Chapter 6

Cubical Homotopy Theorey

6.1 On Compactly Generated Hausdorff Spaces

Proposition

If $A \subseteq X$ is a retract of Hausdorff X , then A is closed.

Compactly generated spaces

For a compactly generated space X , the topology is (compact sets determine all closed sets) :

$$\{C \mid C \subseteq X \text{ is closed}\} = \{C \mid C \cap K \subseteq X \text{ is closed for each compact } K \subseteq X\} \ .$$

Compact-open topology

The mapping space $X^Y = \{l \mid l : Y \longrightarrow X \text{ is continuous}\}$ is a topological space with the compact-open topology :

$$\{f_i\} \subseteq X^Y \text{ is a topology basis.} \iff f_i : C \longrightarrow U \text{ where } C \subseteq Y \text{ is compact , } U \subseteq X \text{ is open.}$$

Exponential law

For unbased X , Y , Z one has :

$$X^{Y \times Z} \cong (X^Y)^Z \ .$$

For based X , Y , Z one has :

$$X^{Y \wedge Z} \cong (X^Y)^Z \ ,$$

$$X^{\Sigma Y} \cong (\Omega X)^Y$$

Half-smash products

For based X and unbased Y , define the half-smash product to be

$$X \wedge Y_+ = X \times Y / \{*\} \times Y \ ,$$

$$X \wedge Y_+ = X \times (Y \sqcup \{*\}) / X \vee (Y \sqcup \{*\}) = (X \times Y \sqcup X \times \{*\}) / X \sqcup Y$$

Proposition

(1) For $f_0 \simeq f_1 : A \longrightarrow Y$, $g_0 \simeq g_1 : A \longrightarrow Y$, one has $f_0 \times g_0 \simeq f_1 \times g_1 : A \longrightarrow X \times Y$.

(2) For $f_0 \simeq f_1 : X \longrightarrow B$, $g_0 \simeq g_1 : Y \longrightarrow B$, one has $f_0 \vee g_0 \simeq f_1 \vee g_1 : X \vee Y \longrightarrow B$.

(3) $[A, X] \times [A, Y] \cong [A, X \times Y]$, $[X, B] \times [Y, B] \cong [X \vee Y, B]$.

Cone and path adjunction

$CX = X \times I/(X \times \{1\} \cup \{*\} \times I)$, $PX = \{l \mid l : I \longrightarrow X$ is continuous, $l(0) = *\}$.
One has

$$[CX, Y] \cong [X, PY] \ .$$

Suspension and loop adjunction

$\Sigma X = X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{*\} \times I)$, $\Omega X = \{l \mid l : S^1 \longrightarrow X$ is continuous, $l(0) = *\}$.
One has

$$[\Sigma X, Y] \cong [X, \Omega Y] \ .$$

Proposition

$f : X \longrightarrow Y$ is nullhomotopic.

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[r, phantom, shift right=4, "\circlearrowright"] & Y \\ CX \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "\tilde{f}"'] \arrow[r, phantom, shift right=4, "\circlearrowright"] & Y \arrow[d] \\ & PY \end{tikzcd}$$

Proposition

$f : X \longrightarrow Y$ is a homotopy equivalence.
$\Longleftrightarrow X^Z \longrightarrow Y^Z$ is a homotopy equivalence for all Z .
$\Longleftrightarrow Z^X \longrightarrow Z^Y$ is a homotopy equivalence for all Z .
$\Longleftrightarrow [X, Z] \cong [Y, Z]$ for all Z .
$\Longleftrightarrow [Z, X] \cong [Z, Y]$ for all Z .

Ω functor : $\mathbf{Ho}(\mathbf{Top}_*) \longrightarrow \mathbf{Ho}(\mathbf{Top}_*)$

Objects : $(X, x) \xrightarrow{f} (Y, y) \dashrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y$
Morphisms :
(1) If $f \simeq g$, then $\Omega f \simeq \Omega g$.
(2) If f is a homotopy equivalence, then Ωf is a homotopy equivalence.

$[-, \Omega X]$ functor (contravariant) : $\mathbf{Ho}(\mathbf{Top}_*) \longrightarrow (\mathbf{Gp})$

Objects : $(A, a) \xrightarrow{f} (B, b) \dashrightarrow [A, \Omega X] \xleftarrow{[f, \mathbb{1}]} [B, \Omega X]$
Morphisms :
(1) For maps $f \simeq g$, $[f, \mathbb{1}] = [g, \mathbb{1}]$ is a homomorphism.
(2) If f is a homotopy equivalence, then $[f, \mathbb{1}]$ is an isomorphism.

Σ functor : $\mathbf{Ho}(\mathbf{Top}_*) \longrightarrow \mathbf{Ho}(\mathbf{Top}_*)$

Objects : $(X,x) \xrightarrow{f} (Y,y) \dashrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y$
Morphisms :
(1) If $f \simeq g$, then $\Sigma f \simeq \Sigma g$.
(2) If f is a homotopy equivalence, then Σf is a homotopy equivalence.

Proposition

(1) $\Omega(X \times Y) \cong \Omega X \times \Omega Y$, $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

(2) $\Sigma(X \wedge Y) \cong \Sigma X \wedge Y \cong X \wedge \Sigma Y$.

Milnor's theorem (1959)

The loop space has the homotopy type of a CW complex.

Proposition

ΣX is a cogroup object in $\mathbf{Ho}(\mathbf{Top}_*)$, ΩX is a group object in $\mathbf{Ho}(\mathbf{Top}_*)$.

Homotopy splitting of CW complexes

For 1-connected CW complexes, consider the cofiber sequence $A \xrightarrow{i} X \xrightarrow{p} C_i$, if there is $s : C_i \longrightarrow X$ such that $s \circ p \simeq \mathbb{1}_{C_i}$, then one has homotopy equivalence $(p \vee s) \circ fold : A \vee C_i \longrightarrow X$.

For 1-connected CW complexes, consider the fiber sequence $M_p \xrightarrow{i} X \xrightarrow{f} Y$, if there is $r : X \longrightarrow M_p$ such that $r \circ i = \mathbb{1}_{M_p}$, then one has homotopy equivalence $(r \times f) \circ \nabla : X \times M_p \longrightarrow Y$.

6.2 Cofibrations and Fibrations

Fibrations (Hurewicz fibrations)

The continuous map $p : E \longrightarrow B$ has the homotopy lifting property with respect to X .
$\Longleftrightarrow$ $\forall$ homotopy $F : X \times I \longrightarrow B$ and contionuous map $G_0 : X \times \{0\} \longrightarrow E$ such that $p \circ G_0 = F_0$,
$\exists$ a homotopy $G : X \times I \longrightarrow E$ such that $p \circ G_t = F_t$ for any $t \in I$.

$$\begin{tikzcd} X \times I \arrow[r, "F_0"] \arrow[r, bend left=60, "F_1"] \arrow[dr, "G_0"'] & B \arrow[d, phantom, "\circlearrowright" near start, shift right=4ex] \\ & E \arrow[u, "p"'] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X \times I \arrow[r, Rightarrow, "F_t"] \arrow[dr, Rightarrow, "G_t"'] & B \arrow[d, phantom, "\circlearrowright" near start, shift right=4ex] \\ & E \arrow[u, "p"'] \end{tikzcd}$$

If the map $p : E \longrightarrow B$ has the homotopy lifting property with respect to any space X , then $p : E \longrightarrow B$ is a fibration.

Cofibrations

The continuous map $i : A \longrightarrow X$ has homotopy extension property with respect to Y .
$\Longleftrightarrow$ $\forall$ homotopy $F : A \times I \longrightarrow Y$ and continonous map $G_0 : X \times \{0\} \longrightarrow Y$ such that $F_0 = G_0 \circ (i \times \mathbb{1})$,
$\exists$ a homotopy $G : X \times I \longrightarrow Y$ such that $F_t = G_t \circ (i \times \mathbb{1})$ for any $t \in I$.

$$\begin{tikzcd} & Y \\ A \times I \arrow[ur, "F_0"] \arrow[ur, bend left=70, "F_1"] \arrow[r, "i \times \mathbb{1}"'] \arrow[ur, phantom, "\circlearrowright", shift right=3ex] & X \times I \arrow[u, "G_0"'] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & Y \\ A \times I \arrow[ur, Rightarrow, "F_t"] \arrow[r, "i \times \mathbb{1}"'] \arrow[ur, phantom, "\circlearrowright", shift right=3ex] & X \times I \arrow[u, Rightarrow, "G_t"'] \end{tikzcd}$$

If the map $i : A \longrightarrow X$ has the homotopy extension property with respect to any space Y , then $i : A \longrightarrow X$ is a cofibration.

Cofibre and fibre sequences

For a cofibration $i : A \longrightarrow X$, $A \longrightarrow X \longrightarrow X/f(A)$ is called a cofibre sequence and $X/f(A) \simeq C_i$.
For a fibration $f : E \longrightarrow X$, $F \longrightarrow E \longrightarrow X$ is called a fibre sequence and $F \simeq M_f$.

Cofibrations and fibrations with homotopy equivalence

For a cofibration and homotopy equivalence $i : A \longrightarrow X$, A is a strong deformation retrack of X .

For a fibration and homotopy equivalence $p : E \longrightarrow X$, one has homotopy inverse $j : X \longrightarrow E$ such that j is a section of p , if j is a cofibration then $j \circ p \simeq \mathbb{1}_X$ rel $j(X)$.

Proposition

(1) If the base space X is 0-connected, then the fibration $p : E \longrightarrow X$ is surjective.

(2) Every cofibration $i : A \longrightarrow X$ is injective and also an embedding.
In the case of Hausdorff space or compactly generated weak Hausdorff space, every cofibratrion is a closed inclusion.

Examples

(1) The inclusion $i : A \longrightarrow X$ of CW pairs (X, A) is a cofibration.
The inclusion $i : N \longrightarrow M$ of smooth closed submanifold is a cofibration.

(2) $ev_0 : X^Y \longrightarrow X$, $f \longmapsto f(*)$ is a fibration.

(3) There are cofibre sequences :

$$X \longrightarrow * \longrightarrow \Sigma X \ ,$$

$$\Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_f \text{ where } f : X \longrightarrow Y \ ,$$

$$X \times Z \longrightarrow Y \times Z \longrightarrow C_f \wedge Z_+ \text{ where } f : X \longrightarrow Y \ ,$$

$$X \wedge Z \longrightarrow Y \wedge Z \longrightarrow C_f \wedge Z \text{ where } f : X \longrightarrow Y \ ,$$

$$\bigvee_i X \longrightarrow \bigvee_i Y \longrightarrow \bigvee_i Z \text{ where } X_i \longrightarrow Y_i \longrightarrow Z_i \text{ is cobifre sequence,}$$

$$C_f \longrightarrow C_{f \circ g} \longrightarrow C_g \text{ where } f : X \longrightarrow Y, g : Y \longrightarrow Z \ .$$

(4) There are fibre sequences :

$$\Omega X \longrightarrow * \longrightarrow X \ ,$$

$$\Omega M_f \longrightarrow \Omega X \longrightarrow \Omega Y \text{ where } f : X \longrightarrow Y \ ,$$

$${M_f}^Z \longrightarrow X^Z \longrightarrow Y^Z \text{ where } f : X \longrightarrow Y \ ,$$

$$Z^{C_f} \longrightarrow Z^Y \longrightarrow Z^X \text{ where } f : X \longrightarrow Y \ ,$$

$$\prod_i X_i \longrightarrow \prod_i Y_i \longrightarrow \prod_i Z_i \text{ where } X_i \longrightarrow Y_i \longrightarrow Z_i \text{ is fibre sequence,}$$

$$M_f \longrightarrow M_{f \circ g} \longrightarrow M_g \text{ where } f : X \longrightarrow Y, g : Y \longrightarrow Z \ .$$

Proposition

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "j_1"] \\ Y \arrow[r, "j_2"'] & P \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A & X \arrow[l, "f'"'] \\ Y \arrow[u, "g'"] & P \arrow[l, "j_2'"] \arrow[u, "j_1'"'] \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

pushout pullback

(1) If f is surjective (or homeomorphism) , then so is j_2 .
If f' is injective (or homeomorphism) , then so is j_2' .

(2) If f is a cofibration, g is a homotopy equivalence (or weak homotopy equivalence) , then j_1 is a homotopy equivalence (or weak homotopy equivalence) .
If f' is a fibration, g' is a homotopy equivalence (or weak homotopy equivalence) , then j_1' is a homotopy equivalence (or weak homotopy equivalence) .

Proof :

Consider the long exact sequence of homotopy groups and five lemma.

Enlarge diagrams

For a homotopy commutative diagram, there exists a map g such that following diagrams commute and commute up to homotopy.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & Y \arrow[r] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & C_f \arrow[r] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r] & C_{f'} \arrow[r] & \Sigma X' \end{tikzcd}$$

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d, "\Omega\alpha"'] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & M_f \arrow[r] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "f"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & X \arrow[d, "\alpha"] \\ \Omega X' \arrow[r] & M_{f'} \arrow[r] & Y \arrow[r, "f'"'] & X' \end{tikzcd}$$

Pushout of cofibrations

For a cofibretion $i : A \longrightarrow Y$ and a continuous map $f : A \longrightarrow B$, the pushout Y_f makes $i' : B \longrightarrow Y_f$ also a cofibration, and their cofibres $Y/i(A)$, $Y_f/i'(A)$ are homeomorphic.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Y/i(A) \arrow[d, "\cong"] \\
B \arrow[r] & Y_f \arrow[r] \arrow[ul, phantom, "\lrcorner", very near start] & Y_f/i'(B)
\end{tikzcd}$$

Homotopy cofibres

For the cofibration $i : X \longrightarrow CX$ and a continuous map $f : X \longrightarrow Y$, the pushout C_f is called the homotopy cofibre of f , and $Y \longrightarrow C_f \longrightarrow \Sigma X$ is a cofibre sequence.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & CX \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\cong"] \\
Y \arrow[r] & C_f \arrow[r] \arrow[ul, phantom, "\lrcorner", very near start] & \Sigma X
\end{tikzcd}$$

Pullback of fibrations

For a fibration $p : E \longrightarrow X$ and a continuous map $f : Y \longrightarrow X$, the pullback E_f makes $p' : E_f \longrightarrow Y$ also a fibration, and their fibres are homeomorphic.

$$\begin{tikzcd}
F \arrow[r] & E \arrow[r] & X \\
F \arrow[r] \arrow[u, "\cong"] & E_f \arrow[r] \arrow[u] \arrow[ur, phantom, "\llcorner", very near start] & Y \arrow[u, "f"']
\end{tikzcd}$$

Homotopy fibres

For the fibration $p : X \longrightarrow PX$ and a continuous map $f : Y \longrightarrow X$, the pullback M_f is called the homotopy fibre of f , and $\Omega X \longrightarrow M_f \longrightarrow X$ is a fibre sequence.

$$\begin{tikzcd}
\Omega X \arrow[r] & PX \arrow[r] & X \\
\Omega X \arrow[r] \arrow[u, "\cong"] & M_f \arrow[r] \arrow[u] \arrow[ur, phantom, "\llcorner", very near start] & Y \arrow[u, "f"']
\end{tikzcd}$$

Principal fibrations

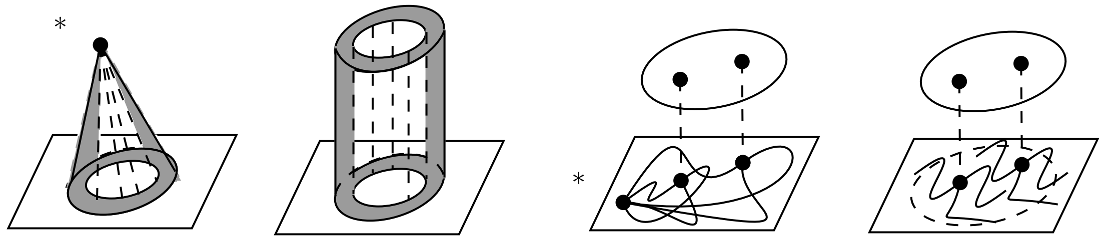
For a map $f : X \longrightarrow Y$ and mapping fibre M_f , one has a fibre sequence

$$\Omega Y \longrightarrow M_f \longrightarrow X$$

called the principal fibration induced by f .

Proposition

For $f \simeq g$, one has $C_f \simeq C_g$, $M_f \simeq M_g$, $I_f \simeq I_g$ (mapping cylinder) , $P_f \simeq P_g$ (mapping path) .
For $f : X \longrightarrow Y$, $g : X' \longrightarrow Y'$, one has $C_f \vee C_g \cong C_{f\vee g}$, $M_f \times M_g \cong M_{f\times g}$.



mapping cone mapping cylinder mapping fibre mapping path

Map decompositions

For a continuous map $f : X \longrightarrow Y$ one has a cofibration $X \longrightarrow I_f$ and a homotopy equivalence $I_f \longrightarrow Y$, such that the cofibre of $X \longrightarrow I_f$ is C_f . Thus there is a coexact sequence

$$X \longrightarrow Y \longrightarrow C_f \ .$$

For a continuous map $f : X \longrightarrow Y$ one has a homotopy equivalence $X \simeq P_f$ and a fibration $P_f \longrightarrow Y$, such that the fibre of $P_f \longrightarrow Y$ is M_f . Thus there is an exact sequence

$$M_f \longrightarrow X \longrightarrow Y \ .$$

Puppe sequences

Consider coexact sequences

$$X \longrightarrow Y \longrightarrow C_f \ , \ Y \longrightarrow C_f \longrightarrow \Sigma X \text{ (cofibration) ,}$$

one has a coexact sequence

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \cdots \ .$$

Consider exact sequences

$$M_f \longrightarrow X \longrightarrow Y \ , \ \Omega Y \longrightarrow M_f \longrightarrow X \text{ (fibration) ,}$$

one has an exact sequence

$$\cdots \longrightarrow \Omega M_f \longrightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \longrightarrow M_f \longrightarrow X \xrightarrow{f} Y \ .$$

Proposition

$$
\begin{tikzcd}
 & & & \mathit{lim}\ (X_{12} \longrightarrow X'_{12} \longleftarrow X'_2) \arrow[dll] \arrow[ddl] & & \\
X_1 \arrow[r] \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowright"] & X_{12} \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowright"] & X_2 \arrow[l, Rightarrow] \arrow[d, Rightarrow] \arrow[ur, dashed, Rightarrow] & & & \mathit{lim}\ \mathcal{X} \arrow[d, dashed, Rightarrow] \\
X'_1 \arrow[r] & X'_{12} & X'_2 \arrow[l, Rightarrow] & \Longrightarrow & & \mathit{lim}\ \mathcal{X}'
\end{tikzcd}
$$

fibrant

For all maps with double line are all fibrations : $X_1 \longrightarrow X'_1$, $X_2 \longrightarrow X'_2$, $X_{12} \longrightarrow X'_{12}$, $X_2 \longrightarrow X_{12}$, $X'_2 \longrightarrow X'_{12}$, $X_2 \longrightarrow \mathit{lim}\ (X_{12} \longrightarrow X'_{12} \longleftarrow X'_2)$, one has an induced fibration

$$\mathit{lim}\ (X_1 \longrightarrow X_{12} \longleftarrow X_2) \longrightarrow \mathit{lim}\ (X'_1 \longrightarrow X'_{12} \longleftarrow X'_2)\ .$$

cofibrant

$$
\begin{tikzcd}
X_1 \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowright"] & X_{12} \arrow[l] \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowright"] & X_2 \arrow[d, Rightarrow] \arrow[ddr] & & & \mathit{colim}\ \mathcal{X} \arrow[d, dashed, Rightarrow] \\
X'_1 & X'_{12} \arrow[l] \arrow[r, Rightarrow] \arrow[drr] & X'_2 & \Longrightarrow & & \mathit{colim}\ \mathcal{X}' \\
 & & & \mathit{colim}\ (X'_{12} \longleftarrow X'_{12} \longrightarrow X_2) \arrow[ul, dashed, Rightarrow] & &
\end{tikzcd}
$$

For all maps with double line are all cofibrations : $X_1 \longrightarrow X'_1$, $X_2 \longrightarrow X'_2$, $X_\emptyset \longrightarrow X'_\emptyset$, $X_\emptyset \longrightarrow X_2$, $X'_\emptyset \longrightarrow X'_2$, $\mathit{colim}\ (X'_\emptyset \longleftarrow X_\emptyset \longrightarrow X_2) \longrightarrow X'_2$, one has an induced cofibration

$$\mathit{colim}\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \longrightarrow \mathit{colim}\ (X'_1 \longleftarrow X'_\emptyset \longrightarrow X'_2)\ .$$

6.3 Homotopy Pushouts and Pullbacks

Homotopy pushouts

pushout in (Top)

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \arrow[dl, phantom, "\lrcorner" near start] \\ X \arrow[r] & X \sqcup Y/\sim \end{tikzcd}$$

$f(a) \sim g(a)$

standrad homotopy pushout in Ho (Top)

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \arrow[dl, phantom, "\simeq\lrcorner" near start] \\ X \arrow[r] & X \sqcup (A \times I) \sqcup Y/\sim \end{tikzcd}$$

$f(a) \sim (a, 0)$, $g(a) \sim (a, 1)$

Homotopy pullbacks

pullback in (Top)

$$\begin{tikzcd} A & Y \arrow[l, "g"'] \\ X \arrow[u, "f"] & \mathrm{eq}(fp_1, gp_2) \arrow[l] \arrow[u] \arrow[ul, phantom, "\lrcorner" near end] \end{tikzcd}$$

$\mathrm{eq}(fp_1, gp_2) = \{(x, y) \mid f(x) = g(y)\}$

standrad homotopy pullback in Ho (Top)

$$\begin{tikzcd} A & Y \arrow[l, "g"'] \\ X \arrow[u, "f"] & P \subseteq X \times A^I \times Y \arrow[l] \arrow[u] \arrow[ul, phantom, "\simeq\lrcorner" near end] \end{tikzcd}$$

$P = \{(x, l, y) \mid f(x) = l(0) \ , \ g(y) = l(1)\}$

Proposition

For $i : A \longrightarrow Y$ and $f : A \longrightarrow B$, the homotopy pushout is Y_f . Then their mapping cones $Y/i(A)$ and $Y_f/i'(B)$ are homotopy equivalent.

For $p : E \longrightarrow X$ and $f : Y \longrightarrow X$, the homotopy pullback is E_f . Then their mapping fibres F_1 and F_2 are homotopy equivalent.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] \arrow[dl, phantom, "\simeq\lrcorner" near start] & Y/i(A) \arrow[d, "\simeq"] \\ B \arrow[r] & Y_f \arrow[r] & Y_f/i'(B) \end{tikzcd}
\qquad
\begin{tikzcd} X & E \arrow[l, "p"'] & F \arrow[l] \\ B \arrow[u, "g"] & E_g \arrow[l] \arrow[u] \arrow[ul, phantom, "\simeq\lrcorner" near end] & F' \arrow[l] \arrow[u, "\simeq"'] \end{tikzcd}$$

Proposition

One has homeomorphism

$$holim\ (X \xrightarrow{f} A \xleftarrow{g} Y) \cong lim\ (P_f \longrightarrow A \xleftarrow{g} Y)\ ,\ (x,l,y) \longmapsto ((x,l),y)\ .$$

Thus there are homeomorphisms

$$holim\ (X \xrightarrow{f} A \xleftarrow{g} Y) \cong lim\ (P_f \longrightarrow A \xleftarrow{g} Y)$$

$$\cong lim\ (P_f \longrightarrow A \longleftarrow P_g) \cong lim\ (X \xrightarrow{f} A \longleftarrow P_g)$$

$$\cong lim\ (lim\ (X \xrightarrow{f} A \xleftarrow{ev_0} A^I) \longrightarrow A^I \longleftarrow lim\ (A^I \xrightarrow{ev_1} A \xleftarrow{g} Y))\ .$$

One has homeomorphism

$$hocolim\ (X \xleftarrow{f} A \xrightarrow{g} Y) \cong colim\ (I_f \longleftarrow A \xrightarrow{g} Y)\ ,\ \begin{pmatrix} y \\ a,t \\ x \end{pmatrix} \longmapsto \begin{pmatrix} y \\ \begin{pmatrix} a,t \\ x \end{pmatrix} \end{pmatrix}\ .$$

Thus there are homeomorphisms

$$hocolim\ (X \xleftarrow{f} A \xrightarrow{g} Y) \cong colim\ (I_f \longleftarrow A \xrightarrow{g} Y)$$

$$\cong colim\ (I_f \longleftarrow A \longrightarrow I_g) \cong colim\ (X \xleftarrow{f} A \longrightarrow I_g)$$

$$\cong colim\ (colim\ (X \xleftarrow{f} A \xrightarrow{i_0} A \times I) \longleftarrow A \times I \longrightarrow colim\ (A \times I \xleftarrow{i_1} A \xrightarrow{g} Y))\ .$$

Matching lemma

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[l, "g"'] \arrow[d, "\simeq"] \\
Y' \arrow[r, "f'"'] & A' & X' \arrow[l, "g'"]
\end{tikzcd}$$

If the vertical maps are homotopy equivalences (or weak homotopy equivalence) , then one has

$holim\ (X \longrightarrow A \longleftarrow Y) \simeq holim\ (X' \longrightarrow A' \longleftarrow Y')$
(or weak equivalence) .

Gluing lemma

$$\begin{tikzcd}
Y \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\simeq"] \\
Y' & A' \arrow[l, "f'"] \arrow[r, "g'"'] & X'
\end{tikzcd}$$

If the vertical maps are homotopy equivalences (or weak homotopy equivalence) , then one has

$hocolim\ (X \longleftarrow A \longrightarrow Y) \simeq hocolim\ (X' \longleftarrow A' \longrightarrow Y')$
(or weak equivalence) .

Proposition

In the pushout square, if g is a cofibration, then it is a homotopy pushout square, one has

$$hocolim\ (X \xleftarrow{f} A \xrightarrow{g} Y) \simeq colim\ (X \xleftarrow{f} A \xrightarrow{g} Y)\ .$$

Proof :

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] & Y \arrow[d] & \\
I_f \arrow[r] \arrow[d, "\simeq"'] \arrow[ur, phantom, "\lrcorner"] & colim\ (I_f \longleftarrow A \longrightarrow Y) \arrow[d] & hocolim\ (X \longrightarrow A \longleftarrow Y) \arrow[l, "\cong"'] \\
X \arrow[r] & colim\ (X \longleftarrow A \longrightarrow Y) &
\end{tikzcd}$$

Then the bottom square is a pushout since

$$colim\ (X \longleftarrow I_f \longrightarrow colim\ (I_f \longleftarrow A \longrightarrow Y)) \cong colim\ (X \longleftarrow A \longrightarrow Y)\ .$$

Proposition

In the pullback square, if g is a fibration, then it is a homotopy pullback square, one has

$$holim\ (X \xrightarrow{f} A \xleftarrow{g} Y) \simeq lim\ (X \xrightarrow{f} A \xleftarrow{g} Y)\ .$$

Proof :

$$\begin{tikzcd}
A & Y \arrow[l, "g"'] & \\
P_f \arrow[u] \arrow[ur, phantom, "\lrcorner"] & lim\ (P_f \longrightarrow A \longleftarrow Y) \arrow[l] \arrow[u] \arrow[r, "\cong"] & holim\ (X \longrightarrow A \longleftarrow Y) \\
X \arrow[u, "\simeq"] & lim\ (X \longrightarrow A \longleftarrow Y) \arrow[l] \arrow[u] &
\end{tikzcd}$$

Then the bottom square is a pullback since

$$lim\ (X \longrightarrow P_f \longleftarrow lim\ (P_f \longrightarrow A \longleftarrow Y)) \cong lim\ (X \longrightarrow A \longleftarrow Y)\ ,\ \ (x, (x', l, y)) \longmapsto (x, y)$$

according to l links $f(x'), g(y)$ is a homeomorphism.

Pullback corner maps

For fibration $p : X \longrightarrow Y$ and cofibration $i : A \longrightarrow B$, one has a fibration

$$X^B \longrightarrow lim\ (X^A \longrightarrow Y^A \longleftarrow Y^B)\ .$$

If either p or i is homotopy equivalence, then

$$X^B \simeq lim\ (X^A \longrightarrow Y^A \longleftarrow Y^B)\ .$$

Proposition

(1)

$$(lim\ (X \longrightarrow A \longleftarrow Y))^E \cong lim\ (X^E \longrightarrow A^E \longleftarrow Y^E)\ .$$

(2)

$$E^{colim\ (X \longleftarrow A \longrightarrow Y)} \cong lim(E^X \longleftarrow E^A \longrightarrow E^Y)\ .$$

Proposition

(1) $hocolim\ (* \longleftarrow X \xrightarrow{f} Y) = C_f$, $hocolim\ (X \longleftarrow X \times Y \longrightarrow Y) = X * Y$, $X * Y \simeq \Sigma(X \wedge Y)$.

(2) $X \longrightarrow X * Y$ and $Y \longrightarrow X * Y$ are nullhomotopic.

(3) For n-connected X and m-connected Y ($m, n \geq 0$) , $X * Y$ is $(m+n)$-connected.

(4)

6.4 On Cofibre and Fibre sequences

Prism theorem

For homotopy commutative diagrams $ABCD$ and $BDEF$ one has :
(1) $ABCD$ is a homotopy pushout, then :

$$BDEF \text{ is a homotopy pushout.} \iff ACEF \text{ is a homotopy pushout.}$$

(2) $BDEF$ is a homotopy pullback, then :

$$ABCD \text{ is a homotopy pullback.} \iff ACEF \text{ is a homotopy pushout.}$$

$$\begin{tikzcd}
& & B \arrow[drr] \arrow[dd] & & \\
A \arrow[urr] \arrow[rrrr] \arrow[dd] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & & & \arrow[d, phantom, "\circlearrowright_{htpy}"] & E \arrow[dd] \\
& & D \arrow[drr] & & \\
C \arrow[urr] \arrow[rrrr] & & & & F
\end{tikzcd}$$

Excision maps

There is a diagram with some commutative triangles.

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] & \Omega Y \arrow[r, "\Omega g"] \arrow[d, "i"'] & \Omega Z \arrow[dl, phantom, "\circlearrowright"] & & \\
& M_f \arrow[ur, dashed, "\gamma"'] \arrow[dr, "w"'] & & M_g \arrow[r] \arrow[d, "v"] \arrow[dl, phantom, "\circlearrowright"] & C_\beta \\
& & X \arrow[ur, dashed, "\beta"] \arrow[r, "f"'] & Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[dl, phantom, "\circlearrowright"] \\
& & M_\alpha \arrow[r] & C_f \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

For a sequence $X \longrightarrow Y \longrightarrow Z$ with $gf = *$, there are three excision maps given by

$$\alpha : (x,t) \longmapsto * \ , \ y \longmapsto f(y) \ ,$$

$$\beta : x \longmapsto (f(x), *) \ ,$$

$$\gamma : (x,l) \longmapsto gl \ .$$

Proposition

There are homotopy pushout $X \times \Omega Z - \Omega Z - M_\alpha - M_g$ and commutative diagrams $X \times \Omega Z - C_{j_1} - C_\beta - M_g$, $X \times \Omega Z - C_{j_1} - \Omega Z$, $M_g - C_\beta - M_\alpha$, since $X - X \times \Omega Z - C_{j_1} - *$ and $X - M_g - C_\beta - *$ are homotopy pushouts, thus $X \times \Omega Z - M_g - C_\beta - C_{j_1}$ and $\Omega Z - C_{j_1} - C_\beta - M_\alpha$ are homotopy pushouts.

$$
\begin{tikzcd}
 & X \arrow[rr, "j_1"] \arrow[ddl] & & X \times \Omega Z \arrow[rr, "\tau"] \arrow[d, "p_2"] \arrow[ddl] & & M_g \arrow[d, "\sigma"] \arrow[ddl] \\
 & & & \Omega Z \arrow[rr, "i"'] & & M_\alpha \\
* \arrow[rr] & & C_{j_1} \arrow[rr, "\tilde{\tau}"'] \arrow[ur, dashed, "\widetilde{p_2}"'] & & C_\beta \arrow[ur, "\tilde{\sigma}"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
\tau &: (x,l) \longmapsto (f(x),l) \\
\sigma &: (y,l) \longmapsto (u(y),l) \\
\tilde{\tau} &: (x,t) \longmapsto (x,t) \\
&\quad (x,l) \longmapsto (f(x),l) \\
\tilde{\sigma} &: (x,t) \longmapsto (f(x),*) \\
&\quad y \longmapsto (g(y),*) \\
&\quad z \longmapsto z \\
&\quad (z,l) \longmapsto (z,l)
\end{aligned}
$$

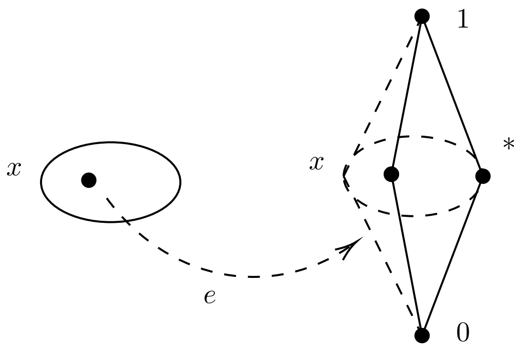
Proposition

For $j_1 : X \longrightarrow X \times \Omega Z$, $\widetilde{p_2} : C_{j_1} \longrightarrow \Omega Z$, one has $C_{\widetilde{p_2}} \simeq X * \Omega Z$.

$$
\begin{tikzcd}
 & & * \arrow[r] & C_{\widetilde{p_2}} \\
X \arrow[r, "j_1"] & X \times \Omega Z \arrow[r] & C_{j_1} \arrow[r] \arrow[u] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowright_{htpy}"] & \Omega Z \arrow[u] \arrow[d] \\
 & & X \times \Omega Z / X \times \{*\} \arrow[r] \arrow[d] & \Omega Z \arrow[d] \\
 & & * \arrow[r] & X * \Omega Z
\end{tikzcd}
$$

James's theorem

For cofibre sequence $X \longrightarrow CX \xrightarrow{p} \Sigma X$, one has a homotopy commutative diagram.



$$\begin{tikzcd} X \arrow[d, "-e"'] \arrow[dr, "\beta"] & \\ \Omega\Sigma X \arrow[r, "\simeq"'] \arrow[ur, phantom, "\circlearrowright_{htpy}"] & M_p \end{tikzcd}$$

For principal fibration $\Omega\Sigma X \longrightarrow M_p \longrightarrow CX \simeq *$ with $\Omega\Sigma X \simeq M_p$, one has $C_\beta \simeq C_e \simeq C_{-e}$. Then one has

$$\Sigma C_\beta \simeq X * \Omega\Sigma X \ .$$

For cofibre sequence $\Omega\Sigma X \simeq M_p \longrightarrow C_\beta \simeq C_e \xrightarrow{r} \Sigma X$, one has $r \simeq *$, thus $\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma C_e$. Then

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma C_e \simeq \Sigma X \vee \Sigma(X * \Omega\Sigma X) \simeq \Sigma X \vee \Sigma(\Sigma(X \wedge \Omega\Sigma X)) \simeq \Sigma X \vee \Sigma(X \wedge \Sigma\Omega\Sigma X) \ .$$

Serre's theorem

For a weak fibration $p : E \longrightarrow X$, $F_{x_0} = p^{-1}(x_0)$ and $f_0 \in F$, the map $p' : (E, F) \longrightarrow (X, x_0)$ with cmmutative diagram induces a bijection $\pi_n(p') : \pi_n(E, F) \longrightarrow \pi_n(X, x_0)$ for all $n \geq 1$.

$$\begin{tikzcd} (E, F) \arrow[dr, "p'"'] & (E, f_0) \arrow[l, "i"'] \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowright"] \\ & (X, x_0) \end{tikzcd}$$

For a weak fibration $p : E \longrightarrow X$, $F = p^{-1}(x_0)$, take the based point $x_0 = p(e_0)$, there is a long exact sequence in category $(\mathbf{Sets}_*)$:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

6.5 Arithmetic Squares

Notice

For a weak homotopy equivalence $f : X \longrightarrow Y$, $f^Z : X^Z \longrightarrow Y^Z$ is not necessarily a weak homotopy equivalence unless Z is a CW complex. In this section, the homotopy cocartesian (or homotopy cartesian) square is defined for CW complexes, we assume that all spaces in this section are CW complexes.

Cocartesian and cartesian squares

$$
\begin{tikzcd}
\mathit{lim}\ (X_1 \longrightarrow X_{12} \longleftarrow X_2) \arrow[drr] \arrow[ddr] & & \\
& X_\emptyset \arrow[ul, dashed, "\cong"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_2 \arrow[d] \\
& X_1 \arrow[r] & X_{12}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X_2 \arrow[d] \arrow[ddr] & \\
X_1 \arrow[r] \arrow[drr] & X_{12} & \\
& & \mathit{colim}\ (X_1 \longleftarrow X \longrightarrow X_2) \arrow[ul, dashed, "\cong"]
\end{tikzcd}
$$

cartesian square　　　　　　　　cocartesian square

By the universal property, a cocartesian square is a pushout, a cartesian square is a pullback.

k-cocartesian and k-cartesian squares

$$
\begin{tikzcd}
\mathit{holim}\ (X_1 \longrightarrow X_{12} \longleftarrow X_2) \arrow[drr] \arrow[ddr] & & \\
& X_\emptyset \arrow[ul, dashed] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner k", very near start] & X_2 \arrow[d] \\
& X_1 \arrow[r] & X_{12}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "k \lrcorner", very near end] & X_2 \arrow[d] \arrow[ddr] & \\
X_1 \arrow[r] \arrow[drr] & X_{12} & \\
& & \mathit{hocolim}\ (X_1 \longleftarrow X \longrightarrow X_2) \arrow[ul, dashed]
\end{tikzcd}
$$

k-cartesian square　　　　　　　　k-cocartesian square

If the induced map is a k-equivalence, then it is k-cocartesian or k-cartesian.
If the induced map is a weak homotopy equivalence, then it is homotopy cocartesian or homotopy cartesian.

Cocartesian and k-cocartesian squares

$$\begin{tikzcd} X \arrow[r, "\mathbb{1}"] \arrow[d, "\mathbb{1}"'] & X \arrow[d, "\mathbb{1}"] \\ X \arrow[r, "\mathbb{1}"'] & X \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathbb{1}"'] & Y \arrow[d] \\ X \arrow[r] & I_f \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & C_f \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd}$$

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X * Y \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd}$$

$\{U, V\}$ is an open cover of X

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & * \arrow[ul, phantom, "\infty \lrcorner" very near start] \end{tikzcd}$$

Cartesian and k-cartesian squares

$$\begin{tikzcd} X & X \arrow[l, "\mathbb{1}"'] \\ X \arrow[u, "\mathbb{1}"] & X \arrow[l, "\mathbb{1}"] \arrow[u, "\mathbb{1}"'] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}
\qquad
\begin{tikzcd} * & X \arrow[l] \\ Y \arrow[u] & X \times Y \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}
\qquad
\begin{tikzcd} X & * \arrow[l] \\ * \arrow[u] & \Omega X \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}$$

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \\ Y \arrow[u, "\mathbb{1}"] & P_f \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}
\qquad
\begin{tikzcd} Y & X \arrow[l, "f"'] \\ * \arrow[u] & M_f \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}
\qquad
\begin{tikzcd} X \times X & X \arrow[l, "\Delta"'] \\ X \arrow[u, "\Delta"] & PX \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner" near start] \end{tikzcd}$$

Proposition

If f is a cofibration, then one has (not only for CW complexes) :

$$
\begin{tikzcd}
X_\emptyset \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X_2 \\
X_1 & X_{12} \arrow[l] \arrow[u]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_\emptyset \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\simeq\ \lrcorner", very near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_\emptyset \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\infty\ \lrcorner", very near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
$$

If f is a fibration, then one has (not only for CW complexes) :

$$
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\lrcorner", very near end] & X_2 \arrow[l, "f"'] \\
X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\simeq\ \lrcorner", very near end] & X_2 \arrow[l, "f"'] \\
X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\infty\ \lrcorner", very near end] & X_2 \arrow[l, "f"'] \\
X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
$$

Proposition

$$
\begin{tikzcd}
X_\emptyset \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\simeq\ \lrcorner", very near end] & X_2 \arrow[d, "h_2"] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
$$

f is a cofibration $\Longrightarrow$

$$
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\simeq\ \lrcorner", very near end] & X_2 \arrow[l, "f"'] \\
X_1 \arrow[u, "h_1"] & X_\emptyset \arrow[l] \arrow[u, "h_2"']
\end{tikzcd}
$$

f is a cofibration $\Longrightarrow$

If h_1 is a homotopy equivalence
(or weak equivalence) , then so is h_2 .

$$
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d, "h_1"'] \arrow[dr, phantom, "\infty\ \lrcorner", very near end] & X_2 \arrow[d, "h_2"] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
\Longrightarrow
$$

$$
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\infty\ \lrcorner", very near end] & X_2 \arrow[l] \\
X_1 \arrow[u, "h_1"] & X_\emptyset \arrow[l] \arrow[u, "h_2"']
\end{tikzcd}
\Longrightarrow
$$

Proposition

$$\begin{tikzcd} X_{12} & X_2 \arrow[l] \\ X_1 \arrow[u, "h_1"] & X_\emptyset \arrow[l] \arrow[u, "h_2"'] \end{tikzcd} \xRightarrow[h_2 \text{ is } k\text{-equivalence}]{h_1 \text{ is } (k+1)\text{-equivalence}} \begin{tikzcd} X_{12} & X_2 \arrow[l] \\ X_1 \arrow[u, "h_1"] & X_\emptyset \arrow[l] \arrow[u, "h_2"'] \arrow[ul, phantom, "k"] \end{tikzcd}$$

$$\Downarrow$$

one can take $k = \infty$

$$\boxed{h_1 \text{ is } k\text{-equivalence.} \implies h_2 \text{ is } k\text{-equivalence.}}$$

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d, "h_1"'] & X_2 \arrow[d, "h_2"] \\ X_1 \arrow[r] & X_{12} \end{tikzcd} \xRightarrow[h_2 \text{ is } k\text{-equivalence}]{h_1 \text{ is } (k-1)\text{-equivalence}} \begin{tikzcd} X_\emptyset \arrow[r] \arrow[d, "h_1"'] & X_2 \arrow[d, "h_2"] \\ X_1 \arrow[r] & X_{12} \arrow[ul, phantom, "k"] \end{tikzcd}$$

$$\Downarrow$$

one can take $k = \infty$

$$\boxed{h_1 \text{ is } k\text{-equivalence.} \implies h_2 \text{ is } k\text{-equivalence.}}$$

With mapping spaces

$$\begin{tikzcd} X_{12} & X_2 \arrow[l] \\ X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u] \arrow[ul, phantom, "k"] \end{tikzcd} \xRightarrow[\text{CW complex } V \text{ with dimension } d]{\text{or take } k = \infty} \begin{tikzcd} (X_{12})^V & (X_2)^V \arrow[l] \\ (X_1)^V \arrow[u] & (X_\emptyset)^V \arrow[l] \arrow[u] \arrow[ul, phantom, "k-d"] \end{tikzcd}$$

$$\begin{tikzcd} X_{12} & X_2 \arrow[l] \\ X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u] \arrow[ul, phantom, "\simeq"] \end{tikzcd} \qquad \begin{tikzcd} (X_{12})^V & (X_2)^V \arrow[l] \\ (X_1)^V \arrow[u] & (X_\emptyset)^V \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty"] \end{tikzcd}$$

$$\xRightarrow[V \text{ is not a CW complex}]{}$$

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_{12} \arrow[ul, phantom, "\simeq"] \end{tikzcd} \qquad \begin{tikzcd} V^{X_\emptyset} & V^{X_2} \arrow[l] \\ V^{X_1} \arrow[u] & V^{X_{12}} \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty"] \end{tikzcd}$$

Proposition

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "k \lrcorner"] & Y \arrow[d] \\
B \arrow[r, "g"'] & Y_f
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & X \arrow[r] \arrow[d] & C_f \arrow[d, "k\text{-equivalence}"] \\
B \arrow[r, "g"'] & Y \arrow[r] & C_g
\end{tikzcd}$$

(or take $k = \infty$)

$$\begin{tikzcd}
X & E \arrow[l, "f"'] \\
Y \arrow[u] \arrow[ur, phantom, "k \lrcorner"] & F \arrow[l, "g"] \arrow[u]
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
X & E \arrow[l, "f"'] & M_f \arrow[l] \\
Y \arrow[u] \arrow[ur, phantom, "k \lrcorner"] & F \arrow[l, "g"] \arrow[u] & M_g \arrow[l] \arrow[u, "k\text{-equivalence}"']
\end{tikzcd}$$

Commutativity of homotopy colimits

$$hocolim\ (V \vee X_1 \longleftarrow V \vee X_\emptyset \longrightarrow V \vee X_2) \cong V \vee hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$hocolim\ (V \times X_1 \longleftarrow V \times X_\emptyset \longrightarrow V \times X_2) \cong V \times hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$hocolim\ (Z \times_Y X_1 \longleftarrow Z \times_Y X_\emptyset \longrightarrow Z \times_Y X_2) \cong Z \times_Y hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$hocolim\ (V \wedge X_1 \longleftarrow V \wedge X_\emptyset \longrightarrow V \wedge X_2) \cong V \wedge hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$hocolim\ (V * X_1 \longleftarrow V * X_\emptyset \longrightarrow V * X_2) \cong V * hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$hocolim\ (\Sigma X_1 \longleftarrow \Sigma X_\emptyset \longrightarrow \Sigma X_2) \cong \Sigma hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2)\ ,$$

$$holim\ (\Omega X_1 \longrightarrow \Omega X_\emptyset \longleftarrow \Omega X_2) \cong \Omega holim\ (X_1 \longrightarrow X_\emptyset \longleftarrow X_2)\ .$$

Homotopy colimit commutes with homotopy cofibers, homotopy limit commutes with homotopy fibers.

$$\begin{tikzcd}
& X_\emptyset \arrow[rr] \arrow[dl] \arrow[dd, "f_\emptyset"'] & & X_2 \arrow[dl] \arrow[dd, "f_2"] & \\
X_1 \arrow[rr] \arrow[dd, "f_1"'] & & hocolim\ \mathcal{X} \arrow[dd, dashed, "\tilde f"'] & & \\
& Y_\emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & X_2 \arrow[dl] \arrow[dd] & \\
Y_1 \arrow[rr] \arrow[dd] & & hocolim\ \mathcal{Y} \arrow[dd, dashed] & & \\
& C_{f_\emptyset} \arrow[rr] \arrow[dl] & & C_{f_2} \arrow[dl] & \\
C_{f_1} \arrow[rr] & & C_{\tilde f} & & hocolim\ \mathcal{C} \arrow[ll, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& X_\emptyset & & X_2 \arrow[ll] & \\
X_1 \arrow[ur] & & holim\ \mathcal{X} \arrow[ll] \arrow[ur] & & \\
& Y_\emptyset \arrow[uu, "f_\emptyset"] & & X_2 \arrow[ll] \arrow[uu, "f_2"'] & \\
Y_1 \arrow[ur] \arrow[uu, "f_1"] & & holim\ \mathcal{Y} \arrow[ll] \arrow[ur] \arrow[uu, dashed, "\tilde f"] & & \\
& M_{f_\emptyset} \arrow[rr] \arrow[dl] \arrow[uu] & & M_{f_2} \arrow[dl] \arrow[uu] & \\
M_{f_1} \arrow[rr] \arrow[uu] & & M_{\tilde f} \arrow[rr, "\cong"'] \arrow[uu, dashed] & & holim\ \mathcal{C}
\end{tikzcd}$$

With products

$$
\begin{tikzcd}
X_\emptyset & X_2 \arrow[l] \\
X_1 \arrow[u] & X_{12} \arrow[l] \arrow[u] \arrow[ul, phantom, "m"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y_\emptyset & Y_2 \arrow[l] \\
Y_1 \arrow[u] & Y_{12} \arrow[l] \arrow[u] \arrow[ul, phantom, "n"]
\end{tikzcd}
$$

$$\xRightarrow[\text{or take } n = \infty]{\text{or take } m = \infty}$$

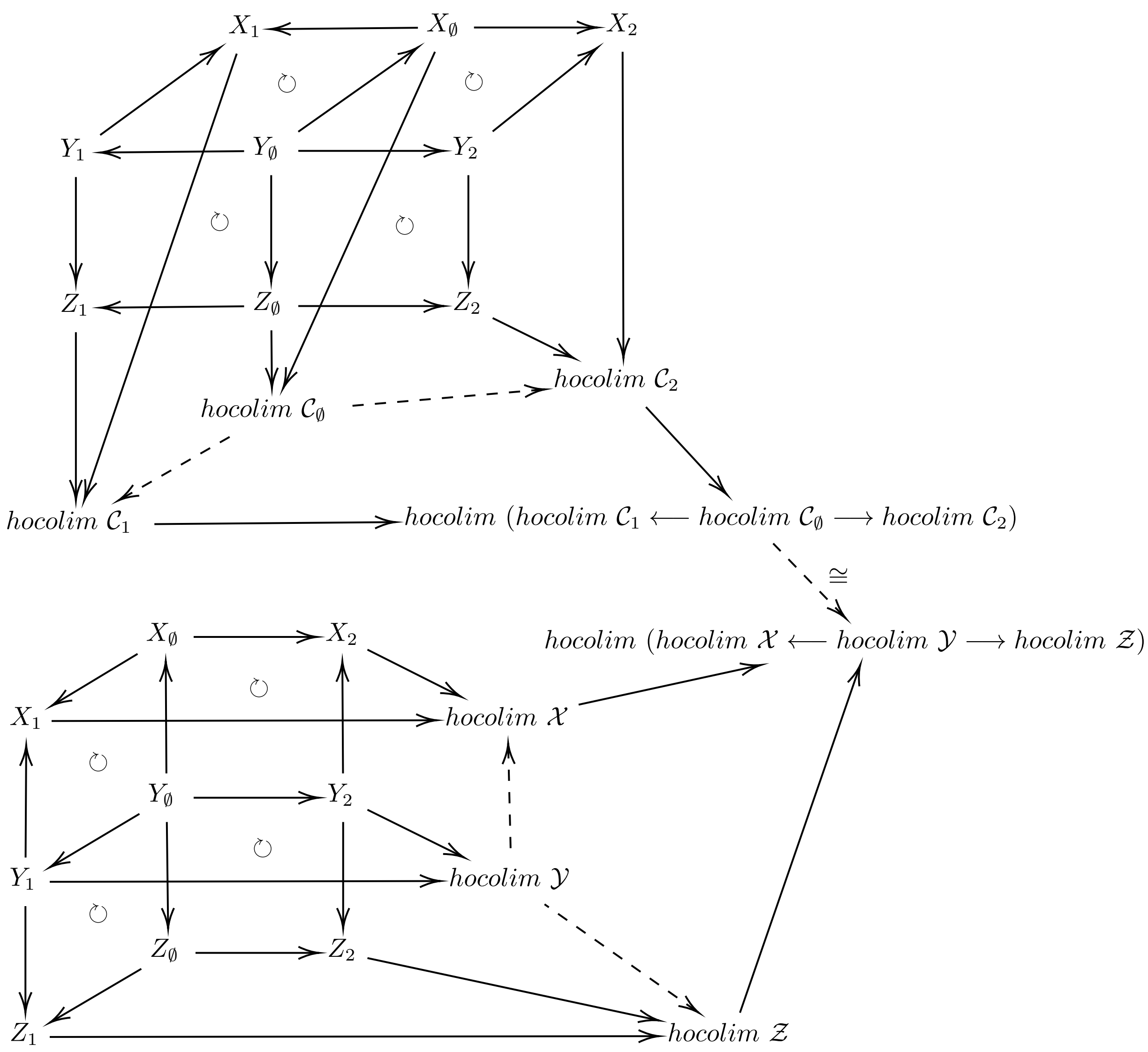
$$
\begin{tikzcd}
X_\emptyset \times Y_\emptyset & X_2 \times Y_2 \arrow[l] \\
X_1 \times Y_1 \arrow[u] & X_{12} \times Y_{12} \arrow[l] \arrow[u] \arrow[ul, phantom, "min\{m, n\}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12} \arrow[ul, phantom, "m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_\emptyset \sqcup Y_\emptyset \arrow[r] \arrow[d] & X_2 \sqcup Y_2 \arrow[d] \\
X_1 \sqcup Y_1 \arrow[r] & X_{12} \sqcup Y_{12} \arrow[ul, phantom, "min\{m, n\}"]
\end{tikzcd}
$$

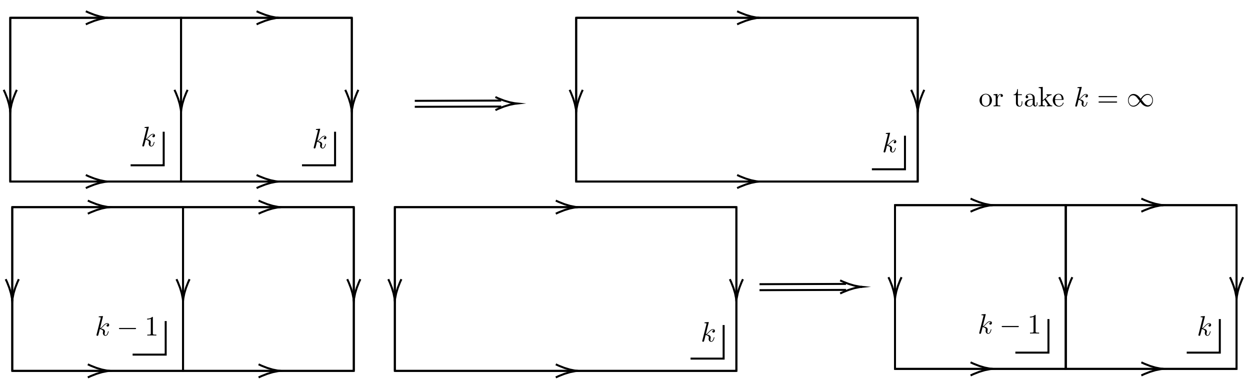
$$
\begin{tikzcd}
Y_\emptyset \arrow[r] \arrow[d] & Y_2 \arrow[d] \\
Y_1 \arrow[r] & Y_{12} \arrow[ul, phantom, "n"]
\end{tikzcd}
\quad
\xRightarrow[\text{or take } n = \infty]{\text{or take } m = \infty}
\quad
\begin{tikzcd}
X_\emptyset \vee Y_\emptyset \arrow[r] \arrow[d] & X_2 \vee Y_2 \arrow[d] \\
X_1 \vee Y_1 \arrow[r] & X_{12} \vee Y_{12} \arrow[ul, phantom, "min\{m, n\}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
V \arrow[r, "\mathbb{1}"] \arrow[d, "\mathbb{1}"'] & V \arrow[d, "\mathbb{1}"] \\
V \arrow[r, "\mathbb{1}"'] & V \arrow[ul, phantom, "\infty"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_\emptyset \times V \arrow[r] \arrow[d] & X_2 \times V \arrow[d] \\
X_1 \times V \arrow[r] & X_{12} \times V \arrow[ul, phantom, "m"]
\end{tikzcd}
$$

Mountain theorem

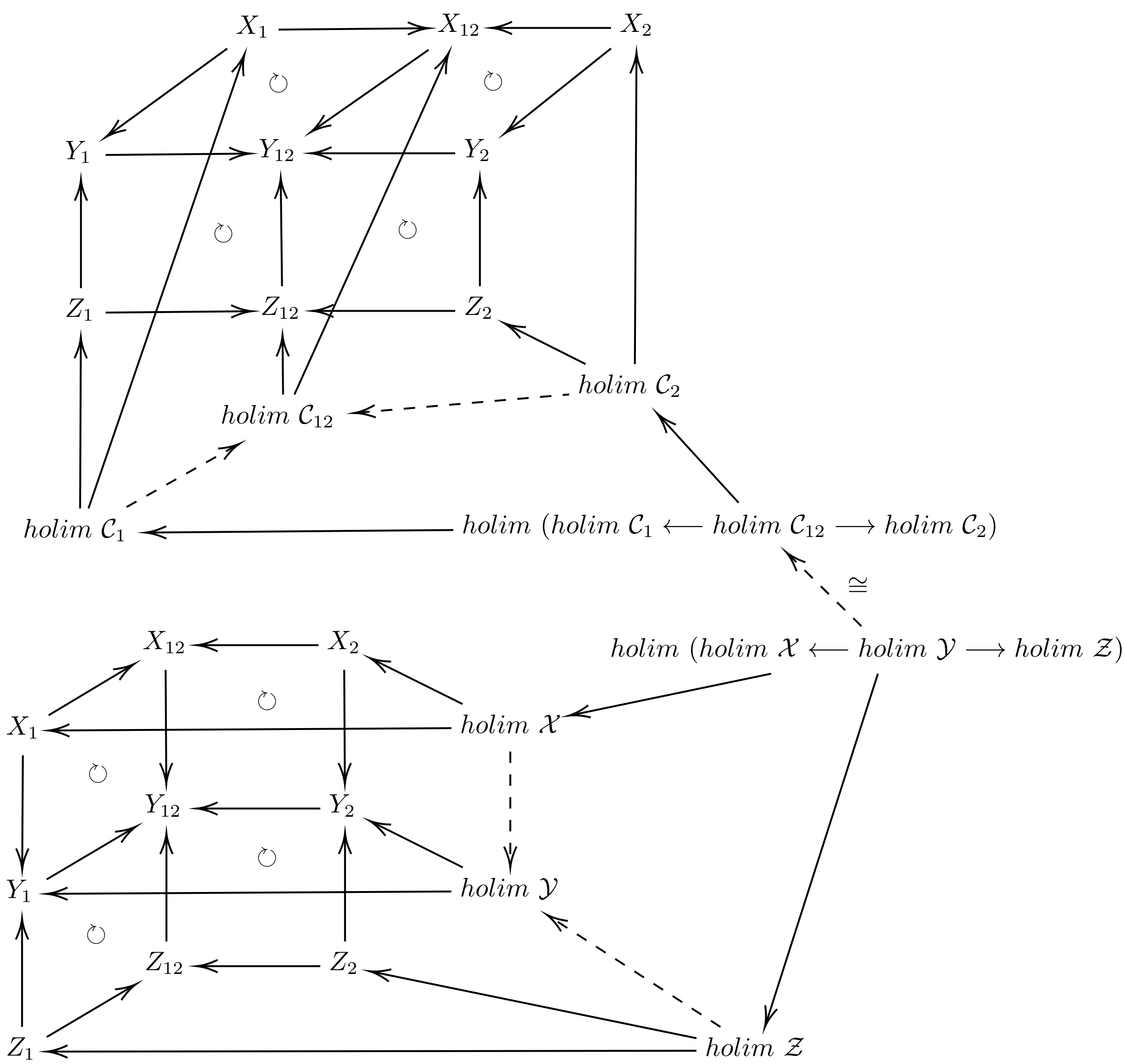


Cocartesian prism theorem

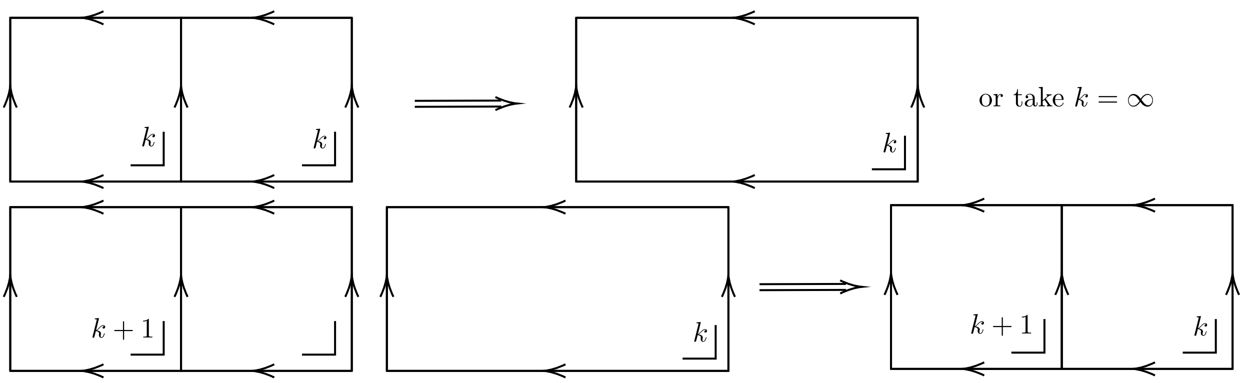


or take $k = \infty$

Valley theorem



Cartesian prism theorem



Proposition

One has weak equivalence $\Sigma(X \wedge Y) \longrightarrow X * Y$.

Proof :

Use moutain theorem.

$$\begin{tikzcd}
* & X \arrow[l] \arrow[r, "\mathbb{1}"] & X \\
* \arrow[u] \arrow[d] \arrow[r, phantom, "\circlearrowright"] & X \vee Y \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[r, phantom, "\circlearrowright"] & X \times Y \arrow[u] \arrow[d] \\
* & Y \arrow[l] \arrow[r, "\mathbb{1}"'] & Y
\end{tikzcd}$$

Proposition

For n-connected X and m-connected Y , $X \wedge Y$ is $(m+n+1)$-connected, $X * Y \simeq \Sigma(X \wedge Y)$ is $(m+n+2)$-connected.

Homotopy fiber and cofiber of constant map

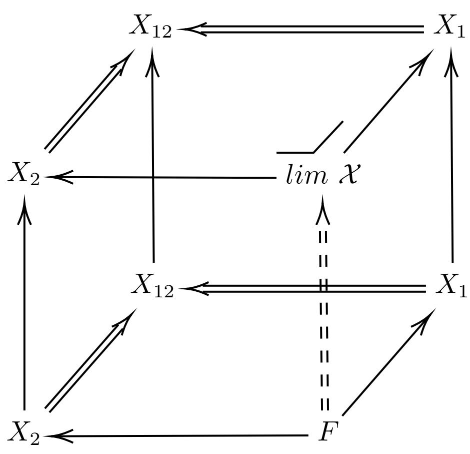
For constant map $X \xrightarrow{f} Y$, one has fiber sequence

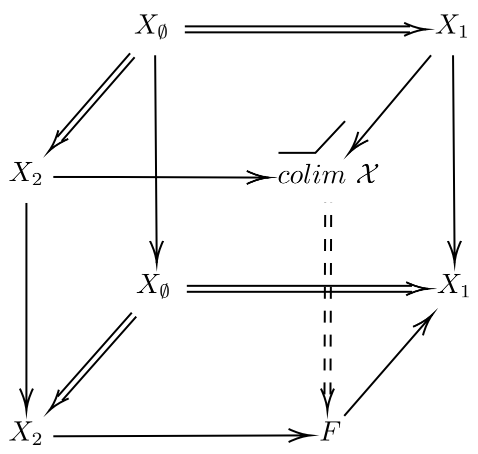
$$X \times \Omega Y \longrightarrow X \longrightarrow Y$$

and cofiber sequence

$$X \longrightarrow Y \longrightarrow Y \vee \Sigma X \ .$$

Cofibrant and fibrant squares



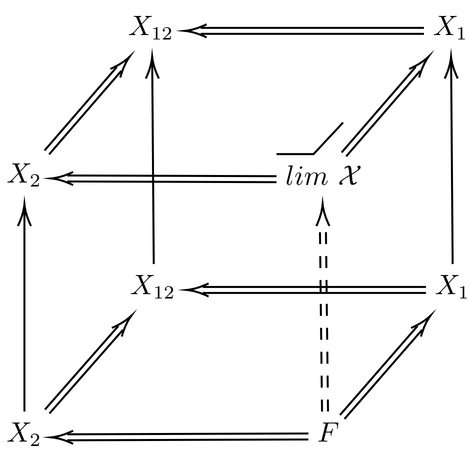


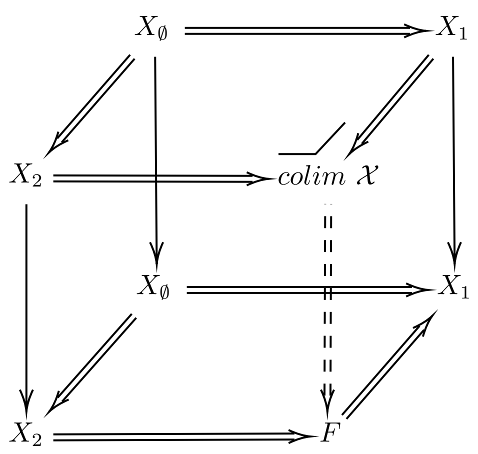
fibrant if all double lines are fibrations

cofibrant if all double lines are cofibrations

pullback keeps fibrations

pushout keeps cofibrations





Fibrant square : all maps are fibrations, induced map is fibration.

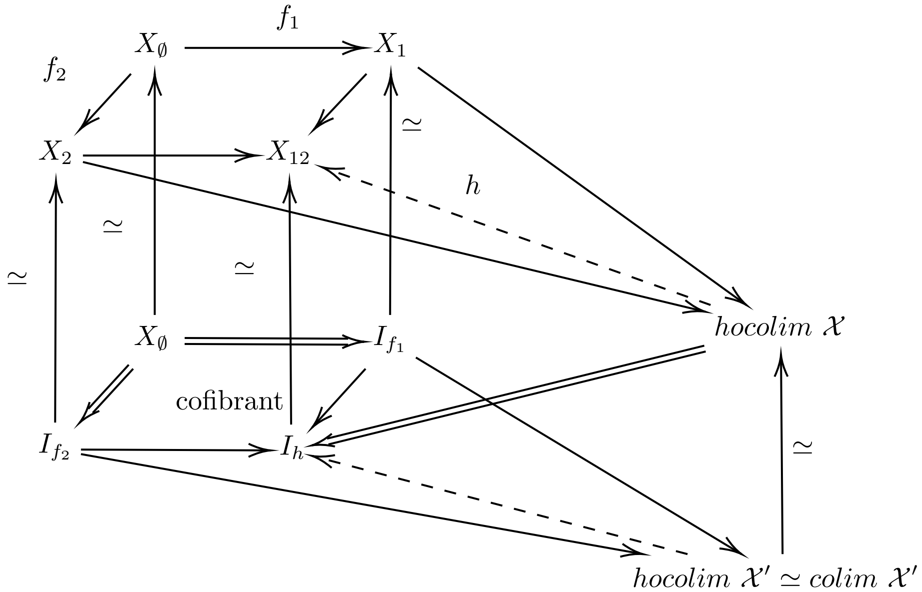
Cofibrant square : all maps are cofibrations, induced map is cofibration.

Cofibrant replacement

Every square $\mathcal{X}$ is homotopy equivalent to a cofibrant square $\mathcal{X}'$ called cofibrant replacement to $\mathcal{X}$.

One has (one can take $k = \infty$) : $\mathcal{X}$ is k-cocartesian. $\iff \mathcal{X}'$ is k-cocartesian.

Proof :

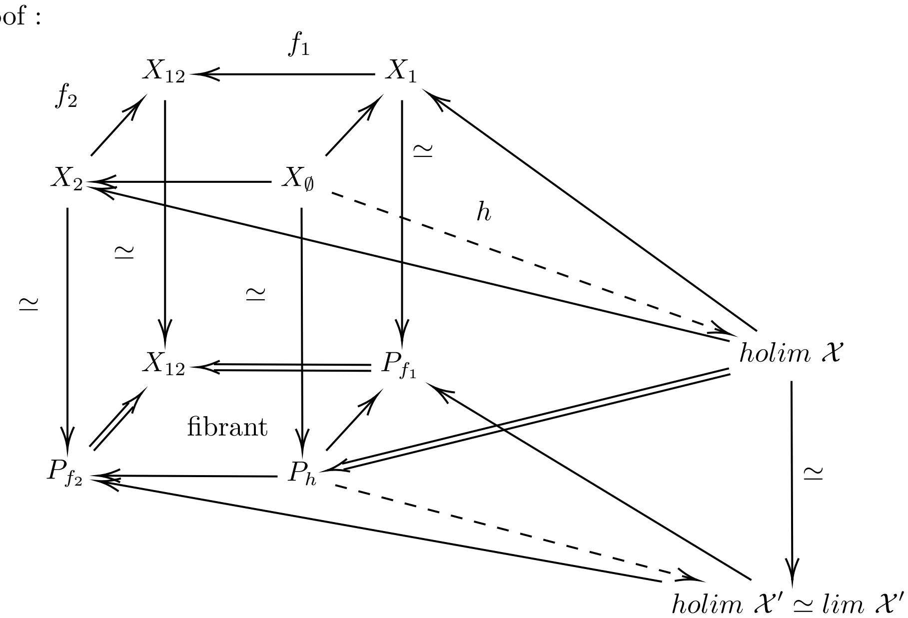


Fibrant replacement

Every square $\mathcal{X}$ is homotopy equivalent to a fibrant square $\mathcal{X}'$.

One has (one can take $k = \infty$) : $\mathcal{X}$ is k-cartesian. $\iff \mathcal{X}'$ is k-cartesian.

Proof :



6.6 Algebraic squares

Notice

In this section, the homotopy cocartesian (or homotopy cartesian) square is defined for CW complexes, we assume that all spaces in this section are CW complexes.

Total cofibers and fibers

$$
\begin{tikzcd}
\mathit{tcofiber}\ \mathcal{X} = C_f & & & & \\
& X_{12} \arrow[ul] & & X_1 \arrow[ll] & \\
& & \mathit{hocolim}\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \arrow[ul, "f"] & & \\
& & \mathit{holim}\ (X_1 \longrightarrow X_{12} \longleftarrow X_2) & & \\
& X_2 \arrow[uuu] & & X_\emptyset \arrow[ll] \arrow[uuu] \arrow[ul, "g"'] & \\
& & & & \mathit{tfiber}\ \mathcal{X} = M_g \arrow[ul]
\end{tikzcd}
$$

Iterated theorem

The total fiber is the iterated homotopy fiber, the total cofiber is the iterated homotopy cofiber.

$$
\begin{tikzcd}
C_\beta = \mathit{tcofiber}\ \mathcal{X} & & & & \\
C_f \arrow[u] \arrow[ddddr, phantom, "\circlearrowleft"] & X_{12} \arrow[l] \arrow[ul] \arrow[ddddrr, phantom, "\circlearrowleft"] & & X_1 \arrow[ll, "f"'] \arrow[ddddr, phantom, "\circlearrowleft"] & M_f \arrow[l] \\
& & \mathit{hocolim}\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \arrow[ul] & & \\
& & & & \\
& & \mathit{holim}\ (X_1 \longrightarrow X_{12} \longleftarrow X_2) & & \\
C_g \arrow[uuuu, dashed, "\beta"] & X_2 \arrow[l] \arrow[uuuu] & & X_\emptyset \arrow[ll, "g"] \arrow[uuuu] \arrow[ul] & M_g \arrow[l] \arrow[uuuu, dashed, "\alpha"'] \\
& & & & M_\alpha = \mathit{tfiber}\ \mathcal{X} \arrow[ul] \arrow[u]
\end{tikzcd}
$$

Proof :

Enlarge with $*$ then use valley theorem and mountain theorem.

Applications of iterated theorem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g \circ f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "g"] \\ Z \arrow[r, "\mathbb{1}"'] & Z \end{tikzcd}$$

fiber sequence $M_f \longrightarrow M_{g\circ f} \longrightarrow M_g$

$$\begin{tikzcd} X \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "g"'] \\ Z \arrow[u, "g \circ f"] & Z \arrow[l, "\mathbb{1}"] \arrow[u, "f"'] \end{tikzcd}$$

cofiber sequence $C_f \longrightarrow C_{g\circ f} \longrightarrow C_g$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathbb{1}"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "g"] \\ X \arrow[r, "\mathbb{1}"'] & X \end{tikzcd}$$

fiber sequence $M_f \longrightarrow * \longrightarrow M_g$

cofiber sequence $C_f \longrightarrow * \longrightarrow C_g$

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{X}\ \circlearrowright"] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X_\emptyset \arrow[d] \\ X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X_{12} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{X}'\ \circlearrowright"] & X_2 \arrow[d] \\ X_\emptyset \arrow[r] & X_1 \arrow[r] & X_\emptyset \end{tikzcd}$$

$tfiber\ \mathcal{X} \simeq \Omega^2 tfiber\ \mathcal{X}'$

$\Sigma^2 tcofiber\ \mathcal{X} \simeq tfiber\ \mathcal{X}'$

where $X_S \longrightarrow X_{S'} \longrightarrow X_S$ is $\mathbb{1}$

Long exact sequence of squares

$\pi_k(\mathcal{X}) = \pi_{k-2}(tfiber\ \mathcal{X})$

$\pi_k(f_1) = \pi_{k-1}(M_{f_1})$

$\pi_k(f_2) = \pi_{k-1}(M_{f_2})$

$$\begin{tikzcd} tfiber\ \mathcal{X} \arrow[d] & & \\ M_{f_1} \arrow[r] \arrow[d] & X_\emptyset \arrow[r, "f_1"] \arrow[d] & X_1 \arrow[d] \\ M_{f_2} \arrow[r] & X_2 \arrow[r, "f_2"'] & X_{12} \arrow[d] \\ & & tcofiber\ \mathcal{X} \end{tikzcd}$$

$H_k(f_1) = \widetilde{H_k}(X_1)$

$H_k(f_2) = \widetilde{H_k}(X_{12})$

$H_k(\mathcal{X}) = H_k(tcofiber\ \mathcal{X})$

One has long exact sequences

$$\cdots \longrightarrow \pi_n(f_1) \longrightarrow \pi_n(f_2) \longrightarrow \pi_{n+2}(\mathcal{X}) \longrightarrow \pi_{n-1}(f_1) \longrightarrow \cdots \longrightarrow \pi_1(f_2)\ ,$$

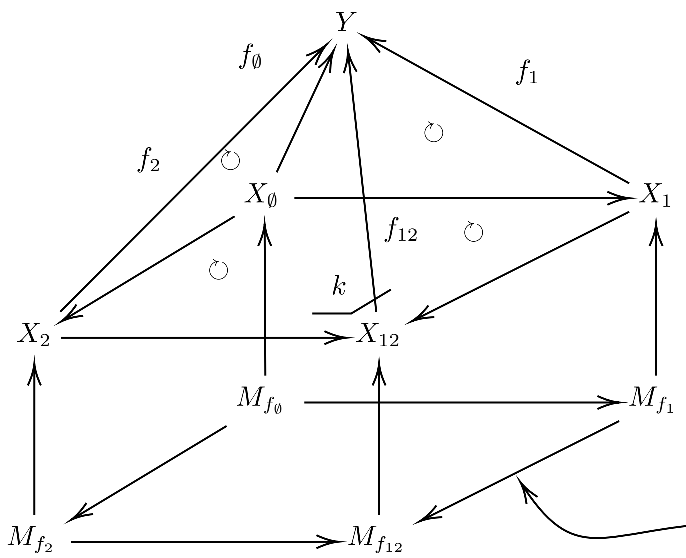
$$\cdots \longrightarrow H_n(f_1) \longrightarrow H_n(f_2) \longrightarrow H_n(\mathcal{X}) \longrightarrow H_{n-1}(f_1) \longrightarrow \cdots\ .$$

House theorem

For four commutative triangles with Y , one has the distributive law with *holim* $(- \longleftarrow Y \longrightarrow *)$ as multiplication and *colim* as addition

$$holim\ (hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \longleftarrow Y \longrightarrow *)$$

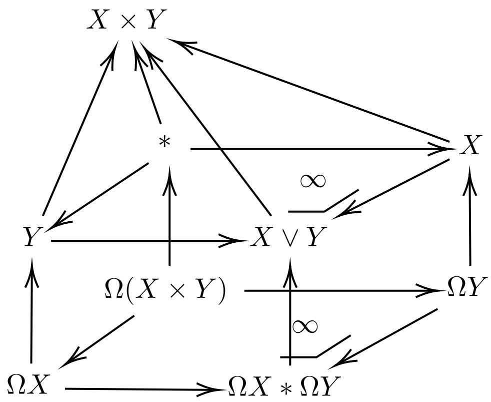
$$\simeq hocolim\ (holim\ (X_1 \longrightarrow Y \longleftarrow *) \longleftarrow holim\ (X_\emptyset \longrightarrow Y \longleftarrow *) \longrightarrow holim\ (X_2 \longrightarrow Y \longleftarrow *))\ .$$



or take $k = \infty$

then k-cocartesian

Applications of house theorem



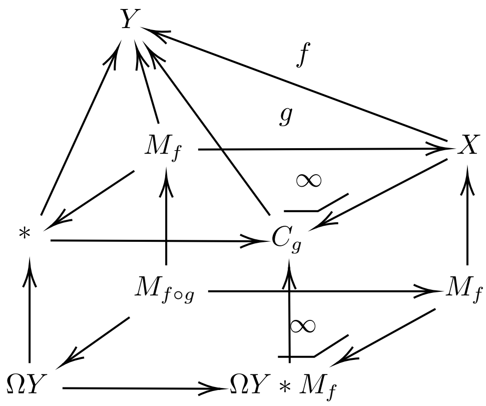
fiber sequence $\Omega X * \Omega Y \longrightarrow X \vee Y \longrightarrow X \times Y$

$\Downarrow$

splitting fiber sequence $\Omega(\Omega X * \Omega Y) \longrightarrow \Omega(X \vee Y) \longrightarrow \Omega(X \times Y)$

$\Downarrow$

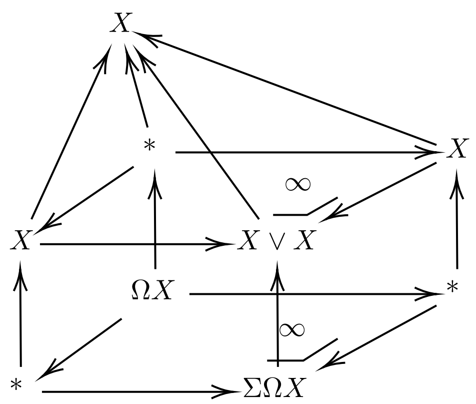
$\Omega(X \vee Y) \simeq \Omega(\Omega X * \Omega Y) \times \Omega(X \times Y)$



fiber sequence $\Omega Y * M_f \longrightarrow C_g \longrightarrow Y$

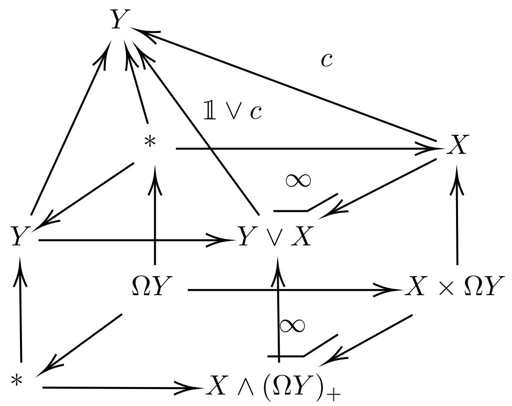
(Ganea's theorem)

$M_{f \circ g} \simeq \Omega Y \times M_f$ since $M_f \longrightarrow Y$ is constant



fiber sequence $\Sigma \Omega X \longrightarrow X \vee X \longrightarrow X$

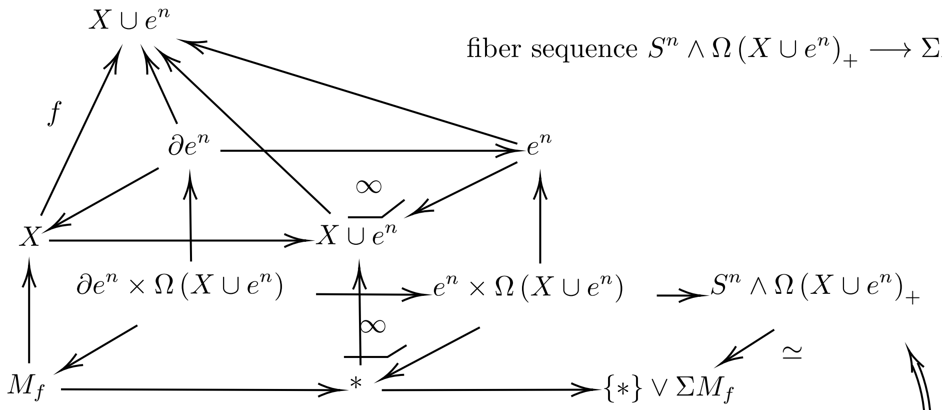
Applications of house theorem



fiber sequence $X \wedge (\Omega Y)_+ \longrightarrow Y \vee X \longrightarrow Y$

$M_{\mathbb{1} \vee c} \simeq C_{c \times \mathbb{1}} = X \times \Omega Y / \{*\} \times \Omega Y = X \wedge (\Omega Y)_+$

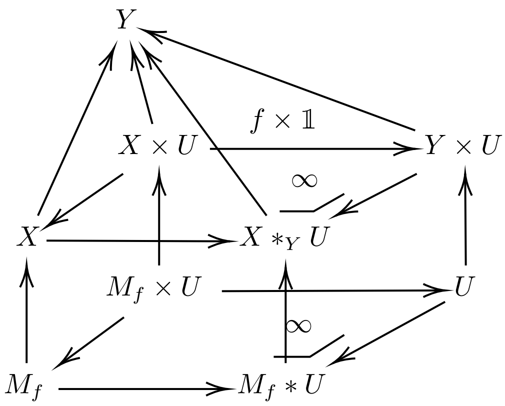
fiber sequence $S^n \wedge \Omega\left(X \cup e^n\right)_+ \longrightarrow \Sigma X \longrightarrow \Sigma\left(X \cup e^n\right)$



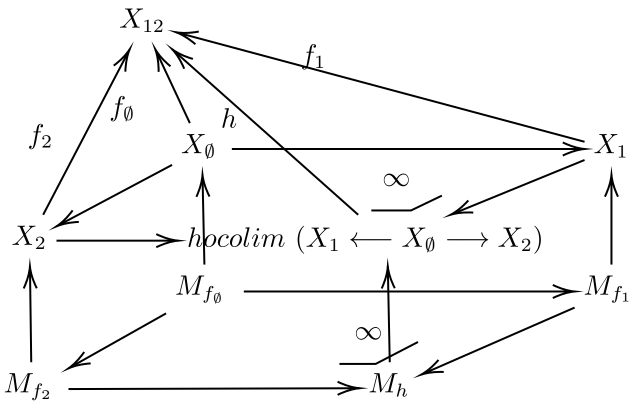
cofiber sequence

$$\begin{tikzcd}
\partial e^n \arrow[r] \arrow[d] & e^n \arrow[d] \\
* \arrow[r] & S^n \arrow[ul, phantom, "\lrcorner"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\partial e^n \times \Omega\left(X \cup e^n\right) \arrow[r] \arrow[d] & e^n \times \Omega\left(X \cup e^n\right) \arrow[r] \arrow[d] & C_i \arrow[d] \\
\{*\} \times \Omega\left(X \cup e^n\right) \arrow[r] & S^n \times \Omega\left(X \cup e^n\right) \arrow[r] \arrow[ul, phantom, "\lrcorner"] & S^n \wedge \Omega\left(X \cup e^n\right)_+
\end{tikzcd}$$

Applications of house theorem



fiber sequence $M_f * U \longrightarrow X *_Y U \longrightarrow Y$



$$X_\emptyset \simeq holim\ (X_1 \longrightarrow X_{12} \longleftarrow X_2)$$

$$\Downarrow$$

$M_{f_\emptyset} \simeq M_{f_1} \times M_{f_2}$ by valley theorem

$$\Downarrow$$

fiber sequence $M_{f_1} * M_{f_2} \longrightarrow hocolim\ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \longrightarrow X_{12}$

Open pushouts

Excisive triads

Homotopy excision theorem

Suppose $\{A, B\}$ is an open cover of X with 0-connected $A \cap B$, then for m-connected pair $(A, A \cap B)$ and n-connected pair $(B, B \cap A)$, one has $(m+n)$-connected inclusion $(A, A \cap B) \longrightarrow (X, B)$.

There is an isomorphism given by Whitehead product

$$\pi_{m+1}(A, A \cap B) \otimes \pi_{n+1}(B, B \cap A) \longrightarrow \pi_{m+n+1}(X; A, B) \ , \ [f] \otimes [g] \longmapsto [f, g] \ .$$

Proposition

For CW complex X , one has an $(m+n-3)$-cartesian square.

$$\begin{tikzcd}
X \cup e^m \cup e^n & X \cup e^n \arrow[l] \\
X \cup e^m \arrow[u] & X \arrow[l] \arrow[u] \arrow[ul, phantom, "m+n-3 \lrcorner"]
\end{tikzcd}$$

For a square with all maps fibrations, one has a quasi-fibration *hocolim* $(X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \longrightarrow X_{12}$.

$$\begin{tikzcd}
X_\emptyset \arrow[r, Rightarrow] \arrow[d, Rightarrow] & X_1 \arrow[d, Rightarrow] & \\
X_2 \arrow[r, Rightarrow] & X_{12} & \\
& & hocolim \ (X_1 \longleftarrow X_\emptyset \longrightarrow X_2) \arrow[ul, "\text{quasi-fibration}"']
\end{tikzcd}$$

Blackers-Massey theorem

$$
\begin{tikzcd}
X_\emptyset \arrow[r, "m\text{-connected}"] \arrow[d, "n\text{-connected}"'] \arrow[dr, phantom, "\infty \lrcorner" near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner m+n-1" near start] \arrow[dr, phantom, "\infty \lrcorner" near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\infty \lrcorner" near end] & X_2 \arrow[l, "m\text{-connected}"'] \\
X_1 \arrow[u, "n\text{-connected}"] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\ulcorner m+n+1" near start] \arrow[dr, phantom, "\infty \lrcorner" near end] & X_2 \arrow[l] \\
X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
$$

generalised version

$$
\begin{tikzcd}
X_\emptyset \arrow[r, "m\text{-connected}"] \arrow[d, "n\text{-connected}"'] \arrow[dr, phantom, "k \lrcorner" near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner min\{m+n-1, k-1\}" near start] \arrow[dr, phantom, "k \lrcorner" near end] & X_2 \arrow[d] \\
X_1 \arrow[r] & X_{12}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "k \lrcorner" near end] & X_2 \arrow[l, "m\text{-connected}"'] \\
X_1 \arrow[u, "n\text{-connected}"] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\ulcorner min\{m+n+1, k+1\}" near start] \arrow[dr, phantom, "k \lrcorner" near end] & X_2 \arrow[l] \\
X_1 \arrow[u] & X_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
$$

Proof :

$$
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_2 \arrow[r] & X_{12} \arrow[ul, phantom, "\infty \lrcorner" near start]
\end{tikzcd}
\xrightarrow[\simeq]{\text{cofibrant replacement} \atop \text{CW approximation}}
\begin{tikzcd}
CW_\emptyset \arrow[r] \arrow[d] & CW_1 \arrow[d] \\
CW_2 \arrow[r] & X_{12} \arrow[ul, phantom, "\infty \lrcorner" near start]
\end{tikzcd}
\quad \text{all maps are inclusions}
$$

$$\Updownarrow$$

$$
\begin{tikzcd}
CW_\emptyset \cup e^k_\alpha \arrow[dr, phantom, "\ulcorner" near start] & CW_\emptyset \cup e_i^{n+1} \cup e_j^{n+2} \cup \cdots \arrow[l] \\
CW_\emptyset \cup e_l^{m+1} \cup e_k^{m+2} \cup \cdots \arrow[u] & CW_\emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\Longleftrightarrow
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\ulcorner" near start] & CW_1 \arrow[l] \\
CW_2 \arrow[u] & CW_\emptyset \arrow[l, "m\text{-connected}"'] \arrow[u, "n\text{-connected}"']
\end{tikzcd}
$$

$$\Updownarrow$$

$$
\begin{tikzcd}
CW_\emptyset \cup e^k_\alpha \arrow[dr, phantom, "\ulcorner" near start] & CW_\emptyset \cup e_i^{n+1} \cup e_j^{n+2} \cup \cdots \arrow[l] & \cdots \arrow[l] & CW_\emptyset \cup e_k^{n+1} \arrow[l] \\
CW_\emptyset \cup e_l^{m+1} \cup e_k^{m+2} \cup \cdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[l] \arrow[u] \\
\vdots \arrow[u] & & & \\
CW_\emptyset \cup e_l^{m+1} \cup e^{n+1} \arrow[u] & \cdots \arrow[l] & CW_\emptyset \cup e^{m+1} \cup e^{n+1} \arrow[l] \arrow[uu] & CW_\emptyset \cup e^{n+1} \arrow[l] \arrow[uu] \arrow[uul, phantom, "m+n-1 \lrcorner" near start] \\
CW_\emptyset \cup e_l^{m+1} \arrow[u] & \cdots \arrow[l] \arrow[u] \arrow[ul, phantom, "m+n-1 \lrcorner" near start] & CW_\emptyset \cup e^{m+1} \arrow[l] \arrow[u] \arrow[ul, phantom, "m+n-1 \lrcorner" near start] & CW_\emptyset \arrow[l] \arrow[u] \arrow[ul, phantom, "m+n-1 \lrcorner" near start]
\end{tikzcd}
$$

by prism theorem

$$
\begin{tikzcd}
CW_\emptyset \cup e^k_\alpha \arrow[dr, phantom, "\ulcorner" near start] & CW_\emptyset \cup e_i^{n+1} \cup e_j^{n+2} \cup \cdots \arrow[l] \\
CW_\emptyset \cup e_l^{m+1} \cup e_k^{m+2} \cup \cdots \arrow[u] & CW_\emptyset \arrow[l] \arrow[u] \arrow[ul, phantom, "m+n-1 \lrcorner" near start]
\end{tikzcd}
$$

by prism theorem (generlised version)

$$
\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner m+n-1" near start] & I_f \arrow[d] \\
X_2 \arrow[r] & hocolim\ \mathcal{X} \arrow[ul, phantom, "\infty \lrcorner" near start]
\end{tikzcd}
\quad
\begin{tikzcd}
I_f \arrow[r, "\simeq"] \arrow[d] & X_1 \arrow[d] \\
hocolim\ \mathcal{X} \arrow[r, "k\text{-connected}"'] & X_{12}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
I_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner k-1" near start] & X_1 \arrow[d] \\
hocolim\ \mathcal{X} \arrow[r] & X_{12}
\end{tikzcd}
$$

for dual version

$$\begin{tikzcd}
X_\emptyset & X_1 \arrow[l] \\
X_2 \arrow[u] & X_{12} \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty \lrcorner"]
\end{tikzcd}$$

fibrant replacement $\longrightarrow$

$$hocolim\ (X_1 \longleftarrow X_{12} \longrightarrow X_2)$$

quasi-fibration

$$\begin{tikzcd}
X_\emptyset & X_1 \arrow[l, Rightarrow] \\
X_2 \arrow[u, Rightarrow] & X_{12} \arrow[l, Rightarrow] \arrow[u, Rightarrow] \arrow[ul, phantom, "\infty \lrcorner"]
\end{tikzcd}$$

all maps are fibrations

$$\Downarrow$$

$$\begin{tikzcd}
X_\emptyset & X_1 \arrow[l] \\
X_2 \arrow[u] & X_{12} = X_1 \times_{X_\emptyset} X_2 \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty \lrcorner"]
\end{tikzcd}$$

take fibers F_1 and F_2 , use the house theorem

m-connected

n-connected

$$\begin{tikzcd}
 & X_\emptyset & & \\
 & X_{12} \arrow[u] \arrow[rr] \arrow[dl] & & X_1 \arrow[ull] \arrow[dl] \\
X_2 \arrow[uur] \arrow[rr] & & hocolim\ (X_1 \longleftarrow X_{12} \longrightarrow X_2) \arrow[uul] & \\
 & M_1 \times M_2 \arrow[uu] \arrow[rr] \arrow[dl] & & M_1 \arrow[uu] \arrow[dl] \\
M_2 \arrow[uu] \arrow[rr] & & M_1 * M_2 \arrow[uu] &
\end{tikzcd}$$

$(m+n)$-connected

homotopy pullback keeps products

$$\begin{tikzcd}
X_\emptyset \times_{X_\emptyset} X_\emptyset & X_1 \times_{X_\emptyset} X_2 \arrow[l] \\
\{*\} \times_{X_\emptyset} \{*\} \arrow[u] & M_1 \times_{X_\emptyset} M_2 \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty \lrcorner"]
\end{tikzcd}$$

it is $M_1 \times M_2$ since $X_\emptyset$ is base space

by prism theorem (generlised version)

$$\begin{tikzcd}
X_\emptyset & X_{12} \arrow[l, "k\text{-connected}"'] \arrow[d] \\
X_2 \arrow[u] & P_f \arrow[l, "\simeq"]
\end{tikzcd}
\quad
\begin{tikzcd}
X_{12} \arrow[dr, phantom, "\ulcorner m+n+1"] & X_1 \arrow[l] \\
P_f \arrow[u] & holim\ \mathcal{X} \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty \lrcorner"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X_\emptyset \arrow[dr, phantom, "\ulcorner k+1"] & X_{12} \arrow[l] \arrow[d] \\
X_2 \arrow[u] & P_f \arrow[l]
\end{tikzcd}$$

Hurewicz theorem

X is n-connected. $\iff \widetilde{H_i}(X)=0$ for $0\le i\le n$, $\widetilde{H_{n+1}}(X)\cong H_{n+1}(X)\cong \pi_{n+1}(X)$.

(X,A) is n-connected with 1-connected A . $\iff \widetilde{H_i}(X,A)=0$ for $0\le i\le n$, $\widetilde{H_{n+1}}(X,A)$.

Proof :

$$\begin{tikzcd}
X & PX \arrow[l, "n\text{-connected}"'] \\
* \arrow[u, "n\text{-connected}"] & \Omega X \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X \arrow[dr, phantom, "2n-1"] & PX \arrow[l] \\
* \arrow[u] & \Omega X \arrow[l] \arrow[u] \arrow[ul, phantom, "\infty"]
\end{tikzcd}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$\pi_i(PX,\Omega X)\cong \pi_i(X)$ $\qquad$ $\widetilde{H_i}(PX,\Omega X)\cong \widetilde{H_i}(X)$ for $i\le 2n$

$\pi_i(X)\cong \pi_{i-1}(\Omega X)$ since $\pi_i(PX)=0$ $\qquad$ $H_i(PX,\Omega X)\cong H_{i-1}(\Omega X)$ since $H_i(PX)=0$

$$\Downarrow$$

$$\begin{tikzcd}
\pi_i(X) \arrow[d] & \pi_i(PX,\Omega X) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \pi_{i-1}(\Omega X) \arrow[d] \\
H_i(X) & H_i(PX,\Omega X) \arrow[l, "\cong"', "i\le 2n"] \arrow[r, "\cong"] & H_{i-1}(\Omega X)
\end{tikzcd}$$

$\pi_1(\Omega X)\cong H_1(\Omega X)$ $\Swarrow$ Hurewizc theorem for 1-connected space $\Nearrow\cdots\Nearrow$ Hurewizc theorem for $(n-1)$-connected space $\Swarrow$ Hurewizc theorem for n-connected space

for relative version, consider

$$\begin{tikzcd}
A \arrow[r, "n\text{-connected}"] \arrow[d, "2\text{-connected}"'] & X \arrow[d] \\
* \arrow[r] & X/A \arrow[ul, phantom, "\infty"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "n+1"] & X \arrow[d] \\
* \arrow[r] & X/A \arrow[ul, phantom, "\infty"]
\end{tikzcd}$$

Applications of Blakers-Massey theorem

$$\begin{aligned} X \text{ is } n\text{-connected} &\Longleftrightarrow (CX, X) \text{ is } (n+1)\text{-connected} \\ &\Longleftrightarrow X \longrightarrow CX \text{ is } n\text{-connected} \end{aligned}$$

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, phantom, "\infty" near end] & * \end{tikzcd} \xrightarrow{\simeq} (min\{n,\infty\}+1)\text{-connected} \qquad \begin{tikzcd} X \vee Y \arrow[r, "(min\{m,\infty\}+1)\text{-connected}"] \arrow[d] & X \vee CY \arrow[d] \\ CX \vee Y \arrow[r] \arrow[ur, phantom, "\infty" near end] & CX \vee CY \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner m+n+1" near start] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, phantom, "\infty" near end] & * \end{tikzcd} \xleftarrow{\simeq} \begin{tikzcd} X \vee Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner m+n+1" near start] & X \vee CY \arrow[d] \\ CX \vee Y \arrow[r] \arrow[ur, phantom, "\infty" near end] & CX \vee CY \end{tikzcd}$$

$$\Downarrow$$

$X \vee Y \longrightarrow X \times Y$ is $(m+n+1)$-connected

$$\begin{tikzcd} X \vee Y \arrow[r, "(m+n+1)\text{-connected}"] \arrow[d, "(min\{m,n\}+1)\text{-connected}"'] & X \times Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\infty" near end] & X \wedge Y \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} \Sigma(\Omega X \wedge \Omega Y) \arrow[r, "\simeq"] & \Omega X * \Omega Y \arrow[r] \arrow[d, "(m+n+min\{m,n\}+1)\text{-connected}"'] & X \vee Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner m+n+min\{m,n\}+1" near start] & X \times Y \arrow[d] \\ & \Omega(X \wedge Y) \arrow[r] & * \arrow[r] \arrow[ur, phantom, "\infty" near end] & X \wedge Y \end{tikzcd}$$

$\Sigma(\Omega X \wedge \Omega Y) \longrightarrow \Omega(X \wedge Y)$ is $m+n+min\{m,n\}+1$

$$\begin{tikzcd} S^n \arrow[r, "n\text{-connected}"] \arrow[d, "n\text{-connected}"'] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\infty\lrcorner"] & S^{n+1} \end{tikzcd} \Longrightarrow \begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner 2n-1"] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\infty\lrcorner"] & S^{n+1} \end{tikzcd}$$

$$\Downarrow$$

$S^n \longrightarrow \Omega\Sigma S^n$ is $(2n-1)$-connected

$X \longrightarrow \Omega\Sigma X$ is $(2n-1)$-connected for $(n-1)$-connected X

(Freudenthal suspension theorem)

$$\begin{tikzcd} X & * \arrow[l, "n\text{-connected}"'] \\ * \arrow[u, "n\text{-connected}"] \arrow[ur, phantom, "\infty\lrcorner"] & \Omega X \arrow[l] \arrow[u] \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[dr, phantom, "\ulcorner 2n+1"] & * \arrow[l] \\ * \arrow[u] \arrow[ur, phantom, "\infty\lrcorner"] & \Omega X \arrow[l] \arrow[u] \end{tikzcd}$$

$$\Downarrow$$

$\Sigma\Omega X \longrightarrow X$ is $(2n+1)$-connected for n-connected X

$$\begin{tikzcd} X \arrow[r, "f \text{ is } n\text{-connected}"] \arrow[d, "k\text{-connected}"'] & Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\infty\lrcorner"] & C_f \end{tikzcd} \Longrightarrow \begin{tikzcd} & X \arrow[r] \arrow[d] & M_i \arrow[d] \\ M_f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner n+k-1"] & Y \arrow[d, "i"] \\ \Omega C_f \arrow[r] & * \arrow[r] \arrow[ur, phantom, "\infty\lrcorner"] & C_f \end{tikzcd}$$

$$\Downarrow$$

$M_f \longrightarrow \Omega C_f$ is $(n+k-1)$-connected

$X \longrightarrow M_i$ is $(n+k-1)$-connected

Ganea's fiber-cofiber construction

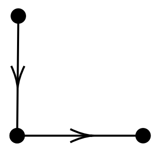
$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & * \arrow[d] \\
* \arrow[r] \arrow[d] & C_i \arrow[d] \\
X \arrow[r, "\mathbb{1}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[d] \arrow[r, phantom, "\circlearrowright"] & C_i \arrow[d] \\
X \arrow[r, "\mathbb{1}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & M_i \arrow[d] \\
* \arrow[d] \arrow[r, phantom, "\circlearrowright"] & C_i \arrow[d, "f_i"] \\
X \arrow[r, "\mathbb{1}"'] & X
\end{tikzcd}
\qquad C_1 = \Sigma\Omega X
$$

$$\Downarrow$$

$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M_3 \arrow[d] \\
* \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Sigma\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C_3 \arrow[d] \\
X \arrow[r, "\mathbb{1}"'] & X \arrow[r, "\mathbb{1}"'] & X \arrow[r, "\mathbb{1}"'] & X
\end{tikzcd}
$$

generated with
first collum by

fibre sequence



cofibre sequence

$$
\begin{tikzcd}
 & X & & \\
 & M_j \arrow[dl] \arrow[rr] & & C_j \arrow[ull] \arrow[dl] \\
* \arrow[uur] \arrow[rr] & & C_{j+1} \arrow[uul] & C_{j+1} \arrow[u] \\
\Omega X \arrow[u] & & \Omega X * M_j \arrow[u] &
\end{tikzcd}
\Longrightarrow
$$

by house theorem one has
fiber sequence $\Omega X * M_j \longrightarrow C_{j+1} \longrightarrow X$

$$\Downarrow$$

$M_j \simeq \Omega X * M_{j-1} \simeq *_{j+1}\Omega X$

M_j is $(jk + j + k - 1)$-connected if X is k-connected

$C_j \longrightarrow X$ is $(jk + j + k)$-connected if X is k-connected

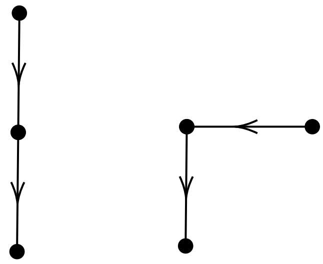
$$PX \simeq * \longrightarrow \Sigma\Omega X \simeq C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow \cdots\cdots$$

has increasing connectivity as up the tower

Dual construction

$$\begin{tikzcd}
X & X \arrow[l, "\mathbb{1}"'] & X \arrow[l, "\mathbb{1}"'] & X \arrow[l, "\mathbb{1}"'] \\
* \arrow[from=u] & \Sigma\Omega X \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"] & C_2 \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"] & C_3 \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"] \\
\Sigma X \arrow[from=u] & M_1 \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"] & M_2 \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"] & M_3 \arrow[l] \arrow[from=u] \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

generated with first collum by



fibre sequence cofibre sequence

$$CX \simeq * \longleftarrow \Sigma\Omega X \simeq M_1 \longleftarrow M_2 \longleftarrow M_3 \longleftarrow \cdots\cdots$$

has increasing connectivity as up the tower

Serre theorem

$$\begin{tikzcd}
X & & Y \arrow[ll, "f \text{ is } n\text{-connected}"'] \\
* \arrow[u, "m\text{-connected}"] & & M_f \arrow[ll] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner"]
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
X & & Y \arrow[ll] \arrow[dll, phantom, "\ulcorner m+n+1"] \\
* \arrow[u] & & M_f \arrow[ll] \arrow[u] \arrow[ul, phantom, "\infty\lrcorner"]
\end{tikzcd}$$

fiber sequence $M_f \xrightarrow{p} Y \xrightarrow{f} X$ induces $(m+n+1)$-connected map $C_p \longrightarrow X$