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The FEniCS Tutorial – Writing State-of-the-Art Finite Element Solvers in Minutes

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Contents

Preface

This book gives a concise and gentle introduction to finite element programming in Python based on the popular FEniCS software library. FEniCS can be programmed in both C++ and Python, but this tutorial focuses exclusively on Python programming, since this is the simplest and most effective approach for beginners. It will also deliver high performance since FEniCS automatically delegates compute-intensive tasks to C++ by help of code generation. After having digested the examples in this tutorial, the reader should be able to learn more from the FEniCS documentation, the numerous demo programs that come with the software, and the comprehensive FEniCS book *Automated Solution of Differential Equations by the Finite element Method* [21]. This tutorial is a further development of the opening chapter in [21].

We thank Johan Hake, Kent-Andre Mardal, and Kristian Valen-Sendstad for many helpful discussions during the preparation of the first version of this tutorial for the FEniCS book [21]. We are particularly thankful to Professor Douglas Arnold for very valuable feedback on early versions of the text. Øystein Sørensen pointed out a lot of typos and contributed with many helpful comments. Many errors and typos were also reported by Mauricio Angeles, Ida Drøsdal, Miroslav Kuchta, Hans Ekkehard Plesser, Marie Rognes, and Hans Joachim Scroll. Ekkehard Ellmann as well as two anonymous reviewers provided a series of suggestions and improvements.

Oslo, May 2016 *Hans Petter Langtangen, Anders Logg*

Watch out for shortcomings!

This book is still in an initial state so the reader is encouraged to send email to the authors on `logg@chalmers.se` about typos, errors, and suggestions for improvements.

Chapter 1
Preliminaries

1.1 The FEniCS Project

The FEniCS Project is a research and software project aiming at creating mathematical methods and software for automated computational mathematical modeling. This means creating easy, intuitive, efficient and flexible software for solving partial differential equations (PDEs) using finite element methods. FEniCS was initially created in 2003 and is developed in collaboration between researchers from a number of universities and research institutes around the world. For more information about FEniCS and the latest updates of the FEniCS software and this tutorial, visit the FEniCS web page at `http://fenicsproject.org`.

FEniCS consists of a number of building blocks (software components) that together form the FEniCS software: DOLFIN, FFC, FIAT, UFL, and a few others. FEniCS users rarely need to think about this internal organization of FEniCS, but since even casual users may sometimes encounter the names of various FEniCS components, we briefly list the components and their main roles in FEniCS. DOLFIN is the computational high-performance C++ backend of FEniCS. DOLFIN implements data structures such as meshes, function spaces and functions, compute-intensive algorithms such as finite element assembly and mesh refinement, and interfaces to linear algebra solvers and data structures such as PETSc. DOLFIN also implements the FEniCS problem-solving environment in both C++ and Python. FFC is the code generation engine of FEniCS (the form compiler), responsible for generating efficient C++ from high-level mathematical abstractions. FIAT is the finite element backend of FEniCS, responsible for generating finite element basis functions, and UFL implements the abstract mathematical language by which users may express variational problems.

1.2 What you will learn

The goal of this tutorial is introduce the concept of programming finite element solvers for PDEs and get you started with FEniCS through a series of simple examples that demonstrate

- how to define a PDE problem as a finite element variational problem,
- how to create (mesh) simple domains,
- how to deal with Dirichlet, Neumann, and Robin conditions,
- how to deal with variable coefficients,
- how to deal with domains built of several materials (subdomains),
- how to compute derived quantities like the flux vector field or a functional of the solution,
- how to quickly visualize the mesh, the solution, the flux, etc.,
- how to solve nonlinear PDEs,
- how to solve time-dependent PDEs,
- how to set parameters governing solution methods for linear systems,
- how to create domains of more complex shape.

1.3 Working with this tutorial

The mathematics of the illustrations is kept simple to better focus on FEniCS functionality and syntax. This means that we mostly use the Poisson equation and the time-dependent diffusion equation as model problems, often with input data adjusted such that we get a very simple solution that can be exactly reproduced by any standard finite element method over a uniform, structured mesh. This latter property greatly simplifies the verification of the implementations. Occasionally we insert a physically more relevant example to remind the reader that the step from solving a simple model problem to a challenging real-world problem is often quite easy with FEniCS.

Using FEniCS to solve PDEs may seem to require a thorough understanding of the abstract mathematical framework of the finite element method as well as expertise in Python programming. Nevertheless, it turns out that many users are able to pick up the fundamentals of finite elements *and* Python programming as they go along with this tutorial. Simply keep on reading and try out the examples. You will be amazed of how easy it is to solve PDEs with FEniCS!

1.4 Obtaining the software

Reading this tutorial obviously requires access to FEniCS. FEniCS is a complex software library, both in itself and due to its many dependencies to state-of-the-art open-source scientific software libraries. Manually building FEniCS and all its dependencies from source can thus be a daunting task. Even for an expert who knows exactly how to configure and build each component, a full build can literally take hours! In addition to the complexity of the software itself, there is an additional layer of complexity in how many different kinds of operating systems (GNU/Linux, Mac OS X, Windows) that may be running on a user's laptop or compute server, with different requirements for how to configure and build software.

For this reason, the FEniCS Project provides prebuilt packages to make the installation easy, fast and foolproof.

FEniCS download and installation

In this tutorial, we highlight the two main options for installing the FEniCS software: Docker containers and Ubuntu packages. While the Docker containers work on all operating systems, the Ubuntu packages only work on Ubuntu-based systems. For more installation options, such as building FEniCS from source, check out the official FEniCS installation instructions at `http://fenicsproject.org/download`.

1.4.1 Installation using Docker containers

A modern solution to the challenge of software installation on diverse software platforms is to use so-called *containers*. The FEniCS Project provides custom-made containers that are controlled, consistent and high-performance software environments for FEniCS programming. FEniCS containers work equally well[1] on all operating systems, including Linux, Mac and Windows.

To use FEniCS containers, you must first install the Docker platform. Docker installation is simple, just follow the instructions from the Docker web page. Once you have installed Docker, just copy the following line into a terminal window:

Terminal

[1] Running Docker containers on Mac and Windows involves a small performance overhead compared to running Docker containers on Linux. However, this performance penalty is typically small and is often compensated for by using the highly tuned and optimized version of FEniCS that comes with the official FEniCS containers, compared to building FEniCS and its dependencies from source on Mac or Windows.

```
Terminal> curl -s http://get.fenicsproject.org | sh
```

Mac and Windows users should make sure to run this command inside the Docker Quickstart Terminal!

The command above will install the program `fenicsproject` on your system. This command lets you easily create FEniCS sessions (containers) on your system:

Terminal

```
Terminal> fenicsproject run
```

This command has several useful options, such as easily switching between the latest release of FEniCS, the latest development version and many more. To learn more, type `fenicsproject help`.

Sharing files with FEniCS containers

When you run a FEniCS session using `fenicsproject run`, it will automatically share your current working directory (the directory from which you run the `fenicsproject` command) with the FEniCS session. When the FEniCS session starts, it will automatically enter into a directory named `shared` which will be identical with your current working directory on your host system. This means that you can easily edit files and write data inside the FEniCS session, and the files will be directly accessible on your host system. It is recommended that you edit your programs using your favorite editor (such as Emacs or Vim) on your host system and use the FEniCS session only to run your program(s).

1.4.2 Installation using Ubuntu packages

For users of Ubuntu GNU/Linux, FEniCS can also be installed easily via the standard Ubuntu package manager `apt-get`. Just copy the following lines into a terminal window:

Terminal

```
Terminal> sudo add-apt-repository ppa:fenics-packages/fenics
Terminal> sudo apt-get update
Terminal> sudo apt-get install fenics
Terminal> sudo apt-get dist-upgrade
```

This will add the FEniCS package archive (PPA) to your Ubuntu computer's list of software sources and then install FEniCS. This step will also automatically install packages for dependencies of FEniCS.

Watch out for old packages!

In addition to being available from the FEniCS PPA, the FEniCS software is also part of the official Ubuntu repositories. However, depending on which release of Ubuntu you are running, and when this release was created in relation to the latest FEniCS release, the official Ubuntu repositories might contain an outdated version of FEniCS. For this reason, it is better to install from the FEniCS PPA.

1.4.3 Testing your installation

Once you have installed FEniCS, you should make a quick test to see that your installation works properly. To do this, type the following command in a FEniCS-enabled[2] terminal:

Terminal

```
Terminal> python -c 'import fenics'
```

If all goes well, you should be able to run this command without any error message (or any other output).

1.5 Obtaining the tutorial examples

In this tutorial, you will learn finite element and FEniCS programming through a number of example programs that demonstrate both how to solve particular PDEs using the finite element method, how to program solvers in FEniCS, and how to create well-designed Python codes that can later be extended to solve more complex problems. All example programs are available from the web page of this book at `http://fenicsproject.org/tutorial`. The programs as well as the source code for this text can also be accessed directly from the Git repository for this book.

[2] For users of FEniCS containers, this means first running the command `fenicsproject run`.

1.6 Background knowledge

1.6.1 Programming in Python

While you can likely pick up basic Python programming by working through the examples in this tutorial, you may want to have some additional material on the language. A natural starting point for beginners is the classical *Python Tutorial* [10], or a tutorial geared towards scientific computing [18]. In the latter, you will also find lots of pointers to other tutorials for scientific computing in Python. Among ordinary books we recommend the general introduction *Dive into Python* [22] as well as texts that focus on scientific computing with Python [13–17].

Python versions

Python comes in two versions, 2 and 3, and these are not compatible. FEniCS has a code base that runs under both versions. All the programs in this tutorial are also developed such that they can be run under both Python 2 and 3. Programs that need to print must then start with

```
from __future__ import print_function
```

to enable the `print` function from Python 3 in Python 2. All use of `print` in the programs in this tutorial consists of function calls, like `print('a:', a)`. Almost all other constructions are of a form that looks the same in Python 2 and 3.

To start a FEniCS Python 3 session, users of FEniCS containers should run the command `fenicsproject run stable-py3`.

1.6.2 The finite element method

There are a large number of books on the finite element method. The books typically fall in either of two categories: the abstract mathematical version of the method and the engineering "structural analysis" formulation. FEniCS builds heavily on concepts from the abstract mathematical exposition. The first author has in development a book that explains all details of the finite element method in an intuitive way, though with the abstract mathematical formulations that FEniCS employ.

The finite element text by Larson and Bengzon [20] is our recommended introduction to the finite element method, with a mathematical notation that goes well with FEniCS. An easy-to-read book, which also provides a

good general background for using FEniCS, is Gockenbach [11]. The book by Donea and Huerta [7] has a similar style, but aims at readers with interest in fluid flow problems. Hughes [12] is also recommended, especially for those interested in solid mechanics and heat transfer applications.

Readers with a background in the engineering "structural analysis" version of the finite element method may find Bickford [3] as an attractive bridge over to the abstract mathematical formulation that FEniCS builds upon. Those who have a weak background in differential equations in general should consult a more fundamental book, and Eriksson *et al* [8] is a very good choice. On the other hand, FEniCS users with a strong background in mathematics and interest in the mathematical properties of the finite element method, will appreciate the texts by Brenner and Scott [5], Braess [4], Ern and Guermond [9], Quarteroni and Valli [23], or Ciarlet [6].

Chapter 2
Fundamentals: Solving the Poisson equation

The goal of this chapter is to show how the Poisson equation, the most basic of all PDEs, can be quickly solved with a few lines of FEniCS code. We introduce the most fundamental FEniCS objects such as `Mesh`, `Function`, `FunctionSpace`, `TrialFunction`, and `TestFunction`, and learn how to write a basic PDE solver, including the specification of the mathematical variational problem, applying boundary conditions, calling the FEniCS solver, and plotting the solution.

2.1 Mathematical problem formulation

Let us start by writing a "Hello, World!" program. In the world of PDEs, this must be a program that solves the Poisson equation:

$$-\nabla^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \text{ in } \Omega, \tag{2.1}$$

$$u(\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \text{ on } \partial\Omega\,. \tag{2.2}$$

Here, $u = u(\boldsymbol{x})$ is the unknown function, $f = f(\boldsymbol{x})$ is a prescribed function, ∇^2 is the Laplace operator (also often written as Δ), Ω is the spatial domain, and $\partial\Omega$ is the boundary of Ω. A stationary PDE like this, together with a complete set of boundary conditions, constitute a *boundary-value problem*, which must be precisely stated before it makes sense to start solving it with FEniCS.

In two space dimensions with coordinates x and y, we can write out the Poisson equation as

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y)\,. \tag{2.3}$$

The unknown u is now a function of two variables, $u = u(x,y)$, defined over a two-dimensional domain Ω.

The Poisson equation arises in numerous physical contexts, including heat conduction, electrostatics, diffusion of substances, twisting of elastic rods, inviscid fluid flow, and water waves. Moreover, the equation appears in numerical splitting strategies of more complicated systems of PDEs, in particular the Navier–Stokes equations.

Solving a PDE such as the Poisson equation in FEniCS consists of the following steps:

1. Identify the computational domain (Ω), the PDE, its boundary conditions, and source terms (f).
2. Reformulate the PDE as a finite element variational problem.
3. Write a Python program which defines the computational domain, the variational problem, the boundary conditions, and source terms, using the corresponding FEniCS abstractions.
4. Call FEniCS to solve the PDE and, optionally, extend the program to compute derived quantities such as fluxes and averages, and visualize the results.

We shall now go through steps 2–4 in detail. The key feature of FEniCS is that steps 3 and 4 result in fairly short code, while most other software frameworks for PDEs require much more code and more technically difficult programming.

2.1.1 Finite element variational formulation

FEniCS is based on the finite element method, which is a general and efficient mathematical machinery for numerical solution of PDEs. The starting point for the finite element methods is a PDE expressed in *variational form*. Readers who are not familiar with variational problems will get a brief introduction to the topic in this tutorial, but getting and reading a proper book on the finite element method in addition is encouraged. Section 1.6.2 contains a list of some suitable books.

The basic recipe for turning a PDE into a variational problem is to multiply the PDE by a function v, integrate the resulting equation over the domain Ω, and perform integration by parts of terms with second-order derivatives. The function v which multiplies the PDE is called a *test function*. The unknown function u to be approximated is referred to as a *trial function*. The terms test and trial function are used in FEniCS programs too. Suitable function spaces must be specified for the test and trial functions. For standard PDEs arising in physics and mechanics such spaces are well known.

In the present case, we first multiply the Poisson equation by the test function v and integrate over Ω:

$$-\int_\Omega (\nabla^2 u) v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x\,. \tag{2.4}$$

We then apply integration by parts to the integrand with second-order derivatives. We find that

$$-\int_\Omega (\nabla^2 u) v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s, \tag{2.5}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot n$ is the derivative of u in the outward normal direction n on the boundary. The test function v is required to vanish on the parts of the boundary where the solution u is known, which in the present problem implies that $v = 0$ on the whole boundary $\partial\Omega$. The second term on the right-hand side of (2.5) therefore vanishes. From (2.4) and (2.5) it follows that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x\,. \tag{2.6}$$

If we require that this equation holds for all test functions v in some suitable space $\hat{V}$, the so-called *test space*, we obtain a well-defined mathematical problem that uniquely determines the solution u which lies in some (possibly different) function space V, the so-called *trial space*. We refer to (2.6) as the *weak form* or *variational form* of the original boundary-value problem (2.1)–(2.2).

The proper statement of our variational problem now goes as follows: Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in \hat{V}\,. \tag{2.7}$$

The trial and test spaces V and $\hat{V}$ are in the present problem defined as

$$V = \{v \in H^1(\Omega) : v = u_0 \text{ on } \partial\Omega\},$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}\,.$$

In short, $H^1(\Omega)$ is the mathematically well-known Sobolev space containing functions v such that v^2 and $|\nabla v|^2$ have finite integrals over Ω (essentially meaning that the functions are continuous). The solution of the underlying PDE must lie in a function space where also the derivatives are continuous, but the Sobolev space $H^1(\Omega)$ allows functions with discontinuous derivatives. This weaker continuity requirement of u in the variational statement (2.7), as a result of the integration by parts, has great practical consequences when it comes to constructing finite element function spaces. In particular, it allows the use of piecewise polynomial function spaces; i.e., function spaces constructed by stitching together polynomial functions on simple domains such as intervals, triangles, or tetrahedrons.

The variational problem (2.7) is a *continuous problem*: it defines the solution u in the infinite-dimensional function space V. The finite element method for the Poisson equation finds an approximate solution of the variational problem (2.7) by replacing the infinite-dimensional function spaces V and $\hat{V}$ by *discrete* (finite-dimensional) trial and test spaces $V_h \subset V$ and $\hat{V}_h \subset \hat{V}$. The discrete variational problem reads: Find $u_h \in V_h \subset V$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in \hat{V}_h \subset \hat{V}\,. \tag{2.8}$$

This variational problem, together with a suitable definition of the function spaces V_h and $\hat{V}_h$, uniquely defines our approximate numerical solution of Poisson's equation (2.1). The mathematical framework may seem complicated at first glance, but the good news is the finite element variational problem (2.8) looks the same as the continuous variational problem (2.7), and FEniCS can automatically solve variational problems like (2.8)!

What we mean by the notation u and V

The mathematics literature on variational problems writes u_h for the solution of the discrete problem and u for the solution of the continuous problem. To obtain (almost) a one-to-one relationship between the mathematical formulation of a problem and the corresponding FEniCS program, we shall drop the subscript $_h$ and use u for the solution of the discrete problem and u_e for the exact solution of the continuous problem, *if* we need to explicitly distinguish between the two. Similarly, we will let V denote the discrete finite element function space in which we seek our solution.

2.1.2 Abstract finite element variational formulation

It turns out to be convenient to introduce the following canonical notation for variational problems:

$$a(u, v) = L(v)\,. \tag{2.9}$$

For the Poisson equation, we have:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \tag{2.10}$$

$$L(v) = \int_\Omega f v \,\mathrm{d}x\,. \tag{2.11}$$

From the mathematics literature, $a(u,v)$ is known as a *bilinear form* and $L(v)$ as a *linear form*. We shall in every linear problem we solve identify the terms with the unknown u and collect them in $a(u,v)$, and similarly collect all terms with only known functions in $L(v)$. The formulas for a and L are then coded directly in the program.

FEniCS provides all the necessary mathematical notation needed to express the variational problem $a(u,v) = L(v)$. To solve a linear PDE in FEniCS, such as the Poisson equation, a user thus needs to perform only two steps:

- Express the PDE as a (discrete) variational problem: find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in \hat{V}$.
- Choose the finite element spaces V and $\hat{V}$ by specifying the domain (the mesh) and the type of function space (polynomial degree and type).

2.1.3 Choosing a test problem

The Poisson equation (2.1) has so far featured a general domain Ω and general functions u_0 and f. For our first implementation, we must decide on specific choices of Ω, u_0, and f. It will be wise to construct a specific problem where we can easily check that the computed solution is correct. Solutions that are lower-order polynomials are primary candidates. Standard finite element function spaces of degree r will exactly reproduce polynomials of degree r. And piecewise linear elements ($r = 1$) are able to exactly reproduce a quadratic polynomial on a uniformly partitioned mesh. This important result can be used to verify our implementation. We just manufacture some quadratic function in 2D as the exact solution, say

$$u_{\mathrm{e}}(x,y) = 1 + x^2 + 2y^2\,. \tag{2.12}$$

By inserting (2.12) into the Poisson equation (2.1), we find that $u_{\mathrm{e}}(x,y)$ is a solution if

$$f(x,y) = -6, \quad u_0(x,y) = u_{\mathrm{e}}(x,y) = 1 + x^2 + 2y^2,$$

regardless of the shape of the domain as long as u_{e} is prescribed along the boundary. We choose here, for simplicity, the domain to be the unit square,

$$\Omega = [0,1] \times [0,1]\,.$$

This simple but very powerful method for constructing test problems is called the *method of manufactured solutions*: pick a simple expression for the exact solution, plug it into the equation to obtain the right-hand side (source term f), then solve the equation with this right-hand side and try to reproduce the exact solution.

Tip: Try to verify your code with exact numerical solutions!

A common approach to testing the implementation of a numerical method is to compare the numerical solution with an exact analytical solution of the test problem and conclude that the program works if the error is "small enough". Unfortunately, it is impossible to tell if an error of size 10^{-5} on a 20×20 mesh of linear elements is the expected (in)accuracy of the numerical approximation or if the error also contains the effect of a bug in the code. All we usually know about the numerical error is its *asymptotic properties*, for instance that it is proportional to h^2 if h is the size of a cell in the mesh. Then we can compare the error on meshes with different h values to see if the asymptotic behavior is correct. This is a very powerful verification technique and is explained in detail in Section **??**. However, if we have a test problem for which we know that there should be no approximation errors, we know that the analytical solution of the PDE problem should be reproduced to machine precision by the program. That is why we emphasize this kind of test problems throughout this tutorial. Typically, elements of degree r can reproduce polynomials of degree r exactly, so this is the starting point for constructing a solution without numerical approximation errors.

2.1.4 FEniCS implementation

A FEniCS program for solving our test problem for the Poisson equation in 2D with the given choices of u_0, f, and Ω may look as follows:

```
from fenics import *

# Create mesh and define function space
mesh = UnitSquareMesh(8, 8)
V = FunctionSpace(mesh, 'P', 1)

# Define boundary conditions
u0 = Expression('1 + x[0]*x[0] + 2*x[1]*x[1]', degree=2)

def u0_boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, u0, u0_boundary)

# Define variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(-6.0)
```

```
a = dot(grad(u), grad(v))*dx
L = f*v*dx

# Compute solution
u = Function(V)
solve(a == L, u, bc)

# Plot solution on the screen
u.rename('u', 'solution')
plot(u)
plot(mesh)

# Dump solution to file in VTK format
vtkfile = File('poisson.pvd')
vtkfile << u

# Compute error in L2 norm
error_L2norm = errornorm(u0, u, 'L2')

# Compute maximum error at vertices
vertex_values_u0 = u0.compute_vertex_values(mesh)
vertex_values_u  = u.compute_vertex_values(mesh)
import numpy as np
error_vertices = np.max(np.abs(vertex_values_u0 - vertex_values_u))

# Print errors
print('error_L2norm   =', error_L2norm)
print('error_vertices =', error_vertices)

# Hold plot
interactive()
```

The complete code can be found in the file `ft01_poisson_flat.py`.

2.1.5 Running the program

The FEniCS program must be available in a plain text file, written with a text editor such as Atom, Sublime Text, Emacs, Vim, or similar.

There are several ways to run a Python program like `ft01_poisson_flat.py`:

- Use a terminal window
- Use an intergrated development environment (IDE), e.g., Spyder
- Use a Jupyter notebook

Terminal window. Open a terminal window, move to the directory containing the program and type the following command:

Terminal

```
Terminal> python ft01_poisson_flat.py
```

Note that this command must be run in a FEniCS-enabled terminal. For users of the FEniCS Docker containers, this means that you must type this command after you have started a FEniCS session using `fenicsproject run`.

When running the above command, FEniCS will run the program to compute the approximate solution u. The approximate solution u will be compared to the exact solution u_e and the error in the maximum norm will be printed. Since we know that our approximate solution should reproduced the exact solution to within machine precision, this error should be small, something on the order of 10^{-15}.

AL 1: Add text here discussing what to expect in terms of plotting. Perhaps we have seamless notebook plotting working soon...

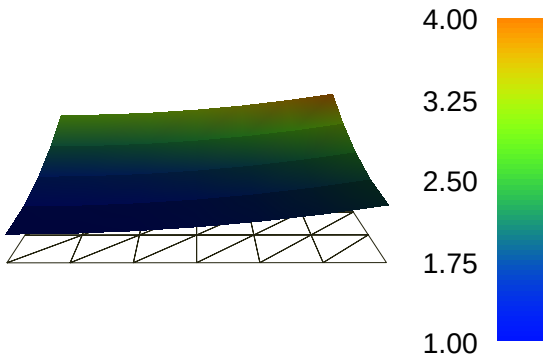


Fig. 2.1 Plot of the solution in the first FEniCS example.

Spyder. Many prefer to work in an integrated development environment where there is an editor for programming, a window for executing code, a window for inspecting objects, etc. The Spyder tool comes with all major Python installations. Just open the file `ft01_poisson_flat.py` and press the play button to run it. We refer to the Spyder tutorial to learn more about working in the Spyder environment. Spyder is highly recommended if you are used to working in the *graphical* MATLAB environment.

Jupyter notebooks. Notebooks make it possible to mix text and executable code in the same document, but you can also just use it to run programs in a web browser. Start `jupyter notebook` from a terminal window, find the **New** pulldown menu in the upper right part of the GUI, choose a new notebook in Python 2 or 3, write `%load ft01_poisson_flat.py` in the blank cell of this notebook, then write Shift+Enter to execute the cell. The file

`ft01_poisson_flat.py` will the be loaded into the notebook. Re-execute the cell (Shift+Enter) to run the program. You may divide the entire program into several cells to examine intermediate results: place the cursor where you want to split the cell and choose **Edit - Split Cell**.

hpl 2: Need to describe this with more care. The first program seems to have some problems with printing the error to the notebook unless we drop the plot commands. Anyway, there should be in-browser plot commands.

2.1.6 Dissection of the program

We shall now dissect this FEniCS program in detail. The program is written in the Python programming language. You may either take a quick look at the official Python tutorial to pick up the basics of Python if you are unfamiliar with the language, or you may learn enough Python as you go along with the examples in the present tutorial. The latter strategy has proven to work for many newcomers to FEniCS. This is because both the amount of abstract mathematical formalism and the amount of Python expertise that is actually needed to be productive with FEniCS is quite limited. And Python is an easy-to-learn language that you will certainly come to love and use far beyond FEniCS programming. Section 1.6.1 lists some relevant Python books.

The listed FEniCS program defines a finite element mesh, a finite element function space V on this mesh, boundary conditions for u (the function u_0), and the bilinear and linear forms $a(u,v)$ and $L(v)$. Thereafter, the unknown trial function u is computed. Then we can compare the numerical and exact solution as well as visualize the computed solution u.

The important first line. The first line in the program,

```
from fenics import *
```

imports the key classes `UnitSquareMesh`, `FunctionSpace`, `Function`, and so forth, from the FEniCS library. All FEniCS programs for solving PDEs by the finite element method normally start with this line.

Generating simple meshes. The statement

```
mesh = UnitSquareMesh(8, 8)
```

defines a uniform finite element mesh over the unit square $[0,1]\times[0,1]$. The mesh consists of *cells*, which in 2D are triangles with straight sides. The parameters 8 and 8 specify that the square should be divided into 8×8 rectangles, each divided into a pair of triangles. The total number of triangles (cells) thus becomes 128. The total number of vertices in the mesh is $9\cdot 9 = 81$. In later chapters, you will learn how to generate more complex meshes.

hpl 3: Note that plot was made by the old partitioning 6×4. Probably no issue.

Defining the finite element function space. Having a mesh, we can define a finite element function space `V` over this mesh:

```
V = FunctionSpace(mesh, 'P', 1)
```

The second argument `'P'` specifies the type of element, while the third argument is the degree of the basis functions of the element. The type of element is here "P", implying the standard Lagrange family of elements. You may also use `'Lagrange'` to specify this type of element. FEniCS supports all simplex element families and the notation defined in the Periodic Table of the Finite Elements [2].

The third argument `1` specifies the degree of the finite element. In this case, the standard P_1 linear Lagrange element, which is a triangle with nodes at the three vertices. Some finite element practitioners refer to this element as the "linear triangle". The computed solution u will be continuous and linearly varying in x and y over each cell in the mesh. Higher-degree polynomial approximations over each cell are trivially obtained by increasing the third parameter to `FunctionSpace`, which will then generate function spaces of type P_2, P_3, and so forth. Changing the second parameter to `'DP'` creates a function space for discontinuous Galerkin methods.

Defining the trial and test functions. In mathematics, we distinguish between the trial and test spaces V and $\hat{V}$. The only difference in the present problem is the boundary conditions. In FEniCS we do not specify the boundary conditions as part of the function space, so it is sufficient to work with one common space `V` for the and trial and test functions in the program:

```
u = TrialFunction(V)
v = TestFunction(V)
```

Defining the boundary and the boundary conditions. The next step is to specify the boundary condition: $u = u_0$ on $\partial\Omega$. This is done by

```
bc = DirichletBC(V, u0, u0_boundary)
```

where `u0` is an expression defining the solution values on the boundary, and `u0_boundary` is a function (or object) defining which points belong to the boundary.

Boundary conditions of the type $u = u_0$ are known as *Dirichlet conditions*. For the present finite element method for the Poisson problem, they are also called *essential boundary conditions*, as they need to be imposed explicitly as part of the trial space (in contrast to being defined implicitly as part of the variational formulation). Naturally, the FEniCS class used to define Dirichlet boundary conditions is named `DirichletBC`.

The variable `u0` refers to an `Expression` object, which is used to represent a mathematical function. The typical construction is

```
u0 = Expression(formula, degree=1)
```

where `formula` is a string containing the mathematical expression. This formula is written with C++ syntax. The expression is automatically turned into an efficient, compiled C++ function. The second argument `degree` is a parameter that specifies how the expression should be treated in computations. FEniCS will interpolate the expression into some finite element space. It is usually a good choice to interpolate expressions into the same space V that is used for the trial and test functions, but in certain cases, one may want to use a more accurate (higher degree) representation of expressions.

The expression may depend on the variables `x[0]` and `x[1]` corresponding to the x and y coordinates. In 3D, the expression may also depend on the variable `x[2]` corresponding to the z coordinate. With our choice of $u_0(x,y) = 1 + x^2 + 2y^2$, the formula string can be written as `1 + x[0]*x[0] + 2*x[1]*x[1]`:

```
u0 = Expression('1 + x[0]*x[0] + 2*x[1]*x[1]', degree=1)
```

String expressions must have valid C++ syntax!

The string argument to an `Expression` object must obey C++ syntax. Most Python syntax for mathematical expressions are also valid C++ syntax, but power expressions make an exception: `p**a` must be written as `pow(p,a)` in C++ (this is also an alternative Python syntax). The following mathematical functions can be used directly in C++ expressions when defining `Expression` objects: `cos`, `sin`, `tan`, `acos`, `asin`, `atan`, `atan2`, `cosh`, `sinh`, `tanh`, `exp`, `frexp`, `ldexp`, `log`, `log10`, `modf`, `pow`, `sqrt`, `ceil`, `fabs`, `floor`, and `fmod`. Moreover, the number π is available as the symbol `pi`. All the listed functions are taken from the `cmath` C++ header file, and one may hence consult the documentation of `cmath` for more information on the various functions.

If/else tests are possible using the C syntax for inline branching. The function

$$f(x,y) = \begin{cases} x^2, & x, y \geq 0 \\ 2, & \text{otherwise} \end{cases}$$

is implemented as

```
f = Expression('x[0] >= 0 && x[1] >= 0? pow(x[0], 2) : 2', degree=1)
```

Parameters in expression strings are allowed, but must be initialized via keyword arguments when creating the `Expression` object. For example, the function $f(x) = e^{-\kappa\pi^2 t}\sin(\pi k x)$ can be coded as

```
f = Expression('exp(-kappa*pow(pi,2)*t)*sin(pi*k*x[0])', degree=1,
               kappa=1.0, t=0, k=4)
```

At any time, parameters can be updated:

```
f.t += dt
f.k = 10
```

The function `u0_boundary` specifies which points that belong to the part of the boundary where the boundary condition should be applied:

```
def u0_boundary(x, on_boundary):
    return on_boundary
```

A function like `u0_boundary` for marking the boundary must return a boolean value: `True` if the given point `x` lies on the Dirichlet boundary and `False` otherwise. The argument `on_boundary` is `True` if `x` is on the physical boundary of the mesh, so in the present case, where we are supposed to return `True` for all points on the boundary, we can just return the supplied value of `on_boundary`. The `u0_boundary` function will be called for every discrete point in the mesh, which allows us to have boundaries where u are known also inside the domain, if desired.

One way to think about the specification of boundaries in FEniCS is that FEniCS will ask you (or rather the function `u0_boundary` which you have implemented) whether or not a specific point `x` is part of the boundary. FEniCS already knows whether the point belongs to the *actual* boundary (the mathematical boundary of the domain) and kindly shares this information with you in the variable `on_boundary`. You may choose to use this information (as we do here), or ignore it completely.

The argument `on_boundary` may also be omitted, but in that case we need to test on the value of the coordinates in `x`:

```
def u0_boundary(x):
    return x[0] == 0 or x[1] == 0 or x[0] == 1 or x[1] == 1
```

Comparing floating-point values using an exact match test with `==` is not good programming practice, because small round-off errors in the computations of the `x` values could make a test `x[0] == 1` become false even though `x` lies on the boundary. A better test is to check for equality with a tolerance, either explicitly

```
def u0_boundary(x):
    return abs(x[0]) < tol or abs(x[1]) < tol \
        or abs((x[0] - 1) < tol or abs(x[1] - 1) < tol
```

or with the `near` command in FEniCS:

```
def u0_boundary(x):
    return near(x[0], 0, tol) or near(x[1], 0, tol) \
        or near(x[0], 1, tol) or near(x[1], 1, tol)
```

Defining the source term. Before defining the bilinear and linear forms $a(u, v)$ and $L(v)$ we have to specify the source term f:

```
f = Expression('-6', degree=1)
```

When f is constant over the domain, `f` can be more efficiently represented as a `Constant`:

```
f = Constant(-6)
```

Defining the variational problem. We now have all the ingredients we need to define the variational problem:

```
a = dot(grad(u), grad(v))*dx
L = f*v*dx
```

In essence, these two lines specify the PDE to be solved. Note the very close correspondence between the Python syntax and the mathematical formulas $\nabla u \cdot \nabla v \, \mathrm{d}x$ and $fv \, \mathrm{d}x$. This is a key strength of FEniCS: the formulas in the variational formulation translate directly to very similar Python code, a feature that makes it easy to specify and solve complicated PDE problems. The language used to express weak forms is called UFL (Unified Form Language) [1, 21] and is an integral part of FEniCS.

Forming and solving the linear system. Having defined the finite element variational problem and boundary condition, we can now ask FEniCS to compute the solution:

```
u = Function(V)
solve(a == L, u, bc)
```

Note that we first defined the variable `u` as a `TrialFunction` and used it to represent the unknown in the form `a`. Thereafter, we redefined `u` to be a `Function` object representing the solution; i.e., the computed finite element function u. This redefinition of the variable `u` is possible in Python and often done in FEniCS applications for linear problems. The two types of objects that `u` refers to are equal from a mathematical point of view, and hence it is natural to use the same variable name for both objects.

Plotting the solution. Once the solution has been computed, it can be visualized by the `plot()` command:

```
plot(u)
plot(mesh)
interactive()
```

Clicking on `Help` or typing `h` in the plot windows brings up a list of commands. For example, typing `m` brings up the mesh. With the left, middle, and right mouse buttons you can rotate, translate, and zoom (respectively) the plotted surface to better examine what the solution looks like. You must click `Ctrl+q` to kill the plot window and continue execution beyond the command

interactive()‘. In the example program, we have therefore placed the call to `interactive()` at the very end. Alternatively, one may use the command `plot(u, interactive=True)` which again means you can interact with the plot window and that execution will be halted until the plot window is closed.

Figure 2.1 displays the resulting u function.

Exporting and post-processing the solution. It is also possible to dump the computed solution to file for post-processing, e.g., in VTK format:

```
vtkfile = File('poisson.pvd')
vtkfile << u
```

The `poisson.pvd` file can now be loaded into any front-end to VTK, in particular ParaView or VisIt. The `plot()` function is intended for quick examination of the solution during program development. More in-depth visual investigations of finite element solutions will normally benefit from using highly professional tools such as ParaView and VisIt.

Prior to plotting and storing solutions to file it is wise to give `u` a proper name by `u.rename('u', 'solution')`. Then `u` will be used as name in plots (rather than the more cryptic default names like `f_7`).

Once the solution has been stored to file, it can be opened in Paraview by choosing **File - Open**. Find the file `poisson.pvd`, and click the green **Apply** button to the left in the GUI. A 2D color plot of $u(x, y)$ is then shown. You can save the figure to file by **File - Export Scene...** and choosing a suitable filename. For more information about how to install and use Paraview, see the `http://www.paraview.org/`.

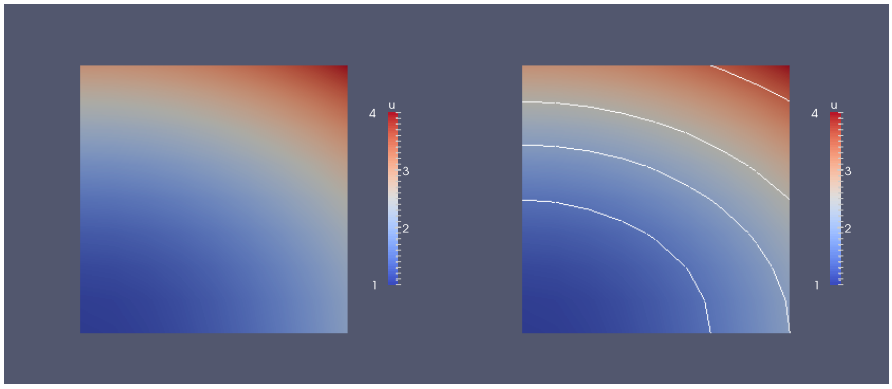


Fig. 2.2 Visualization of test problem in ParaView, with contour lines added in the right plot.

Computing the error. Finally, we compute the error to check the accuracy of the solution. We do this by comparing the finite element solution `u` with the exact solution `u0`, which in this example happens to be the same as the

`Expression` used to set the boundary conditions. We compute the error in two different ways. First, we compute the L^2 norm of the error, defined by

$$E = \sqrt{\int_\Omega (u_0 - u)^2 \, \mathrm{d}x}.$$

Since the exact solution is quadratic and the finite element solution is piecewise linear, this error will be nonzero. To compute this error in FEniCS, we simply write

```
error_L2norm = errornorm(u0, u, 'L2')
```

The `errornorm()` function can also compute other error norms such as the H^1 norm. Type `pydoc fenics.errornorm` in a terminal window for details.

We also compute the maximum value of the error at all the vertices of the finite element mesh. As mentioned above, we expect this error to be zero to within machine precision for this particular example. To compute the error at the vertices, we first ask FEniCS to compute the value of both `u0` and `u` at all vertices, and then subtract the results:

```
vertex_values_u0 = u0.compute_vertex_values(mesh)
vertex_values_u  = u.compute_vertex_values(mesh)
import numpy as np
error_vertices = np.max(np.abs(vertex_values_u0 - vertex_values_u))
```

We have here used maximum and absolute value functions from `numpy`, because these are much more efficient for large arrays (a factor of 30) than Python's built-n `max` and `abs` functions.

How to check that the error vanishes?

With inexact arithmetics, as we always have on a computer, the maximum error at the vertices is not zero, but should be a small number. The machine precision is about 10^{-16}, but in finite element calculations, rounding errors of this size may accumulate, to produce an error larger than 10^{-16}. Experiments show that increasing the number of elements and increasing the degree of the finite element polynomials increases the error. For a mesh with $2 \times (20 \times 20)$ cubic Lagrange elements (degree 3) the error is about $2 \cdot 10^{-12}$, while for 81 linear elements the error is about $2 \cdot 10^{-15}$.

2.1.7 Degrees of freedom and vertex values

A finite element function like u is expressed as a linear combination of basis functions ϕ_j, spanning the space V:

$$u = \sum_{j=1}^{N} U_j \phi_j \,. \tag{2.13}$$

By writing `solve(a == L, u, bc)` in the program, a linear system will be formed from a and L, and this system is solved for the $U_1, \ldots, U_N$ values. The $U_1, \ldots, U_N$ values are known as the *degrees of freedom* ("dofs") or *nodal values* of u. For Lagrange elements (and many other element types) U_j is simply the value of u at the node with global number j. The location of the nodes and cell vertices coincide for linear Lagrange elements, while for higher-order elements there are additional nodes associated with the facets, edges and sometimes also the interior of cells.

Having `u` represented as a `Function` object, we can either evaluate `u(x)` at any point `x` in the mesh (expensive operation!), or we can grab all the degrees of freedom values U directly by

```
u_nodal_values = u.vector()
```

The result is a `Vector` object, which is basically an encapsulation of the vector object used in the linear algebra package that is used to solve the linear system arising from the variational problem. Since we program in Python it is convenient to convert the `Vector` object to a standard `numpy` array for further processing:

```
u_array = u_nodal_values.array()
```

With `numpy` arrays we can write MATLAB-like code to analyze the data. Indexing is done with square brackets: `u_array[j]`, where the index `j` always starts at `0`. If the solution is computed with piecewise linear Lagrange elements (P_1), then the size of the array `u_array` is equal to the number of vertices, and each `u[j]` is the value at some vertex in the mesh. However, the degrees of freedom are not necessarily numbered in the same way as the vertices of the mesh, see Section **??** for details. If we therefore want to know the values at the vertices, we need to call the function `u.compute_vertex_values()`. This function returns the values at all the vertices of the mesh as a `numpy` array with the same numbering as for the vertices of the mesh, for example:

```
u_vertex_values = u.compute_vertex_values()
```

Note that `u_array` and `u_vertex_values` are arrays of the same length and containing the same values, albeit in different order.

2.2 Deflection of a membrane

AL 4: I AM HERE

The previous problem and code targeted a simple test problem where we can easily verify the implementation. Now we turn the attention to a more physically relevant problem, in a non-trivial geometry, and that results in solutions of somewhat more exciting shape.

We want to compute the deflection $D(x,y)$ of a two-dimensional, circular membrane, subject to a load p over the membrane. The appropriate PDE model is

$$-T\nabla^2 D = p(x,y) \quad \text{in } \Omega = \{(x,y)\,|\,x^2+y^2 \leq R\}\,. \tag{2.14}$$

Here, T is the tension in the membrane (constant), and p is the external pressure load. The boundary of the membrane has no deflection, implying $D=0$ as boundary condition. A localized load can be modeled as a Gaussian function:

$$p(x,y) = \frac{A}{2\pi\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-x_0}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{y-y_0}{\sigma}\right)^2\right). \tag{2.15}$$

The parameter A is the amplitude of the pressure, (x_0,y_0) the localization of the maximum point of the load, and σ the "width" of p.

2.2.1 Scaling

The localization of the pressure, (x_0,y_0), is for simplicity set to $(0,R_0)$. There are many physical parameters in this problem, and we can benefit from grouping them by means of scaling. Let us introduce dimensionless coordinates $\bar{x}=x/R$, $\bar{y}=y/R$, and a dimensionless deflection $w=D/D_c$, where D_c is a characteristic size of the deflection. Introducing $\bar{R}_0=R_0/R$, we get

$$\frac{\partial^2 w}{\partial \bar{x}^2} + \frac{\partial^2 w}{\partial \bar{y}^2} = \alpha\exp\left(-\beta^2(\bar{x}^2+(\bar{y}-\bar{R}_0)^2)\right) \text{ for } \bar{x}^2+\bar{y}^2<1,$$

where

$$\alpha = \frac{R^2A}{2\pi T D_c\sigma}, \quad \beta = \frac{R}{\sqrt{2}\sigma}\,.$$

With an appropriate scaling, $\bar{w}$ and its derivatives are of size unity, so the left-hand side of the scaled PDE is about unity in size, while the right-hand side has α as its characteristic size. This suggest choosing α to be unity, or around unit. We shall in particular choose $\alpha=4$. With this value, the solution is $w(\bar{x},\bar{y}) = 1-\bar{x}^2-\bar{y}^2$. (One can also find the analytical solution in scaled coordinates and show that the maximum deflection $D(0,0)$ is D_c if we choose $\alpha=4$ to determine D_c.) With $D_c = AR^2/(8\pi\sigma T)$ and dropping the bars we get the scaled problem

$$\nabla^2 w = 4\exp\left(-\beta^2(x^2+(y-R_0)^2)\right), \tag{2.16}$$

to be solved over the unit circle with $w = 0$ on the boundary. Now there are only two parameters to vary: the dimensionless extent of the pressure, β, and the localization of the pressure peak, $R_0 \in [0,1]$. As $\beta \to 0$, we have a special case with solution $w = 1 - x^2 - y^2$.

Given a computed w, the physical deflection is given by

$$D = \frac{AR^2}{8\pi\sigma T} w\,.$$

Just a few modifications are necessary in our previous program to solve this new problem.

2.2.2 Defining the mesh

A mesh over the unit circle can be created by the `mshr` tool in FEniCS:

```
from mshr import *
domain = Circle(Point(0.0, 0.0), 1.0)
n = 20
mesh = generate_mesh(domain, n)
plot(mesh, interactive=True)
```

The `Circle` shape from `mshr` takes the center and radius of the circle as the two first arguments, while `n` is the resolution, here the suggested number of cells per radius.

2.2.3 Defining the load

The right-hand side pressure function is represented by an `Expression` object. There are two physical parameters in the formula for f that enter the expression string and these parameters must have their values set by keyword arguments:

```
beta = 8
R0 = 0.6
p = Expression(
    '4*exp(-pow(beta,2)*(pow(x[0], 2) + pow(x[1]-R0, 2)))',
    beta=beta, R0=R0)
```

The coordinates in `Expression` objects *must* be a vector with indices 0, 1, and 2, and with the name `x`. Otherwise we are free to introduce names of parameters as long as these are given default values by keyword arguments.

All the parameters initialized by keyword arguments can at any time have their values modified. For example, we may set

```
p.beta = 12
p.R0 = 0.3
```

2.2.4 Variational form

We may introduce `w` instead of `u` as primary unknown and `p` instead of `f` as right-hand side function:

```
w = TrialFunction(V)
v = TestFunction(V)
a = dot(grad(w), grad(v))*dx
L = p*v*dx

w = Function(V)
solve(a == L, w, bc)
```

2.2.5 Visualization

It would be of interest to visualize p along with w so that we can examine the pressure force and the membrane's response. We must then transform the formula (`Expression`) to a finite element function (`Function`). The most natural approach is to construct a finite element function whose degrees of freedom are calculated from p. That is, we interpolate p:

```
p = interpolate(p, V)
```

Note that the assignment to `p` destroys the previous `Expression` object `p`, so if it is of interest to still have access to this object, another name must be used for the `Function` object returned by `interpolate`.

We can now plot `w` and `p` on the screen as well as dump the fields to file in VTK format:

```
plot(w, title='Deflection')
plot(p, title='Load')

vtkfile1 = File('membrane_deflection.pvd')
vtkfile1 << w
vtkfile2 = File('membrane_load.pvd')
vtkfile2 << p
```

Figure 2.3 shows the result of the `plot` commands.

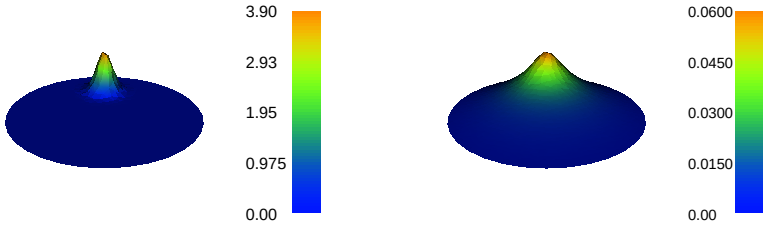


Fig. 2.3 Load (left) and resulting deflection (right) of a circular membrane.

2.2.6 Curve plots through the domain

The best way to compare the load and the deflection is to make a curve plot along the line $x = 0$. This is just a matter of defining a set of points along the line and evaluating the finite element functions `w` and `p` at these points:

```
# Curve plot along x=0 comparing p and w
import numpy as np
import matplotlib.pyplot as plt
tol = 1E-8  # Avoid hitting points outside the domain
y = np.linspace(-1+tol, 1-tol, 101)
points = [(0, y_) for y_ in y]  # 2D points
w_line = np.array([w(point) for point in points])
p_line = np.array([p(point) for point in points])
plt.plot(y, 100*w_line, 'r-', y, p_line, 'b--') # magnify w
plt.legend(['100 x deflection', 'load'], loc='upper left')
plt.xlabel('y');  plt.ylabel('$p$ and $100u$')
```

(Remember a `plt.show()` at the end to show the plot on the screen.) The resulting curve plot appears in Figure 2.4. It is seen how the localized input (p) is heavily damped and smoothened in the output (w). This reflects a typical property of the Poisson equation.

2.2.7 Running ParaView

ParaView is a very strong and well-developed tool for visualizing scalar and vector fields, including those computed by FEniCS.

Our program file writes w and p to file as finite element functions. The default filenames are `membrane_deflection.pvd` for w and `membrane_load.vtu` for p. These files are in VTK format and their data can be visualized in ParaView.

1. Start the ParaView application.

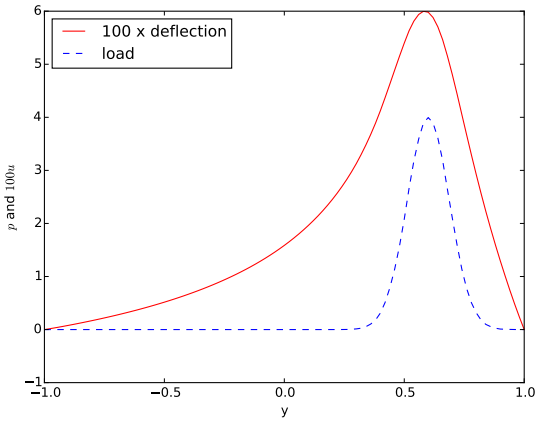


Fig. 2.4 Comparison of membrane load and deflection.

2. Open a file with **File - Open...**. You will see a list of `.pvd` and `.vtu` files. More specifically you see `membrane_deflection.pvd`. Choose this file.
3. Click on **Apply** to the left (*Properties* pane) in the GUI, and ParaView will visualize the contents of the file, here as a color image.
4. To get rid of the axis in the lower left corner of the plot area and axis cross in the middle of the circle, find the *Show Orientation Axis* and *Show Center* buttons to the right in the second row of buttons at the top of the GUI. Click on these buttons to toggle axis information on/off.
5. If you want a color bar to explain the mapping between w values and colors, go to the *Color Map Editor* in the right of the GUI and use the *Show/hide color legend* button. Alternatively, find *Coloring* in the lower left part of the GUI, and toggle the *Show* button.
6. The color map, by default going from blue (low values) to red (high values), can easily be changed. Find the *Coloring* menu in the left part of the GUI, click *Edit*, then in the *Color Map Editor* double click at the left end of the color spectrum and choose another color, say yellow, then double click at the right and of the spectrum and choose pink, scroll down to the bottom of the dialog and click *Update*. The color map now goes from yellow to pink.
7. To save the plot to file, click on **File - Export Scene...**, fill in a filename, and save. See Figure 2.5 (middle).
8. To change the background color of plots, choose **Edit - Settings...**, **Color** tab, click on **Background Color**, and choose it to be, e.g., white. Then choose **Foreground Color** to be something different.
9. To plot the mesh with colors reflecting the size of w, find the *Representation* drop down menu in the left part of the GUI, and replace *Surface* by *Wireframe*.

10. To overlay a surface plot with a wireframe plot, load w and plot as surface, then load w again and plot as wireframe. Make sure both icons in the *Pipeline Browser* in the left part of the GUI are *on* for the `membrane_deflection.pvd` files you want to display. See Figure 2.5 (left).
11. Redo the surface plot. Then we can add some contour lines. Press the semi-sphere icon in the third row of buttons at the top of the GUI (the so-called *filters*). A set of contour values can now be specified at in a dialog box in the left part of the GUI. Remove the default contour (0.578808) and add 0.01, 0.02, 0.03, 0.04, 0.05. Click *Apply* and see an overlay of white contour lines. In the *Pipeline Browser* you can click on the icons to turn a filter on or off.
12. Divide the plot window into two, say horizontally, using the top right small icon. Choose the **3D View** button. Open a new file and load `memberane_load.pvd`. Click on **Apply** to see a plot of the load.

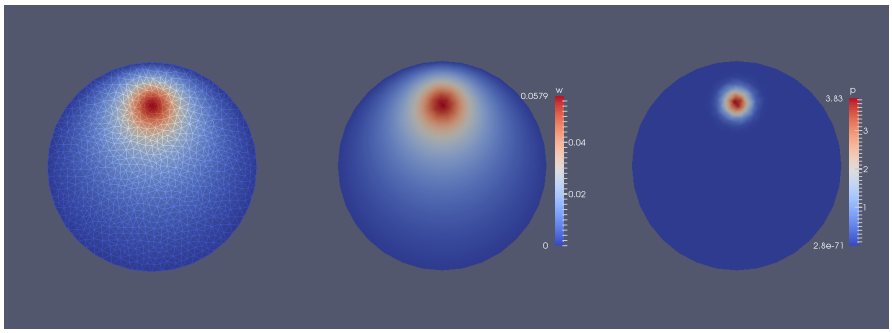


Fig. 2.5 Default visualizations in ParaView: deflection (left, middle) and pressure load (right).

A particularly useful feature of ParaView is that you can record GUI clicks (**Tools - Start/Stop Trace**) and get them translated to Python code. This allows you automate the visualization process. You can also make curve plots along lines through the domain, etc.

For more information, we refer to The ParaView Guide [24] (free PDF available) and to the ParaView tutorial as well as an instruction video.

2.2.8 Using the built-in visualization tool

This section explains some useful visualization features of the built-in visualization tool in FEniCS. The `plot` command applies the VTK package to

visualize finite element functions in a very quick and simple way. The command is ideal for debugging, teaching, and initial scientific investigations. The visualization can be interactive, or you can steer and automate it through program statements. More advanced and professional visualizations are usually better created with advanced tools like Mayavi, ParaView, or VisIt.

The `plot` function can take additional arguments, such as a title of the plot, or a specification of a wireframe plot (elevated mesh) instead of a colored surface plot:

```
plot(mesh, title='Finite element mesh')
plot(w, wireframe=True, title='Solution')
```

Axes can be turned on by the `axes=True` argument, while `interactive=True` makes the program hang at the plot command - you have to type `q` in the plot window to terminate the plot and continue execution.

The left mouse button is used to rotate the surface, while the right button can zoom the image in and out. Point the mouse to the `Help` text down in the lower left corner to get a list of all the keyboard commands that are available.

The plots created by pressing `p` or `P` are stored in filenames having the form `dolfin_plot_X.png` or `dolfin_plot_X.pdf`, where `X` is an integer that is increased by one from the last plot that was made. The file stem `dolfin_plot_` can be set to something more suitable through the `hardcopy_prefix` keyword argument to the `plot` function, for instance, `plot(f, hardcopy_prefix='pressure')`.

Plots stored in PDF format need to be rotated 90 degrees before inclusion in documents. This can be done by the `convert -rotate 90` command (from the ImageMagick utility), but the resulting file has then no more high-resolution PDF vector graphics. A better solution is therefore to use `pdftk` to preserve the vector graphics:

Terminal

```
Terminal> pdftk dolfin_plot_1.pdf cat 1-endnorth output out.pdf
```

For making plots in batch, we can do the following:

```
viz_w = plot(w, interactive=False)
viz_w.elevate(-10)  # adjust (lift) camera from the default view
viz_w.plot(w)       # bring new settings into action
viz_w.write_png('deflection')  # make deflection.png
viz_w.write_pdf('deflection')  # make deflection.pdf
# Rotate pdf file (right) from landscape to portrait
import os
os.system('pdftk deflection.pdf cat 1-endnorth output w.pdf')
```

The ranges of the color scale can be set by the `range_min` and `range_max` keyword arguments to `plot`. The values must be `float` objects. These arguments are important to keep fixed for animations in time-dependent problems.

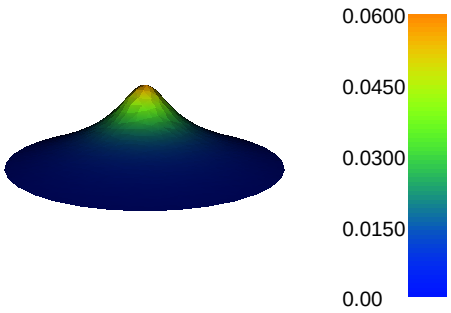


Fig. 2.6 Plot of the deflection of a membrane.

Exercise 2.1: Visualize a solution in a cube

Solve the problem $-\nabla^2 u = f$ on the unit cube $[0,1]\times[0,1]\times[0,1]$ with $u_0 = 1+x^2+2y^2-4z^2$ on the boundary. Visualize the solution. Explore both the built-in visualization tool and ParaView.

Solution. As hinted by the filename in this exercise, a good starting point is the `solver` function in the program `ft04_poisson_func.py`, which solves the corresponding 2D problem. Only two lines in the body of `solver` needs to be changed (!): `mesh = ...`. Replace this line with

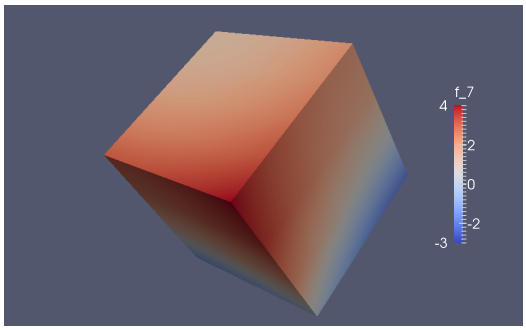
```
mesh = UnitCubeMesh(Nx, Ny, Nz)
```

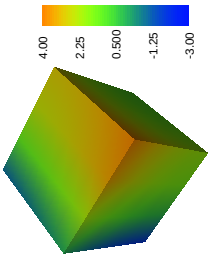
and add `Nz` as argument to `solver`. We implement the new u_0 function in `application_test` and realize that the proper $f(x,y,z)$ function in this new case is 2.

```
u0 = Expression('1 + x[0]*x[0] + 2*x[1]*x[1] - 4*x[2]*x[2]')
f = Constant(2.0)
u = solver(f, u0, 6, 4, 3, 1)
```

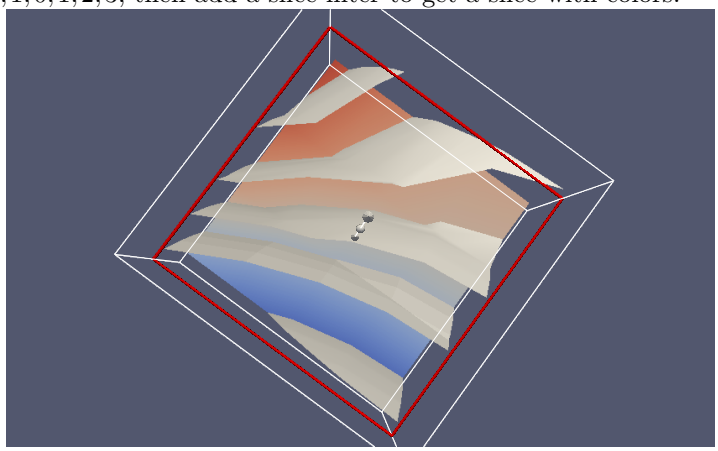
The numerical solution is without approximation errors so we can reuse the unit test from 2D, but it needs an extra `Nz` parameter.

The variation in u is only quadratic so a coarse mesh is okay for visualization. Below is plot from the ParaView (left) and the built-in visualization tool (right). The usage is as in 2D, but now one can use the mouse to rotate the 3D cube.





We can in ParaView add a contour filter and define contour surfaces for $u = -2, 1, 0, 1, 2, 3$, then add a slice filter to get a slice with colors:



Filename: `poissin_3d_func`.

Chapter 3
A Gallery of finite element solvers

The goal of this chapter is to show how a range of important PDEs from science and engineering can be quickly solved with a few lines of FEniCS code. We start with the time-dependent diffusion equation and continue with a nonlinear Poisson equation, the vector PDE for linear elasticity, and the Navier-Stokes equations.

We derive the variational formulation and put together basic objects such as `Mesh`, `Function`, `FunctionSpace`, `TrialFunction`, and `TestFunction` to express in variational formulation in Python code in a way that closely resembles the mathematics.

3.1 The time-dependent diffusion equation

The examples in Section 2.1.4 illustrate that solving linear, stationary PDE problems with the aid of FEniCS is easy and requires little programming. FEniCS clearly automates the spatial discretization by the finite element method. One can use a separate, one-dimensional finite element method in the domain as well, but very often, it is easier to just use a finite difference method, or to formulate the problem as an ODE system and leave the time-stepping to an ODE solver.

hpl 5: Should exemplify all three approaches? With emphasis on simple finite differences?

3.1.1 Variational formulation

Our model problem for time-dependent PDEs reads

$$\frac{\partial u}{\partial t} = \nabla^2 u + f \text{ in } \Omega, \tag{3.1}$$

$$u = u_0 \text{ on } \partial\Omega, \tag{3.2}$$

$$u = I \text{ at } t = 0\,. \tag{3.3}$$

Here, u varies with space and time, e.g., $u = u(x,y,t)$ if the spatial domain Ω is two-dimensional. The source function f and the boundary values u_0 may also vary with space and time. The initial condition I is a function of space only.

A straightforward approach to solving time-dependent PDEs by the finite element method is to first discretize the time derivative by a finite difference approximation, which yields a recursive set of stationary problems, and then turn each stationary problem into a variational formulation.

Let superscript k denote a quantity at time t_k, where k is an integer counting time levels. For example, u^k means u at time level k. A finite difference discretization in time first consists in sampling the PDE at some time level, say k:

$$\frac{\partial}{\partial t}u^k = \nabla^2 u^k + f^k\,. \tag{3.4}$$

The time-derivative can be approximated by a finite difference. For simplicity and stability reasons we choose a simple backward difference:

$$\frac{\partial}{\partial t}u^k \approx \frac{u^k - u^{k-1}}{\Delta t}, \tag{3.5}$$

where Δt is the time discretization parameter. Inserting (3.5) in (3.4) yields

$$\frac{u^k - u^{k-1}}{\Delta t} = \nabla^2 u^k + f^k\,. \tag{3.6}$$

This is our time-discrete version of the diffusion PDE (3.1).

We may reorder (3.6) so that the left-hand side contains the terms with the unknown u^k and the right-hand side contains computed terms only. The result is a recursive set of spatial (stationary) problems for u^k (assuming u^{k-1} is known from computations at the previous time level):

$$u^0 = I, \tag{3.7}$$

$$u^k - \Delta t\nabla^2 u^k = u^{k-1} + \Delta t f^k, \quad k = 1,2,\ldots \tag{3.8}$$

Given I, we can solve for u^0, u^1, u^2, and so on.

An alternative to (3.8), which can be convenient in implementations, is to collect all terms on one side of the equality sign:

$$F(u;v) = u^k - \Delta t\nabla^2 u^k - u^{k-1} - \Delta t f^k = 0, \quad k = 1,2,\ldots \tag{3.9}$$

We use a finite element method to solve (3.7) and either of the equations (3.8) or (3.9). This requires turning the equations into weak forms. As usual, we multiply by a test function $v \in \hat{V}$ and integrate second-derivatives by parts. Introducing the symbol u for u^k (which is natural in the program), the resulting weak form arising from formulation (3.8) can be conveniently written in the standard notation:

$$a(u,v) = L(v),$$

where

$$a(u,v) = \int_\Omega (uv + \Delta t \nabla u \cdot \nabla v)\,\mathrm{d}x, \tag{3.10}$$

$$L(v) = \int_\Omega \left(u^{k-1} + \Delta t f^k\right) v\,\mathrm{d}x\,. \tag{3.11}$$

The alternative form (3.9) has an abstract formulation

$$F(u;v) = 0,$$

where

$$F = \int_\Omega \left(uv + \Delta t \nabla u \cdot \nabla v - \left(u^{k-1} - \Delta t f^k\right) v\right)\mathrm{d}x\,. \tag{3.12}$$

The initial condition (3.7) can also be turned into a weak form,

$$a_0(u,v) = L_0(v),$$

with

$$a_0(u,v) = \int_\Omega uv\,\mathrm{d}x, \tag{3.13}$$

$$L_0(v) = \int_\Omega Iv\,\mathrm{d}x\,. \tag{3.14}$$

The alternative is to construct u_0 by just interpolating I (which is also a much cheaper operation since no linear system is involved).

The continuous variational problem is to find $u^0 \in V$ such that $a_0(u^0,v) = L_0(v)$ holds for all $v \in \hat{V}$, and then find $u^k \in V$ such that $a(u^k,v) = L(v)$ for all $v \in \hat{V}$, or alternatively, $F(u^k,v) = 0$ for all $v \in \hat{V}$, $k = 1,2,\ldots$.

Approximate solutions in space are found by restricting the functional spaces V and $\hat{V}$ to finite-dimensional spaces, exactly as we have done in the Poisson problems. We shall use the symbol u for the finite element approximation at time t_k. In case we need to distinguish this space-time discrete approximation from the exact solution of the continuous diffusion problem,

we use u_{e} for the latter. By u^{k-1} we mean the finite element approximation of the solution at time t_{k-1}.

Instead of solving (3.7) by a finite element method, i.e., projecting I onto V via the problem $a_0(u,v) = L_0(v)$, we could simply interpolate u^0 from I. That is, if $u^0 = \sum_{j=1}^{N} U_j^0 \phi_j$, we simply set $U_j = I(x_j, y_j)$, where (x_j, y_j) are the coordinates of node number j. We refer to these two strategies as computing the initial condition by either projecting I or interpolating I. Both operations are easy to compute through one statement, using either the `project` or `interpolate` function.

3.1.2 A simple implementation

Our program needs to implement the time stepping explicitly, but can rely on FEniCS to easily compute a_0, L_0, F, a, and L, and solve the linear systems for the unknowns.

Test problem. Before starting the coding, we shall construct a problem where it is easy to determine if the calculations are correct. The simple backward time difference is exact for linear functions, so we decide to have a linear variation in time. Combining a second-degree polynomial in space with a linear term in time,

$$u = 1 + x^2 + \alpha y^2 + \beta t, \tag{3.15}$$

yields a function whose computed values at the nodes will be exact, regardless of the size of the elements and Δt, as long as the mesh is uniformly partitioned. By inserting (3.15) in the PDE problem (3.1), it follows that u_0 must be given as (3.15) and that $f(x,y,t) = \beta - 2 - 2\alpha$ and $I(x,y) = 1 + x^2 + \alpha y^2$.

The code. A new programming issue is how to deal with functions that vary in space *and time*, such as the boundary condition u_0 given by (3.15). A natural solution is to apply an `Expression` object with time t as a parameter, in addition to the parameters α and β:

```
alpha = 3; beta = 1.2
u0 = Expression('1 + x[0]*x[0] + alpha*x[1]*x[1] + beta*t',
                alpha=alpha, beta=beta, t=0)
```

This function expression has the components of `x` as independent variables, while `alpha`, `beta`, and `t` are parameters. The parameters can later be updated as in

```
u0.t = t
```

The essential boundary conditions, along the entire boundary in this case, are set in the usual way,

```
def boundary(x, on_boundary):  # define the Dirichlet boundary
    return on_boundary

bc = DirichletBC(V, u0, boundary)
```

We shall use `u` for the unknown u at the new time level and `u_1` for u at the previous time level. The initial value of `u_1`, implied by the initial condition on u, can be computed by either projecting or interpolating I. The $I(x,y)$ function is available in the program through `u0`, as long as `u0.t` is zero. We can then do

```
u_1 = interpolate(u0, V)
# or
u_1 = project(u0, V)
```

Projecting versus interpolating the initial condition

To actually recover the exact solution (3.15) to machine precision, it is important not to compute the discrete initial condition by projecting I, but by interpolating I so that the degrees of freedom have exact values at $t = 0$ (projection results in approximative values at the nodes).

We may either define a or L according to the formulas above, or we may just define F and ask FEniCS to figure out which terms that go into the bilinear form a and which that go into the linear form L. The latter is convenient, especially in more complicated problems, so we illustrate that construction:

```
dt = 0.3      # time step

u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)

F = u*v*dx + dt*dot(grad(u), grad(v))*dx - (u_1 + dt*f)*v*dx
a, L = lhs(F), rhs(F)
```

Finally, we perform the time stepping in a loop:

```
u = Function(V)   # the unknown at a new time level
T = 2             # total simulation time
t = dt

while t <= T:
    u0.t = t
    solve(a == L, u, bc)

    t += dt
    u_1.assign(u)
```

Remember to update expression objects with the current time!

Inside the time loop, observe that `u0.t` must be updated before the `solve` statement to enforce computation of Dirichlet conditions at the current time level. (The Dirichlet conditions look up the `u0` object for values.)

The time loop above does not contain any comparison of the numerical and the exact solution, which we must include in order to verify the implementation. As in the Poisson equation example in Section 2.1.6, we compute the difference between the array of nodal values of `u` and the array of the interpolated exact solution. The following code is to be included inside the loop, after `u` is found:

```
u_e = interpolate(u0, V)
error = np.abs(u_e.vector().array() -
               u.vector().array()).max()
print('error, t=%.2f: %-10.3g' % (t, max_error))
```

The complete program code for this time-dependent case goes as follows:

```
from fenics import *
import numpy as np

# Create mesh and define function space
nx = ny = 4
mesh = UnitSquareMesh(nx, ny)
V = FunctionSpace(mesh, 'P', 1)

# Define boundary conditions
alpha = 3; beta = 1.2
u0 = Expression('1 + x[0]*x[0] + alpha*x[1]*x[1] + beta*t',
                alpha=alpha, beta=beta, t=0)

def boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, u0, boundary)

# Initial condition
u_1 = interpolate(u0, V)
#project(u0, V) will not result in exact solution at the nodes!

dt = 0.3      # time step

# Define variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)

F = u*v*dx + dt*dot(grad(u), grad(v))*dx - (u_1 + dt*f)*v*dx
```

```
a, L = lhs(F), rhs(F)

# Compute solution
u = Function(V)   # the unknown at a new time level
T = 1.9           # total simulation time
t = dt
while t <= T:
    print('time =', t)
    u0.t = t
    solve(a == L, u, bc)

    # Verify
    u_e = interpolate(u0, V)
    error = np.abs(u_e.vector().array() -
                   u.vector().array()).max()
    print('error, t=%.2f: %-10.3g' % (t, error))

    t += dt
    u_1.assign(u)
```

The code is available in the file `ft02_diffusion_flat1.py`.

3.1.3 Diffusion of a Gaussian function

The mathematical problem. Now we want to solve a more physical problem, namely the diffusion of a Gaussian hill. It means that the initial condition is given by

$$I(x,y) = e^{-ax^2-ay^2}$$

on a domain $[-2,2]\times[2,2]$. A possible value of a is 5.

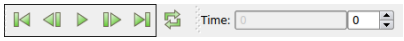
Implementation. What are the necessary changes to the previous program?

1. The domain is not the unit square and it needs much higher resolution: `mesh = RectangleMesh(Point(-2,-2), Point(2,2), 30, 30)`.
2. The boundary condition is zero everywhere: `DirichletBC(V, Constant(0), boundary)`.
3. The initial condition is different: `I = Expression('exp(...)')`.
4. The time step should be sufficiently small: `dt = 0.01` or `dt = 0.05`.
5. The right-hand side function `f` is zero: `f = Constant(0)` (just `0` will given an error as functions in FEniCS must be `Expression`, `Function` (over a mesh) or `Constant`).
6. The end time for the simulation must be longer: `T = 0.8`.
7. The initial condition and the solution inside the time loop should be stored to file in VTK format for visualization: `vtkfile << (u, t)`.
8. We can add a `plot(u)` command inside the time loop as well.

The complete program appears below.

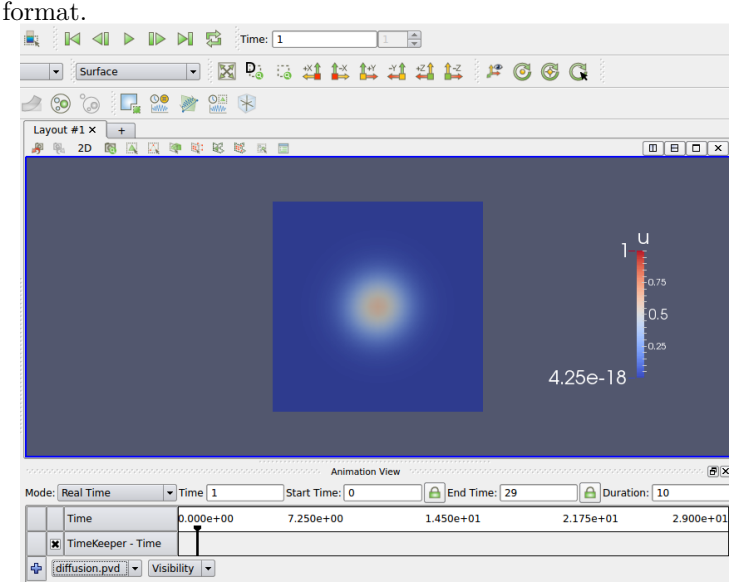
```
from fenics import *
import time

# Create mesh and define function space
nx = ny = 30
mesh = RectangleMesh(Point(-2,-2), Point(2,2), nx, ny)
V = FunctionSpace(mesh, 'P', 1)

# Define boundary conditions
def boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, Constant(0), boundary)

# Initial condition
I = Expression('exp(-a*pow(x[0],2)-a*pow(x[1],2))', a=5)
u_1 = interpolate(I, V)
u_1.rename('u', 'initial condition')
vtkfile = File('diffusion.pvd')
vtkfile << (u_1, 0.0)
#project(u0, V) will not result in exact solution at the nodes!

dt = 0.01    # time step

# Define variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(0)

F = u*v*dx + dt*dot(grad(u), grad(v))*dx - (u_1 + dt*f)*v*dx
a, L = lhs(F), rhs(F)

# Compute solution
u = Function(V)             # the unknown at a new time level
u.rename('u', 'solution')   # name and label for u
T = 0.5                     # total simulation time
t = dt
while t <= T:
    print('time =', t)
    solve(a == L, u, bc)
    vtkfile << (u, float(t))
    plot(u)
    time.sleep(0.3)

    t += dt
    u_1.assign(u)
```

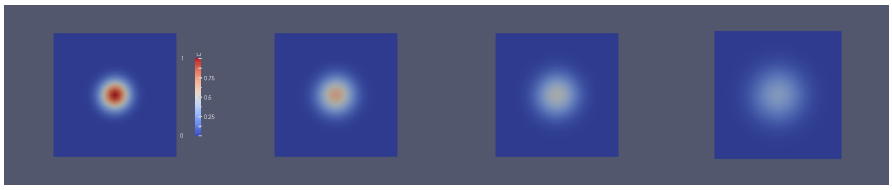
Visualization in ParaView. Start ParaView, choose **File - Open**, open the file `diffusion.pvd`, click the green **Apply** button on the left to see the initial condition being plotting. Choose **View - Animation View**. Click on the play button or (better) the next frame button in the row of buttons at the top of the GUI to see the evolution of the scalar field you just have computed:



The cross in the middle of the plot can be turned off by the **Show Center** button:

Choose **File - Save Animation...** to save the animation to the OGG video format.





3.2 A nonlinear Poisson equation

3.2.1 Variational formulation

Now we shall address how to solve nonlinear PDEs in FEniCS. Our sample PDE for implementation is taken as a nonlinear Poisson equation:

$$-\nabla\cdot(q(u)\nabla u) = f, \tag{3.16}$$

in Ω, with $u = u_0$ on the boundary $\partial\Omega$. The coefficient $q(u)$ makes the equation nonlinear (unless $q(u)$ is constant in u).

The variational formulation of our model problem reads: Find $u \in V$ such that

$$F(u;v) = 0 \quad \forall v \in \hat{V}, \tag{3.17}$$

where

$$F(u;v) = \int_\Omega (q(u)\nabla u\cdot\nabla v + fv)\,\mathrm{d}x, \tag{3.18}$$

and

$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial Omega\},$$
$$V = \{v \in H^1(\Omega) : v = u_0 \text{ on } \partial\Omega\}.$$

The discrete problem arises as usual by restricting V and $\hat{V}$ to a pair of discrete spaces. As usual, we omit any subscript on discrete spaces and simply say V and $\hat{V}$ are chosen finite dimensional according to some mesh with some element type. Similarly, we let u from now on be the discrete solution.

The discrete nonlinear problem is then written as: find $u \in V$ such that

$$F(u;v) = 0 \quad \forall v \in \hat{V}, \tag{3.19}$$

with $u = \sum_{j=1}^{N} U_j\phi_j$. Since F is a nonlinear function of u, the variational statement gives rise to a system of nonlinear algebraic equations in the unknowns $U_1,\ldots,U_N$.

3.2.2 A simple implementation

Overview. A working solver for the nonlinear Poisson equation is as easy to implement as a solver for the corresponding linear problem. All we need to do is the state the formula for F and call `solve(F == 0, u, bc)` instead of `solve(a == L, u, bc)` as we did in the linear case. Here is a minimalistic code:

```
from fenics import *

def q(u):
    """Nonlinear coefficient in the PDE."""
    return 1 + u**2
```

```
mesh = UnitSquareMesh(60, 40)
V = FunctionSpace(mesh, 'P', 1)
u0 = Expression(...)

def u0_boundary(x, on_boundary):
    return on_boundary

bc = DirichletBC(V, u0, u0_boundary)

# Define variational problem
u = Function(V)
v = TestFunction(V)
f = Expression(...)
F = dot(q(u)*grad(u), grad(v))*dx - f*v*dx

# Compute solution
solve(F == 0, u, bc)
```

The major difference from a linear problem is that the unknown function `u` in the variational form is in the nonlinear case a `Function`, not a `TrialFunction`.

The `solve` function takes the nonlinear equations and derives symbolically the Jacobian matrix and runs a Newton method.

Constructing a test problem with SymPy. Let us do a specific computation. We then need choices for f and u_0. Previously, we have worked with manufactured solutions that can be reproduced without approximation errors. This is more difficult in nonlinear problems, and the algebra is more tedious. However, we may utilize SymPy for symbolic computing and integrate such computations in the FEniCS solver. This allows us to easily experiment with different manufactured solutions. The forthcoming code with SymPy requires some basic familiarity with this package (here, defining symbols, `diff` for differentiation, `ccode` for C/C++ code generation).

We try out a two-dimensional manufactured solution that is linear in the unknowns:

```
# Warning: from fenics import * imports f, q, and sym
# (which overwrites our own f and q (function) objects
# and also sym if we do import sympy as sym).
# Therefore, do fenics import first and then overwrite
from fenics import *

def q(u):
    """Nonlinear coefficient in the PDE."""
    return 1 + u**2

# Use sympy to compute f given manufactured solution u
import sympy as sym
x, y = sym.symbols('x[0] x[1]')
u = 1 + x + 2*y
f = - sym.diff(q(u)*sym.diff(u, x), x) - \
```

```
      sym.diff(q(u)*sym.diff(u, y), y)
f = sym.simplify(f)
```

Define symbolic coordinates as required in `Expression` objects

Note that we would normally write `x, y = sym.symbols('x y')`, but if we want the resulting expressions to be have valid syntax for `Expression` objects, and then x reads `x[0]` and y must be `x[1]`. This is easily accomplished with `sympy` by defining the names of `x` and `y` as `x[0]` and `x[1]`: `x, y = sym.symbols('x[0] x[1]')`.

Turning the expressions for `u` and `f` into C or C++ syntax for `Expression` objects needs two steps. First we ask for the C code of the expressions,

```
u_code = sym.printing.ccode(u)
f_code = sym.printing.ccode(f)
```

Sometimes we need some editing of the result to match the required syntax of `Expression` objects, but not in this case. (The primary example is that `M_PI` for π in C/C++ must be replaced by `pi` for `Expression` objects.) In our case here, the output of `c_code` and `f_code` is

```
x[0] + 2*x[1] + 1
-10*x[0] - 20*x[1] - 10
```

After having defined the mesh, the function space, and the boundary, we define the boundary values, `u0`, as

```
u0 = Expression(u_code)
```

Similarly, we define the right-hand side function as

```
f = Expression(f_code)
```

The complete code is found in the file `ft03_poisson_flat_nonlinear.py`.

Name clash between `fenics` and program variables

In a program like the one above, strange errors may occur due to name clashes. If you define `sym`, `q`, and `f` prior to doing `from fenics import *`, the latter statement will also import variables with the names `sym`, `q`, and `f` and overwrite the objects you had! This may lead to strange errors. The best solution is to do `import fenics as fe` and prefix all FEniCS object names by `fe`. The next best solution is to do the `from fenics import *` first and then define our own variables that overwrite those imported from `fenics`. This is acceptable if we do not need `f`, `q`, and `sym` from `fenics`.

Running the code gives output that tells how the Newton iteration progresses. With $2(6\times 4)$ cells we get convergence in 7 iterations with a tolerance of 10^{-9}, and the error in the numerical solution is about 10^{-11}. Using more elements, e.g., $2(16\times 14)$, brings the error down to about 10^{-15}, which provides evidence for a correct implementation.

The current example shows how easy it is to solve a nonlinear problem in FEniCS. However, experts on numerical solution of nonlinear PDEs know very well that automated procedures may fail in nonlinear problems, and that it is often necessary to have much more manual control of the solution process than what we have in the current case. Therefore, we return to this problem in Chapter **??** in [19] and show how we can implement our own solution algorithms for nonlinear equations and also how we can steer the parameters in the automated Newton method used above. You will then realize how easy it is to implement tailored solution strategies for nonlinear problems in FEniCS.

3.3 The equations of linear elasticity

Analysis of structures is one of the major activities in modern engineering, thus making the PDEs for deformation of elastic bodies most likely the most popular PDE model in the world. It just takes a page of code to solve the equations of 2D or 3D elasticity in FEniCS, and the details follows below.

3.3.1 Variational formulation

The equations governing small elastic deformations of a body Ω can be written as

$$\nabla\cdot\sigma = \varrho f \text{ in } \Omega, \tag{3.20}$$

$$\sigma = \lambda \operatorname{tr}\varepsilon I + 2\mu\varepsilon, \tag{3.21}$$

$$\varepsilon = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \tag{3.22}$$

where σ is the stress tensor, ϱ is the density of the material, f is the body force, λ and μ are Lame's elasticity coefficients for the material in Ω I is the identity tensor, tr is the trace operator on a tensor, ε is the strain tensor, and u is the displacement vector field.

We shall combine (3.21) and (3.22) to

$$\sigma = \lambda\nabla\cdot u I + \mu(\nabla u + (\nabla u)^T). \tag{3.23}$$

Note that (3.20)-(3.22) can easily be transformed to a vector PDE for u, which is the governing PDE for the unknown u. In the derivation of the variational formulation, however, the splitting of the equations as done above is convenient.

The variational formulation of (3.20)-(3.22) consists of forming the inner product of (3.20) and a *vector* test function $v \in \hat{V}$, where $\hat{V}$ is a test vector function space, and integrating over the domain Ω:

$$\int_\Omega (\nabla \cdot \sigma) \cdot v \,\mathrm{d}x = \int_\Omega \varrho f \cdot v \,\mathrm{d}x\,.$$

Since $\nabla \cdot \sigma$ contains second-order derivatives of the primary unknown u, we integrate this term by parts:

$$\int_\Omega (\nabla \cdot \sigma) \cdot \nabla v \,\mathrm{d}x - \int_\Omega \sigma : \nabla v \,\mathrm{d}x + \int_{\partial\Omega} (\sigma \cdot n) \cdot v \,\mathrm{d}s,$$

where the colon operator is the inner product between tensors, and n is the outward unit normal at the boundary. The quantity $\sigma \cdot n$ is known as the *traction* or stress vector at the boundary, and is often prescribed as a boundary condition. We assume that it is prescribed at a part $\partial\Omega_T$ of the boundary and set $T = \sigma \cdot n$. We then have

$$\int_\Omega (\sigma : \nabla v + \varrho f \cdot v) = \int_{\partial\Omega_T} T \cdot v \,\mathrm{d}s\,.$$

Inserting (3.23) for σ gives the variational form with u as unknown.

We can now summarize the variational formulation as find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in \hat{V}, \tag{3.24}$$

where

$$a(u,v) = \int_\Omega \sigma(u) : \nabla v \,\mathrm{d}x, \tag{3.25}$$

$$\sigma(u) = \lambda \nabla \cdot u I + \mu(\nabla u + (\nabla u)^T), \tag{3.26}$$

$$L(v) = -\int_\Omega \varrho f \cdot v \,\mathrm{d}x + \int_{\partial\Omega_T} T \cdot v \,\mathrm{d}s\,. \tag{3.27}$$

One can show that the inner product of a symmetric tensor A and a non-symmetric tensor B vanishes. If we express ∇v as a sum of its symmetric and non-symmetric parts, only the symmetric part will survive in the product $\sigma : \nabla v$ since σ is a symmetric tensor. This gives rise to the slightly different variational form

$$a(u,v) = \int_\Omega \sigma(u) : \varepsilon(v) \,\mathrm{d}x, \tag{3.28}$$

where $\varepsilon(v)$ is the symmetric part of v:

$$\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T).$$

3.3.2 A simple implementation

Test problem. As test example, we may look at a clamped beam deformed under its own weight. Then $f = (0,0,-g)$ is the body force with g as the acceleration of gravity. The beam is box-shaped with length L and square cross section of width W. We set $u = (0,0,0)$ at the clamped end, $x = 0$. The rest of the boundaries is traction free.

Let us scale the problem. **hpl 6**: This was meant to simplify the problem so we don't need values for λ, μ, ϱ, etc for a specific material, but the scaling requires some care. In the equation for u, arising from inserting (3.21) and (3.22) in (3.20),

$$\nabla\cdot(\lambda\nabla\cdot u) + \mu\nabla^2 u = \varrho f,$$

we insert coordinates made dimensionless by L, and $\bar{u} = u/u_c$, which results in the dimensionless governing equation

$$\bar{\nabla}\cdot(\bar{\nabla}\cdot\bar{u}) + \beta\bar{\nabla}^2\bar{u} = \bar{f}, \quad \bar{f} = (0,0,\gamma),$$

where $\beta = \mu/\lambda$ is a dimensionless elasticity parameter and

$$\gamma = \frac{\varrho g L^2}{u_c\lambda}.$$

Sometimes, one will argue to chose u_c to make γ unity ($u_c = \varrho g L^2/\lambda$). This is often the reasoning for getting a $\bar{u}$ that is of order unity. However, in elasticity, this leads us to displacements of the size of the geometry, which looks very strange in plots. We therefore want the characteristic displacement to be a small fraction of the characteristic length of the geometry. Actually, for a clamped beam, one has a deflection formula which gives $u_c = \frac{3}{2}\varrho g L^2\delta^2/E$, where $\delta = L/W$. Thus, the dimensionless parameter δ is very important in the problem (as expected: $\delta \gg 1$ is what gives beam theory). Taking E to be of the same order of λ, we realize that $\gamma \sim \delta^{-2}$. Experiments with the code point to $\gamma = 0.25\delta^{-2}$ as an appropriate choice. We implement the code with physical parameters, λ, μ, ϱ, g, L, and W, but set these to achieve the solution of the scaled problem: $\lambda = \varrho = L = 1$, W as W/L, $g = \gamma$, and $\mu = \beta$.

Code. hpl 7: Must explain the code. New concepts here, though not many.

```
from fenics import *
```

```
# Scaled variables
L = 1; W = 0.2
lambda_ = 1
rho = 1
delta = W/L
gamma = 0.25*delta**2
beta = 0.8
mu = beta
g = gamma

# Create mesh and define function space
mesh = BoxMesh(Point(0,0,0), Point(L,W,W), 10, 3, 3)
V = VectorFunctionSpace(mesh, 'P', 1)

# Define boundary conditions
tol = 1E-14

def clamped_boundary(x, on_boundary):
    return on_boundary and (x[0] < tol)

bc = DirichletBC(V, Constant((0,0,0)), clamped_boundary)

def epsilon(u):
    return 0.5*(nabla_grad(u) + nabla_grad(u).T)
    #return sym(nabla_grad(u))

def sigma(u):
    return lambda_*nabla_div(u)*Identity(d) + 2*mu*epsilon(u)

# Define variational problem
u = TrialFunction(V)
d = u.geometric_dimension()  # no of space dim
v = TestFunction(V)
f = rho*Constant((0,0,g))
T = Constant((0,0,0))
a = inner(sigma(u), epsilon(v))*dx
L = -dot(f, v)*dx + dot(T, v)*ds

# Compute solution
u = Function(V)
solve(a == L, u, bc)

# Plot solution and mesh
plot(u, title='Displacement', mode='displacement')

von_Mises = inner(sigma(u), sigma(u)) - div(u)
V = FunctionSpace(mesh, 'P', 1)
von_Mises = project(von_Mises, V)
plot(von_Mises, title='Stress intensity', mode='displacement')
u_magnitude = sqrt(dot(u,u))
u_magnitude = project(u_magnitude, V)
plot(u_magnitude, 'Displacement magnitude', mode='displacement')
print('min/max u:', u_magnitude.vector().array().min(),
      u_magnitude.vector().array().max())
```

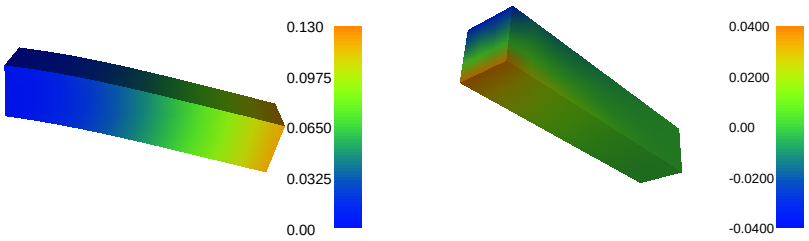


Fig. 3.1 Gravity-induced deformation of a clamped beam: deflection (left) and stress intensity (right).

3.4 The Navier–Stokes equations

Should we here also include coupling to a transport equation? It shows multi-physics capabilities.

3.4.1 Variational formulation

3.4.2 A simple implementation

Chapter 4
Mesh generation, subdomains and boundary conditions

In this chapter, we focus on a fundamental step in the solution of many PDE problems: the generation of a mesh, and the specification of subdomains and boundary conditions. Our starting point is the convection-diffusion equation, which extends the Poisson equation (the diffusion equation) from the previous chapter to take into account the effects of convection.

4.1 Physical problem formulation

We will simulate the conduction of heat in an insulated pipe via both diffusion and convection (transport). The inner diameter of the pipe is 4 cm, its thickness is 4mm, the thickness of the surrounding insulation is 10 mm, and its length is 50 cm. We assume that the temperature of the water at the inlet is 42 degrees centigrade and the temperature of the surrounding air is 22 degrees centigrade. We also assume that the pipe transmits 0.1 liters of water per second. A sketch of the pipe is given in Figure X.

AL 8: Add nice 3D graphics here...

To compute the temperature distribution in the pipe, we need to set boundary conditions, both at the inflow and outflow, as well as on the boundary of the insulating layer which is exposed to the surrounding air. We do this by assuming that the temperature at the inlet is 42 degrees, for the water as well as for the pipe and the insulating layer. We similarly assume the temperature to be 22 degrees in all three layers at the outflow. The boundary of the insulating layer is also assumed to have the temperature 22 degrees.

To compute the temperature distribution, we also need to know the value of the thermal conductivity λ $[\mathrm{W} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1}]$ for water, pipe, and insulation. From a book of physical tables or from the internet, we find the following values that we will use for our simulation: $\lambda_{\text{water}} = 0.6\,\mathrm{W} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1}$, $\lambda_{\text{pipe}} = 18\,\mathrm{W} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1}$ (stainless steel), and $\lambda_{\text{insul}} = 0.035\,\mathrm{W} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1}$ (styrofoam). We will also need to know the density and specific heat of wa-

ter, which we set to $\rho = 1\,\mathrm{kg/dm^3}$ and $c = 4.184\,\mathrm{kJ/(kg \cdot K)}$ (one kcal per $\mathrm{kg \cdot K}$), respectively.

Finally, we need to know the velocity field for the water flowing through the pipe. We know the total flow rate of water (0.1 liters per second). If the flow is laminar (Poiseuille flow), we know that the velocity profile is a quadratic function in the radius with its maximum at the center and zero velocity on the boundary. We may thus compute the velocity profile from the dimensions of the pipe. In particular, we know that the velocity profile takes the form

$$\beta(x) = (0, 0, C(a - r)^2),$$

where $r = \sqrt{x^2 + y^2}$. The flow rate is then

$$Q = \int_0^a C(a-r)^2\, 2\pi r\, \mathrm{d}r = 2\pi C a^4 \int_0^1 s(1-s)^2\, \mathrm{d}s = \pi C a^4/6$$

Knowing that $Q = 0.1\,\mathrm{dm^3/s}$, we can solve for C and find $C = 0.6\,\mathrm{dm^3}/(\pi a^4)$.

4.2 Mathematical problem formulation

We will model the conduction of heat in the pipe using the standard convection-diffusion equation:

$$-\nabla \cdot (\lambda \nabla u) + \nabla \cdot (c\rho\beta u) = f. \tag{4.1}$$

Here, u denotes the temperature, λ is the thermal conductivity, c is the specific heat, ρ is the density, β is the velocity field, and f is the source term. Since our problem does not have a source term, we will set $f = 0$.

The convection-diffusion equation is often stated in the following slightly modified form:

$$-\nabla \cdot (\lambda \nabla u) + c\rho\beta \cdot \nabla u = f.$$

This formulation is equivalent to (4.1) if the velocity field β is divergence free; that is, if $\nabla \cdot \beta = 0$: $\nabla \cdot (\beta u) = (\nabla \cdot \beta)u + \beta \cdot \nabla u = \beta \cdot \nabla u$.

4.3 Scaling the equation

Before we can solve the PDE, we must first introduce dimensionless quantities since, strictly speaking, our program can not work with units, only with numbers. We let L be a reference length, let T be a reference time length, let U be a reference temperature, and let P be a reference power (energy per unit time). We then introduce the following dimensionless quantities:

$$\bar{x} = \frac{x}{L},\ \bar{y} = \frac{y}{L},\ \bar{z} = \frac{z}{L},\ \bar{u} = \frac{u}{U},\ \bar{\beta} = \frac{\beta}{LT^{-1}}, \bar{f} = \frac{f}{PL^{-3}}.$$

The dimensionless coordinates $\bar{x}$, $\bar{y}$ and $\bar{z}$ also lead to dimensionless derivatives: $\bar{\nabla} = (\partial/\partial\bar{x}, \partial/\partial\bar{y}, \partial/\partial\bar{z}) = L\nabla$. Inserting $u = U\bar{u}$, $\beta = L^{-1}T\bar{\beta}$, and $f = L^{-3}P\bar{f}$ into the convection-diffusion equation (4.1) and using $\nabla = L^{-1}\bar{\nabla}$, we obtain

$$-L^{-1}\bar{\nabla}\cdot(\lambda L^{-1}\bar{\nabla}(U\bar{u})) + L^{-1}\bar{\nabla}\cdot(c\rho LT^{-1}\bar{\beta}U\bar{u}) = L^{-3}P\bar{f}.$$

Rearranging the factors L, U, T, P, we obtain

$$-\bar{\nabla}\cdot(LUP^{-1}\lambda\bar{\nabla}\bar{u}) + \bar{\nabla}\cdot\left(L^3UT^{-1}P^{-1}c\rho\bar{\beta}\bar{u}\right) = \bar{f}.$$

Finally, we identify the two dimensionless parameters $\bar{\lambda}$ and $\bar{c}$ given by

$$\bar{\lambda} = LUP^{-1}\lambda, \quad \bar{c} = L^3UT^{-1}P^{-1}c\rho.$$

Let's double-check that these are indeed dimensionless quantities. We have

$$\begin{aligned}[\bar{\lambda}] &= [LUP^{-1}\lambda] = [LUP^{-1}]\cdot[\lambda] = [LUP^{-1}]\cdot[\mathrm{W}\cdot\mathrm{m}^{-1}\cdot\mathrm{K}^{-1}] \\ &= [LUP^{-1}]\cdot[L^{-1}U^{-1}P] = [LUP^{-1}\cdot L^{-1}U^{-1}P] = [1].\end{aligned}$$

Similarly, we have

$$\begin{aligned}[L^3UT^{-1}P^{-1}c\rho] &= [L^3UT^{-1}P^{-1}]\cdot[c]\cdot[\rho] \\ &= [L^3UT^{-1}P^{-1}]\cdot[\mathrm{kJ}/(\mathrm{kg}\cdot\mathrm{K})]\cdot[\mathrm{kg}/\mathrm{dm}^3] \\ &= [L^3UT^{-1}P^{-1}]\cdot[PTU^{-1}\cdot L^{-3}] = [1].\end{aligned}$$

We thus obtain the dimensionless and fully scaled convection-diffusion equation

$$-\bar{\nabla}\cdot(\bar{\lambda}\bar{\nabla}\bar{u}) + \bar{\nabla}\cdot\left(c\bar{\beta}\bar{u}\right) = \bar{f}.$$

The reference quantities L, T, U and P may be chosen arbitrarily. If working with very large or very small quantities, one may want to choose for example L to obtain a domain of unit size. However, for our problem this is not necessary. We will therefore make the most straightforward choice which is to use standard SI units. We thus take $L = 1\,\mathrm{m}$, $T = 1\,\mathrm{s}$, $U = 1\,\mathrm{K}$, and $P = 1\,\mathrm{W}$. The scaled variables and parameters $\bar{u}$, $\bar{\beta}$, $\bar{\lambda}$ and so on may then be easily computed by expressing everything in standard SI units and then simply dropping the units.

In the following we will simply write u in place of $\bar{u}$ but remember that the u we compute is actually $\bar{u}$ and the actual temperature will be $u\,\mathrm{K}$:

$$-\nabla\cdot(\lambda\nabla u) + \nabla\cdot(c\beta u) = f. \tag{4.2}$$

Furthermore, since only derivatives of u appear in the equation, adding a constant offset to u does not change the equation. We may therefore work in Kelvin (K) as well as in degrees centigrade without needing to rescale the equation.

4.4 Finite element variational formulation

The finite element variational formulation of the convection diffusion equation is obtained in the same way as for the Poisson equation in the previous chapter, by multiplying the equation with a test function v, integrating over the domain Ω, and integrating terms with second derivatives by parts. For the (scaled) convection-diffusion equation (4.2), we have one such term, namely $-\nabla\cdot(\lambda u)$. For this term, integration by parts gives

$$-\int_\Omega (\nabla\cdot(\lambda\nabla u))\,v\,\mathrm{d}x = \int_\Omega \lambda\nabla u\cdot\nabla v\,\mathrm{d}x - \int_{\partial\Omega}\lambda\nabla u\cdot n\,v\,\mathrm{d}s.$$

Since we assume Dirichlet boundary conditions on the entire boundary, we take the test function v to be zero on $\partial\Omega$ and thus obtain the following variational problem: find $u\in V$ such that

$$\int_\Omega \lambda\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega \nabla\cdot(c\beta u)\,v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x,$$

for all test functions $v\in V$.

The variational problem can be stated in FEniCS as follows:

```
a = lmdba*dot(grad(u), grad(v))*dx + div(c*beta*u)*v*dx
L = f*v*dx
```

Note the intentional misspelling of `lmbda` to avoid a name clash with the built-in Python keyword `lmbda`.

4.5 Mesh generation

4.6 Subdomain markers

4.7 The complete program

```
from fenics import *
from mshr import *
```

```
# Parameters for geometry
a = 0.04
b = a + 0.004
c = a + 0.01
L = 0.5

# Define cylinders
cylinder_a = Cylinder(Point(0, 0, 0), Point(0, 0, L), a, a)
cylinder_b = Cylinder(Point(0, 0, 0), Point(0, 0, L), b, b)
cylinder_c = Cylinder(Point(0, 0, 0), Point(0, 0, L), c, c)

# Define domain and set subdomains
domain = cylinder_c
domain.set_subdomain(1, cylinder_b)
domain.set_subdomain(2, cylinder_a)

# Generate mesh
mesh = generate_mesh(domain, 16)

xmlfile = File('pipe.xml')
xmlfile << mesh

vtkfile = File('pipe.pvd')
vtkfile << mesh
```

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Index