A Case Study on Optimizing Accurate Half Precision Average

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numscale
Unlocked software performance

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A Very Average Presentation

- Average is fundamental component of some ML algorithms
  - k-means, meanshift, average pooling

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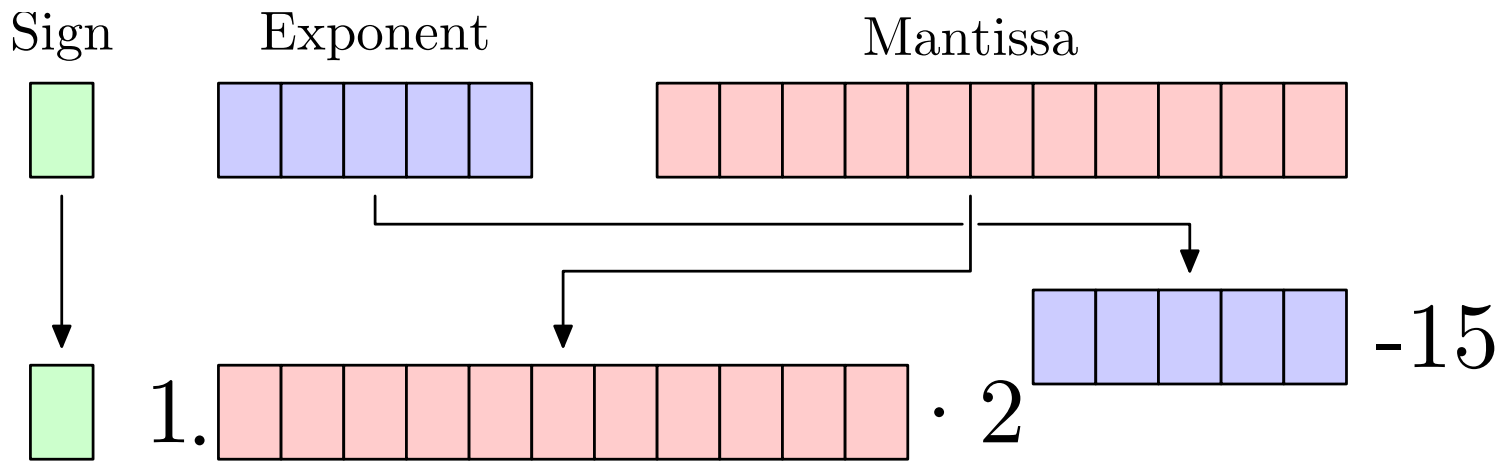
  - k-means, meanshift, average pooling
- FP16 hardware support incoming
  - Pascal GPU, ARM SVE
- FP16 precision imposes serious limitations
- Performance is important

Floating Point Format - Specifications



Precision	Exponent	Mantissa	Max	Min
Half	5	10	65504	$6.1 \cdot 10^{-5}$
Single	8	23	$3.4 \cdot 10^{38}$	$1.2 \cdot 10^{-38}$
Double	11	52	$1.8 \cdot 10^{308}$	$2.2 \cdot 10^{-308}$

Limitations

- Overflow - too big

  $65504 + 32 = \infty$

  $256 \cdot 256 = \infty$

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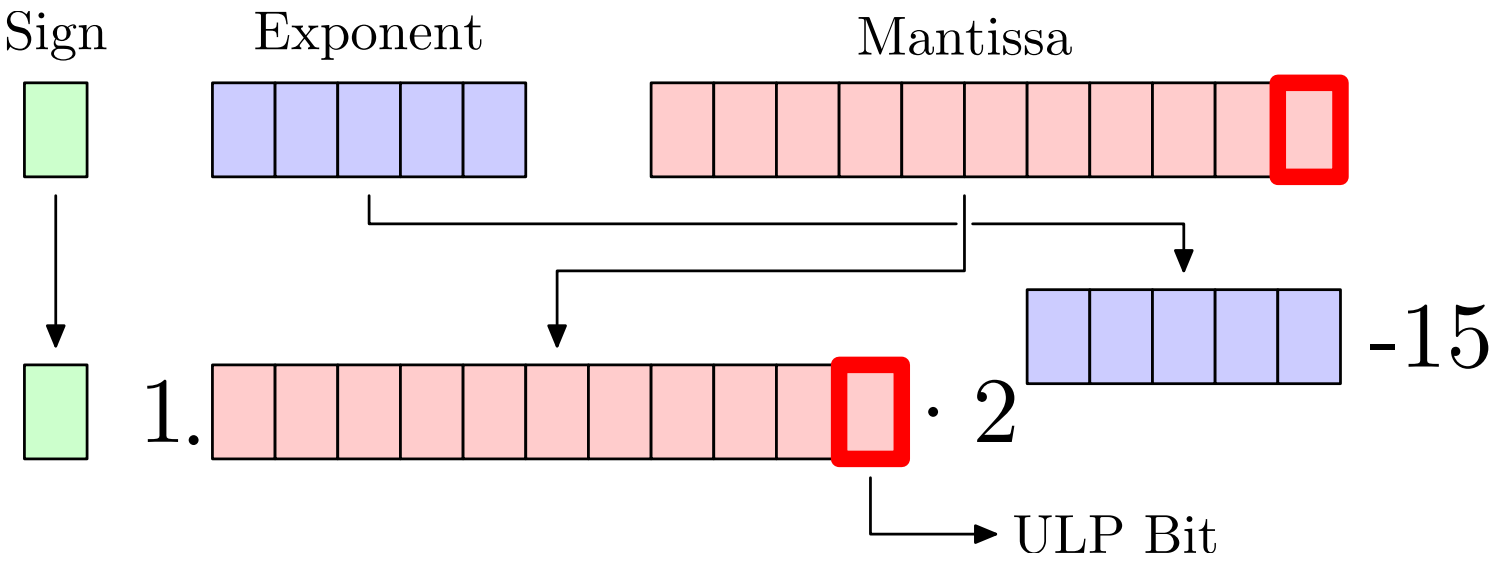
  $2048 + 1 = 2048$

  $2048 + 3.5 = 2050$

Limitations

- Overflow - too big
  $65504 + 32 = \infty$
  $256 \cdot 256 = \infty$
- Underflow/Subnormals - too small
  $1 \div 66000 = 0$
  $0.0625/64992 = 9.53674e - 07$
- Exponent (mis)alignment - juuuuuuust wrong
  $2048 + 1 = 2048$
  $2048 + 3.5 = 2050$
- Limited hardware support for FP16
  Emulated via `half` C++ library

Unit of Least Precision (ULP)

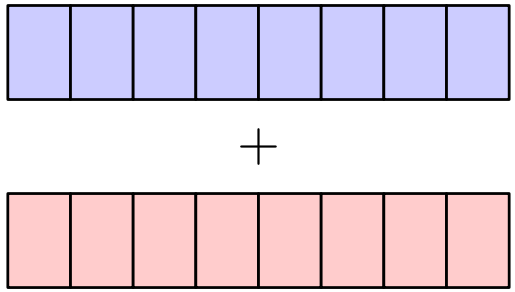


Base Value	Half ULP	Float ULP	Double ULP
2^{-10}	2^{-20}	2^{-33}	2^{-63}
1	2^{-10}	2^{-23}	2^{-53}
2^{10}	1	2^{-13}	2^{-43}

Table: Value of 1 ULP at different base values for half, single, and double precisions.

Parallelization (SIMD)

Single Instruction Single Data



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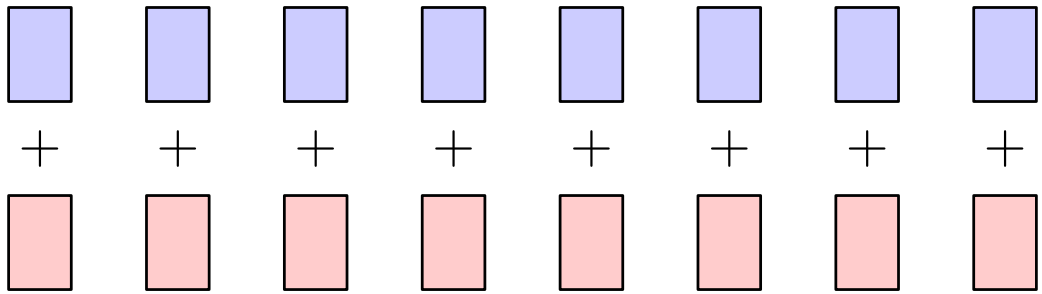

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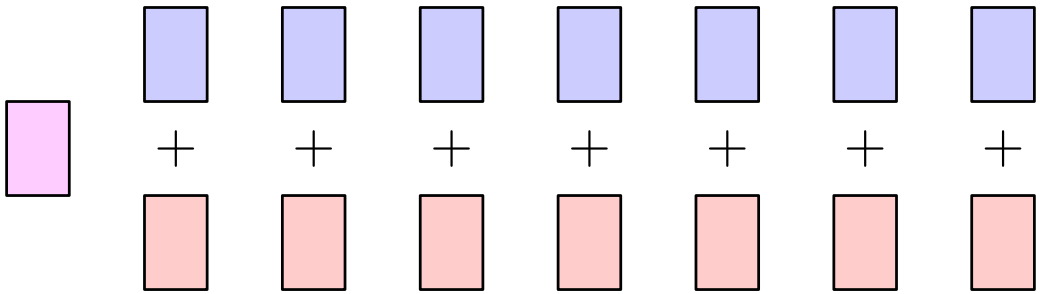

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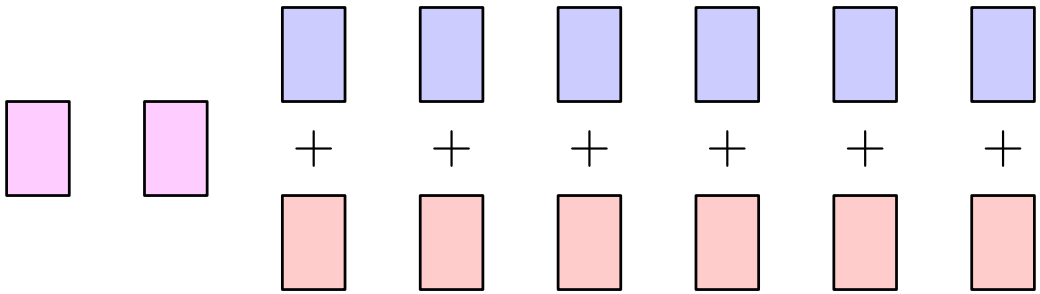

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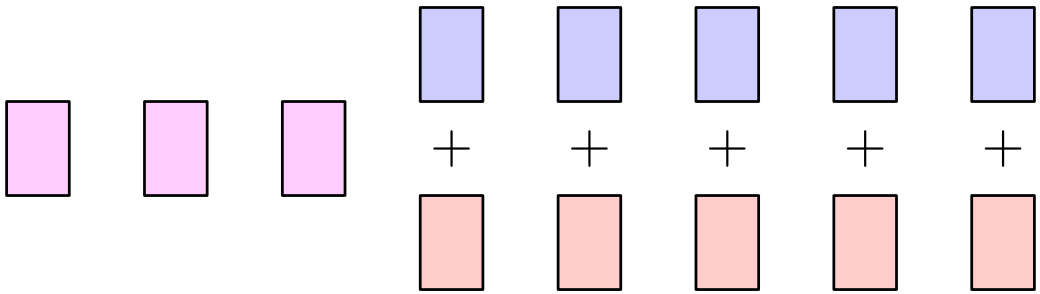

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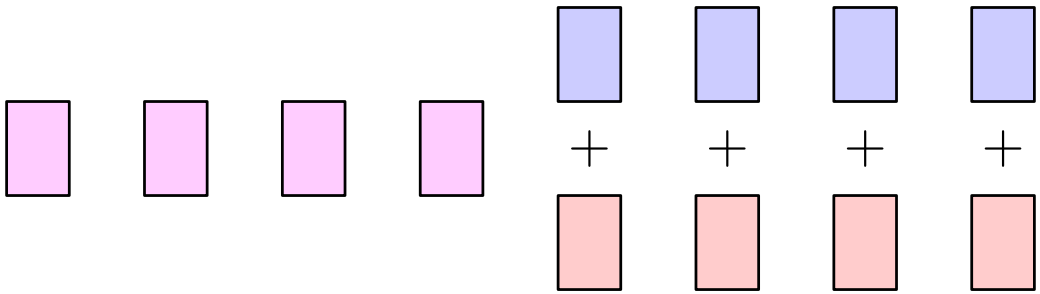

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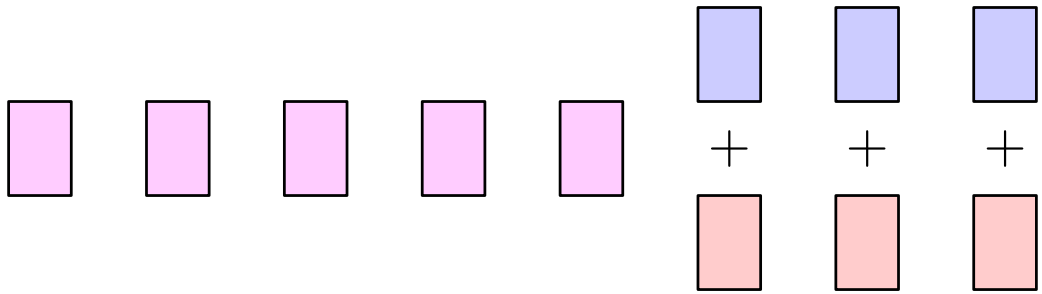

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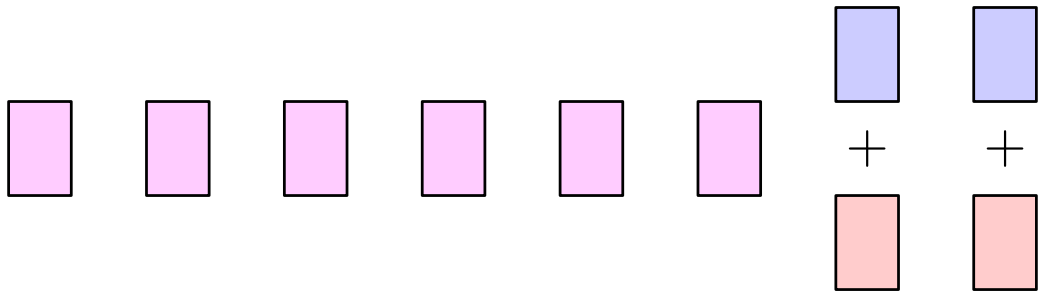

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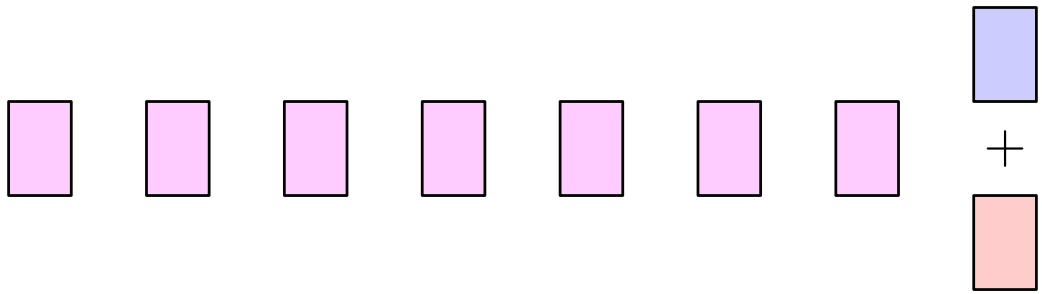

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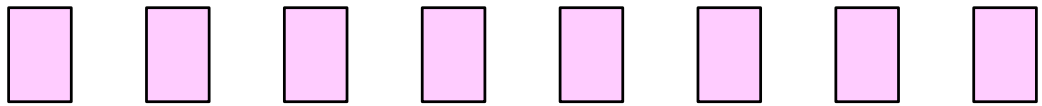

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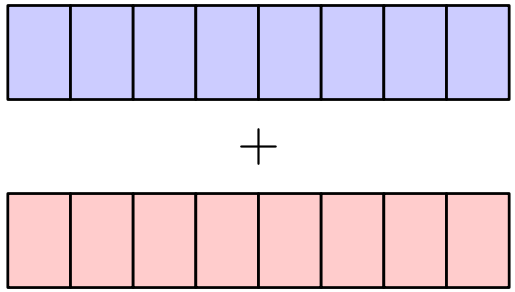
Single Instruction Multiple Data



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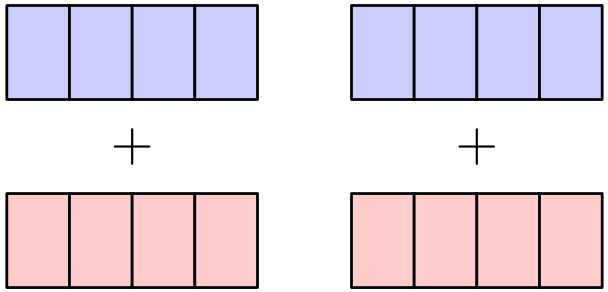

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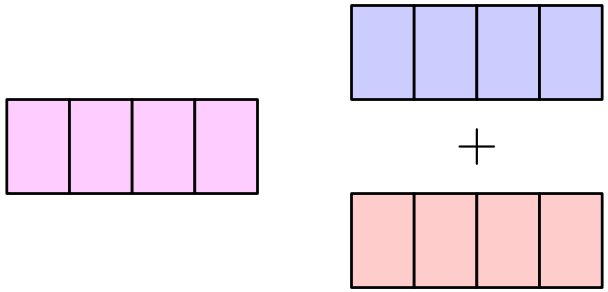

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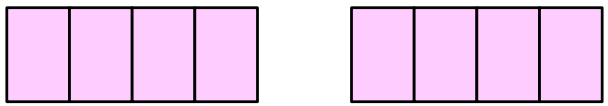


Parallelization (SIMD)

Single Instruction Multiple Data

SIMD Technologies

- GPU - CUDA, OpenCL
- Multithread - OpenMP, MPI
- CPU Vectorization - AVX, NEON, PPC
  - No communication/transfer
  - Fixed register sizes
  - Not portable (without NSIMD)

Vectorization - NSIMD

```
// AVX
__m256 a = _mm256_load_ps( &(A[i]) );
__m256 b = _mm256_load_ps( &(B[i]) );
__m256 c = _mm256_add_ps( a , b );
_mm256_store_ps( &(C[i]) , c );
```

Vectorization - NSIMD

```
// NEON
float32x4_t a = vld1q_s32( &(A[i]) );
float32x4_t b = vld1q_s32( &(B[i]) );
float32x4_t c = vaddq_f32( a , b );
vst1q_f32( &(C[i]) , c );
```

Vectorization - NSIMD

```
// VMX
__vector float a = vec_ld( 0 , &(A[i]) );
__vector float b = vec_ld( 0 , &(B[i]) );
__vector float c = vec_add( a , b );
vec_st( c , 0 , &(C[i]) );
```

Vectorization - NSIMD

```
// NSIMD (All architectures)
nsimd::pack<float> b = nsimd::load( &(B[i]) );
nsimd::pack<float> c = nsimd::load( &(C[i]) );
nsimd::pack<float> a = b * c;
nsimd::store( &(A[i]) , a );
```

Performance Considerations

- Number of Operations/Instructions
- Instruction Latency
- Data Types (SIMD consideration)

Operation	Integer	Float	SIMD Float	SIMD Double
Load/Store	1	1	1	1
Addition	1	3	3	3
Subtraction	1	3	3	3
Multiplication	3	5	5	5
Division	22	10	10	10

Table: Minimum cycles required to perform basic arithmetic operations on an Intel Haswell CPU. (Agner Fog)

Calculating the Average

Seems simple, right?

$$\text{Avg}(X) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Not with half precision!

Precision Benchmarks

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
$\text{s} = 1$	100 1000 10000					
$\text{s} = 0.001$	100 1000 10000					
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
$\text{r} = \text{N}/2$	100 1000 10000					
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
$\text{d} = \text{N}/2$	100 1000 10000					
Random	$\text{Input}[i] = \mathit{rand}()$					
$[0, 1]$ $[0, 10]$	1000 1000					
Fixed	$\text{Input}[i] = \text{c}$					
$\text{c} = 10$	1000 10000 1000000					
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300 3000 30000					
Image1 Image2	2073600 2073600					

Performance Benchmarks

Processor Used	Clock Speed	RAM
Intel i7-4790S	3.20GHz	16GB
Power8 8348-21C	2.061GHz	64GB
ARM Cortex-A57r1	1.91GHz	4GB

Benchmark	Compiler Used	Compilation Flags
Scalar	All of below	`-O3`
SSE	GCC 6.3.1	`-O3 -msse4.2`
AVX	GCC 6.3.1	`-O3 -mavx2`
NEON	GCC 5.4.1	`-O3 -march=armv8-a+simd`
Altivec	GCC 6.3.0	`-O3 -maltivec`

Performance measured using float instead of half

Naive Average

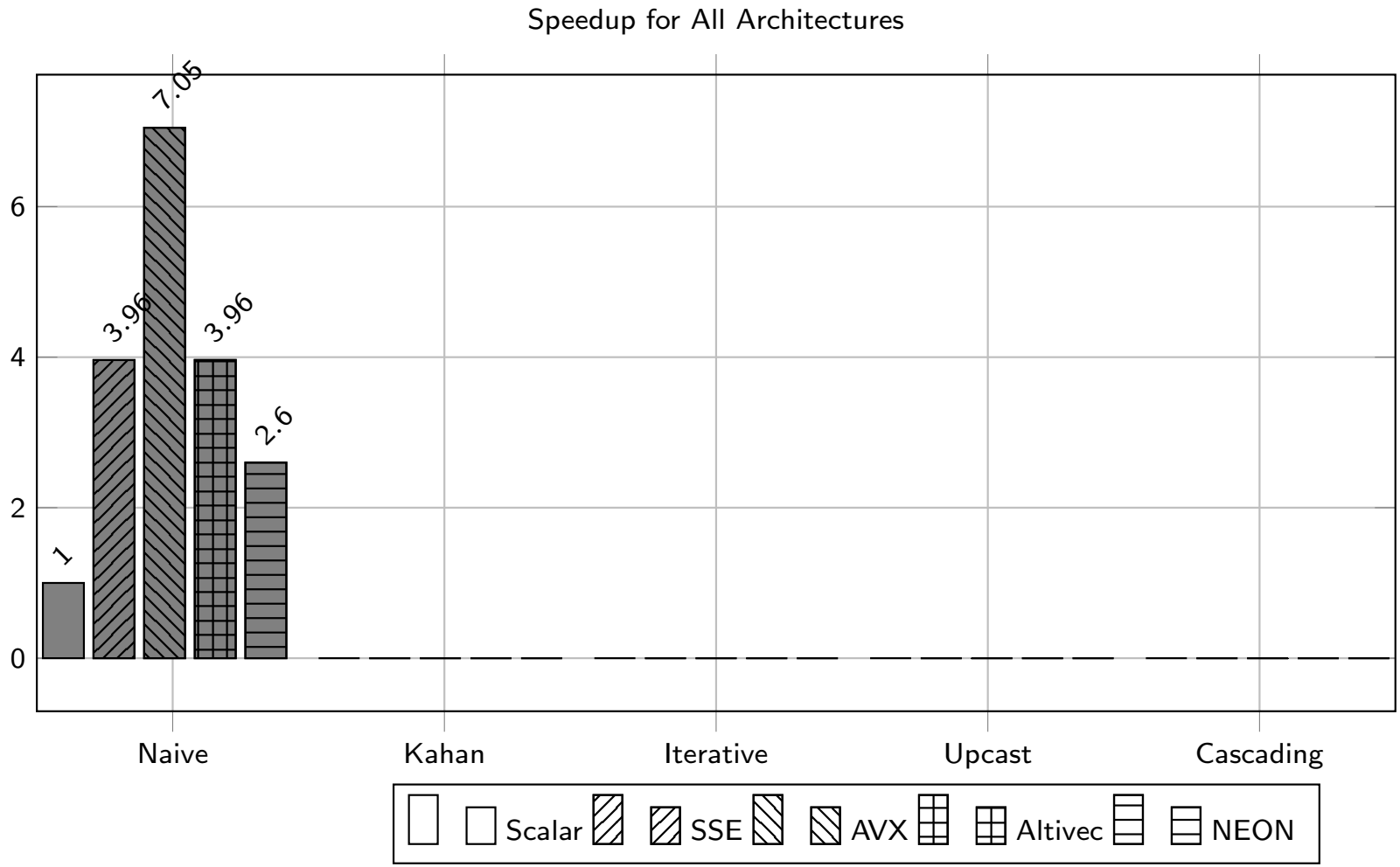
```
1: function NAIVE AVERAGE(Array)
2:     sum = 0
3:     for a in Array do
4:         sum += a
5:     end for
6:     avg = sum / length(Array)
7: end function
```

- Rounding errors gets worse and worse
- Susceptible to overflow
- Computationally simple

Naive Average

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
s = 1	100	9				
	1000	fail				
	10000	fail				
s = 0.001	100	16				
	1000	198				
	10000	fail				
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
r = N/2	100	1				
	1000	fail				
	10000	fail				
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
d = N/2	100	13				
	1000	fail				
	10000	fail				
Random	$\text{Input}[i] = rand()$					
[0, 1]	1000	271				
[0, 10]	1000	fail				
Fixed	$\text{Input}[i] = \text{c}$					
c = 10	1000	152				
	10000	1070				
	1000000	1277				
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300	17				
	3000	709				
	30000	1998				
Image1	2073600	fail				
Image2	2073600	1761				

Naive Average



Average Using Kahan Sum

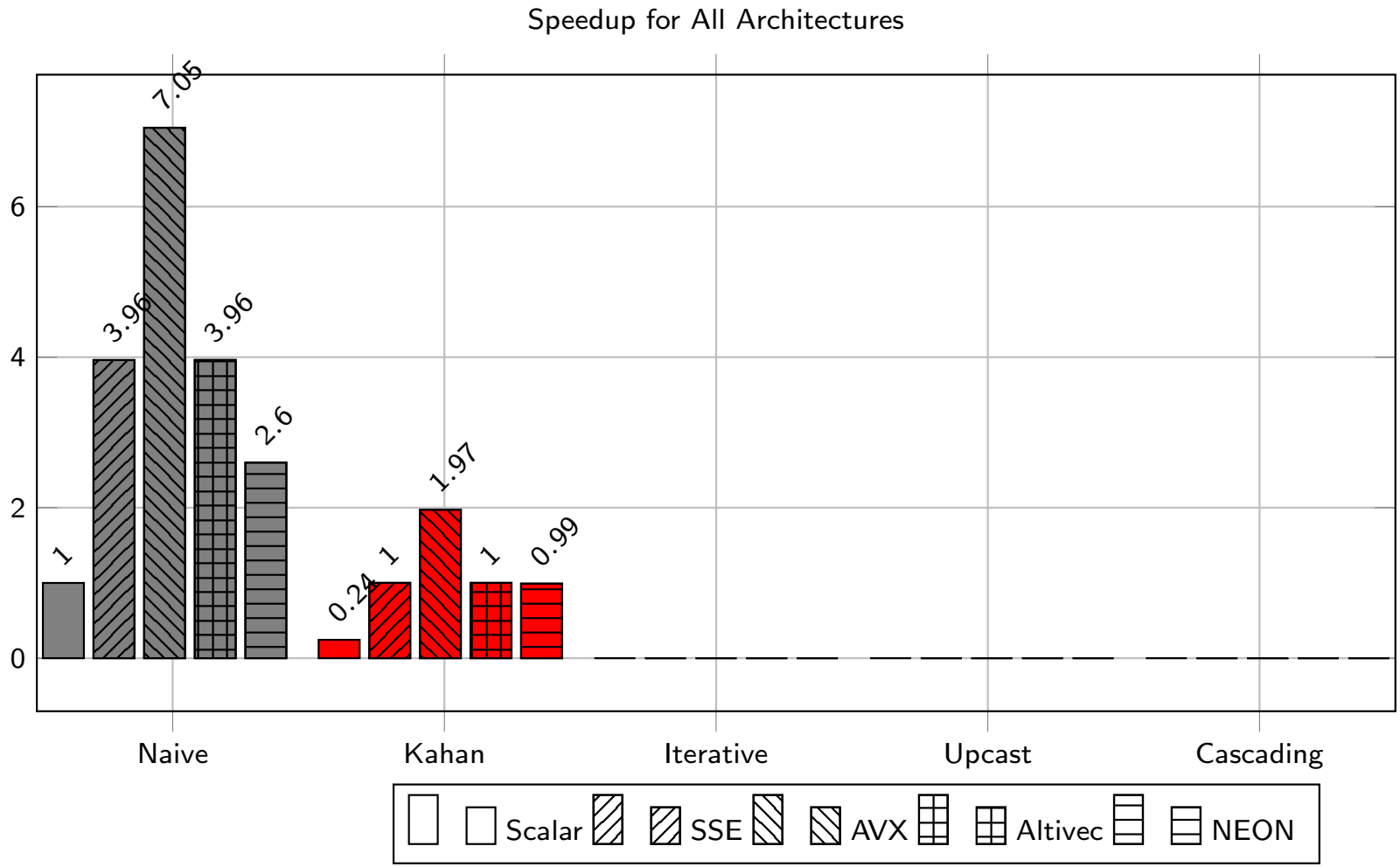
```
 1: function KAHAN AVERAGE(Array)
 2:     sum = 0
 3:     rem = 0
 4:     for a in Array do
 5:         y = a - rem
 6:         t = sum + y
 7:         rem = ( t - sum ) - y
 8:         sum = t
 9:     end for
10:     avg = sum / length(Array)
11: end function
```

- Mostly compensates rounding errors
- Still susceptible to overflow and misalignments

Average Using Kahan Sum

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
s = 1	100	9	1			
	1000	fail	fail			
	10000	fail	fail			
s = 0.001	100	16	1			
	1000	198	1			
	10000	fail	fail			
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
r = N/2	100	1	1			
	1000	fail	fail			
	10000	fail	fail			
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
d = N/2	100	13	1			
	1000	fail	fail			
	10000	fail	fail			
Random	$\text{Input}[i] = \textit{rand}()$					
[0, 1]	1000	271	0			
[0, 10]	1000	fail	fail			
Fixed	$\text{Input}[i] = \text{c}$					
c = 10	1000	152	0			
	10000	1070	fail			
	1000000	1277	fail			
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300	17	0			
	3000	709	1			
	30000	1998	fail			
Image1	2073600	fail	fail			
Image2	2073600	1761	391			

Average Using Kahan Sum



Iterative Average

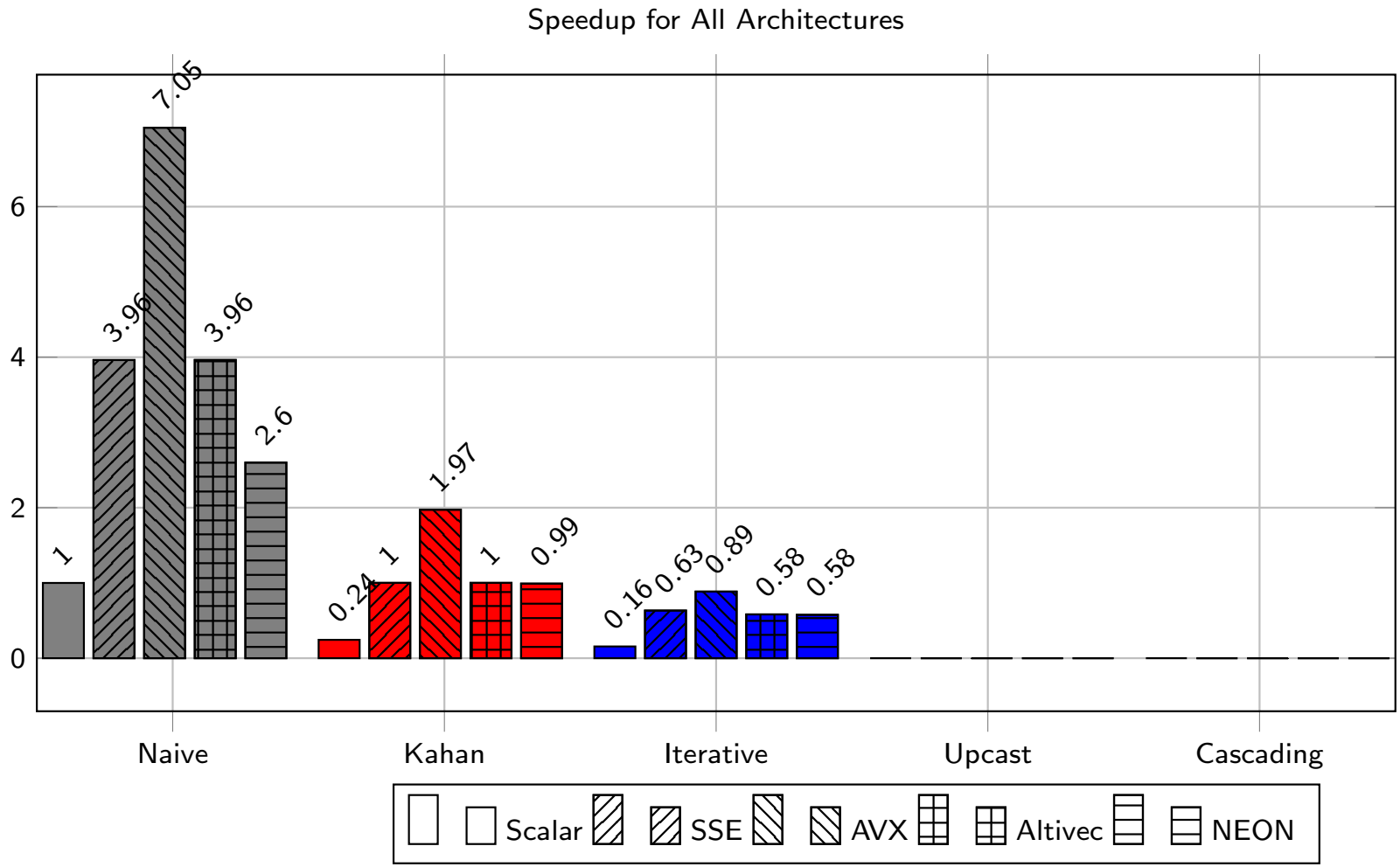
```
1: function ITERATIVE AVERAGE(Array)
2:     avg = 0
3:     for i in 1 → length(Array) do
4:         avg += (avg - Array[i]) / i
5:     end for
6: end function
```

- Removes risk of overflow
- Misalignment, underflow, and subnormals get progressively worse
- $$\lim_{i \to \infty} \frac{(x_i - \text{Avg}_i)}{i} = 0$$

Iterative Average

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
$\text{s} = 1$	100	9	1	0		
	1000	fail	fail	0		
	10000	fail	fail	993		
$\text{s} = 0.001$	100	16	1	20		
	1000	198	1	249		
	10000	fail	fail	1486		
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
$\text{r} = \text{N}/2$	100	1	1	17		
	1000	fail	fail	3		
	10000	fail	fail	994		
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
$\text{d} = \text{N}/2$	100	13	1	23		
	1000	fail	fail	227		
	10000	fail	fail	737		
Random	$\text{Input}[i] = \mathit{rand}()$					
$[0, 1]$	1000	271	0	249		
$[0, 10]$	1000	fail	fail	261		
Fixed	$\text{Input}[i] = \text{c}$					
$\text{c} = 10$	1000	152	0	0		
	10000	1070	fail	0		
	1000000	1277	fail	0		
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300	17	0	74		
	3000	709	1	128		
	30000	1998	fail	128		
Image1	2073600	fail	fail	1037		
Image2	2073600	1761	391	1701		

Iterative Average



Increased Precision

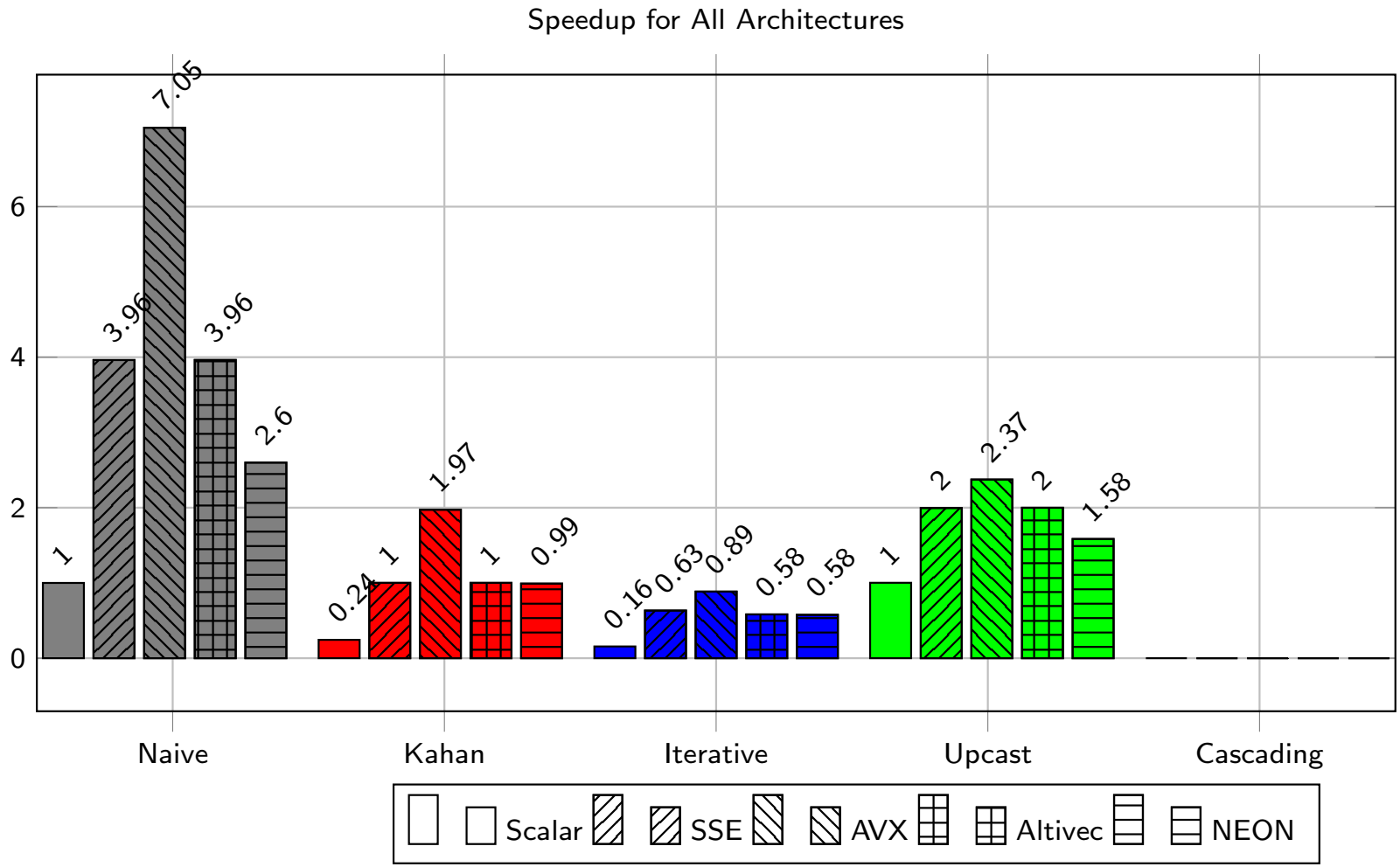
```
1: function UPCAST AVERAGE(Array)
2:     sum = (upcast)0
3:     for a in Array do
4:         sum += (upcast)a
5:     end for
6:     avg = sum / length(Array)
7: end function
```

- Same fundamental problems as naive average
- Limitations less problematic

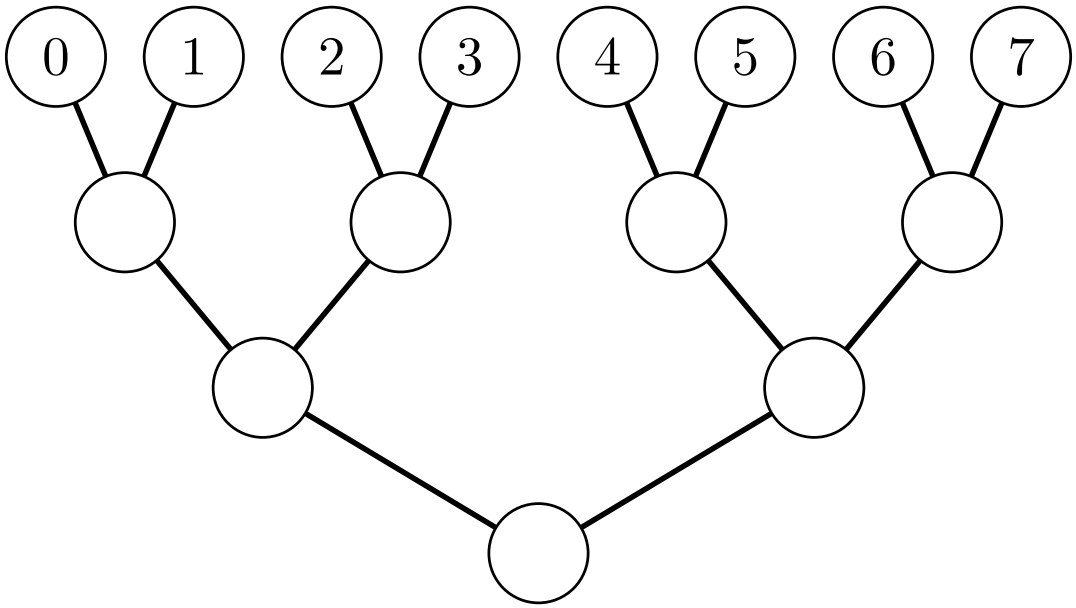
Increased Precision

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
s = 1	100	9	1	0	0	
	1000	fail	fail	0	0	
	10000	fail	fail	993	0	
s = 0.001	100	16	1	20	0	
	1000	198	1	249	0	
	10000	fail	fail	1486	0	
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
r = N/2	100	1	1	17	0	
	1000	fail	fail	3	0	
	10000	fail	fail	994	0	
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
d = N/2	100	13	1	23	0	
	1000	fail	fail	227	0	
	10000	fail	fail	737	0	
Random	$\text{Input}[i] = rand()$					
[0, 1]	1000	271	0	249	0	
[0, 10]	1000	fail	fail	261	0	
Fixed	$\text{Input}[i] = \text{c}$					
c = 10	1000	152	0	0	0	
	10000	1070	fail	0	0	
	1000000	1277	fail	0	0	
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300	17	0	74	0	
	3000	709	1	128	0	
	30000	1998	fail	128	0	
Image1	2073600	fail	fail	1037	1	
Image2	2073600	1761	391	1701	12	

Increased Precision



Cascading Average



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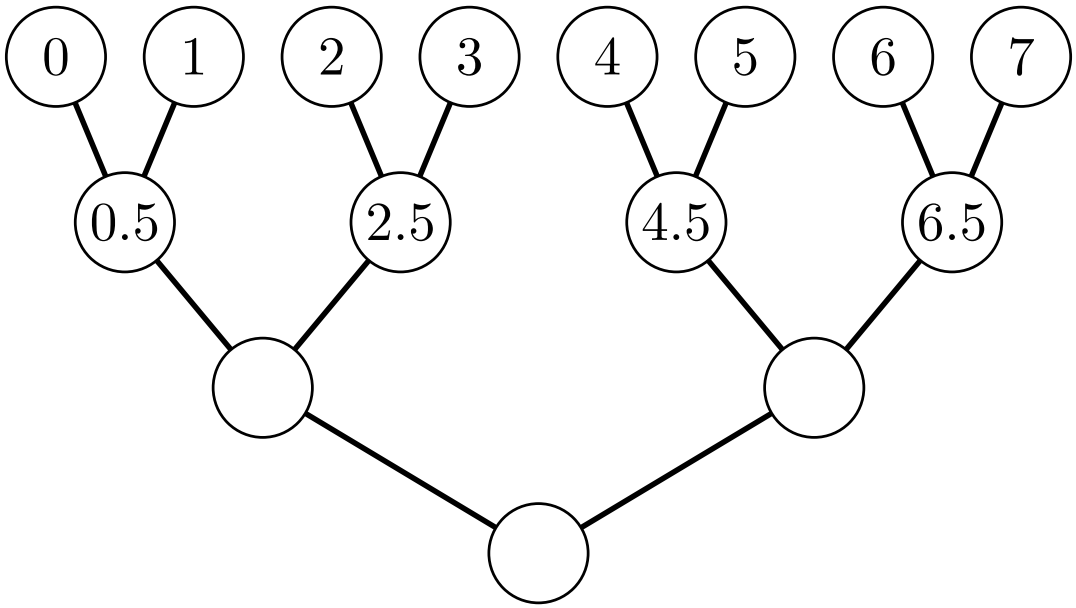

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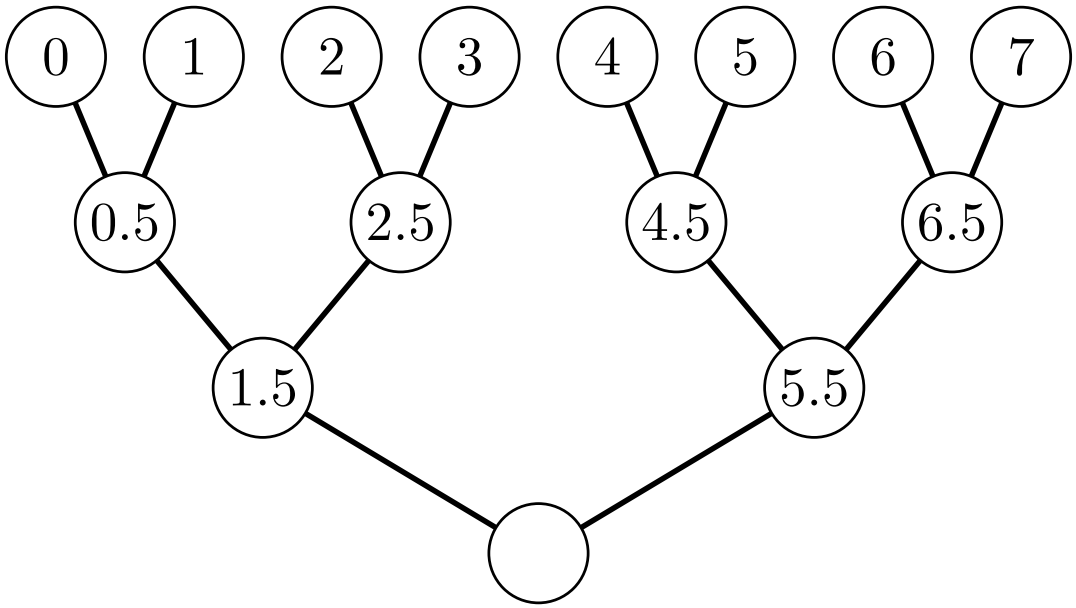

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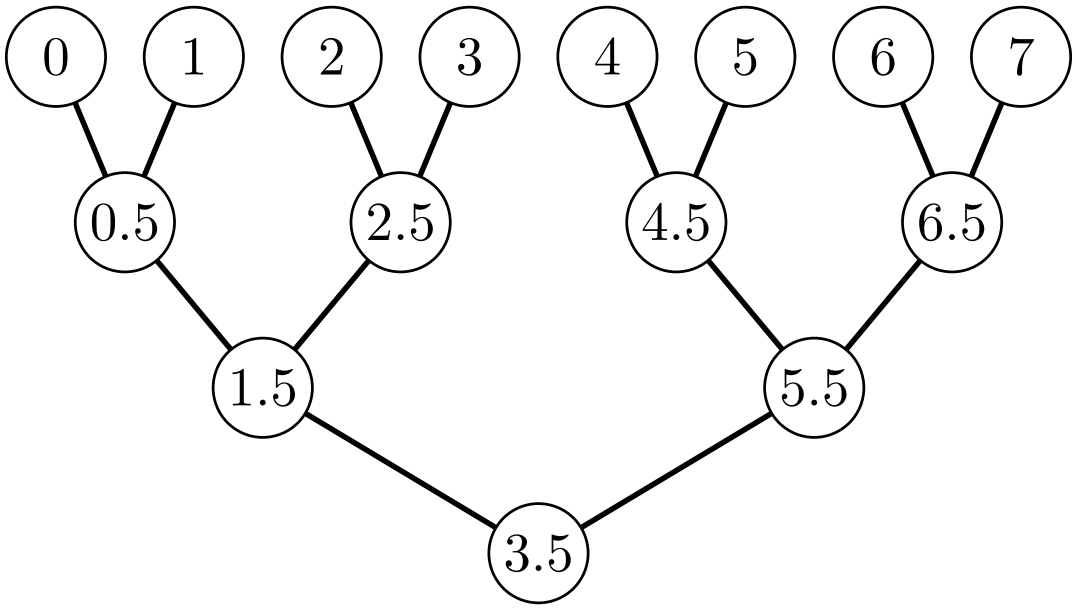
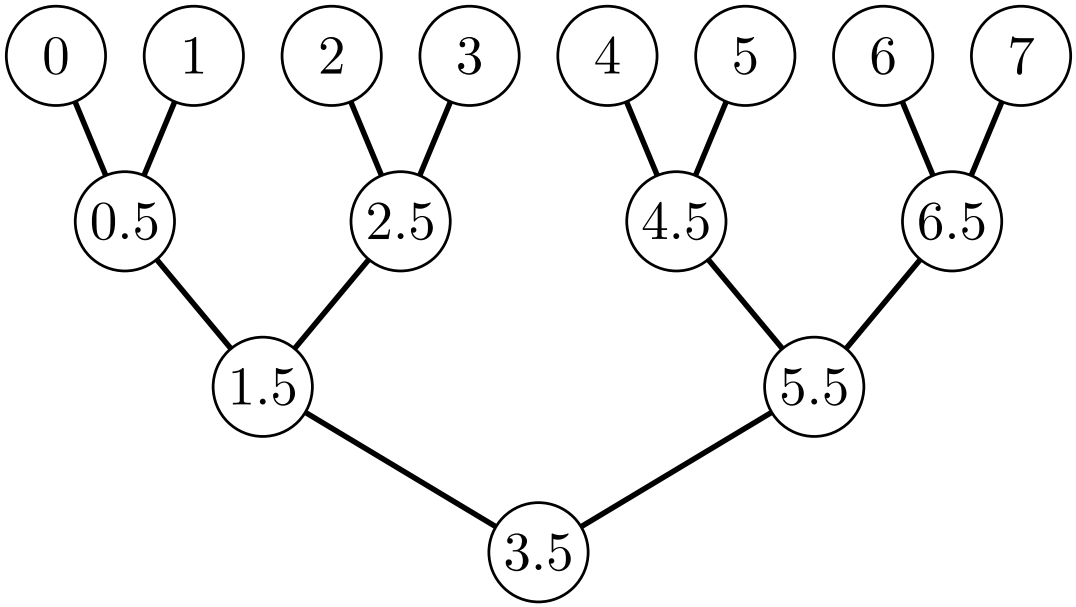


Cascading Average



- Pairwise operations
- No accumulator = no overflow
- Rounding errors from input variation
- $\leq 2N$ operations

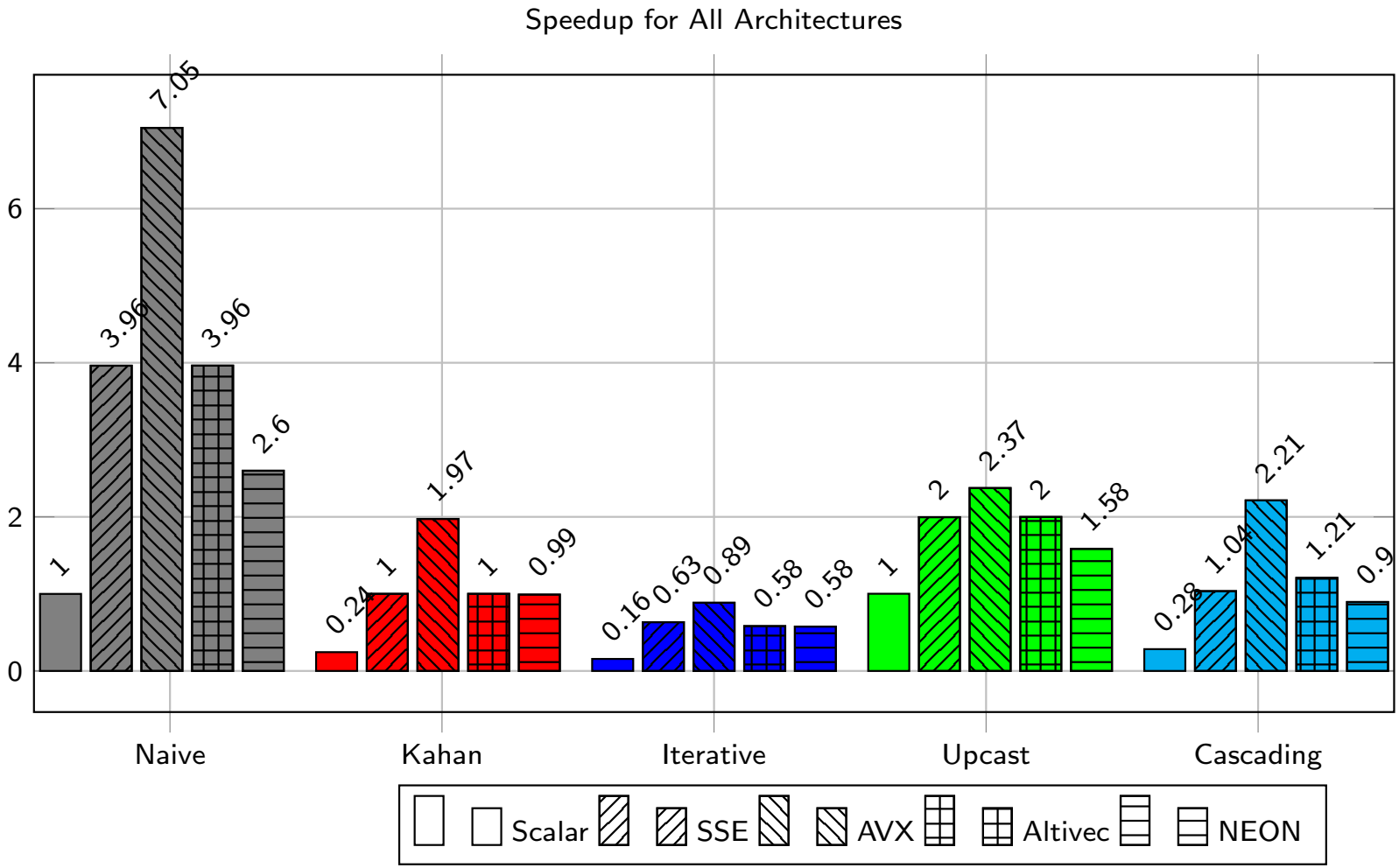
Cascading Average

```
1: function CASCADING AVERAGE(Array)
2:     n = length(Array)
3:     if n == 1 then
4:         return Array[0]
5:     else if n == 2 then
6:         return ( Array[0] + Array[1] ) / 2
7:     else
8:         return ( Cascading Average( Array[1:n/2] ) + Cascading
   Average( Array[n/2:n] ) ) / 2
9:     end if
10: end function
```

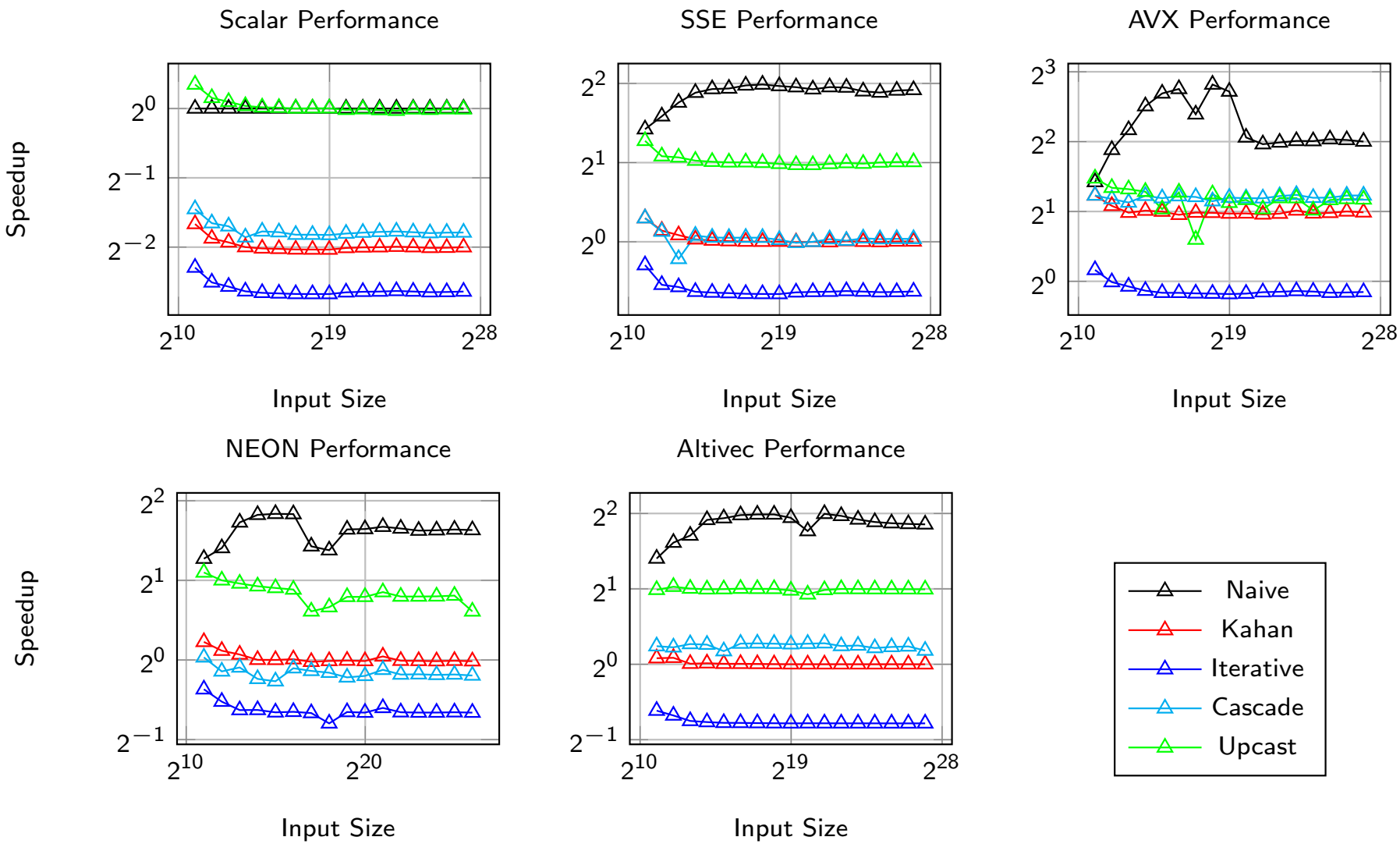
Cascading Average

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	$\text{Input}[i] = \text{s} \cdot i$					
s = 1	100	9	1	0	0	0
	1000	fail	fail	0	0	0
	10000	fail	fail	993	0	1
s = 0.001	100	16	1	20	0	2
	1000	198	1	249	0	4
	10000	fail	fail	1486	0	4
Fixed Ratio	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
r = N/2	100	1	1	17	0	0
	1000	fail	fail	3	0	0
	10000	fail	fail	994	0	1
Fixed Diff	$\text{Input}[i] = \begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
d = N/2	100	13	1	23	0	0
	1000	fail	fail	227	0	0
	10000	fail	fail	737	0	0
Random	$\text{Input}[i] = rand()$					
[0, 1]	1000	271	0	249	0	3
[0, 10]	1000	fail	fail	261	0	4
Fixed	$\text{Input}[i] = \text{c}$					
c = 10	1000	152	0	0	0	0
	10000	1070	fail	0	0	0
	1000000	1277	fail	0	0	0
Repeating	$\text{Input}[i] = 10 + (i\%3)$					
10 - 12	300	17	0	74	0	1
	3000	709	1	128	0	1
	30000	1998	fail	128	0	1
Image1	2073600	fail	fail	1037	1	3
Image2	2073600	1761	391	1701	12	7

Cascading Average



Performance - Overall



Analysis

- Sum then divide methods overflow easily
- Iterative average method fails for large N
- Kahan sum (with divide when necessary) is the most precise when it works
- Naive method is fast enough to increase precision and still be faster (with SIMD)
- Cascading Average is never a bad choice thanks to its robustness

Conclusion

- Numerical precision is complicated
- Performance requires effort
- Even simple things can need research

Future Works

- Similar study using 8 bit floats
- Similar study with exotic numerical formats
- Mix algorithms intelligently
- Look into other "solved" algorithms under half precision

THE END

Full Precision Results

Input	N	Naive	Kahan	Iterative	Upcast	Cascade
Sequential	Input[i] = s $\cdot$ i					
s = 1	100	9	1	0	0	0
	1000	fail	fail	0	0	0
	10000	fail	fail	993	0	1
s = 0.001	100	16	1	20	0	2
	1000	198	1	249	0	4
	10000	fail	fail	1486	0	4
s = -1	100	9	1	0	0	0
	1000	fail	fail	0	0	0
	10000	fail	fail	993	0	1
Fixed Ratio	Input[i] = $\begin{cases} \text{even: } i \div 2 \\ \text{odd: } i \div (2 \cdot \text{r}) \end{cases}$					
r = N/2	100	1	1	17	0	0
	1000	fail	fail	3	0	0
	10000	fail	fail	994	0	1
Fixed Diff	Input[i] = $\begin{cases} \text{even: } i \div 2 \\ \text{odd: } (i \div 2) + \text{d} \end{cases}$					
d = N/2	100	13	1	23	0	0
	1000	fail	fail	227	0	0
	10000	fail	fail	737	0	0
Random	Input[i] = $rand()$					
[0, 1]	1000	271	0	249	0	3
[0, 10]	1000	fail	fail	261	0	4
[0, 100]	1000	fail	fail	246	0	2
[0, 1000]	1000	fail	fail	253	0	3
Fixed	Input[i] = c					
	1000	152	0	0	0	0
c = 10	10000	1070	fail	0	0	0
	100000	1259	fail	0	0	0
	1000000	1277	fail	0	0	0
Repeating	Input[i] = 10 + (i%3)					
10 - 12	300	17	0	74	0	1
	3000	709	1	128	0	1
	30000	1998	fail	128	0	1
	300000	1401	fail	128	0	1
Image1	2073600	fail	fail	1037	1	3
Image2	2073600	1761	391	1701	12	7