Particle Filtering for Nonlinear State Space Models

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Outline

- What I have done
- State Space Models
- Particle Filters
  - Kalman Filter
  - Sequential Importance Resampling (SIR)
  - Continuous Sequential Importance Resampling (CSIR)
  - Importance Sampling Particle Filter
- Evaluation
- Illustration
  - Trivariate Local Level Model
  - Hierarchical Dynamic Poisson Model

State Space Models

Local Level Model

Formulation

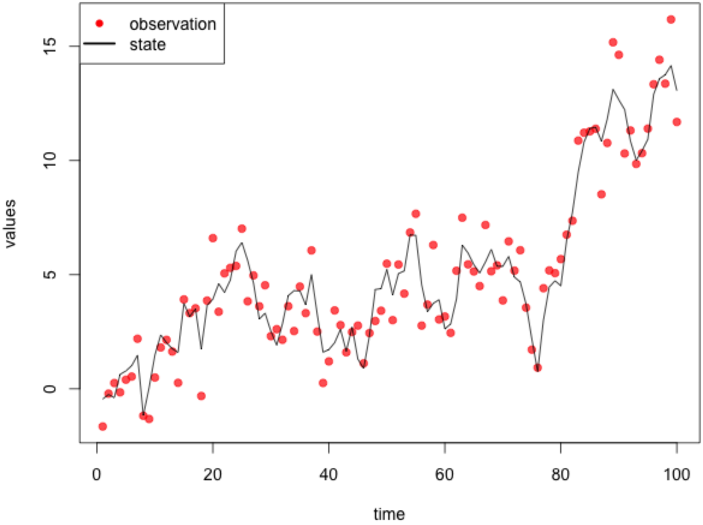
$$\begin{aligned} \text{observation:} \quad & y_t = x_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \\ \text{state:} \quad & x_{t+1} = x_t + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta^2) \end{aligned}$$

$$\boldsymbol{\theta} = [\sigma_\eta^2, \sigma_\epsilon^2]^T$$

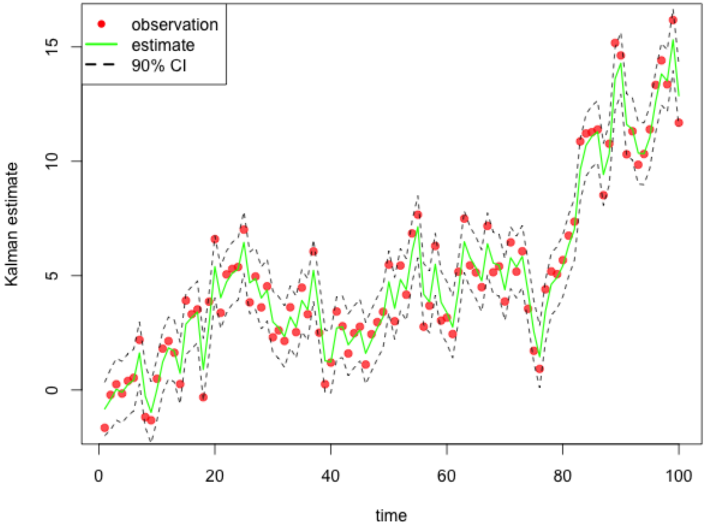
$$\begin{aligned} \text{transition density:} \quad & x_{t+1} | x_t, \boldsymbol{\theta} \quad \sim N(x_t, \sigma_\epsilon^2) \\ \text{measurement density:} \quad & y_t | x_t, \boldsymbol{\theta} \quad \sim N(x_t, \sigma_\eta^2) \end{aligned}$$

Local Level Realization

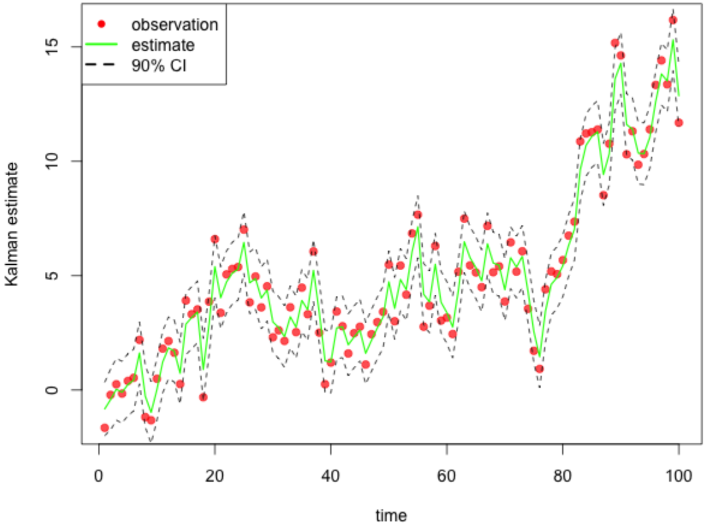
$\sigma^2_\eta = 1.4$, $\sigma^2_\epsilon = 1.0$



Latent State Inference



Parameter Inference



Particle Filtering

Parameters of the dynamic programming algorithm

- State of the system:
  - x_i: yards to the goal line
  - y_i: yards to the first down
  - d: number of downs

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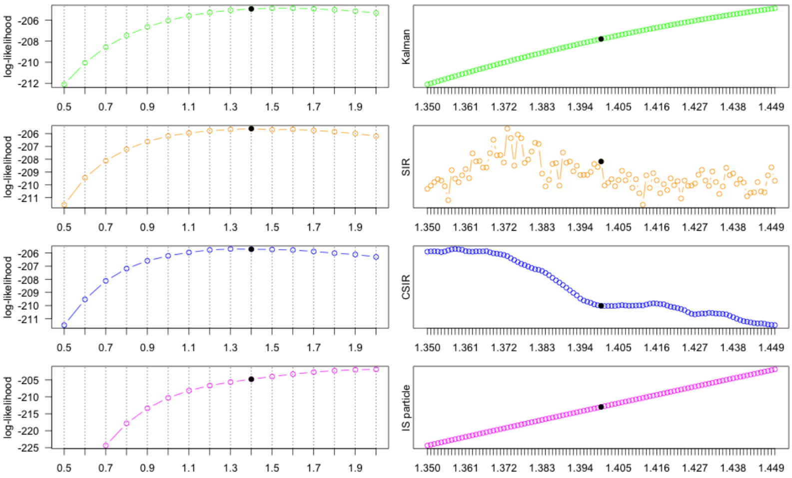
  - U: punt
  - K: kick
- Rewards:
  - Touchdown: 6.8
  - Field goal: 3
  - Safety: -2
  - Opposition score $= -\frac{6.8x}{100}$

Evaluation

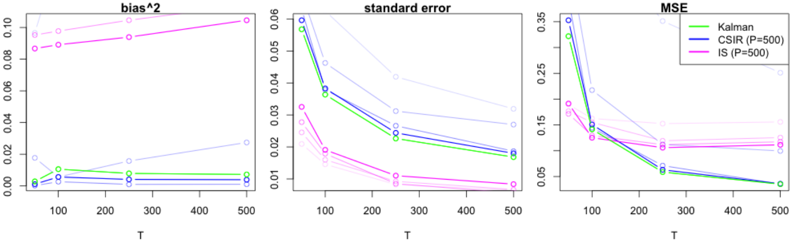
Method Comparison



Method Comparison

Filter	Latent state	Parameter	Comment
Kalman	x	x	linear Gaussian models only
SIR	x		
CSIR	x	x	univariate models only
IS		x	

Monte Carlo Simulations



Illustration

Trivariate Local Level Model

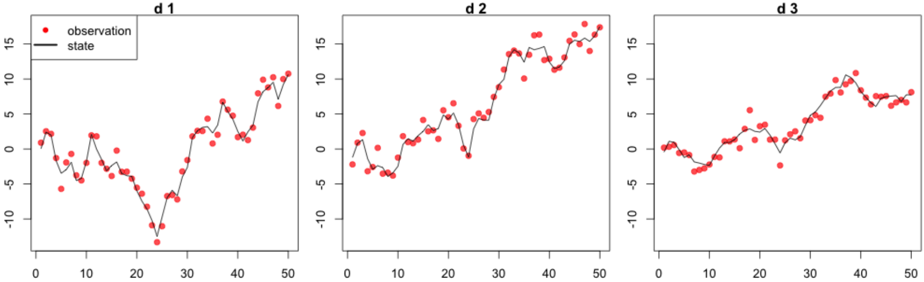
Formulation

$$\begin{aligned} \text{observation:} \quad \boldsymbol{y}_t &= \boldsymbol{x}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \sigma_\epsilon^2 I_3) \\ \text{state:} \quad \boldsymbol{x}_{t+1} &= \boldsymbol{x}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \Sigma_\eta) \end{aligned}$$

$$\Sigma_\eta = \begin{bmatrix} \sigma_{\eta 1}^2 & \rho\sigma_{\eta 1}\sigma_{\eta 2} & \rho\sigma_{\eta 1}\sigma_{\eta 3} \\ \rho\sigma_{\eta 1}\sigma_{\eta 2} & \sigma_{\eta 2}^2 & \rho\sigma_{\eta 2}\sigma_{\eta 3} \\ \rho\sigma_{\eta 1}\sigma_{\eta 3} & \rho\sigma_{\eta 2}\sigma_{\eta 3} & \sigma_{\eta 3}^2 \end{bmatrix}$$

$$\boldsymbol{\theta} = [\rho, \sigma_{\eta 1}^2, \sigma_{\eta 2}^2, \sigma_{\eta 3}^2, \sigma_\epsilon^2]^T$$

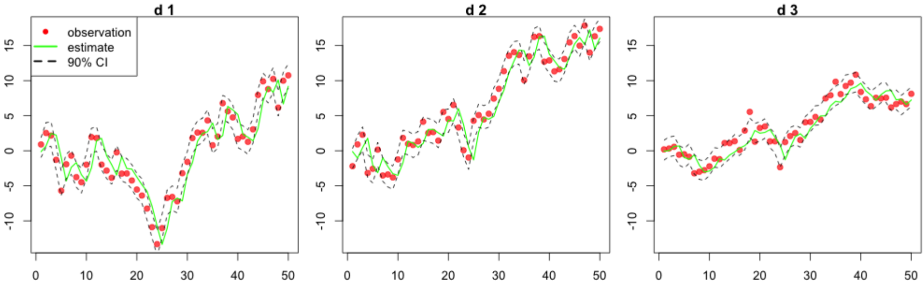
Trivariate Local Level Realization



$$\boldsymbol{\theta} = [\rho = 0.7, \sigma^2_{\eta 1} = 4.2, \sigma^2_{\eta 2} = 2.8, \sigma^2_{\eta 3} = 0.9, \sigma^2_{\epsilon} = 1.0]^T$$

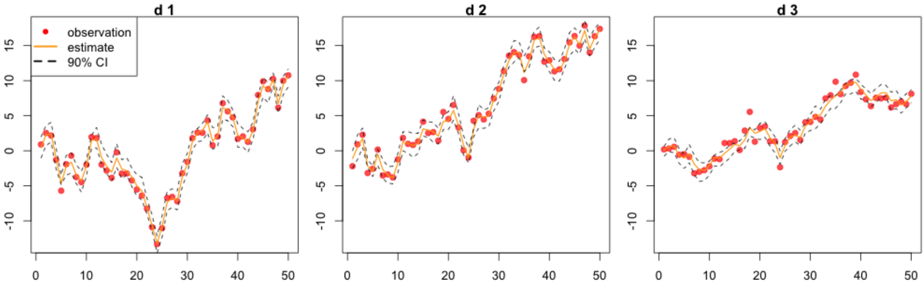
Latent State Inference

Kalman filter



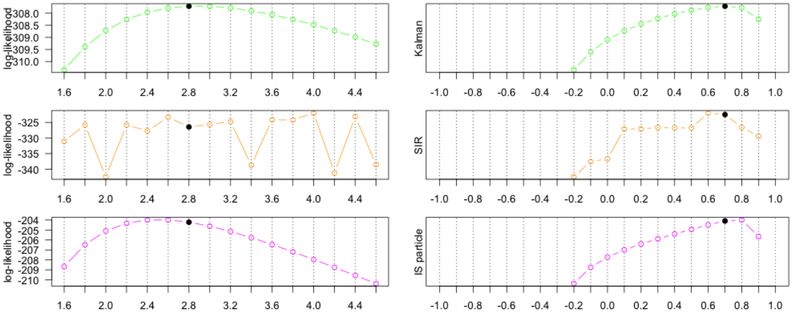
Latent State Inference

SIR particle filter



Parameter Inference

Log-likelihood plots for $\sigma^2_{\eta 2}$ and ρ



Parameter Inference

Results

	$\sigma^2_{\eta 1}$	$\sigma^2_{\eta 2}$	$\sigma^2_{\eta 3}$	ρ	true $\log \mathcal{L}$	MLE $\log \mathcal{L}$
True	4.20	2.80	0.90	0.70		
Kalman	4.96	3.10	1.01	0.73	-307.712	-307.459
SIR	2.27	1.53	1.29	0.52	-313.466	-336.291
IS	2.69	2.09	1.06	0.42		

Hierarchical Dynamic Poisson Model

Formulation

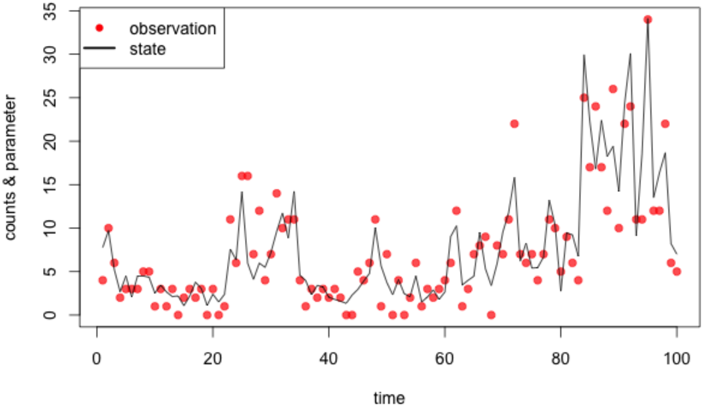
$$
\begin{aligned}
\text{observation:} \quad y_{m,n} &\sim \text{Poisson}(\lambda_{m,n}) \\
\text{state:} \quad \log \lambda_{m,n} &= \log \lambda_m^{(D)} + \log \lambda_{m,n}^{(I)} + \log \lambda_n^{(P)}
\end{aligned}
$$

$$
\begin{aligned}
\text{daily:} \quad \log \lambda_{m+1}^{(D)} &= \phi_0^{(D)} + \phi_1^{(D)} \log \lambda_m^{(D)} + \eta_m^{(D)} & \eta_t \sim N(0, \sigma^2_{(D)}) \\
\text{intra-daily:} \quad \log \lambda_{m,n+1}^{(I)} &= \phi_1^{(I)} \log \lambda_{m,n}^{(I)} + \eta_{m,n}^{(I)} & \eta_{m,n} \sim N(0, \sigma^2_{(I)}) \\
\text{periodic:} \quad \log \lambda_n^{(P)} &= \phi_1^{(P)} \sin(\pi(n-1)/M)
\end{aligned}
$$

$$
\boldsymbol{\theta} = [\phi_0^{(D)}, \phi_1^{(D)}, \sigma^2_{(D)}, \phi_1^{(I)}, \sigma^2_{(I)}, \phi_1^{(P)}]^T
$$

Hierarchical Dynamic Poisson Realization

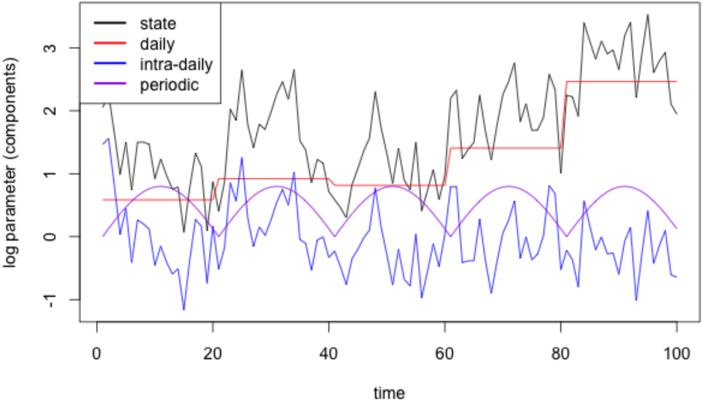
$N = 5,\ M = 20$



$$\boldsymbol{\theta} = [\phi_0^{(D)} = 0.7, \phi_1^{(D)} = 0.6, \sigma^2_{(D)} = 0.6, \phi_1^{(I)} = 0.3, \sigma^2_{(I)} = 0.2, \phi_1^{(P)} = 0.8]^T$$

Hierarchical Dynamic Poisson Realization

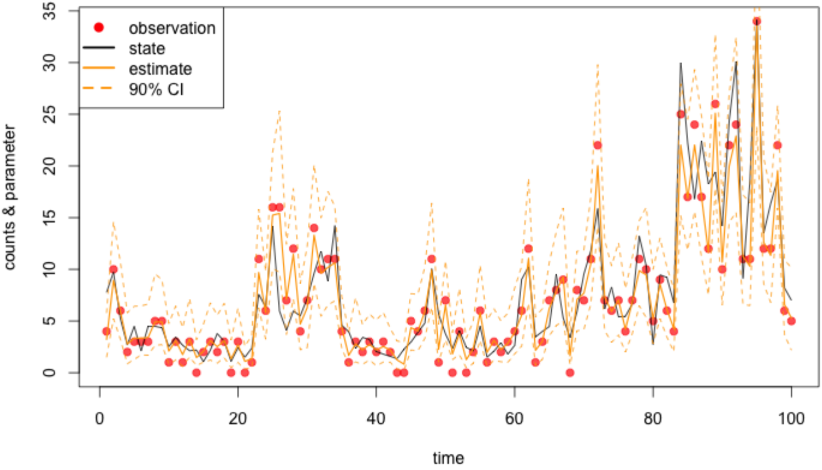
Components



$$\boldsymbol{\theta} = [\phi_0^{(D)} = 0.7, \phi_1^{(D)} = 0.6, \sigma^2_{(D)} = 0.6, \phi_1^{(I)} = 0.3, \sigma^2_{(I)} = 0.2, \phi_1^{(P)} = 0.8]^T$$

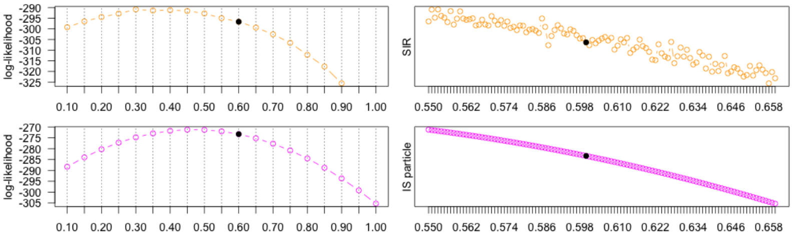
Latent State Inference

SIR particle filter



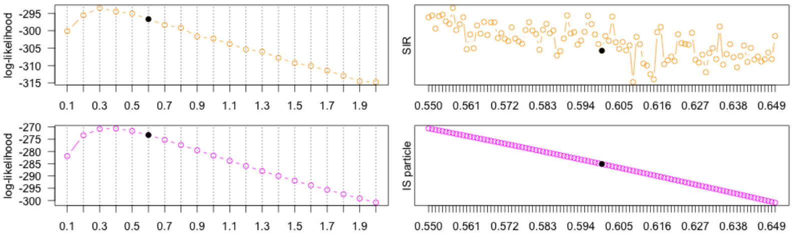
Parameter Inference

Log-likelihood plots for $\phi_1^{(D)}$



Parameter Inference

Log-likelihood plots for $\sigma^2_{(D)}$



Parameter Inference

Results

	$\phi_0^{(D)}$	$\phi_1^{(D)}$	$\sigma^2_{(D)}$	$\phi_1^{(I)}$	$\sigma^2_{(I)}$	$\phi_1^{(P)}$
True	0.70	0.60	0.30	0.80	0.60	0.20
SIR	0.76	0.56	0.85	0.48	0.83	1.13
IS	0.65	0.59	0.40	0.63	0.35	0.31

Q & A