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THE WAKE OF AN UNSTEADY ACTUATOR DISC

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. dr. ir. T. H. J. J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op dinsdag 3 april 2018 om 15:00 uur

door

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geboren te Hubei, China.

2

STATE OF THE ART OF UNSTEADY ROTOR AERODYNAMICS

Research is what I'm doing when I don't know what I'm doing.

Wernher von Braun (1912-1977)

This chapter briefly introduces unsteady rotor aerodynamics and the history of dynamic inflow. The state of the art of research on dynamic inflow is presented separately for aspects of modelling and experimental work. This chapter introduces the following models: Blade Element Momentum and the actuator-disc wake model for steady rotor aerodynamics, the advanced vortex model and computational fluid dynamics (CFD) for unsteady rotor aerodynamics. The state of the art of dynamic-inflow engineering models used in the wind energy field is also introduced: the Pitt-Peters model (Pitt and Peters, 1981), the Øye model (Øye, 1986, 1990) and the ECN model (Schepers, 2012). These models are validated and benchmarked against experimental and numerical results later in this thesis. Finally, the experimental work on dynamic-inflow problem in the wind energy field are presented.

2.1. INTRODUCTION TO UNSTEADY AERODYNAMICS

Understanding aerodynamics is essential for the design of efficient and reliable wind turbines. According to Huyer et al. (1996), underestimating power output and load leads to the failure of generators, gear-boxes, and even turbine blades. This in turn increases operational and maintenance costs.

Furthermore, wind turbines are subjected to non-stationary environment. Turbulence, wind shear, the flexible blade structure of wind turbines, and the passive and active aerodynamic control strategies, such as yaw, pitch control and smart rotor control, leads to a wind turbine operating in a highly dynamic state. All these factors enhance the complexity of the flow field of a wind turbine. To accurately predict the unsteady flow field of a wind turbine is challenging.

The various aerodynamic sources which contribute to the unsteady aerodynamic load on a wind turbine are summarized in Figure 2.1, including the variation of wind speed or direction, turbulence, blade deformation, pitch, yaw, wake dynamic, etc. Only the effect of change due to change in load, not change in inflow is the subject of this thesis, which can be caused by a rotor-thrust change resulting from a pitch action, rotor speed variation, etc.

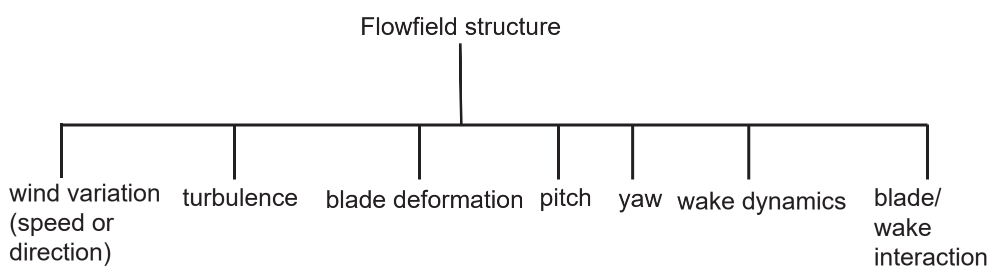

Figure 2.1: Summary of the various aerodynamic sources that contribute to the unsteady airloads on a wind turbine (modified from Leishman (2002)).

According to the length scale, the unsteady aerodynamics of a wind turbine can be divided into three main scales: airfoil scale, blade scale and rotor scale. This thesis focuses on the rotor scale, the phenomenon is also called 'Dynamic inflow' or 'Dynamic wake' or 'Dynamic induction' in the literature.

Dynamic inflow refers to the time lagging in the response of the induced velocity field of a rotor following rapid changes in the rotor operating state (Hansen and Butterfield, 1993). The characteristic time scale for dynamic inflow is the ratio of the rotor diameter and the speed of incoming flow, represented by D/V_0 (Snel and Schepers, 1995). The wake can not reach a new equilibrium state instantaneously to a change in rotor loading because of the inertia of the air in the wake. This can be observed after a change in blade pitch angle or change in rotor thrust.

2.2. HISTORY OF DYNAMIC INFLOW RESEARCH

Dynamic inflow has gained much attention in rotary-wing analysis, relevant to the design and analysis of both rotorcraft and wind turbine.

2.2.1. DYNAMIC INFLOW IN ROTORCRAFT

Dynamic-inflow phenomena influence the unsteady rotor load. It has maintained a dominant position in the real-time flight simulation, stability computations, and flight mechanics and control (Peters, 2009). The history of dynamic-inflow development in rotorcraft application was summarized in Pitt (1980), Gaonkar and Peters (1986) and Peters (2009).

The dynamic-inflow studies in rotorcraft flight dynamics started in the 1950's. Amer (1950) developed a theory to predict the pitch and roll damping of a rotor. The response of a hovering rotor to rapid changes in collective pitch was measured by Carpenter and Fridovich (1953), and they addressed the time lag of inflow by introducing the 'apparent mass' concept. They assumed that the uniform induced velocity of the initial flow field is analogous to the flow field produced by an impermeable disc moved normal to its plane. Sissingh (1952) derived the inflow formulas for both hover and forward flight. It is the first prototype of a dynamic-inflow model, albeit quasi-steady, Peters (2009) thought that it includes enough physics to explain the effects.

The research boomed in the 1970's and 1980's. The unsteady dynamic rotor hovering wake was modeled as an approximate steady-state wake with a time lag by Crews and Hohenemser (1973). Crews' unsteady inflow theory (Crews and Hohenemser, 1973) was extended by Peters (1974) to encompass more general inflow models. Pitt and Peters (1981) used the potential functions of Mangler and Squire (Mangler and Squire, 1950) to develop a closed-form representation of the induced flow matrix [L] and the rotor response matrix [M]. This was the development of the Pitt-Peters dynamic inflow model, which became one of the most important models in flight simulations and rotor response analysis of rotorcraft.

2.2.2. DYNAMIC INFLOW IN WIND TURBINE

Snel and Schepers (1995) pointed out that the wind turbine operational conditions are different from the conditions experienced by helicopters. Helicopters are designed for high thrust with minimum power consumption at low induction factors, while wind turbines are designed for maximum power extraction at high induction factors close to 1/3. The dynamic-inflow effects enhance with higher induction factors, due to the interrelationship between the dynamic-inflow problem and the induction field. Therefore, some of the assumptions and simplifications used in the helicopter related models may not be applicable for wind turbine aerodynamics, which needs further justification.

The importance of dynamic inflow in wind energy applications has been realized since the 1990's, and subsequently it became one of the most interesting topics of wind turbine research.

A joint investigation of dynamic-inflow effect supported by six European organizations was initiated in the 1990's, the JOULE project I (Snel and Schepers, 1995) and II (Schepers and Snel, 1995). Several dynamic-inflow models were proposed and tested in these research. The Pitt-Peters dynamic-inflow model (Pitt and Peters, 1981) for rotorcraft was modified to annular sections to apply to wind turbines by Suzuki (2000). Detailed measurements on dynamic-inflow characteristics under a series of unsteady operational conditions were conducted during the NREL/NASA Ames wind tunnel tests (Hand et al., 2001) on a 10m rotor. Dynamic inflow measurements are also part of the goal of ongoing EU FP5 projects on the MEXICO rotor in phase I (Schepers and Snel, 2009) and phase II (Schepers et al., 2014).

The importance of including a dynamic-inflow model when estimating the free mean wind speed and the induced velocities in aero-servo-elastic modelling was demonstrated by Henriksen et al. (2013). To consider dynamic-inflow effects for designing a controller of wind turbines was also found important by van Engelen and van der Hooft (2004) and Hansen et al. (2005).

2.3. DYNAMIC INFLOW MODELLING

The most rigorous way of calculating the unsteady flow field of a rotor is to solve the time-dependent incompressible Navier-Stokes equations (Snel, 1998). However, due to the computational cost of CFD, it is used more often as an analysis method rather than a design method presently.

Vortex modelling by using the Euler equations instead of the Navier-Stokes equations is an intermediate method between the CFD and BEM method, in terms of the computational cost, complexity and accuracy. Free wake vortex models can intrinsically handle the unsteady aerodynamic problem. Even though the computational cost of the vortex models is considerably less than that of CFD, it is still much higher than Blade Element Momentum method.

The current widely used design and analysis tool for rotor aerodynamics is still BEM method. However, due to the quasi-steady wake assumption, BEM can only account for the steady rotor aerodynamics. A practical solution is to integrate BEM with dynamic-inflow engineering models.

BEM method and the state of the art dynamic-inflow engineering models of Pitt-Peters, Øye and ECN are introduced in the following first two subsections. The basic actuator-disc model for steady wake and more advanced models for dynamic wake of a rotor are introduced in the last two subsections, respectively.

2.3.1. BLADE ELEMENT MOMENTUM THEORY

Since the first introduction of BEM method to wind turbine design (Wilson and Lissaman, 1974) four decades ago, it has remained the main tool for wind turbine rotor aerodynamic design and aero-servo-elastic load simulation. The major reasons behind the predominance can be summarized as

- It is computationally cheap and easily integrated with a servo-elastic model.
- This approach allows fundamental understanding of the effects of varying geometrical and aerodynamic parameters on the performance of a wind turbine (Leishman, 2002), by taking into account the sectional aerodynamics of airfoil.
- In spite of the assumptions and oversimplifications made in BEM theory, the method often predicts rotor performance with acceptable accuracy (Hansen and Butterfield, 1993).

The blade element momentum theory applied for wind turbine analysis is reported in ample literature, e.g. the textbooks by Burton et al. (2001) and Leishman (2006). The basic ideas of momentum theory and blade element theory and their main assumptions which challenge the application to dynamic inflow study are introduced concisely below.

AXIAL MOMENTUM THEORY

The momentum theory is the most basic way to analyze a wind turbine, which is also called "actuator-disc theory" or "one-dimensional slip-stream theory". It was first developed by Rankine (1865) and Froude (1889). The history of the development of momentum theory can be found in van Kuik et al. (2015). Sørensen (2016) summarized the recent development of momentum theory on application to HAWT.

In the axi-symmetric axial flow, a 1-D model of the control volume of a wind turbine is shown in Figure 2.2. The rotor is represented by a hypothetical permeable actuator disc, which exerts a force on the flow. The boundary surface which separates the affected flow from undisturbed flow is extended upstream and downstream to form a streamtube. The flow within the streamtube slows down, the cross-sectional area of the streamtube must expand to accommodate it. The undisturbed wind speed is V_0, the axial induced velocity at the plane of the turbine is denoted as v_i, with the axial induced velocity in the far wake being v_w. The axial induced velocity is the velocity reduction resulting from the energy extracting of the turbine. Therefore, the net velocity at the plane of the turbine is $V_0 - v_i$, the net velocity in the far wake is $V_0 - v_w$.

- `Mass conservation`

  By applying the principle of the conservation of mass, the fluid mass flow rate, $\dot{m}$, through the disc is

$$\dot{m} = \rho A(V_0 - v_i). \tag{2.1}$$

- `Momentum conservation`

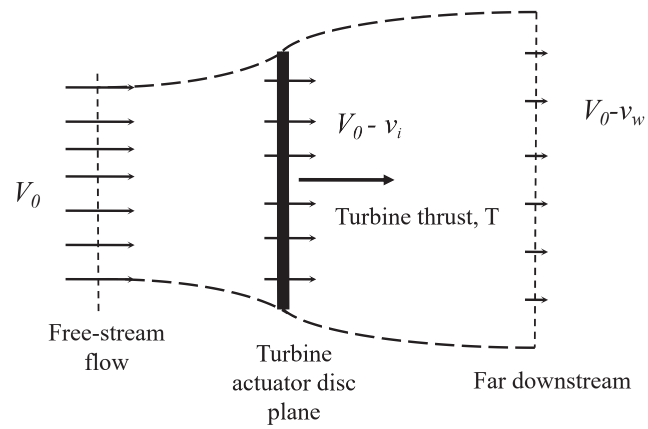

Figure 2.2: Flow model used for the momentum theory analysis of a wind turbine.

The thrust on the disc can be represented by the change of momentum of the flow across the disc

$$T = \dot{m}V_0 - \dot{m}(V_0 - v_w). \tag{2.2}$$

After expanding the above equation, a relationship between the thrust on the turbine and the velocity deficit in the far downstream can be obtained,

$$T = \dot{m}V_0 - \dot{m}V_0 + \dot{m}v_w = \dot{m}v_w. \tag{2.3}$$

- `Energy conservation`

  By applying the law of the conservation of energy, the power extracted from the flow is

$$P = \frac{1}{2}\dot{m}V_0^2 - \frac{1}{2}\dot{m}(V_0 - v_w)^2 = \dot{m}v_w(V_0 - \frac{1}{2}v_w). \tag{2.4}$$

The energy absorbed by the disc can also be obtained by

$$P = T(V_0 - v_i). \tag{2.5}$$

Substituting Equation 2.3 into Equation 2.5 yields

$$P = \dot{m}v_w(V_0 - v_i). \tag{2.6}$$

Under the assumption of no viscosity and no other loss, the kinetic energy absorbed by the disc should equal to the power extracted from the flow (ie. Equation 2.4 equals to Equation 2.6), which gives $v_i = v_w/2$.

In the analysis of a wind turbine, it is usual to define an independent parameter, the induction factor $a = v_i/V_0$. Accordingly, $v_i = aV_0$, $v_w = 2aV_0$.

Combining Equation 2.3 and 2.1, the thrust is

$$T = \rho A(V_0 - v_i)v_w. \tag{2.7}$$

Substituting $v_i = aV_0$, $v_w = 2aV_0$ into Equation 2.7 gives

$$T = \rho A(V_0 - V_0 a)2V_0 a = 2\rho AV_0^2(1-a)a. \tag{2.8}$$

The thrust coefficient is defined by

$$C_t = \frac{T}{\frac{1}{2}\rho AV_0^2}. \tag{2.9}$$

Consequently,

$$C_t = \frac{2\rho AV_0^2(1-a)a}{\frac{1}{2}\rho AV_0^2} = 4a(1-a). \tag{2.10}$$

Combining Equation 2.6 and 2.1 gives

$$P = \dot{m}v_w(V_0 - v_i) = \rho A(V_0 - v_i)v_w(V_0 - v_i) = \rho A(V_0 - v_i)^2 v_w. \tag{2.11}$$

Substituting $v_i = aV_0$, $v_w = 2aV_0$ into Equation 2.11 results in

$$P = \rho A(V_0 - aV_0)^2 2aV_0 = 2\rho AV_0^3(1-a)^2 a. \tag{2.12}$$

The power coefficient is defined as

$$C_p = \frac{P}{\frac{1}{2}\rho AV_0^3}. \tag{2.13}$$

Hence

$$C_p = \frac{2\rho AV_0^3(1-a)^2 a}{\frac{1}{2}\rho AV_0^3} = 4(1-a)^2 a. \tag{2.14}$$

The momentum theory itself only gives the averaged velocity of the actuator disc, but not the velocity distribution at the disc. It is valid for inviscid, incompressible, steady flow.

STEADY ASSUMPTION OF THE AXIAL MOMENTUM THEORY

The momentum theory is only applicable when the flow field reaches an equilibrium state. It assumes that the velocity and pressure field follows the load changes instantaneously. However, during the operation of a wind turbine, the load on the disc varies frequently due to the sources in Figure 2.1.

Figure 2.3 shows the change of the streamtube and the velocity when the load on the disc changes from a low thrust to a higher one. The flow will be decelerated due to the increased load. It takes a certain time for the flow to accomplish the change due to the inertia of the streamtube flow volume. To understand the time delay of the development of the new induction flow field is essential to model the dynamic inflow effect correctly by adding it into the steady BEM simulation.

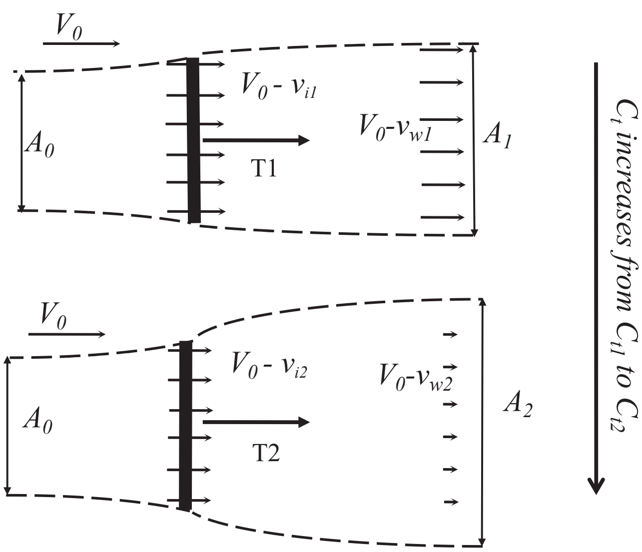

Figure 2.3: Change of the streamtube and velocity when the load change on the rotor.

BLADE ELEMENT THEORY

The momentum theory considers the averaged force and velocity at the actuator disc plane. In practice, the equations are applied to annuli, which is swept by an independent element of the blade. The induced velocity at each element is attained by performing a momentum balance for an annular control volume containing the blade element and the air bounded by the stream surfaces extending upwind and downwind of the element. The aerodynamic forces on the element are calculated using two-dimensional lift and drag coefficients at the local angle of attack.

ASSUMPTION OF THE BLADE ELEMENT THEORY

The main assumption of the blade element theory is that they are independent from each other. This is challenged when applying it to the analysis of the blade with abrupt changes of load in radial direction. According to Sørensen and Kock (1995), this assumption is acceptable, except for regions where load changes abruptly, for example, at the root and the tip regions. The annuli were shown to be not independent and the pressure at the boundary of the annuli was suggested by van Kuik and Lignarolo (2016) to be included in the momentum theory to remedy the effect of annuli independent assumption. Another situation where this assumption might not be applicable is when distributed aerodynamic control is applied. For instance, the application of trailing edge flaps or micro-tabs can create a non-uniform distributed loading along the blade span.

2.3.2. DYNAMIC-INFLOW ENGINEERING MODELS

Various dynamic-inflow engineering models have been proposed and applied in the wind energy field. The widely used ones are the models of Pitt-Peters (Pitt and Peters, 1981), Øye (1986, 1990), and ECN (Schepers, 2012), which are introduced in the following in detail. All the models describe the distribution of the inflow in the form of ordinary equations, with time constants representing the dynamic lag in the build-up of the inflow. The theory behind the models are different. The model of Pitt and Peters (1981) is based on the 'apparent mass ' theory. The latter two are developed based on vortex models.

THE PITT-PETERS DYNAMIC INFLOW MODEL

THE Pitt-Peters dynamic inflow model (Pitt and Peters, 1981) was developed for an actuator disc with an assumed inflow distribution across the disc. Based on the assumption that the equation of Pitt-Peters can be applied to a blade element or actuator annulus level, the dynamic inflow equation for each annular ring becomes

$$\frac{1}{\rho A_j V_0^2/2}\left[\frac{8}{3\pi}\rho A_j r_j \frac{dv_j}{dt} + 2\rho A_j v_j (V_0 + v_j)\right] = C_{tj}. \tag{2.15}$$

where j indicates the j^{th} annular ring, A_j and C_{tj} are the area and thrust coefficient of the j^{th} annulus and v_j is its azimuthal averaged induced velocity. The first term inside the outer bracket of Equation 2.15 represents the additional force on the rotor disc resulting from the accelerating or decelerating inflow (Leishman, 2002), while the second term results from the static pressure difference across the actuator disc.

THE ØYE DYNAMIC INFLOW MODEL

IN the dynamic inflow model of Øye (1986, 1990), the induced velocity is estimated by filtering the quasi-steady values through two first-order differential equations

$$v_{int} + \tau_1 \frac{dv_{int}}{dt} = v_{qs} + b\tau_1 \frac{dv_{qs}}{dt}. \tag{2.16}$$

$$v_z + \tau_2 \frac{dv_z}{dt} = v_{int}. \tag{2.17}$$

where v_{qs} is the quasi-steady value from BEM, v_{int} is an intermediate value and the final filtered value v_z is treated as the induced velocity. After calibration using a vortex ring model (Øye, 1990), the two time constants are recommended as follows (Snel and Schepers, 1995)

$$\tau_1 = \frac{1.1}{(1-1.3a)}\frac{R}{V_0}. \tag{2.18}$$

$$\tau_2 = (0.39 - 0.26(\frac{r_j}{R})^2)\tau_1. \tag{2.19}$$

where a is the axial induction factor, R is the rotor radius, r_j is the radius of j^{th} annulus, and b is a constant value of 0.6.

2

THE ECN DYNAMIC INFLOW MODEL

THE dynamic inflow model developed by Schepers (2012), was derived from an integral relation of the streamtube model (see more details of the streamtube model in subsection 2.3.3). For the condition of constant wind speed, the equation is

$$\frac{R}{V_w} f_a \frac{da}{dt} + a(1-a) = C_{tj}/4. \tag{2.20}$$

where C_{tj} is the axial force coefficient on the rotor annulus j. The term f_a is a function of the radial position, defined as

$$f_a = 2\pi / \int_0^{2\pi} \frac{[1-(r/R)cos\Phi_r]}{[1+(r/R)^2 - 2(r/R)cos\Phi_r]^{3/2}} d\Phi_r. \tag{2.21}$$

These models have been applied to different aero-elastic codes. The Pitt-Peters model has been applied in the open source code from NREL — Aerodyn (Laino and Hansen, 2002), the Øye's model has been applied to DTU's code HAWC2 (Larsen and Hansen, 2007), the ECN's model has been applied to the code PHATAS (Lindenburg, 2005). The improvements over quasi-steady models was reported. However, to what extent they can reflect the real dynamic inflow problem is unknown, and their comparative performance is not fully evaluated.

The use of BEM with engineering add-ons has become popular in the design frame of a wind turbine, due to its favorable implementation within an aero-servo-elastic program and the modest computational time. In addition to the engineering models, more advanced models, which can account for more physics, are introduced in the following sections.

2.3.3. ACTUATOR-DISC MODEL FOR STEADY WAKE

The vortex model was introduced by Joukowsky (1912, 1914, 1915, 1918), consisting of blade bound vortices, a root vortex and a vortex at the tip of each blade. The application of the actuator disc with infinite number of blades leads to a simple wake model, which is illustrated in Figure 2.4. Details of a rotor flow field can be calculated from such a vortex model.

It results in a linear actuator-disc model representation as shown in Figure 2.5, when the wake expansion is neglected. Analytical formulae of the velocity induced by a right cylinder were independently obtained by Callaghan and Maslen (1960), Gibson (1974), van Kuik and Lignarolo (2016), and Branlard and Gaunaa (2014) using different approaches. Callaghan and Maslen (1960) derived the magnetic field of a finite solenoid based on vector potential, which is applicable to velocity field calculation of a right vortex cylinder with tangential vorticity in rotor aerodynamics. Gibson (1974) obtained all three components of the velocity field induced by a semi-infinite vortex cylinder based on a lemma and integration by parts and proved the feasibility of applying the semi-infinite vortex cylinder to actuator disc flow. van Kuik and Lignarolo (2016) obtained the analytical axial velocity of a semi-infinite vortex tube using the gradient of the solid angle. Branlard and Gaunaa (2014) obtained the analytical equations of the velocity field induced by a finite or semi-infinite vortex tube using direct integration of the Biot-Savart law.

Using a superposition of a system of coaxial vortex cylinders to calculate the induced velocity field of non-uniform disc loading was applied by Heyson and Katzoff (1956) and Branlard and Gaunaa (2014).

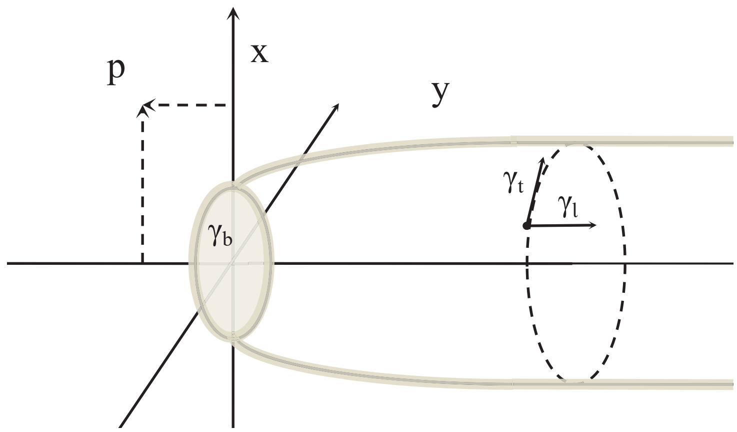

Figure 2.4: A nonlinear actuator disc model, representing the wake of a wind turbine.

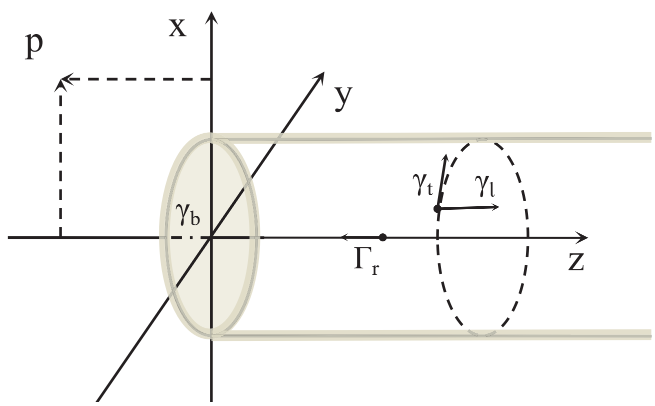

Figure 2.5: A linear actuator disc model, representing the wake of a wind turbine.

2.3.4. ADVANCED MODELS FOR DYNAMIC WAKE

The free tracking vortex-based model and CFD models inherently can account for the unsteady aerodynamic effects of a rotor. These advanced models are more physically representative than the dynamic-inflow engineering models, at the cost of more computational time.

VORTEX MODELLING

Vortex models have become popular in the analysis of a wide range of aerodynamic problems (Cottet and Koumoutsakos, 2000; Lewis, 1991). The historical and recent developments of the vorticity-based methods for the study of wind turbine aerodynamics are detailed in Branlard (2017). The vortex models applied to rotor wake problems can be divided into two categories : the "prescribed" and "free" vortex techniques (Leishman, 2002).

In the prescribed vortex models, positions of the vortical elements are specified beforehand based on semi-empirical rules. The prescribed vortex model in wind turbine application is constrained by the limited documented wake vortices positions, which has to be obtained by experiments encompassing a wide range of rotor geometric (e.g., blade shape and twist) and operating states (wind speed, yaw angle, etc.) (Leishman, 2002).

In the free vortex models, the vortex elements are convected and deformed freely under the action of the local velocity field to force-free locations, which are pertinent for the dynamic inflow study. The free vortex model has been applied to the helicopter analysis since 1970s (Clark D.R., 1970). The application has been transmitted to wind turbines later on. The theoretical development of a method for prediction of the aerodynamic performance of horizontal axis wind turbines was presented by Jeng and Keith (1982). Various vortex tools were applied in the JOULE projects (Snel and Schepers, 1995; Schepers and Snel, 1995). Among them, the AWSM code (van Garrel, 2003) is a free wake lifting line model developed in ECN; the ROVLM (Bareiss and Wagner, 1993) is a free/hybrid wake code, in which the flow field around a solid lift producing surface is modelled by superposition of the singularities of a source and a doublet on the panel with the free stream velocity. This is also called panel method or panel model. The code GENUVP (Voutsinas, 2006) is also a free wake panel code based on a vortex particles (vortex blobs) approximation of the wake. Time constants in the Øye 's dynamic-inflow engineering model (Snel and Schepers, 1995) was also tuned using a hybrid wake model represented by discretized vortex rings (Øye, 1990).

A 3D unsteady free wake panel model was applied to a vertical axis wind turbine (VAWT) by Ferreira (2009) to understand the dynamics of the 3D near wake. A 3D unsteady, potential flow panel model was also used by Micallef (2012) to study the 3D flow field near a horizontal axis wind turbine (HAWT) rotor under yawing. Recently, a free wake lifting line model was developed and applied to HAWT under yawing and pitching conditions. A good agreement with experiments of three different turbine models in terms of blade load, the rotor torque and the locations of the tip vortex cores in the wake was obtained (Qiu et al., 2014).

The needed memory and computational time increases exponentially with the increase of number of discrete elements per vortex filament, which makes the technique of free wake vortex modelling very expensive. However, it is still considerably less expensive than using CFD methods. Due to the artificial viscosity, CFD tends to cause concentrated vorticity to decay (Peters, 2009). This explains why the free wake models are more suitable for induced field calculation than CFD. The artificial viscosity is needed for numerical stability in CFD.

The application of the GPU technique makes it practical to run the simulations in this thesis using free wake modelling on a personal computer.

COMPUTATIONAL FLUID DYNAMICS

In the application of CFD to rotor aerodynamics, the continuous flow domain of the wake of a wind turbine is governed by the RANS or LES equations. The equations are solved in a discrete manner. It is divided into two categories based on the different representation of the blades — the generalized actuator disc approach and the direct approach (Sanderse et al., 2011). In the former, the blades are represented by a body force; in the latter, the actual blades are directly discretized on a computational mesh.

Depending on the different ways of representing the blades, the generalized actuator disc approaches are divided into actuator disc, actuator line and actuator surface. Rajagopalan and Rickerl (1990) were one of the first to apply the actuator-type approach in a CFD code. Although the actuator disc methods are only for steady load conditions, Sørensen and Myken (1992) made it suitable for unsteady computations by representing the axisymmetric Euler equations in a vorticity-stream function formulation. Sørensen and Kock (1995) employed this model to calculate the unsteady flow past the 2 MW Tjæreborg wind turbine. As an extension of the actuator disc approach, the actuator line approach, where the force term is represented by a actuator line for the wake calculation of a rotor , was introduced by Sørensen and Shen (2002). The method was implemented in the CFD code of EllipSys3D by Mikkelsen (2003), which has been applied in various subsequent researches. The actuator line was further extended to actuator surface approach by Shen et al. (2007, 2009).

The CFD methods are increasingly applied to wind turbine aerodynamics and wind farm aerodynamics. However, the demanding computational requirements limit it for analysis tool rather than design tool at current stage.

2.4. EXPERIMENTS ON DYNAMIC INFLOW

Most of the experimental research on dynamic inflow of wind turbines were conducted in the projects mentioned in subsection 2.2.2. Some more details are given in this section.

Øye (1986) observed the delay of response of flapwise moment and power from change of pitch angle on Nibe B HAWT. In the project JOULE I (Snel and Schepers, 1995), both axisymmetric and yawed cases were tested. Two turbines were utilized for the test. The measurement on the full scale turbine of Tjæreborg focused on pitching transient and yawed flow conditions (Øye, 1991a,b). However, only the blade bending moments were successfully measured. The measurement on a wind turbine model tested in the TUDelft open jet wind tunnel focused on the wind gust and yawed flow. The velocity at three fixed positions just downstream the rotor plane was measured by a hot wire for some specific test cases (Snel and Schepers, 1995). The wind gust was not simulated successfully, because the generation of gust was slower than the dynamic inflow time scale. The velocity measured at downstream planes was not representative as the azimuthal position relative to the rotating blade also changed in this set-up where the measurement positions were fixed. In the project JOULE II (Schepers and Snel, 1995), a big supplement to the project JOULE I was that partial pitch transient load cases were tested.

In the NREL/NASA Ames wind tunnel test (Hand et al., 2001), a series of operational conditions ranging from upwind axial operation, down wind operation, pitch step, yaw operation and variation in angle of attack were tested. An important improvement of this test was that the instantaneous pressure at five radial positions instead of only blade bending moment as in previous projects was measured. This provides the information for studying the radial dependency of the time constant along the blade (Schepers, 2007).

In the MEXICO project phase I (Schepers et al., 2012), pitching transient and variation of incoming flow were tested on a 3 bladed turbine with 4.5m diameter in the Large Low-speed Facility (LLF) of DNW in the Netherlands. However, the dynamic inflow effects were too small to be used for validation due to a slow change in pitch and flow (Schepers et al., 2014). Therefore, in the second phase (Schepers et al., 2014), the test cases in phase I were reconducted, and also yawed conditions were tested.

Full scale turbines operating in real atmospheric flow conditions were further tested in DANAERO experiment (Madsen et al., 2010; Troldborg et al., 2013). Measurements were performed on a 2MW NM80 turbine with an 80m rotor undergoing pitch transient at the Tjæreborg Enge site (Madsen et al., 2010) . However, Schepers et al. (2014) thought that it was not sufficient for analysis of dynamic inflow effects due to the atmospheric turbulence effect, because only 10 min data was acquired. Consequently, a new campaign was performed on a Siemens 3.6MW turbine at the Høvsøre test site in Denmark (Madsen et al., 2010). The blade bending moment and flow at five radial locations along the blade were obtained under pitch transient with a pitch rate of 1 degree per second.

The above mentioned experiments were performed with different size of turbine models, ranging from wind tunnel models to full-scale wind turbines. Basically, two different operational cases were tested: the axi-symmetric case, including step pitch and variation in incoming flow; the axi-asymmetric case, yawed conditions. The blade bending moment was the focus for most cases. The distributed pressure and inflow velocity at different radial locations along the blade were measured at some specific tasks. The velocity in the wake was only measured once in the test case of wind gust during project JOULE I (Snel and Schepers, 1995). As mentioned, the first problem of this test was that the generation of gust was too slow. The second problem was that the velocity was measured at fixed location in the wake of the rotor while the blade rotates.

In summary, the previous experimental work on dynamic inflow can be improved at the following two aspects.

- The unsteady airfoil aerodynamics is intertwined in the dynamic inflow problem in the past tested cases, where the blades of a full-scale wind turbine or a wind turbine model are pitching. New experimental set-ups should be designed to decouple the effects.

- Previous research mainly focused on the load and velocity measurement at the rotor plane, not in the wake. The wake determines the induction on the rotor and it also determines the inflow conditions for the downstream rotors, but the wake development under unsteady load is unknown and requires new research.

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