Assignment-17

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***Abstract*—In this document, we solve for basis of the vector space of a transformation matrix.**

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_Assignments/tree/master/Assignment_17

1 Problem Statement

Let $\mathbf{P}_3$ be the vector space of polynomials with real coefficients and of at most degree 3. Consider the linear map $T : \mathbf{P}_3 \to \mathbf{P}_3$ defined by $T(p(x)) = p(x+1) + p(x-1)$. Which of the following does the matrix of T (with respect to the standard basis $\mathbf{B} = \{1, x, x^2, x^3\}$ of $\mathbf{P}_3$) satisfy?

1) $\det(T) = 0$
2) $(T - 2\mathbf{I})^4 = 0$ but $(T - 2\mathbf{I})^3 \neq 0$
3) $(T - 2\mathbf{I})^3 = 0$ but $(T - 2\mathbf{I})^2 \neq 0$
4) 2 is an eigenvalue with multiplicity 4.

2 Solution

Given

$$T(p(x)) = p(x+1) + p(x-1). \tag{2.0.1}$$

The matrix of T with respect to the standard basis $\mathbf{B} = \{1, x, x^2, x^3\}$ is given by:

$$p(x) = 1 \implies T(1) = 1 + 1$$
$$= 2 \tag{2.0.2}$$
$$p(x) = x \implies T(x) = x + 1 + x - 1$$
$$= 2x \tag{2.0.3}$$
$$p(x) = x^2 \implies T(x^2) = (x+1)^2 + (x-1)^2$$
$$= 2 + 2x^2 \tag{2.0.4}$$
$$p(x) = x^3 \implies T(x^3) = (x+1)^3 + (x-1)^3$$
$$= 6x + 2x^3 \tag{2.0.5}$$

Hence, matrix of T is:

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{2.0.6}$$

3 Answers for different cases

$\det(T) = 0$	**False**. From (2.0.6), it is found that the determinant is not zero as the eigenvalues are nonzero.
$(T - 2\mathbf{I})^4 = 0$ but $(T - 2\mathbf{I})^3 \neq 0$	**False**. $(T - 2\mathbf{I}) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\implies (T - 2\mathbf{I})^2 = 0$ and hence $(T - 2\mathbf{I})^4 = 0$ and $(T - 2\mathbf{I})^3 = 0$
$(T - 2\mathbf{I})^3 = 0$ but $(T - 2\mathbf{I})^2 \neq 0$	**False**. Because $(T - 2\mathbf{I})^3 = 0$ and $(T - 2\mathbf{I})^2 = 0$
2 is an eigenvalue with multiplicity 4.	**True**. It is noted that the matrix of T is an upper triangular matrix having the value 2 along its principal diagonal and hence 2 is an eigenvalue with algebraic multiplicity 4.