Assignment-18

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Abstract—In this document, we explore the properties of eigenvalues of non-diagonalizable matrices.

Download all latex-tikz codes from

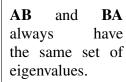
https://github.com/poojah15/EE5609_Assignments/tree/master/Assignment 18

1 Problem Statement

Let **A** and **B** be $n \times n$ matrices over **C**. Then,

- 1) **AB** and **BA** always have the same set of eigenvalues.
- 2) If AB and BA have the same set of eigenvalues then AB = BA
- 3) If A^{-1} exists, then AB and BA are similar
- 4) The rank of **AB** is always the same as the rank of **BA**.

2 Answers for different cases



True.

Let λ be an eigenvalue of AB, and \mathbf{x} be a corresponding eigenvector.

Then

$$ABx = \lambda x$$

Left-multiplying by **B**:

$$\mathbf{B}(\mathbf{A}\mathbf{B})\mathbf{x} = \mathbf{B}(\lambda \mathbf{x})$$

$$(\mathbf{B}\mathbf{A})\mathbf{B}\mathbf{x} = \lambda(\mathbf{B}\mathbf{x})$$
 (by associativity of multiplication)

 $\implies \lambda$ is an eigenvalue of **BA** with **Bx** as the corresponding eigenvector, assuming **Bx** is not a null vector.

If **Bx** is null, then **B** is singular, so that both **AB** and **BA** are singular, and $\lambda = 0$. Since both the products are singular, 0 is an eigenvalue of both.

Example:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$

Then

$$\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$

Since AB and BA results with the same characteristic equation,

$$\lambda^2 + 2\lambda = 0$$

they will have same set of eigenvalues that is $\lambda_1 = 0, \lambda_2 = -2$

If **AB** and **BA** have the same set of eigenvalues then **AB** = **BA**

False.

Counter example:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$

then

$$\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$

 \implies Same eigenvalues $(\lambda_1 = 0, \lambda_2 = -2)$, but $\mathbf{AB} \neq \mathbf{BA}$

If A^{-1} exists, then AB and BA are similar

True.

Given that A^{-1} exists and hence,

$$\mathbf{AB} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{BA} = \mathbf{BA}.$$

Hence, $AB \simeq BA$

Example:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$

then

$$\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A}$$
$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$
$$= \mathbf{BA}$$

The rank of **AB** is always the same as the rank of **BA**.

False.

Counter example:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{BA} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the above AB and BA, it is noted that the rank(AB) = 2 and rank(BA)=1. Hence the rank of AB need not always be same as rank of BA.