The Statistical Evaluation of Directional Spectrum Estimates Derived from Pitch/Roll Buoy Data

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ABSTRACT

Estimates of the ocean wave directional spectrum may be extracted from observations of surface vertical acceleration and slope made with a pitch/roll buoy. The analysis requires the specification of a parametrical model of the spectrum and a procedure by which the parameters are fixed. The statistical validity and variability of the result must then be examined. This is accomplished by formulating the hypothesis that the model spectrum is the true spectrum; the hypothesis is then rejected if the difference between observations and data computed from the model is improbably large. Otherwise, the model is accepted as statistically valid. Model variability may then be computed in terms of the variances of model parameters. One particular parametrical model and analysis scheme has received wide application in recent years; this paper examines the statistical validity and variability of results obtained with this "conventional" procedure. Explicit formulas for the data covariance matrix, which summarizes the statistics of the observations and forms the core of the statistical analysis, are presented as are formulas for the variances of derived spectral parameters.

1. Introduction

A large amount of data on ocean wave directional spectra has been collected in recent years using pitch/roll buoys (in, for example, such programs as JONSWAP, GATE, JASIN, the Gulf of Alaska Experiment, the Marineland experiment, etc.). These instruments provide time series of vertical acceleration and slope of the water surface from which the directional spectrum of the wave field may be estimated (Longuet-Higgins *et al.*, 1963). While a variety of techniques are being used to analyze the data in special cases, most of the observations are being routinely processed using a computationally convenient procedure based on a particular parametrical model for the directional spectrum F, specifically,

$$F(\theta,f) = E(f)S(\theta,f), \qquad (1)$$

where

$$S(\theta,f) = \kappa(f) \cos^{2P(f)}\tfrac{1}{2}[\theta - \theta_0(f)], \qquad (2)$$

S is the spreading function, E the wave frequency spectrum, f frequency (Hz), θ the wave propagation direction and κ is a normalization factor such that

$$\int_0^{2\pi} d\theta S = 1.$$

The parameters of the spreading function model are the mean wave direction θ_0 and the exponential factor P. The disadvantage of this model is that it cannot accurately represent a frequency band which is multiple-lobed or strongly skewed in direction. Its advantages are that it seems to fit simple wind wave spectra well and each parameter has an immediate physical interpretation. The purpose of the present work is to examine the statistical significance and variability of directional spectrum estimates obtained from pitch/roll buoy observations with this "conventional" analysis procedure.

The methods to be used in this examination are those of Olbers *et al.* (1976). The statistical significance of a model directional spectrum is examined by formulating a test of the hypothesis that the model spectrum is the true (expectation) value of the directional spectrum. If the test is failed at a specified level of confidence, the hypothesis must be rejected; otherwise, the model is accepted as a statistically valid estimate of the true directional spectrum. The variability of the estimate may then be defined in terms of the variances of model parameters.

In the next section, we review the analysis problem presented by pitch/roll buoy data as it is generally formulated and establish some useful notation. In Section 3, the test of model validity is developed. At the core of this test is the data covariance matrix **V**, which specifies the statistics of the buoy observations. Explicit formulas for the elements of **V** are provided, and the application of the validity test to the results of the conventional analysis procedure is discussed. The variability of model parameters is examined in Section 4, and explicit formulas are obtained for the conventional analysis procedure. Finally, illustrative examples are presented in Section 5, and a brief summary and some additional comments are provided in Section 6.

2. Analysis of pitch/roll buoy data

Pitch/roll buoys record time series of the vertical acceleration and two orthogonal components of the slope of the sea surface at a fixed horizontal position. Estimation of the wave directional spectrum from these data begins with computing estimates of all possible autospectra and cross spectra between records; this accomplishes a frequency decomposition of the analysis problem, permitting each wave frequency band to be treated independently. The nonvanishing expectation values of this set of spectra are given by the following integral equations (Longuet-Higgins *et al.*, 1963):

$$C_{11}(f) = \int_0^{2\pi} d\theta (2\pi f)^4 F(\theta,f)$$

$$C_{22}(f) = \int_0^{2\pi} d\theta k^2 \cos^2\theta F(\theta,f)$$

$$C_{33}(f) = \int_0^{2\pi} d\theta k^2 \sin^2\theta F(\theta,f)$$

$$Q_{12}(f) = \int_0^{2\pi} d\theta k (2\pi f)^2 \cos\theta F(\theta,f)$$

$$Q_{13}(f) = \int_0^{2\pi} d\theta k (2\pi f)^2 \sin\theta F(\theta,f)$$

$$C_{23}(f) = \int_0^{2\pi} d\theta k^2 \sin\theta \cos\theta F(\theta,f).$$

(C_{ij} indicates the cospectrum and Q_{ij} the quadrature spectrum; the subscript 1 denotes acceleration, 2 denotes slope in the north direction, and 3 denotes slope in the east direction; θ is measured clockwise from north.) These equations can be reduced to a linearly independent set by factoring the directional spectrum F as in Eq. (1), then rearranging the results to obtain the equalities

$$E(\mathbf{x}) = \frac{C_{11}}{(2\pi f)^4} = \frac{C_{22} + C_{33}}{k^2} = \frac{[C_{11}(C_{22} + C_{33})]^{1/2}}{(2\pi f)^2 k} \qquad (3)$$

and the set of four "data equations"

$$\mathbf{d} = \int_0^{2\pi} d\theta S(\theta) \mathbf{b}(\theta), \qquad (4)$$

where the vectors

$$\mathbf{x} = \begin{bmatrix} C_{11} \\ C_{22} \\ C_{33} \\ Q_{12} \\ Q_{13} \\ C_{23} \end{bmatrix},$$

$$\mathbf{d}(\mathbf{x}) = \begin{bmatrix} Q_{12}/[C_{11}(C_{22} + C_{33})]^{1/2} \\ Q_{13}/[C_{11}(C_{22} + C_{33})]^{1/2} \\ (C_{22} - C_{33})/(C_{22} + C_{33}) \\ 2C_{23}/(C_{22} + C_{33}) \end{bmatrix}, \qquad (5)$$

$$\mathbf{b}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

(the function argument f will henceforward be suppressed).

This formulation has several advantages: Eqs. (4) governing S are independent of the wavenumber k, hence of the dispersion characteristics of the wave field; further, estimates of $\mathbf{d}$ obtained by using estimates of $\mathbf{x}$ in (5) are relatively insensitive to gain calibration errors in the buoy instrumentation; and the estimation and variability of E and S may be examined independently. For the moment, we focus our attention on S, delaying further consideration of E until Section 4.

Cross-spectral estimates obtained from the buoy time series are subject to statistical error. Hence, in general,

$$\mathbf{x} = \tilde{\mathbf{x}} - \delta\tilde{\mathbf{x}},$$

where the tilde indicates a particular realization of a random variable. As a result, an estimate of $\mathbf{d}$, computed by using $\tilde{\mathbf{x}}$ for $\mathbf{x}$ in (5), will also contain random error, whence

$$\mathbf{d} = \tilde{\mathbf{d}} - \delta\tilde{\mathbf{d}},$$

where we use $\mathbf{d}$ for $\mathbf{d}(\mathbf{x})$ and $\tilde{\mathbf{d}}$ for $\mathbf{d}(\tilde{\mathbf{x}})$. The statistics of $\delta\tilde{\mathbf{x}}$ are known (Jenkins and Watts, 1968), and the statistics of $\delta\tilde{\mathbf{d}}$ may be computed approximately from those of $\delta\tilde{\mathbf{x}}$.

In terms of physically realizable observations, then, the data equations (4) should be written

$$\delta\tilde{\mathbf{d}} = \tilde{\mathbf{d}} - \int_0^{2\pi} d\theta S \mathbf{b}. \qquad (6)$$

The analysis task now reduces to extracting an S which is consistent with this expression. This task is not trivial for two reasons: first, the set of equations (6) is underdetermined; and second, the unknown errors $\delta\tilde{\mathbf{d}}$ must be properly accounted for.

The indeterminacy of the set of equations can be dealt with only by invoking additional, external constraints. One way to do this is to choose a parametrical model [such as Eq. (2)] for S which has a number of free parameters equal to or less than the number of equations in (6); the model parameters are then adjusted until $\delta\tilde{\mathbf{d}}$ is, in some appropriate sense, minimized. The residual $\delta\tilde{\mathbf{d}}$ is charged off to random variability in the observations. If $\delta\tilde{\mathbf{d}}$ proves to be improbably large, the model must be rejected; otherwise, it is accepted as statistically valid.

It is possible, of course, to reduce $\delta\tilde{\mathbf{d}}$ to zero by using any appropriate model with the same number of free parameters as there are equations in (6). Such a procedure is undesirable because, even though the result is automatically statistically valid, an exact fit to noisy data may produce a grossly distorted version of the true spectrum. Some data-independent, *a priori* knowledge or intuition must be invoked in order to select the structure and degrees of freedom of the parametrical model.

Long and Hasselmann (1979) have proposed a procedure which quantifies this selection process by defining an integral property of the model spectrum which, on external grounds, is considered undesirable (its "nastiness"). This property is then minimized, subject to the constraint that $\delta\tilde{\mathbf{d}}$ does not become improbably large. Their procedure simultaneously defines the optimal model (in the minimum nastiness sense) and assures its statistical validity.

The conventional analysis procedure has been selected on more arbitrary grounds. It employs model (2) and proceeds as follows:

Substituting (2) for S in (6) gives, on carrying out the integrations over θ,

$$\begin{bmatrix}\tilde{d}_1 - \delta\tilde{d}_1 \\ \tilde{d}_2 - \delta\tilde{d}_2\end{bmatrix} = \frac{P}{P+1}\begin{bmatrix}\cos\theta_0 \\ \sin\theta_0\end{bmatrix},$$

$$\begin{bmatrix}\tilde{d}_3 - \delta\tilde{d}_3 \\ \tilde{d}_4 - \delta\tilde{d}_4\end{bmatrix} = \frac{P(P-1)}{(P+1)(P+2)}\begin{bmatrix}\cos2\theta_0 \\ \sin2\theta_0\end{bmatrix}.$$

The most rational way to fix θ_0 and P would be to select them such that some appropriate measure of the magnitude of $\delta\tilde{\mathbf{d}}$ (e.g., ρ^2 as defined in the next section) is minimized; instead, for the sake of computational simplicity, the conventional procedure computes alternative pairs of parameters by setting either $\delta\tilde{d}_1 = \delta\tilde{d}_2 = 0$ or $\delta\tilde{d}_3 = \delta\tilde{d}_4 = 0$ (Cartwright and Smith, 1964; Mitsuyasu *et al.*, 1975; J. Ewing, personal communication; D. Hasselmann, personal communication; W. McLeish, personal communication). Then

$$\left.\begin{aligned}\theta_0 &= \arctan\left(\frac{\tilde{d}_2}{\tilde{d}_1}\right) \\ P &= \frac{\tilde{K}_1}{1-\tilde{K}_1}\end{aligned}\right\}, \tag{7}$$

or alternatively,

$$\left.\begin{aligned}\theta_0 &= \left[\tfrac{1}{2}\arctan\left(\frac{\tilde{d}_4}{\tilde{d}_3}\right) + 0 \text{ or } \pi\right] \equiv \theta_0' \\ P &= \left[\frac{1 + 3\tilde{K}_2 + (\tilde{K}_2^2 + 14\tilde{K}_2 + 1)^{1/2}}{2(1-\tilde{K}_2)}\right] \equiv P'\end{aligned}\right\}, \tag{8}$$

where $\tilde{K}_1 = (\tilde{d}_1^2 + \tilde{d}_2^2)^{1/2}$ and $\tilde{K}_2 = (\tilde{d}_3^2 + \tilde{d}_4^2)^{1/2}$.

In this way, the conventional procedure provides two estimates of the spreading function S; to insure that these results are meaningful, they must be verified as statistically consistent with the observations. Once this is done, the statistics of the model parameters may be calculated to serve as measures of the reliability of the estimates.

3. Testing model validity

The statistical validity of a given parametrical model is established as follows (Olbers *et al.*, 1976):

We form the hypothesis that the model $S(\theta,\boldsymbol{\alpha})$, where the parameter vector $\boldsymbol{\alpha}$ has been fixed in some appropriate fashion, is the true (expectation) value of the spreading function; therefore, it may be substituted for S in (6). The resulting $\delta\tilde{\mathbf{d}}$ must then be due to random variability in $\tilde{\mathbf{d}}$. The magnitude of $\delta\tilde{\mathbf{d}}$ is defined generally by

$$\rho^2 = \delta\tilde{\mathbf{d}}^{\mathrm{T}}\mathbf{M}\delta\tilde{\mathbf{d}}, \tag{9}$$

where $\mathbf{M}$ is a symmetric matrix of weights. If $\mathbf{M}$ were the identity matrix, for example, ρ^2 would be the Euclidian norm of $\delta\tilde{\mathbf{d}}$. To the extent that the elements of $\tilde{\mathbf{d}}$ are jointly Gaussian, a more appropriate choice for the weight matrix is

$$\mathbf{M} = \mathbf{V}^{-1},$$

where $\mathbf{V}$ is the matrix of covariances

$$V_{ij} = \mathrm{cov}\{\delta\tilde{d}_i,\delta\tilde{d}_j\}$$

(Olbers *et al.*, 1976; Long and Hasselmann, 1979). The ensemble of all possible data realizations $\tilde{\mathbf{d}}$ corresponding to the hypothetical true spectrum then defines, through Eqs. (6) and (9), an ensemble of realizations of ρ^2, the members of which are distributed as χ_μ^2, where μ is the number of elements in the vector $\tilde{\mathbf{d}}$ (four in the present case). For $\mu = 4$, 80% of all data realizations will have $\rho^2 < 6.2$ if the hypothesis is true. Thus, at the 80% level of confidence, we reject any model for which the observed

$\rho^2 > 6.2$; otherwise, we label the model valid, i.e., consistent with the observed data.

The use of the 80% rejection confidence level, rather than 90 or 95%, is somewhat arbitrary. It has been chosen in order to make the validity test reasonably discriminating and implies that we are willing to accept a 20% probability of rejecting the hypothesis when it was, in fact, true. Other choices of confidence level may be invoked as long as the inevitable trade-off between discrimination and statistical confidence is kept in mind.

Statistical validity as defined here should be considered a necessary but not sufficient condition for model "goodness"; recall the example of the four-parameter model which is able to fit the observations exactly; this model is then automatically valid, no matter how distorted ("bad") the result may be. If the parameter-fixing algorithm employs the maximum likelihood least-squares method (i.e., adjusts $\boldsymbol{\alpha}$ to minimize the available realization of ρ^2), a more discriminating statistical test can be constructed which takes into account model structure and the fact that it has been fitted to the observations (P. Lemke, personal communication). However, for arbitrary fitting algorithms, the corresponding statistical tests are not well established. Consequently, we adopt the present definition of statistical validity and rely on additional, data-independent constraints to define goodness [e.g., the minimum nastiness principle of Long and Hasselmann (1979) or the arbitrary specification of a parametrical model as with the conventional pitch/roll buoy analysis procedure].

The covariance matrix $\mathbf{V}$ is determined by the statistics of the spectral estimates $\tilde{\mathbf{x}}$ as follows: Expanding the functions (5) in Taylor series about the expectation value of $\tilde{\mathbf{x}}$ gives

$$\tilde{\mathbf{d}} \equiv \mathbf{d}(\tilde{\mathbf{x}}) = \mathbf{d}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\delta\tilde{\mathbf{x}} + \cdots,$$

where

$$D_{ij} = \frac{\partial d_i}{\partial x_j}.$$

To lowest order in $\delta\tilde{\mathbf{x}}$, the expectation value of $\tilde{\mathbf{d}}$ is given by

$$\langle\tilde{\mathbf{d}}\rangle = \mathbf{d}$$

(the angle braces indicating ensemble average), and the covariance matrix

$$\mathbf{V} = \mathbf{D}\mathbf{U}\mathbf{D}^{\mathrm{T}}, \tag{10}$$

where

$$\mathbf{U} = \langle\delta\tilde{\mathbf{x}}\delta\tilde{\mathbf{x}}^{\mathrm{T}}\rangle$$

is the covariance matrix for $\tilde{\mathbf{x}}$. Each element of $\mathbf{U}$ is given, to a high level of accuracy, by one of the following forms:

TABLE 1. Elements of the data covariance matrix $\mathbf{V}$ for pitch/roll buoy observations.

$$V_{11} = \tfrac{1}{2}[d_1^2 z_1 - 2d_1d_2d_4 - d_3(2d_1^2 - 1) + 1]\nu^{-1}$$
$$V_{22} = \tfrac{1}{2}[d_2^2 z_1 - 2d_1d_2d_4 + d_3(2d_2^2 - 1) + 1]\nu^{-1}$$
$$V_{12} = \tfrac{1}{2}[d_1d_2z_1 - d_4(d_1^2 + d_2^2 - 1)]\nu^{-1}$$
$$V_{13} = [d_1d_3z_2 - d_1(d_1^2 - d_2^2 + d_3^2 - 1) - d_2d_4(d_3 + 1)]\nu^{-1}$$
$$V_{23} = [d_2d_3z_2 - d_2(d_1^2 - d_2^2 - d_3^2 + 1) - d_1d_4(d_3 - 1)]\nu^{-1}$$
$$V_{14} = [d_1d_4(z_2 - d_3) - d_2(2d_1^2 + d_4^2 - d_3 - 1)]\nu^{-1}$$
$$V_{24} = [d_2d_4(z_2 + d_3) - d_1(2d_2^2 + d_4^2 + d_3 - 1)]\nu^{-1}$$
$$V_{33} = (d_3^2 - 1)(d_3^2 + d_4^2 - 1)\nu^{-1}$$
$$V_{44} = (d_4^2 - 1)(d_3^2 + d_4^2 - 1)\nu^{-1}$$
$$V_{34} = d_3d_4(d_3^2 + d_4^2 - 1)\nu^{-1}$$
$$V_{ij} = V_{ji}.$$

Here

$$z_1 = 2(d_1^2 + d_2^2 - 1) + \tfrac{1}{2}(d_3^2 + d_4^2 - 1)$$
$$z_2 = (d_1^2 + d_2^2 - 1) + \tfrac{1}{2}(d_3^2 + d_4^2 + 1).$$

$$\left.\begin{aligned} \operatorname{cov}\{\tilde{C}_{pq},\tilde{C}_{rs}\} &= (C_{pr}C_{qs} + Q_{pr}Q_{qs} \\ &\qquad + C_{ps}C_{qr} + Q_{ps}Q_{qr})\nu^{-1} \\ \operatorname{cov}\{\tilde{C}_{pq},\tilde{Q}_{rs}\} &= (C_{pr}Q_{qs} - Q_{pr}C_{qs} \\ &\qquad - C_{ps}Q_{qr} + Q_{ps}C_{qr})\nu^{-1} \\ \operatorname{cov}\{\tilde{Q}_{pq},\tilde{Q}_{rs}\} &= (C_{pr}C_{qs} + Q_{pr}Q_{qs} \\ &\qquad - C_{ps}C_{qr} - Q_{ps}Q_{qr})\nu^{-1} \end{aligned}\right\} \tag{11}$$

(Jenkins and Watts, 1968), where ν is the number of degrees of freedom of the spectral estimates. If ν is sufficiently large for (10) to hold (>25, say), the elements of $\delta\tilde{\mathbf{x}}$ are approximately jointly Gaussian, whence so are the elements of $\delta\tilde{\mathbf{d}}$.

Eq. (10) can be evaluated explicitly in terms of the elements of $\mathbf{d}$; the derived formulas are listed in Table 1.

For the special case of the conventional analysis procedure, the parameter pair (7) implies that

$$\left.\begin{aligned} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} &= \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} \\ \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} &= \left[\frac{P(P-1)}{(P+1)(P+2)}\right]\begin{bmatrix} \cos 2\theta_0 \\ \sin 2\theta_0 \end{bmatrix} \end{aligned}\right\}, \tag{12}$$

whence

$$\delta\tilde{\mathbf{d}} = \begin{bmatrix} 0 \\ 0 \\ \tilde{d}_3 - d_3 \\ \tilde{d}_4 - d_4 \end{bmatrix},$$

and

$$\rho^2 = M_{33}(\delta\tilde{d}_3)^2 + 2M_{34}\delta\tilde{d}_3\delta\tilde{d}_4 + M_{44}(\delta\tilde{d}_4)^2, \tag{13}$$

where $\mathbf{M} = \mathbf{V}^{-1}$, and $\mathbf{V}$ is computed using (12) in the formulas of Table 1. Alternatively, Eq. (8) implies that

$$\left.\begin{aligned}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} &= \frac{P'}{P'+1}\begin{bmatrix} \cos\theta_0' \\ \sin\theta_0' \end{bmatrix} \\ \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} &= \begin{bmatrix} \tilde{d}_3 \\ \tilde{d}_4 \end{bmatrix}\end{aligned}\right\}, \tag{14}$$

$$\delta\tilde{\mathbf{d}} = \begin{bmatrix} \tilde{d}_1 - d_1 \\ \tilde{d}_2 - d_2 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\rho^2 = M_{11}(\delta\tilde{d}_1)^2 + 2M_{12}\delta\tilde{d}_1\delta\tilde{d}_2 + M_{22}(\delta\tilde{d}_2)^2, \tag{15}$$

where $\mathbf{M} = \mathbf{V}^{-1}$, and $\mathbf{V}$ is computed using (14) in the formulas of Table 1. If (13) or (15) is less than 6.2, the hypothesis that the corresponding model is the true spectrum cannot be rejected at the 80% level of confidence; hence, the model is accepted as statistically valid.

4. Model variability

The variability of a model directional spectrum may be examined by posing the following question: Given that a specified model is the true directional spectrum, what are the statistics of the ensemble of all possible realizations of its characteristic parameters? These parameters include, but are not limited to, those specific to individual frequency bands (e.g., the set of spreading function parameters $\boldsymbol{\alpha}$ and the frequency spectrum E) and such global properties of the directional spectrum as mean squared surface displacement due to waves or, equivalently, significant wave height.

Let $S(\theta,\tilde{\boldsymbol{\alpha}})$ be a particular realization of the spreading function, which has expectation value $S(\theta,\boldsymbol{\alpha})$. The parameter-fixing algorithm by which $\tilde{\boldsymbol{\alpha}}$ is determined from the observations defines a one-to-one correspondence between $\tilde{\boldsymbol{\alpha}}$ and the corresponding data realization $\tilde{\mathbf{d}}$ which may be written formally as $\tilde{\boldsymbol{\alpha}} = \mathbf{g}(\tilde{\mathbf{d}})$; then, expanding about the expectation value of $\tilde{\mathbf{d}}$ gives, to linear order,

$$\tilde{\boldsymbol{\alpha}} = \mathbf{g}(\mathbf{d}) + G(\mathbf{d})\delta\tilde{\mathbf{d}}, \tag{16}$$

where $G_{ij} = \partial g_i/\partial \tilde{d}_j$. The expectation value of $\tilde{\boldsymbol{\alpha}}$ is then $\mathbf{g}(\mathbf{d})$ (because $\langle\delta\tilde{\mathbf{d}}\rangle = 0$), and the parameter covariance matrix is given by

$$W = \langle\delta\tilde{\boldsymbol{\alpha}}\delta\tilde{\boldsymbol{\alpha}}^{\mathrm{T}}\rangle = \mathbf{GVG}^{\mathrm{T}}, \tag{17}$$

where, as before, $\mathbf{V} = \langle\delta\tilde{\mathbf{d}}\delta\tilde{\mathbf{d}}^{\mathrm{T}}\rangle$ is the covariance matrix for $\tilde{\mathbf{d}}$. Since $\tilde{\mathbf{d}}$ is approximately jointly Gaussian, to the extent that (16) holds, so is $\tilde{\boldsymbol{\alpha}}$, whence $\mathbf{W}$ completely summarizes the statistics of its elements.

For the conventional analysis procedure, realizations of the parameter vector

$$\boldsymbol{\alpha} = \begin{bmatrix} \theta_0 \\ P \end{bmatrix}$$

are given explicitly in terms of $\tilde{\mathbf{d}}$ by (7) or (8). Using these for $\mathbf{g}(\tilde{\mathbf{d}})$ in (17) gives for the diagonal elements of $\mathbf{W}$

$$W_{11} = \mathrm{var}\{\tilde{\theta}_0\} = K_1^{-4}[K_1^2 - d_3(d_1^2 - d_2^2) - 2d_1d_2d_4](1/2\nu), \tag{18}$$

$$W_{22} = \mathrm{var}\{\tilde{P}\} = \frac{1}{(1-K_1)}\{K_1^4 + \tfrac{1}{4}K_1^2(K_2^2 - 1) + (\tfrac{1}{2}K_1^{-2} - 1)[K_1^2 + d_3(d_1^2 - d_2^2) + 2d_1d_2d_4]\}\nu^{-1}, \tag{19}$$

from (7) or, alternatively,

$$W_{11} = \mathrm{var}\{\tilde{\theta}_0'\} = K_2^{-2}(1 - K_2^2)(1/4\nu), \tag{20}$$

$$W_{22} = \mathrm{var}\{\tilde{P}'\} = \left\{\frac{1+K_2}{1-K_2} \times \left[\frac{2(1+K_2)}{(K_2^2 + 14K_2 + 1)^{1/2}} + 1\right]\right\}^2 \frac{4}{\nu}, \tag{21}$$

from (8), where $K_1 = (d_1^2 + d_2^2)^{1/2}$ and $K_2 = (d_3^2 + d_4^2)^{1/2}$. The off-diagonal elements are calculated similarly, but since they are of less immediate utility, we shall not do so here. The diagonal elements provide direct estimates of the accuracy with which the parameters may be fixed; $(W_{ii})^{1/2}$ is one standard deviation of the Gaussian random variable $\tilde{\alpha}_i$ (considered independently of $\tilde{\alpha}_j$, $i \neq j$).

Other properties of the individual frequency bands may be similarly treated. For example, the parameters

$$\theta_s = (2 - 2K_1)^{1/2}, \tag{22}$$

$$R_{\mathrm{ex}} = \frac{1}{g}\left(\frac{C_{11}}{C_{22} + C_{33}}\right)^{1/2}, \tag{23}$$

have been reported; θ_s is approximately equal to root-mean-square directional spread, while the dispersion factor R_{ex} has expectation value $(2\pi f)^2/gk$ ($=1$ in the deep water limit; here, g is the acceleration of gravity). The three alternative estimates of E defined by using observations in Eqs. (3) also may be incorporated into the same calculation. Formally, we consider $\boldsymbol{\alpha}(d_1,d_2,d_3,d_4)$ to be an extended parameter vector including θ_s as an additional element; carrying out the computations implied by (17) gives

$$\mathrm{var}\{\theta_s\} = \tfrac{1}{2}(1 - K_1)^3\,\mathrm{var}\{\tilde{P}\} \tag{24}$$

for the additional diagonal element of the parameter covariance matrix $\mathbf{W}$. We may likewise extend the data vector to include R_{ex} and the three alternative forms of $E(\mathbf{x})$, *viz.*,

$$\begin{bmatrix} d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} = \begin{bmatrix} \frac{1}{g}\left(\frac{C_{11}}{C_{22}+C_{33}}\right)^{1/2} \\ \frac{C_{11}}{(2\pi f)^4} \\ \frac{C_{22}+C_{33}}{k^2} \\ \frac{[C_{11}(C_{22}+C_{33})]^{1/2}}{k(2\pi f)^2} \end{bmatrix}.$$

Then, carrying out the computations implied by (10) gives

$$\left.\begin{aligned} \operatorname{var}\{\tilde{d}_5\} &= \frac{(2\pi f)^4}{g^2k^2}(3 - 4K_1^2 + K_2^2)\frac{1}{4\nu} \\ \operatorname{var}\{\tilde{d}_6\} &= \frac{2}{\nu}E^2 \\ \operatorname{var}\{\tilde{d}_7\} &= (1 + K_2^2)\frac{1}{\nu}E^2 \\ \operatorname{var}\{\tilde{d}_8\} &= (3 + 4K_1^2 + K_2^2)\frac{1}{4\nu}E^2 \end{aligned}\right\}, \quad (25)$$

for the additional diagonal elements of the data covariance matrix $\mathbf{V}$.

Deriving the variability of estimates of global properties of the spectrum, such as mean-squared surface displacement due to waves

$$E_{\text{tot}} = \int_0^\infty dfE(f) \quad (26)$$

or, equivalently, significant wave height

$$H_{1/3} = 4(E_{\text{tot}})^{1/2}, \quad (27)$$

requires some additional calculation. Standard spectral analysis procedures replace the continuous frequency spectrum with a set of discrete bands of width Δf, each designated by its center frequency f_i. If the records are discretely sampled in time, the upper frequency limit is the Nyquist frequency f_N. The integral of (26) reduces in this case to the sum

$$E_{\text{tot}} = \sum_{i=1}^{N} \Delta f E_i, \quad (28)$$

where $E_i \equiv E(f_i)$. An estimate $\tilde{E}_{\text{tot}}$ is obtained by using estimates $\tilde{E}_i$ in (28); then,

$$\operatorname{var}\{\tilde{E}_{\text{tot}}\} = \sum_{i=1}^{N}\sum_{j=1}^{N} \Delta f^2 \langle \delta\tilde{E}_i \delta\tilde{E}_j \rangle.$$

The factor $\langle \delta\tilde{E}_i \delta\tilde{E}_j \rangle$ is the (i,j)th element of the matrix of covariances between spectral density estimates at two frequencies. This matrix is diagonally dominant (estimates well separated in frequency are uncorrelated), and it is reasonable to ignore the off-diagonal elements (Jenkins and Watts, 1968). Then

$$\operatorname{var}\{\tilde{E}_{\text{tot}}\} = \sum_{i=1}^{N} \Delta f^2 \operatorname{var}\{\tilde{E}_i\rangle. \quad (29)$$

Given sufficient degrees of freedom in the estimates $\tilde{E}_i$ ($\nu > 25$, say), these are approximately Gaussian, whence so is $\tilde{E}_{\text{tot}}$. (More precisely, $\tilde{E}_i$ is distributed as χ_ν^2, while $\tilde{E}_{\text{tot}}$ is distributed approximately as χ_η^2 with $\eta \geqslant \nu$; hence, for $\nu > 25$ or so, the normal approximation is quite satisfactory for both variables.)

$H_{1/3}$ is estimated by using $\tilde{E}_{\text{tot}}$ in (27); then, expanding about E_{tot} gives, to linear order,

$$\tilde{H}_{1/3} = H_{1/3}(E_{\text{tot}}) + \frac{dH_{1/3}}{d\tilde{E}_{\text{tot}}}(E_{\text{tot}})\delta\tilde{E}_{\text{tot}},$$

thus the expectation value of $\tilde{H}_{1/3}$ is $H_{1/3}(E_{\text{tot}})$ and

$$\operatorname{var}\{\tilde{H}_{1/3}\} = \frac{4}{E_{\text{tot}}} \operatorname{var}\{\tilde{E}_{\text{tot}}\}, \quad (30)$$

with $\operatorname{var}\{\tilde{E}_{\text{tot}}\}$ given by (29). To the extent that the approximations leading to (30) hold, $\tilde{H}_{1/3}$, like $\tilde{E}_{\text{tot}}$, is also approximately Gaussian.

Eqs. (29) and (30) apply in general. For pitch/roll buoy data, it is customary to use $\tilde{d}_6 = \tilde{C}_{11}(f_i)/(2\pi f_i)^4$ for $\tilde{E}_i$, since this estimate is independent of the dispersion characteristics of the wave field. In this case, $\operatorname{var}\{\tilde{E}_i\}$ is given by the second of Eqs. (25), and Eqs. (29) and (30) can be rearranged to read

$$\left.\begin{aligned} &\operatorname{var}\left\{\frac{\tilde{H}_{1/3}}{H_{1/3}}\right\} = \tfrac{1}{4} \operatorname{var}\left\{\frac{\tilde{E}_{\text{tot}}}{E_{\text{tot}}}\right\} \\ &\text{where} \\ &\operatorname{var}\left\{\frac{\tilde{E}_{\text{tot}}}{E_{\text{tot}}}\right\} = \frac{2}{\nu}\frac{\sum_{i=1}^{N}(E_i)^2}{(\sum_{i=1}^{N} E_i)^2} \end{aligned}\right\}. \quad (31)$$

5. Examples

To illustrate the foregoing, we consider the following examples:

First, let the true spreading function S be given by (2) with $\theta_0 = 137°$ and $P = 10$ (see Fig. 1a). A synthetic "data realization" is generated by adding a statistically consistent error vector ϵ to the true data vector

$$\mathbf{d} = \int_0^{2\pi} d\theta S\mathbf{b},$$

assuming 30° of freedom in the cross spectral estimates $\tilde{\mathbf{x}}$. (The errors are generated by representing ϵ as a linear combination of the statistically orthog-

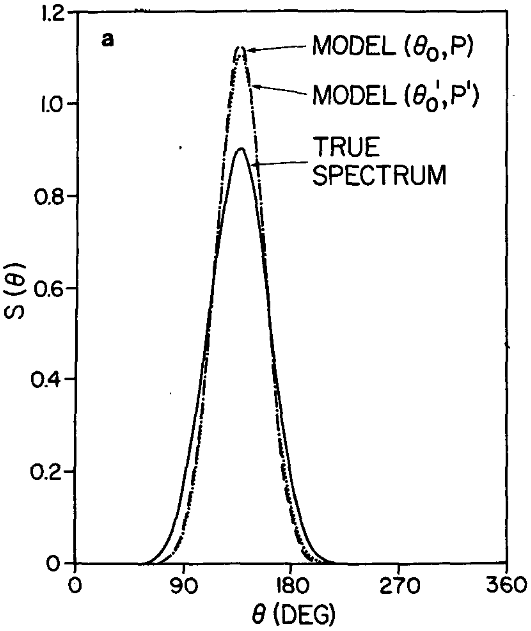

FIG. 1a. Comparison of directional spectrum with estimates extracted from synthetic data using the parametrical model of Eq. (2) and the conventional procedure. True spectrum is given by (2) with $\theta = 137°$ and $P = 10$. Models are extracted from data containing artificial noise statistically consistent with $\nu = 30$ in the buoy time series cross spectra.

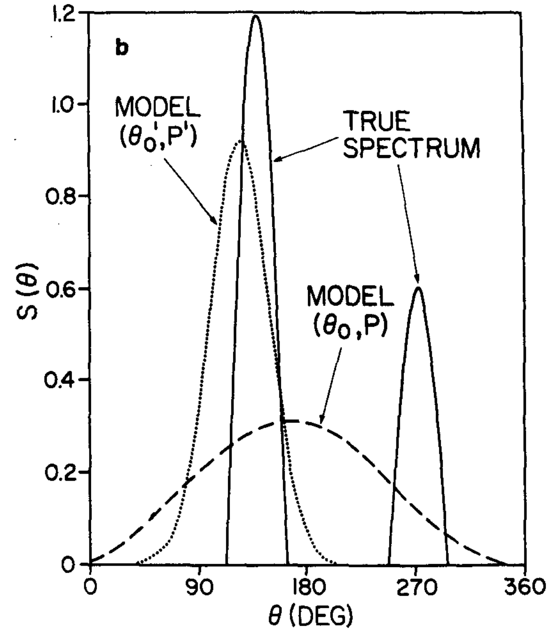

FIG. 1b. Bimodal true spectrum with peaks at $\theta = 137°$ and 272°. Models are extracted from noise-free data.

onal, normalized eigenvectors of the covariance matrix **V**; the coefficient of each eigenvector is determined by selecting a random number from a Gaussian population with zero mean and variance equal to the corresponding eigenvalue.) The parameters θ_0, P, θ_0', P' and θ_s are calculated from the noisy synthetic data according to (7), (8) and (22). The statistical validity of the derived spectral models are then checked according to (13) and (15), and the anticipated variability of the parameters calculated using (18) through (21), (24) and (25). The results, in Table 2, show good agreement between the parameter sets (θ_0,P) and (θ_0',P'). Both sets correspond to statistically valid models. Parameter variances computed from the true spectrum do not differ radically from those based on the derived models, and the differences between true and derived parameters are consistent with both sets of variances (though sometimes larger than one standard deviation).

TABLE 2. Example 1: True spectrum—model (2) (see Fig. 1a). Observed data contains noise consistent with $\nu = 30$ in $\dot{\mathbf{x}}$.

Parameters	From True data		From Observed data		
	Value	St. dev	Value	Error	St. dev
θ_0	137.0°	4.6°	137.4°	0.4°	3.8°
θ_0'	137.0°	5.6°	137.7°	0.7°	4.3°
P	10.0	3.4	15.7	5.7	5.5
P'	10.0	3.9	15.0	5.0	5.7
θ_s	24.4°	2.0°	19.8°	−4.6°	3.3°
R_{ex}	—	0.04	—	—	0.02
$E = d_6$	—	$0.26E$	—	—	$0.26E$
$E = d_7$	—	$0.22E$	—	—	$0.23E$
$E = d_8$	—	$0.24E$	—	—	$0.24E$
$\rho^2(\theta_0,P)$	5.2*	—	4.5**	—	—
$\rho^2(\theta_0',P')$	5.2*	—	4.7**	—	—

* Observed data displacement from true data.
** Observed data displacement from model data.

In the above example, the true spectrum is from a class that is well represented by the model spectrum (which, in the absence of noise, would reproduce the true spectrum exactly). Consider now an example which is poorly represented by the model, one comprising two directional peaks, at $\theta = 137°$ and $\theta = 272°$ (see Fig. 1b); the first peak is scaled to be twice as high as the second. We omit adding noise to the true data, equivalent to having, by chance, acquired a data realization which is the same as its expectation value. The results in Table 3 show that the model fails the validity test at $\nu = 30°$ of freedom using either set of parameters. Moreover, the lack of agreement between θ_0 and θ_0' and between P and P' is an additional indication of the inadequacy of the conventional procedure in this case.

Finally, Fig. 2 shows a wave frequency spectrum estimated from pitch/roll buoy data acquired during GATE (on 10 September 1974). These results were derived from acceleration autospectral estimates having 64 degrees of freedom; accordingly, the second of Eqs. (25) describes the variance of the wave spectral estimates, and Eqs. (31) describe the anticipated variance of mean-square surface displacement and significant wave height estimates,

TABLE 3. Example 2: True spectrum—bimodal model (See Fig. 1b). Observed data = True data.

Parameter	Value	Standard deviation
θ_0	165.7°	14.8°
θ_0'	123.7°	5.4°
P	0.9	0.4
P'	10.5	16.4
θ_s	58.3°	9.3°
R_{ex}	—	0.15
$E = d_6$	—	0.26E
$E = d_7$	—	0.22E
$E = d_8$	—	0.19E
$\rho^2(\theta_0,P)$	27.4*	—
$\rho^2(\theta_0',P')$	59 × 10³*	—

* Assuming $\nu = 30$ in $\tilde{\mathbf{x}}$.

TABLE 4. Example 3: Data source—GATE P/R buoy observations of 10 September 1974; 64 degrees of freedom in spectral estimates of $\tilde{\mathbf{x}}$.

Frequency band parameters	Wind sea peak		Swell peak	
	Value	Standard deviation	Value	Standard deviation
θ_0	54.5°	7.2°	28.6°	4.9°
θ_0'	60.0°	12.6°	24.9°	3.6°
P	1.50	0.37	1.31	0.40
P'	2.78	0.94	10.85	2.87
θ_s	51.2°	3.8°	53.3°	4.7°
R_{ex}	1.19	0.08	1.00	0.09
E (=d_6)	0.88 m² Hz⁻¹	0.18E	2.00 m² Hz⁻¹	0.18E
$\rho^2(\theta_0,P)$	8.6	—	84.3	—
$\rho^2(\theta_0',P')$	110.4	—	56 × 10⁴	—

Other properties	Value	Standard deviation
E_{tot}	0.144 m²	0.062E_{tot}
$H_{1/3}$	1.52 m	0.031$H_{1/3}$

given that the observed spectrum is true (see Table 4). The frequency spectrum exhibits two peaks; presumably caused by swell (at 0.078 Hz) and a local wind sea (at 0.156 Hz); applying the conventional analysis procedure to these frequency bands gives the parameters shown in Table 4. Of the two bands, the wind sea is clearly better fitted by model (2) than the swell. Although none of the derived parameter sets yield statistically valid models at the 80% rejection confidence level, the unprimed pair (θ_0,P) comes very close for the wind sea band. The results for the swell band, on the other hand, are reminiscent of those for the second example (Fig. 2), where the true spectrum was sharply bimodal. Clearly, the conventional procedure is inadequate for estimating

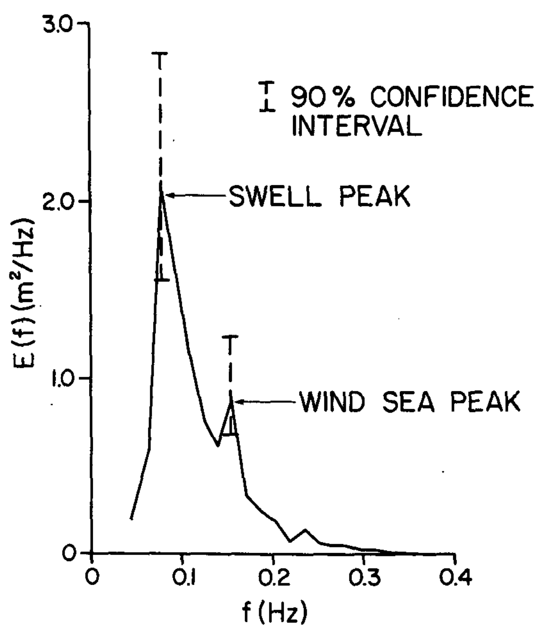

FIG. 2. GATE frequency spectrum acquired with a pitch/roll buoy on 10 September 1974. Estimates have 64 degrees of freedom.

the directional properties of the swell band. The standard deviation of the frequency spectrum estimate $\tilde{\mathbf{E}}$ $(=\tilde{\mathbf{d}}_6)$ is $0.18E$. Thus, for example, 90% of all realizations will fall in the interval $(1 \pm 1.64 \times 0.18)$ times the observed value, given that the observed value equals its expectation value. Since the ratio $(\mathrm{var}\{\tilde{\mathbf{d}}_6\})^{1/2}/E = (2/\nu)^{1/2}$ is independent of E, an alternative statement is also possible: Given a particular realization $\tilde{\mathbf{E}}$, the expectation value E lies with 90% probability in the interval $\tilde{E}/1.29 < E < \tilde{E}/0.71$; this is the 90% confidence interval shown on Fig. 2. Such a statement is not possible for any of the other parameters considered here because, in every other case, the corresponding ratio cannot be evaluated without the hypothesis that the derived model spectrum is the true spectrum. Note that the variability of $\tilde{E}_{tot}$ is proportionately much less than that of $\tilde{E}$, reflecting the effective increase in degrees of freedom resulting from the summation of Eq. (28).

6. Summary and conclusions

Pitch/roll buoys provide time series of the vertical acceleration and slope of the sea surface at a point. The set $\tilde{\mathbf{x}}$ of autospectral and cross-spectral estimates computed from the time series represents integral properties of the surface wave directional spectrum. Given sufficient degrees of freedom, these estimates are approximately jointly Gaussian with statistics summarized by the known covariance matrix $\mathbf{U}$. The measured directional properties of the wave field are then defined by the data vector $\tilde{\mathbf{d}}$, whose elements are normalized combinations of the elements of $\tilde{\mathbf{x}}$. The statistics of $\tilde{\mathbf{d}}$ are also approximately jointly Gaussian and are summarized by the data covariance matrix **V**, which may be calculated from knowledge of **U**.

Extracting a directional spectrum estimate from a set of observations $\tilde{\mathbf{d}}$ requires the specification of a parametrical model $S(\theta,\boldsymbol{\alpha})$ for the directional distribution of spectral density at each frequency and a procedure by which the parameters $\boldsymbol{\alpha}$ are to be fixed. To examine the significance of the results obtained, a test of statistical validity must be imposed. The test consists of verifying that the residual differences $\delta\tilde{\mathbf{d}}$ between the observed data and data computed from the fitted model are not improbably large; improbable, in this case, means inconsistent at some specified level of confidence with the hypothesis that the fitted model is the true spectrum. The appropriate measure of the magnitude of $\delta\tilde{\mathbf{d}}$ is

$$\rho^2 = \delta\tilde{\mathbf{d}}^T\mathbf{V}^{-1}\delta\tilde{\mathbf{d}}.$$

The random variable ρ^2 is distributed as $\chi_\mu{}^2$, where $\mu = 4$ for pitch/roll buoy data. Thus, for example, at the 80% level of confidence, a model implying $\rho^2 > 6.2$ must be rejected; otherwise, it is accepted as a statistically valid estimate of the directional spectrum.

Given a statistically valid model, its variability is defined by the statistics of those properties which characterize it. These include (but are not limited to) the spreading function model parameters $\boldsymbol{\alpha}$, the frequency spectrum E, the rms directional spread θ_s, the dispersion factor R_{ex}, mean-square surface displacement E_{tot} and significant wave height $H_{1/3}$. The statistics of these quantities may be computed approximately from the covariance matrices **U** and **V**.

A large volume of pitch/roll buoy data has been processed in recent years using the spreading function model (2) with alternative sets of parameters fixed by constraining pairs of elements of $\delta\tilde{\mathbf{d}}$ to vanish. We have applied the above principles to this "conventional analysis procedure" and derived explicit formulas for validity testing and for computing the statistics of derived characteristic parameters. Several examples have been provided as illustration.

Note that, while the formulas ultimately derived are specific to pitch/roll buoy data formulated as in Eq. (5) and analyzed with the conventional procedure, the principles are relevant to any problem involving fitting a parametrical model to data subject to statistical variability and representing integral properties of the function being modeled. Further, the formulas of Table 1 for the covariance matrix **V** of pitch/roll buoy observations are applicable to any directional spectrum model and analysis scheme as long as the data are formulated as in Eq. (5).

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