

Assignment-11

Pooja H

Abstract—In this document, we find the basis for the space \mathbf{V}

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_Assignments/tree/master/Assignment_11

1 PROBLEM STATEMENT

Let \mathbf{V} be the space of 2×2 matrices over \mathbf{F} . Find a basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ for \mathbf{V} such that $\mathbf{A}_j^2 = \mathbf{A}_j$ for each j

2 SOLUTION

Every 2×2 matrix may be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.1)$$

This shows that

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (2.0.2)$$

can be the basis for the space \mathbf{V} of all 2×2 matrices. However \mathbf{A}_2 and \mathbf{A}_3 doesn't satisfy the property of $\mathbf{A}^2 = \mathbf{A}$. Consider $b = 0$ and $c = 0$, then the matrix

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad (2.0.3)$$

can't be a basis as it is the linear combination of \mathbf{A}_1 and \mathbf{A}_4 . Hence either b or c or both must be non zero. Hence,

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad (2.0.5)$$

Here, $\mathbf{A}_2^2 = \mathbf{A}_2$ and $\mathbf{A}_3^2 = \mathbf{A}_3$. Therefore the basis can be

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (2.0.6)$$

$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ forms the basis, iff they are linearly independent and the linear combination of them span the space \mathbf{V} . To show that they are linearly independent, we show that the equation has a trivial solution.

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.7)$$

$$\implies a + b = 0 \quad (2.0.8)$$

$$b = 0 \quad (2.0.9)$$

$$c = 0 \quad (2.0.10)$$

$$c + d = 0 \quad (2.0.11)$$

The corresponding matrix form is $\mathbf{Ax} = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.12)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_3]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.0.14)$$

Therefore, $a = b = c = d = 0$. Hence the matrices are linearly independent. To show that the linear combination of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ span the space \mathbf{V} , consider an arbitrary matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad (2.0.15)$$

Compute a, b, c, d such that

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.16)$$

$$= \begin{pmatrix} a+b & c \\ b & c+d \end{pmatrix} \quad (2.0.17)$$

Equating the entries, this produces system of linear equations,

$$a + b = w, y = b, x = c, z = c + d \quad (2.0.18)$$

$$\implies a = w - y \quad (2.0.19)$$

$$b = y \quad (2.0.20)$$

$$c = x \quad (2.0.21)$$

$$d = z - x \quad (2.0.22)$$

In particular, there exists atleast one solution regardless of the values of w, x, y, z . For example, consider the following matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \quad (2.0.23)$$

Here, $a = 5, b = -2, c = 4, d = 3$. Using (2.0.16), we get

$$5 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \quad (2.0.24)$$

Hence $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ forms the basis for the given space \mathbf{V} .