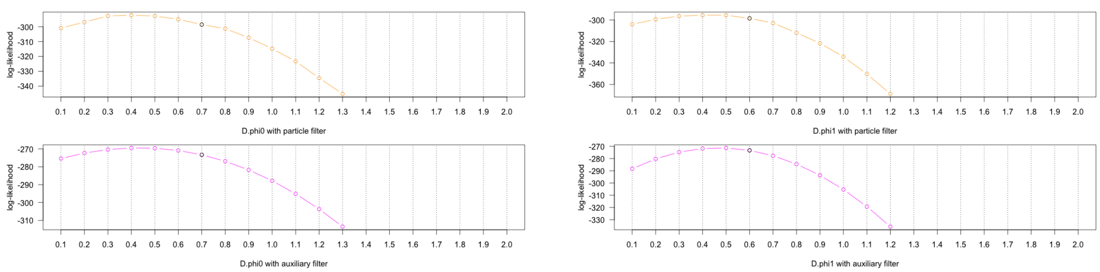

Figure 4.3: Belief convergence without misinformation after 300 and 2000 iterations

5 Conclusion

by Etessami et al.[?]