Assignment-18

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***Abstract*—In this document, we explore the properties of eigenvalues of non-diagonalizable matrices.**

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_Assignments /tree/master/Assignment_18

1 Problem Statement

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over $\mathbf{C}$. Then,

1) $\mathbf{AB}$ and $\mathbf{BA}$ always have the same set of eigenvalues.
2) If $\mathbf{AB}$ and $\mathbf{BA}$ have the same set of eigenvalues then $\mathbf{AB} = \mathbf{BA}$
3) If $\mathbf{A}^{-1}$ exists, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar
4) The rank of $\mathbf{AB}$ is always the same as the rank of $\mathbf{BA}$.

2 Answers for different cases

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<td>AB and BA always have the same set of eigenvalues.</td>
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True.

Let λ be an eigenvalue of **AB**, and **x** be a corresponding eigenvector.
Then

$$\mathbf{ABx} = \lambda\mathbf{x}$$

Left-multiplying by **B**:

$$\mathbf{B}(\mathbf{AB})\mathbf{x} = \mathbf{B}(\lambda\mathbf{x})$$
$$(\mathbf{BA})\mathbf{Bx} = \lambda(\mathbf{Bx}) \text{ (by associativity of multiplication)}$$

$\implies$ λ is an eigenvalue of **BA** with **Bx** as the corresponding eigenvector, assuming **Bx** is not a null vector.
If **Bx** is null, then **B** is singular, so that both **AB** and **BA** are singular, and $\lambda = 0$. Since both the products are singular, 0 is an eigenvalue of both.

Example:
Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$

Then

$$\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$

Since **AB** and **BA** results with the same characteristic equation,

$$\lambda^2 + 2\lambda = 0$$

they will have same set of eigenvalues that is $\lambda_1 = 0, \lambda_2 = -2$

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<tr>
<td>If AB and BA have the same set of eigenvalues then AB = BA</td>
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False.

Counter example:
Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$

then

$$\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$

$\implies$ Same eigenvalues ($\lambda_1 = 0, \lambda_2 = -2$), but $\mathbf{AB} \neq \mathbf{BA}$

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If $\mathbf{A}^{-1}$ exists, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar	**True**. Given that $\mathbf{A}^{-1}$ exists and hence, $\mathbf{AB} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{BA} = \mathbf{BA}.$ Hence, $\mathbf{AB} \simeq \mathbf{BA}$ Example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A}$ $= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$ $= \mathbf{BA}$
The rank of $\mathbf{AB}$ is always the same as the rank of $\mathbf{BA}$.	**False**. Counter example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ From the above $\mathbf{AB}$ and $\mathbf{BA}$, it is noted that the rank($\mathbf{AB}$) = 2 and rank($\mathbf{BA}$)=1. Hence the rank of $\mathbf{AB}$ need not always be same as rank of $\mathbf{BA}$.