Assignment-9

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***Abstract*—In this document, we check whether the given lines intersect. If they don't, then find the closest points using SVD.**

Download all Python codes from

https://github.com/poojah15/EE5609_Assignments/tree/master/Assignment_9

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_Assignments/tree/master/Assignment_9

1 Problem Statement

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{1.0.1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{1.0.2}$$

find the value of k

2 Solution

In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}-3\\2k\\2\end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix}3\\1\\6\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}3k\\1\\-5\end{pmatrix} \tag{2.0.1}$$

To find the value of k, let's assume that the given lines are perpendicular to each other. Then the dot product of their direction vectors should be 0. i.e.,

$$\mathbf{m}_1\mathbf{m}_2 = 0 \tag{2.0.2}$$

$$\implies \begin{pmatrix}-3\\2k\\2\end{pmatrix}\begin{pmatrix}3k\\1\\-5\end{pmatrix} = 0 \tag{2.0.3}$$

$$\implies k = -\frac{10}{7} \tag{2.0.4}$$

The lines will intersect if

$$\mathbf{A}_1 + \lambda_1\mathbf{m}_1 = \mathbf{A}_2 + \lambda_2\mathbf{m}_2 \tag{2.0.5}$$

$$\implies \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}-3\\2k\\2\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} + \lambda_2\begin{pmatrix}3k\\1\\-5\end{pmatrix} \tag{2.0.6}$$

$$\implies \lambda_1\begin{pmatrix}-3\\2k\\2\end{pmatrix} - \lambda_2\begin{pmatrix}3k\\1\\-5\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \tag{2.0.7}$$

$$\implies \begin{pmatrix}-3 & 3k\\2k & 1\\2 & -5\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix} \tag{2.0.8}$$

$$\implies \begin{pmatrix}-3 & -\frac{30}{7}\\-\frac{20}{7} & 1\\2 & -5\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix} \tag{2.0.9}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}-3 & -\frac{30}{7} & 2\\-\frac{20}{7} & 1 & -1\\2 & -5 & 3\end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2+\frac{20}{7}R_1]{R_1\leftarrow -\frac{R_1}{3}} \begin{pmatrix}1 & \frac{10}{7} & -\frac{2}{3}\\0 & \frac{249}{49} & -\frac{61}{21}\\2 & -5 & 3\end{pmatrix} \tag{2.0.10}$$

$$\xleftrightarrow[R_2\leftarrow \frac{49}{249}R_2]{R_3\leftarrow R_3-2R_1} \begin{pmatrix}1 & \frac{10}{7} & -\frac{2}{3}\\0 & \frac{249}{49} & -\frac{61}{21}\\2 & -5 & 3\end{pmatrix} \tag{2.0.11}$$

$$\xleftrightarrow[R_2\leftarrow \frac{49}{249}R_2]{R_3\leftarrow R_3-2R_1} \begin{pmatrix}1 & \frac{10}{7} & -\frac{2}{3}\\0 & 1 & -\frac{427}{747}\\0 & -\frac{55}{7} & \frac{13}{3}\end{pmatrix} \tag{2.0.12}$$

$$\xleftrightarrow[R_3\leftarrow -\frac{747}{118}R_3]{R_3\leftarrow R_3+\frac{55}{7}R_2} \begin{pmatrix}1 & \frac{10}{7} & -\frac{2}{3}\\0 & 1 & -\frac{427}{747}\\0 & 0 & 1\end{pmatrix} \tag{2.0.13}$$

$$\xleftrightarrow[R_1\leftarrow R_1+\frac{2}{3}R_3-\frac{10}{7}R_2]{R_2\leftarrow R_2+\frac{427}{47}R_3} \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{2.0.14}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect which implies that the given lines are skew lines. To find the closest points using

SVD, consider the equation (2.0.9) which can be expressed as

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.15}$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.16}$$

where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{MM}^T = \begin{pmatrix} \frac{1341}{49} & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 \end{pmatrix} \tag{2.0.18}$$

2.1 *To get* $\mathbf{V}$ *and* $\mathbf{S}$

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1037}{49} - \lambda & 0 \\ 0 & \frac{2174}{49} - \lambda \end{vmatrix} = 0 \tag{2.1.1}$$

$$\implies \lambda^2 - \frac{286699}{637}\lambda + \left[\frac{1037 \times 2174}{49^2}\right] = 0 \tag{2.1.2}$$

The eigenvalues are the roots of equation 2.1.2 is given by

$$\lambda_{11} = \frac{2174}{49} \tag{2.1.3}$$

$$\lambda_{12} = \frac{1037}{49} \tag{2.1.4}$$

The corresponding eigen vectors are,

$$\mathbf{u}_{11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{u}_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.1.6}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.7}$$

$\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \tag{2.1.8}$$

2.2 *To get* $\mathbf{U}$

The characteristic equation of $\mathbf{MM}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1341}{49} - \lambda & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} - \lambda & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 - \lambda \end{vmatrix} = 0 \tag{2.2.1}$$

$$\implies -\lambda^3 + \frac{3211}{49}\lambda^2 - \frac{2254438}{2401}\lambda = 0 \tag{2.2.2}$$

The eigenvalues are the roots of equation 2.2.2 is given by

$$\lambda_{21} = \frac{2174}{49} \tag{2.2.3}$$

$$\lambda_{22} = \frac{1037}{49} \tag{2.2.4}$$

$$\lambda_{23} = 0 \tag{2.2.5}$$

The corresponding eigen vectors are ,

$$\mathbf{u}_{21} = \begin{pmatrix} -\frac{6}{7} \\ \frac{1}{5} \\ -1 \end{pmatrix}, \mathbf{u}_{22} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{10}{7} \\ 1 \end{pmatrix}, \mathbf{u}_{23} = \begin{pmatrix} -\frac{602}{747} \\ \frac{384}{249} \\ 1 \end{pmatrix} \tag{2.2.6}$$

Normalizing the eigen vectors,

$$||\mathbf{u}_{21}|| = \sqrt{\left(\frac{-6}{7}\right)^2 + \left(\frac{1}{5}\right)^2 + 1} = \frac{\sqrt{2174}}{35} \tag{2.2.7}$$

$$\implies \mathbf{u}_{21} = \begin{pmatrix} -\frac{210}{7\sqrt{2174}} \\ \frac{35}{5\sqrt{2176}} \\ -\frac{35}{\sqrt{2174}} \end{pmatrix} \tag{2.2.8}$$

$$||\mathbf{u}_{22}|| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-10}{7}\right)^2 + 1} = \frac{\sqrt{1037}}{14} \tag{2.2.9}$$

$$\implies \mathbf{u}_{22} = \begin{pmatrix} -\frac{42}{2\sqrt{1037}} \\ -\frac{20}{\sqrt{1037}} \\ \frac{14}{\sqrt{1037}} \end{pmatrix} \tag{2.2.10}$$

$$||\mathbf{u}_{23}|| = \sqrt{\left(\frac{-602}{747}\right)^2 + \left(\frac{384}{249}\right)^2 + 1} = \frac{\sqrt{4027743}}{1000} \tag{2.2.11}$$

$$\implies \mathbf{u}_{23} = \begin{pmatrix} -\frac{602000}{747\sqrt{4027743}} \\ \frac{384000}{249\sqrt{4027743}} \\ \frac{1000}{\sqrt{4027743}} \end{pmatrix} \tag{2.2.12}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \tag{2.2.13}$$

2.3 To get $\mathbf{x}$

Using (2.0.16) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \tag{2.3.1}$$

By substituting the equation (2.0.16) in equation (2.0.15) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.3.2}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.3.3}$$

where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{7}{\sqrt{2174}} & 0 & 0 \\ 0 & \frac{7}{\sqrt{1037}} & 0 \end{pmatrix} \tag{2.3.4}$$

From (2.3.3) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-172}{\sqrt{2174}} \\ \frac{20}{\sqrt{1037}} \\ \frac{-115000}{747\sqrt{4027743}} \end{pmatrix} \tag{2.3.5}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-602}{1087} \\ \frac{140}{1037} \end{pmatrix} \tag{2.3.6}$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{140}{1037} \\ \frac{-602}{1087} \end{pmatrix} \tag{2.3.7}$$

2.4 Verification of $\mathbf{x}$

Verifying the solution of (2.3.7) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.4.1}$$

Evaluating the R.H.S in (2.4.1) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \tag{2.4.2}$$

$$\implies \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \tag{2.4.3}$$

Solving the augmented matrix of (2.4.3) we get,

$$\begin{pmatrix} \frac{1037}{49} & 0 & \frac{20}{7} \\ 0 & \frac{2174}{49} & -\frac{172}{7} \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{49}{2174}R_2]{R_1 \leftarrow \frac{49}{1037}R_1} \begin{pmatrix} 1 & 0 & \frac{140}{1037} \\ 0 & 1 & -\frac{602}{1087} \end{pmatrix} \tag{2.4.4}$$

Hence, Solution of (2.4.1) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{140}{1037} \\ \frac{-602}{1087} \end{pmatrix} \tag{2.4.5}$$

Comparing results of $\mathbf{x}$ from (2.3.7) and (2.4.5) we conclude that the solution is verified.