CSCI567 Machine Learning (Fall 2016)

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Outline

1. Generative versus discriminative
   - Contrast Naive Bayes and logistic regression
   - Another example: Gaussian discriminant analysis

Naive Bayes and logistic regression: two different modeling paradigms

- Setup of the learning problem
  Suppose the training data is from an *unknown* joint probabilistic model $p(\boldsymbol{x}, y)$
- Differences in *assuming* models for the data
  - the generative approach requires we specify the model for the joint distribution (such as Naive Bayes), and thus, maximize the *joint* likelihood $\sum_n \log p(\boldsymbol{x}_n, y_n)$
  - the discriminative approach (discriminative) requires only specifying a model for the conditional distribution (such as logistic regression), and thus, maximize the *conditional* likelihood $\sum_n \log p(y_n|\boldsymbol{x}_n)$

Naive Bayes and logistic regression: two different modeling paradigms

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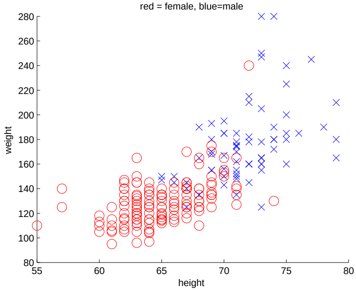
  - the discriminative approach (discriminative) requires only specifying a model for the conditional distribution (such as logistic regression), and thus, maximize the *conditional* likelihood $\sum_n \log p(y_n|\boldsymbol{x}_n)$
- Differences in computation
  - Sometimes, modeling by discriminative approach is easier
  - Sometimes, parameter estimation by generative approach is easier

Determining sex (man or woman) based on measurements

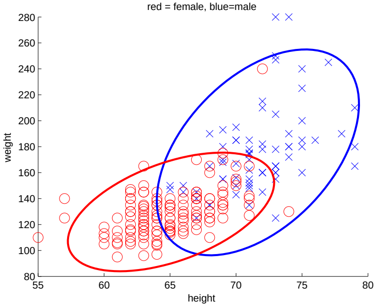


Generative approach

Propose a model of the joint distribution of ($x =$ height, $y =$sex)

our data

Sex	Height
1	$6'$
2	$5'2"$
1	$5'6"$
1	$6'2"$
2	$5.7"$
$\cdots$	$\cdots$



Intuition: we will model how heights vary (according to a Gaussian) in each sub-population (male and female).

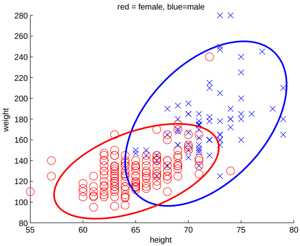
Note: This is similar to Naive Bayes for detecting spam emails.

Model of the joint distribution

$$p(x, y) = p(y)p(x|y) \tag{1}$$

$$= \begin{cases} p_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} & \text{if } y = 1 \\ p_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}} & \text{if } y = 2 \end{cases} \tag{2}$$

where $p_1 + p_2 = 1$ represents two *prior* probabilities that x is given the label 1 or 2 respectively. $p(x|y)$ is called *class distributions*, which we have assumed to be Gaussians.



Parameter estimation

Likelihood of the training data $\mathcal{D} = \{(x_n, y_n)\}_{n=1}^N$ with $y_n \in \{1, 2\}$

$$\begin{aligned}
\log P(\mathcal{D}) &= \sum_n \log p(x_n, y_n) \\
&= \sum_{n: y_n = 1} \log \left(p_1 \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_n - \mu_1)^2}{2\sigma_1^2}} \right) \\
&+ \sum_{n: y_n = 2} \log \left(p_2 \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_n - \mu_2)^2}{2\sigma_2^2}} \right)
\end{aligned}$$

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Maximize the likelihood function

$$(p_1^*, p_2^*, \mu_1^*, \mu_2^*, \sigma_1^*, \sigma_2^*) = \arg\max \log P(\mathcal{D})$$

Decision boundary

As before, the Bayes optimal one under the assumed joint distribution depends on

$$p(y = 1|x) \geq p(y = 2|x)$$

which is equivalent to

$$p(x|y = 1)p(y = 1) \geq p(x|y = 2)p(y = 2)$$

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Namely,

$$-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \log\sqrt{2\pi}\sigma_1 + \log p_1 \geq -\frac{(x-\mu_2)^2}{2\sigma_2^2} - \log\sqrt{2\pi}\sigma_2 + \log p_2$$

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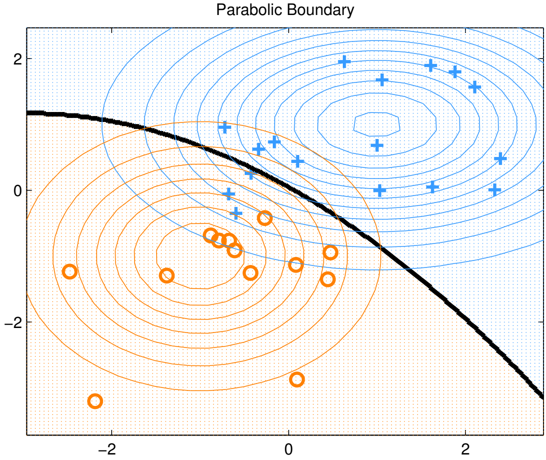
$\Rightarrow ax^2 + bx + c \geq 0$ $\quad\leftarrow$ the decision boundary not *linear*!

Example of nonlinear decision boundary



Note: the boundary is characterized by a quadratic function, giving rise to the shape of parabolic curve.

A special case: what if we assume the two Gaussians have the same variance?

We will get a linear decision boundary

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with $\sigma_1 = \sigma_2$, we have

$$bx + c \geq 0$$

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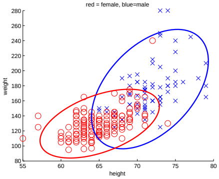
Note: equal variances across two different categories could be a very strong assumption.



For example, from the plot, it does seem that the *male* population has slightly bigger variance (i.e., bigger eclipse) than the *female* population. So the assumption might not be applicable.

Mini-summary

Gaussian discriminant analysis

- A generative approach, assuming the data modeled by

$$p(x, y) = p(y)p(x|y)$$

  where $p(x|y)$ is a Gaussian distribution.
- Parameters (of those Gaussian distributions) are estimated by maximizing the likelihood
  - Computationally, estimating those parameters are very easy — it amounts to computing sample mean vectors and covariance matrices
- Decision boundary
  - In general, nonlinear functions of x — in this case, we call the approach *quadratic discriminant analysis*
  - In the special case we assume equal variance of the Gaussian distributions, we get a linear decision boundary — we call the approach *linear discriminant analysis*

So what is the discriminative counterpart?

Intuition

The decision boundary in Gaussian discriminant analysis is

$$ax^2 + bx + c = 0$$

Let us model the conditional distribution analogously

$$p(y|x) = \sigma[ax^2 + bx + c] = \frac{1}{1 + e^{-(ax^2+bx+c)}}$$

Or, even simpler, going after the decision boundary of linear discriminant analysis

$$p(y|x) = \sigma[bx + c]$$

Both look very similar to logistic regression — i.e. we focus on writing down the *conditional* probability, *not* the joint probability.

Does this change how we estimate the parameters?

First change: a smaller number of parameters to estimate

Our models are only parameterized by a, b and c. There is no prior probabilities (p_1, p_2) or Gaussian distribution parameters (μ_1, μ_2, σ_1 and σ_2).

Second change: we need to maximize the conditional likelihood $p(y|x)$

$$(a^*, b^*, c^*) = \arg\min -\sum_n \left\{ y_n \log \sigma(ax_n^2 + bx_n + c) \right. \tag{3}$$

$$\left. + (1 - y_n) \log[1 - \sigma(ax_n^2 + bx_n + c)] \right\} \tag{4}$$

Computationally, much harder!

How easy for our Gaussian discriminant analysis?

Example

$$\hat{p}_1 = \frac{\#\text{ of training samples in class 1}}{\#\text{ of training samples}} \tag{5}$$

$$\hat{\mu}_1 = \frac{\sum_{n:y_n=1} x_n}{\#\text{ of training samples in class 1}} \tag{6}$$

$$\hat{\sigma}_1^2 = \frac{\sum_{n:y_n=1} (x_n - \mu_1)^2}{\#\text{ of training samples in class 1}} \tag{7}$$

Note: detailed derivation is in the books. They can be generalized rather easily to multi-variate distributions as well as multiple classes.

Generative versus discriminative: which one to use?

There is no fixed rule

- Selecting which type of method to use is dataset/task specific
- It depends on how well your modeling assumption fits the data
- Recent trend: big data is always useful for both!
  - Apply very complex discriminative models, such as deep learning methods, for building classifiers
  - Apply very complex generative models, such as nonparametric Bayesian methods, for modeling data