Chapter 9

Manifold Theory

9.1 Smooth Structures

Differentiable functions

$f:\mathbb{R}\longrightarrow\mathbb{R}$ is differentiable at $x\in\mathbb{R}$.

$\Longleftrightarrow f'(x)=\underset{h\to 0}{lim}\frac{f(x+h)-f(x)}{h}$ exists.

$\Longleftrightarrow f(x+h)-f(x)=f'(x)h+o(h)$ as $||h||\to 0$.

$f:\mathbb{R}^n\longrightarrow\mathbb{R}^k$ is differentiable at $x\in\mathbb{R}^n$.

$\Longleftrightarrow Df(x):\mathbb{R}^n\longrightarrow\mathbb{R}^k$ exists such that $\underset{||h||\to 0}{lim}\frac{||f(x+h)-f(x)-Df(x)h||}{||h||}=0$

where $Df(x)=\begin{pmatrix}\frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^n}(x)\\ \vdots & & \vdots\\ \frac{\partial f^k}{\partial x^1}(x) & \cdots & \frac{\partial f^k}{\partial x^n}(x)\end{pmatrix}$ is the Jacobi matrix of f , the total derivative.

$\Longleftrightarrow f(x+h)-f(x)=Df(x)h+o(||h||)$ as $||h||\to 0$.

Locally euclidean spaces

A topological space M is locally Euclidean space if for every $p\in M$, the map $\varphi:U_p\longrightarrow\mathbb{R}^n$ is a homeomorphism from U_p onto an open set in $\mathbb{R}^n$.

The pair (U_p,φ) is a chart, U_p is a coordinate neighbourhood or coordinate open set.

The chart (U_p,φ) is centred at p if $\varphi(p)=0\in\mathbb{R}^n$.

A topological manifold is a Hausdorff, second countable and locally euclidean topological space.

Smooth atlases

The two charts (U_p,φ) and (V_p,ψ) are C^∞-compatible if $\varphi\circ\psi^{-1}$ and $\psi\circ\varphi^{-1}$ are smooth.

An atlas (smooth atlas or C^∞ atlas) of locally Euclidean M is a collection $\mathcal{A}=\{(U_{p_i},\varphi_i)\}$ of C^∞-compatible charts covering M , $M=\bigcup_i U_{p_i}$.

The chart (U,φ) as a map is a diffeomorphism between manifolds U and $\varphi(U)$.

Smooth structures

If the two charts (V_{p_1},ψ_1) and (V_{p_2},ψ_2) are both C^∞-compatible with the atlas $\mathcal{A}$, then (V_{p_1},ψ_1) and (V_{p_2},ψ_2) are C^∞-compatible.

The maximal atlas containing $\mathcal{A}$ is a unique smooth structure determined by the atlas $\mathcal{A}$.

Smooth functions

M is a smooth n-manifold, we say $f : M \longrightarrow \mathbb{R}$ is a smooth function at $p \in M$ if there is a chart (U_p, φ) such that $f \circ \varphi^{-1} : \varphi(U_p) \longrightarrow \mathbb{R}$ is smooth at $\widetilde{p} = \varphi(p)$.

If $f : M \longrightarrow \mathbb{R}$ is smooth at every $p \in M$, then it is a smooth function on M .

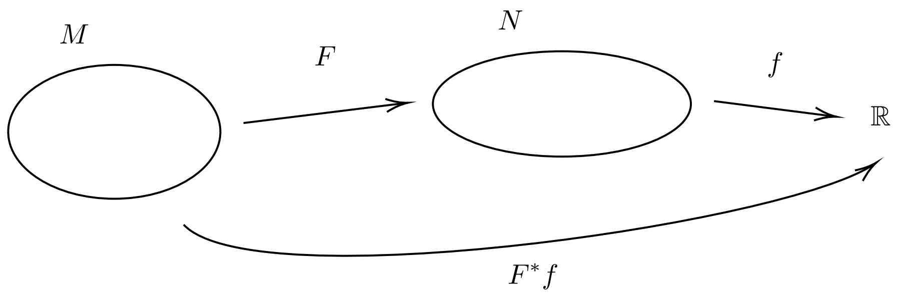
Proposition

(1) Any atlas $\mathcal{A}$ for M is in an unique maximal smooth atlas which is called the smooth structure determined by atlas $\mathcal{A}$.

(2) Two atlases for M are in the same smooth structure. $\Longleftrightarrow$ Their union is a smooth atlas.

The pullback of a function f on N

Let $F : M \longrightarrow N$ be a map, $f : N \longrightarrow \mathbb{R}$ is a function on N , then $F^*f = f \circ F$ is a pullback of f by F .



The smooth structure of $\mathbb{R}\mathrm{P}^n$

$\mathbb{R}\mathrm{P}^n = \mathbb{R}^{n+1}_{\times}/(x \sim kx)$.

Take

$$U_0 = \{[x^0, x^1, \cdots, x^n] \in \mathbb{R}\mathrm{P}^n \mid x^0 \neq 0\} \ , \ \varphi_0 : [x^0, x^1, \cdots, x^n] \longmapsto (\frac{x^1}{x^0}, \cdots, \frac{x^n}{x^0}) \ .$$

Then (φ_0, U_0) is a chart since $\varphi_0^{-1} : (x^1, \cdots, x^n) \longmapsto [1, x^1, \cdots, x^n]$, φ_0 is a homeomorphism from U_0 onto an open set $\varphi_0(U_0)$ in $\mathbb{R}\mathrm{P}^n$.

Take

$$U_i = \{[x^0, \cdots, x^i, \cdots, x^n] \in \mathbb{R}\mathrm{P}^n \mid x^i \neq 0\} \ , \ \varphi_i : [x^0, \cdots, x^i, \cdots, x^n] \longmapsto (\frac{x^0}{x^i}, \cdots, \frac{\hat{x^i}}{x^i}, \cdots, \frac{x^n}{x^i}) \ .$$

Then $\{(U_i, \varphi_i)\}$ is an atlas of $\mathbb{R}\mathrm{P}^n$ since $\varphi_i \circ \varphi_j^{-1}$ is smooth, given the smooth structure determined by this atlas , $\mathbb{R}\mathrm{P}^n$ is a smooth manifold.

Proposition

(1) $\mathrm{Int}(M)$ is an open subset of M and a topological n-manifold without boundary.

(2) ∂M is a closed subset of M and a topological $(n-1)$-manifold with boundary.

(3) M is a topological manifold. $\Longleftrightarrow \partial M = \emptyset$.

(4) If M is a topological 0-manifold and $\partial M = \emptyset$, then it is a 0-manifold.

(5) Suppose $M_1, \cdots, M_k$ are smooth manifolds without boundary, N is a smooth manifold with boundary, then $M_1 \times \cdots \times M_k \times N$ is a manifold with boundary, $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.

(6) If $F : M \longrightarrow N$ is a diffeomorphism, then $F(\partial M) = \partial N$ and $F|_{\mathrm{Int}(M)}$ is also a diffeomorphism.

Computations in coordinates

$\varphi : U \longrightarrow \mathbb{R}^n$ is a chart on smooth manifold M , then as a map $\varphi : U \longrightarrow \widetilde{U}$ is a diffeomorphism, the differential $d\varphi_p : \mathbf{T}_p M \longrightarrow \mathbf{T}_{\widetilde{p}}\mathbb{R}^n$ is an isomorphism.

One has $\mathbf{T}_p M \cong \mathbf{T}_{\widetilde{p}}\mathbb{R}^n$ given by

$$d\varphi_p(\frac{\partial}{\partial x^i}|_p)f = \frac{\partial}{\partial x^i}|_p(f \circ \varphi) = \frac{\partial}{\partial x^i}|_{\widetilde{p}}\widetilde{f} \ .$$

The representation of differentials by matrices

Take $F : M \longrightarrow N$, their tangent spaces at p and $F(p)$ are $\mathbf{T}_p M =< \frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^m}|_p >$, $\mathbf{T}_{F(p)} N =< \frac{\partial}{\partial y^1}|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}|_{F(p)} >$. Then dF_p is a $n \times m$ matrix :

$$dF_p = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^m}(p) \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \cdots & \frac{\partial F^n}{\partial x^m}(p) \end{pmatrix}$$

(the Jacobian matrix of F at p)

since

$$\sum_{k=1}^{m} \frac{\partial F^k}{\partial x^i}(p) \cdot \frac{\partial}{\partial x^k}|_p = \frac{\partial F^1}{\partial x^i}(p) \cdot \frac{\partial}{\partial x^1}|_p + \cdots + \frac{\partial F^m}{\partial x^i}(p) \cdot \frac{\partial}{\partial x^m}|_p = \frac{\partial}{\partial y^i}|_{F(P)} \ .$$

9.2 Tangent and Cotangent Spaces

Derivations

A linear map $D : \mathbf{C}^{\infty}(M) \longrightarrow \mathbb{R}$ is a derivation of manifold M at p if it satisfies the product rule $D(f \cdot g) = Df \cdot g(x) + f(x) \cdot Dg$ and f is constant $\Longrightarrow Df = 0$.

Derivatives

Define the smooth map $\gamma : (-\epsilon, \epsilon) \longrightarrow M$, $0 \longmapsto p$ to be a (smooth) curve at p .

The directional derivative of f at p along γ is $\frac{d}{dt}(f \circ \gamma)|_{t=0}$ where $f : M \longrightarrow \mathbb{R}^n$.

Proposition

Suppose there are two curves γ_1 , γ_2 at p with $\gamma_1(0) = \gamma_2(0) = p$.

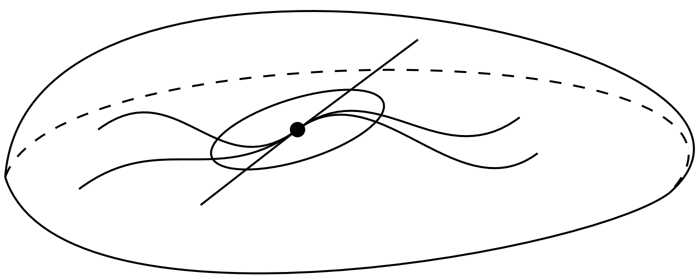
If for any chart φ at p one has $\frac{d}{dt}(\varphi \circ \gamma_1)|_{t=0} = \frac{d}{dt}(\varphi \circ \gamma_2)|_{t=0}$, then for any smooth function $f : M \longrightarrow \mathbb{R}^n$

one has $\frac{d}{dt}(f \circ \gamma_1)|_{t=0} = \frac{d}{dt}(f \circ \gamma_2)|_{t=0}$ (their derivatives along γ_1 and γ_2 are the same) .

Tangent and cotangent vectors

A tangent vector v_p of manifold M at p is a derivation at p . The tangent space is denoted by $\mathbf{T}_p(M)$.
A cotangent vector of manifold M at p is a map $\omega_p : \mathbf{T}_p M \longrightarrow \mathbb{R}$.
The cotangent space is the dual space of $\mathbf{T}_p M$, denoted by $\mathbf{T}_p^* M$.

Tangent spaces as equivalences of smooth curve



$$\frac{d}{dt}(\varphi \circ l_v)|_{t=0} = \frac{d}{dt}(\varphi \circ \gamma_1)|_{t=0} = \frac{d}{dt}(\varphi \circ \gamma_2)|_{t=0}$$

$$l_v(0) = \gamma_1(0) = \gamma_2(0) = p$$

For the tangent line l_v in the equivalence class $\gamma_1 \sim \gamma_2$, the directional derivative of $f : M \longrightarrow \mathbb{R}^n$ at p is $\frac{d}{dt}(f \circ l_v)|_{t=0} = D_v f(p)$ and $D_v|_p$ is also a derivation on M thus a tangent vector.

The bases of tangent and cotangent spaces

Take $\varphi \circ l_v(t) = (0, \cdots, v_i(t), \cdots, 0)$, then $D_{v_i} f(p) = D_i f(p) = \begin{pmatrix} \frac{\partial f^1}{\partial x^j}(p) \\ \vdots \\ \frac{\partial f^k}{\partial x^j}(p) \end{pmatrix} = \frac{\partial}{\partial x^i} f(p)$. Thus the tangent space at p has a basis of derivations $\{D_1, \cdots, D_n\}$ (that is $\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^n}|_p\}$) .

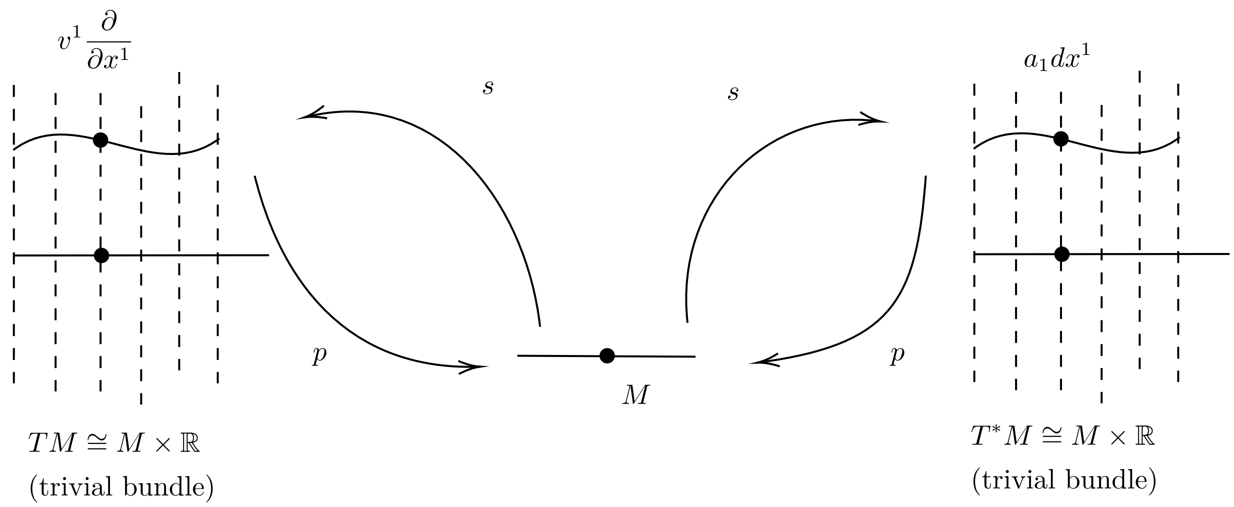
Take $f = x^i : M \longrightarrow \mathbb{R}$, $p \longmapsto (\widetilde{p})^i$ where $\widetilde{p}$ is the coordinate of p , then one has $dx^i(\frac{\partial}{\partial x^j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Thus as a vector space $\mathbf{T}_p M$ with basis $\{\frac{\partial}{\partial x^1|_p}, \cdots, \frac{\partial}{\partial x^n}|_p\}$, its dual space $\mathbf{T}_p^* M$ has basis $\{dx_p^1, \cdots, dx_p^n\}$ as the dual basis.

Vector fields and covector fields

For tangent bundle $p : \mathbf{T}M \longrightarrow M$, a (smooth) vector field on M is a (smooth) section $s : M \longrightarrow \mathbf{T}M$.

For cotangent bundle $p : \mathbf{T}^* M \longrightarrow M$, a (smooth) covector field (or a (smooth) differential 1-form) on M is a (smooth) section $s : M \longrightarrow \mathbf{T}^* M$.



Given a chart $(U, \varphi) = (U, x^1, \cdots, x^n)$ on M , there is an induced chart $(\mathbf{T}U, \varphi) = (\mathbf{T}U, x^1, \cdots, x^n, v^1, \cdots, v^n)$ on $\mathbf{T}M$ and a chart $(\mathbf{T}^*U, \varphi) = (\mathbf{T}^*U, x^1, \cdots, x^n, a_1, \cdots, a_n)$ on $\mathbf{T}^*M$ where

$$X_p = \sum v^i \frac{\partial}{\partial x^i}|_p \ , \ \omega_p = \sum a_i dx^i|_p \ .$$

If every v^i is smooth on U , then the vector field $X = \sum v^i \frac{\partial}{\partial x^i}$ is smooth, denote $X \in \mathcal{F}(U)$.

If every a_i is smooth on U , then the covector field $\omega = \sum a_i dx^i$ is smooth, denote $\omega \in \Omega^1(U)$.

Proposition

$\mathcal{F}(U)$ or $\Omega^1(U)$ is a module over ring $\mathbf{C}^\infty(U)$ by (for $f \in \mathbf{C}^\infty(U)$) :

$$fX = \sum f(v^i) \cdot \frac{\partial}{\partial x^i} \ , \ f\omega = \sum f(a_i) dx^i$$

By the way $Xf = \sum v^i \cdot \frac{\partial f}{\partial x^i}$ and Xf satisfies the product rule $X(fg) = (Xf)g + f(Xg)$.

Differentials

For a smooth map $F : M \longrightarrow N$, $dF_p : \mathbf{T}_p M \longrightarrow \mathbf{T}_{F(p)} N$ is the differential of F at p .

For $v_p \in \mathbf{T}_p M$, $dF_p(v_p)$ acts on $f \in \mathbf{C}^\infty(N)$ by $dF_p(v_p)f = v_p(f \circ F)$, of course $dF_p(v_p) : \mathbf{C}^\infty(N) \longrightarrow \mathbb{R}$ is a derivation at $F(p)$.

If F is a diffeomorphism, then the differential $dF_p : \mathbf{T}_p M \longrightarrow \mathbf{T}_{F(p)} N$ is an isomorphism and $(dF_p)^{-1} = (dF^{-1})_{F(p)}$. For another smooth map $G : N \longrightarrow K$, the differential of the composition $d(G \circ F)_p : \mathbf{T}_p M \longrightarrow \mathbf{T}_{G \circ F(p)} K$ is the composition of the differential , $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

The representation of differentials by curves

Take $F : M \longrightarrow N$, $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ is a curve at $p \in M$. The velocity X_p of γ at p is a tangent vector at $p \in M$.

One has $dF_p(X_p) = \frac{d}{dt}(F \circ \gamma)|_{t=0}$.

Proposition

(1) For $p \in M$ and $v_p \in \mathbf{T}_p M$, if $f, g \in \mathbf{C}^\infty(M)$ agree on some neighbourhood of p , then $v_p f = v_p g$.

(2) Let $U \subseteq M$ be an open subset of M , then for every $p \in U$, the differential $di_p : \mathbf{T}_p U \longrightarrow \mathbf{T}_p M$ induced by the inclusion $i : U \longrightarrow M$ is an isomorphism.

(3) If M is an n-dimension smooth manifold with or without boundary, then for each $p \in M$, the tangent space $\mathbf{T}_p M$ is an n-dimension vector space.

1-forms as the differential of smooth functions

For a smooth function $f : M \longrightarrow \mathbb{R}$, the differential is given by

$$df_p : \mathbf{T}_p M \longrightarrow \mathbf{T}_p \mathbb{R} \ , \ X_p \longmapsto df_p(X_p) = u^1 \cdot \frac{\partial}{\partial x^1} \in \mathbf{T}_p \mathbb{R} \ ,$$

one has $u^1 = df_p(X_p)(x^1) = X_p(x^1 \circ f)$, since $f : M \longrightarrow \mathbb{R}$, $x^1 \circ f = f$, then $u^1 = X_p f$.

If think of the basis of $\mathbb{R}$ as 1 but not $\frac{\partial}{\partial x^1} \in \mathbf{T}_p \mathbb{R}$ (they are isomorphic) , then the differential df_p can be an 1-form by defining $df_p(X_p) = X_p f$.

Proposition

(1) X is a smooth vector field on M .

$\Longleftrightarrow$ On any chart (U,φ) , the coefficient function v^i of $X=\sum v^i\frac{\partial}{\partial x^i}$ are all smooth relative to the frame $\frac{\partial}{\partial x^i}$.

$\Longleftrightarrow$ There is an atlas where the coefficient function v^i of $X=\sum v^i\frac{\partial}{\partial x^i}$ are all smooth relative to the frame $\frac{\partial}{\partial x^i}$ for any chart.

$\Longleftrightarrow$ For any smooth funcion $f:M\longrightarrow\mathbb{R}$, Xf is smooth on M .

(2) X is a smooth covector field on M .

$\Longleftrightarrow$ On any chart (U,φ) , the coefficient function a^i of $\omega=\sum a^i dx^i$ are all smooth relative to the coframe dx^i .

$\Longleftrightarrow$ There is an atlas where the coefficient function a^i of $\omega=\sum a^i dx^i$ are all smooth relative to the coframe dx^i for any chart.

$\Longleftrightarrow$ For any smooth vector field X , the function $\omega(X)$ is smooth on M .

(3) If f is a smooth function on M , then the differential df is a smooth 1-form (covecetor field) on M .

(4) f is a smooth function, X is a vector field, ω is a covector field, then one has $\omega(fX)=f\omega(X)$.

F-related vector fields

Given the smooth map $F:M\longrightarrow N$, for a vector field X_p at p , $dF_p(X_p)$ is a tangent vector at $F(p)$ but not the well defined vector field at all.

If for two vector fields X and Y on M and N , one has $dF_p(X_p)=Y_{F(p)}$ for every $p\in M$, then X and Y are F-related.

Proposition

If X and Y are smooth vector fields on M and N respectively, then one has :

X and Y are F-related.

$\Longleftrightarrow$ For every smooth function $f:U\longrightarrow\mathbb{R}$ on open subset $U\subseteq N$, $X(f\circ F)=Yf\circ F$.

The pushforward of a vector field X on M

If $F : M \longrightarrow N$ is a diffeomorphism, then for every smooth vector field X on M , there is a unique smooth vector field Y on N such that they are F-related.

Denote this smooth vector field by F_*X by $(F_*X)_{F(p)} = dF_p(X_p) = F_*(X_p)$.

The pullback of a 1-form ω on M

For a smooth map $F : M \longrightarrow N$, there is a dual differential $F^* : \mathbf{T}^*_{F(p)}N \longrightarrow \mathbf{T}^*_pM$ given by $F^*(\omega_{F(p)})(X_p) = \omega_{F(p)}(dF_p(X_p))$.

If ω is a (smooth) covector field on N , then there is a pullback $F^*\omega$ (F is not diffeomorphism necessarily) which is a (smooth) covector on M given by $(F^*\omega)_p = F^*(\omega_{F(p)})$.

If F is a diffeomorphism, then one has $(F^*\omega)_p(X_p) = \omega_{F(p)}(F_*X_p)$.

Proposition

Suppose $F : M \longrightarrow N$ is smooth. One has : Commucation with differential of smooth function $f : N \longrightarrow \mathbb{R}$: $F^*(df) = d(F^*f)$. With sum and product of $\omega, \tau \in \Omega^1(N)$: $F^*(\omega + \tau) = F^*(\omega) + F^*(\tau)$, $F^*(g\omega) = F^*(g)F^*(\omega)$ where $g : M \longrightarrow \mathbb{R}$ is smooth.

Restricting on submanifolds

For an immersed or embedded submanifold S of M , the vector field X_p on M is not a vector field on S necessarily.

For an embedded submanifold S : X is a vector field on S . $\Longleftrightarrow$ for every smooth function $f : M \longrightarrow \mathbb{R}$, $S \longrightarrow 0$, one has $Xf : S \longrightarrow 0$.

For an immersed submanifold S : X is a vector field on S . $\Longrightarrow$ there is a unique vector field on S denoted by $X|_S$ such that they are i-related.

For an immersed submanifold S : ω is a 1-form on S . $\Longrightarrow$ the pullback $i^*\omega = \omega|_S$.

9.3 Submanifolds

Ranks of smooth maps

For the smooth map $F : M \longrightarrow N$, the rank of F is the rank of the Jacobian matrix of F at p , which is the rank of linear map $dF_p : \mathbf{T}_pM \longrightarrow \mathbf{T}_{F(p)}N$. If F has the same rank at every point, we say that F has constant rank. Surjective : $\dim\ M \geq \dim\ N = r(dF_p)$. Injective : $r(dF_p) = \dim\ M \leq \dim\ N$.

Submersions

If the differential dF_p is surjective at every point $p \in M$, then F is a submersion.

If the differential dF_p is surjective at $p \in M$, then there is a neighbourhood U of p such that $F|_U$ is a submersion.

Immersions

If the differential dF_p is injective at every point $p \in M$, then F is an immersion.

If the differential dF_p is injective at $p \in M$, then there is a neighbourhood U of p such that $F|_U$ is an immersion.

Rank theorem

$F : M \longrightarrow N$ is a smooth map between smooth manifolds with dimension m and n respectively and a smooth map with constant rank r .

For each $p \in M$, there are a chart (φ, U) for M and a chart (ψ, V) for N such that $F(U) \subseteq V$, in which F has a coordinate representation of the form $\widetilde{F} = \psi \circ F \circ \varphi : (x^1, \cdots, x^m) \longmapsto (x^1, \cdots, x^r, 0, \cdots, 0)$.

If F is a submersion, then one has $\widetilde{F} : (x^1, \cdots, x^n, \cdots, x^m) \longmapsto (x^1, \cdots, x^n)$.

If F is an immersion, then one has $\widetilde{F} : (x^1, \cdots, x^m) \longmapsto (x^1, \cdots, x^m, 0, \cdots, 0)$.

The Global Rank Theorem

For a smooth map $F : M \longrightarrow N$ between smooth manifolds with dimension m and n respectively, and F is a smooth map with constant rank r .

(1) If F is surjective, then it is a submersion.

(2) If F is injective, then it is an immersion.

(3) If F is bijective, then it is a diffeomorphism.

Embeddings

If F is an immersion and $M \cong F(M)$ (also diffeomorphism) , then F is an embedding.

Local embedding theorem

The smooth map $F : M \longrightarrow N$ is an immersion.

$\Longleftrightarrow$ For every point $p \in M$, there is a neighbourhood U such that $F|_U : U \longrightarrow N$ is an embedding.

Proposition

If $M \longrightarrow N$ is an injective immersion, and any of the following holds, then F is a embedding :

(1) F is an open or closed map.

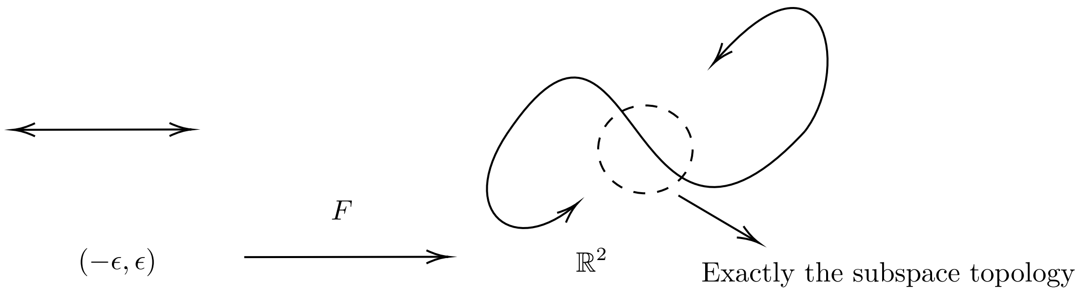
(2) F is a proper map (For any compact set $K \subseteq N$, $F^{-1}(K)$ is compact) .

(3) M is compact.

(4) $\partial M = \emptyset$, $\dim M = \dim N$.

Embedded submanifolds

An embedded (or regular) submanifold S of M is a subset as manifold with the subspace topology, and $i : S \longrightarrow M$ is an embedding.

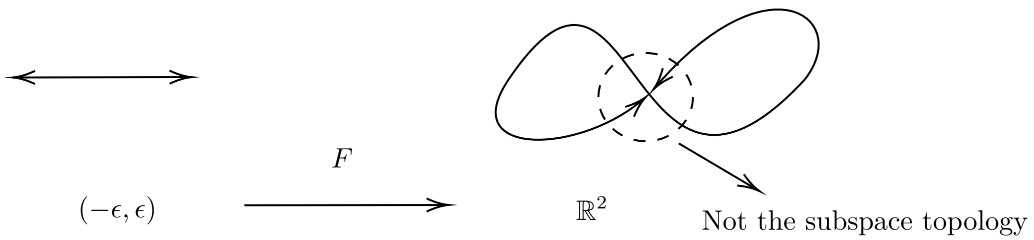


Denote $\dim M - \dim S$ to be the codimension of S in M . M is called the ambient manifold of S .

An embedded submanifold of codimension 0 is an open submanifold in M . An embedded submanifold of codimension 1 is an embedded hypersurface.

Immersed submanifolds

An immersed (or smooth) submanifold S of M is a subset as manifold with a topology (not the subspace topology necessarily, so it might not be a manifold) , and $i : S \longrightarrow M$ is an immersion.

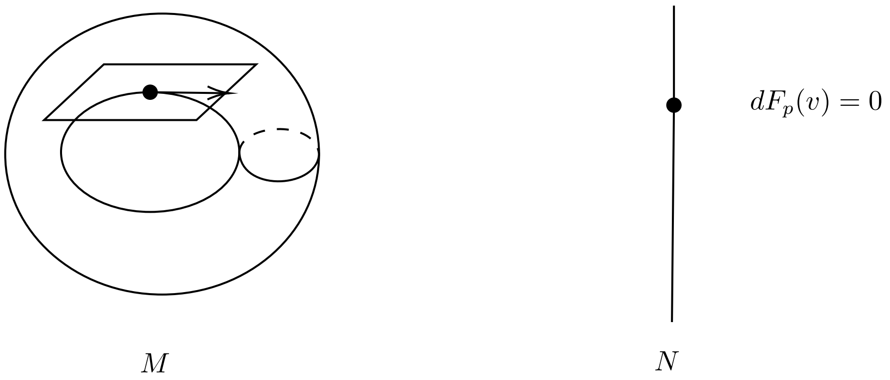


Denote $\dim M - \dim S$ to be the codimension of S in M . A smooth (immersed) hypersurface is an immersed submanifold of codimension 1 .

Critical and regular points

If dF_p fails to be surjective, then p is a critical point of $F : M \longrightarrow N$.
The image of critical point in N is a critical value, otherwise it is a regular value even it is out of $\mathcal{I}m(f)$.

If dF_p is surjective, then p is a regular point of $F : M \longrightarrow N$.



The Sard theorem

For the smooth map $F : M \longrightarrow N$, the set of the critical values of F in N has measure zero in N .

Level sets

A level set of $F : M \longrightarrow N$ is $F^{-1}(c) = \{p \in M \mid F(p) = c, c \in N\}$.
If c is a regular value, then $F^{-1}(c)$ is a regular level set.

$F^{-1}(c)$ is a non-empty regular level set. $\Longleftrightarrow$ $F : M \longrightarrow N$ is a submersion at $p \in F^{-1}(c)$.
If $N = \mathbb{R}^n$, then $F^{-1}(0)$ is a zero set of F .

Regular level set theorem

$F : M \longrightarrow N$, dim $M = m$ and dim $N = n$, then the non-empty regular level set $F^{-1}(c)$ is an embedded submanifold of M with dimension $m - n$.

Constant-rank level set theorem

$F : M \longrightarrow N$ has the constant rank k in a neighbourhood of level set $F^{-1}(c)$ in M , then the level set $F^{-1}(c)$ is an embedded submanifold of M with codimension k .

Proposition

Considering the smooth map $\det : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $\det(M) = m_{1,1}A_{1,1} + \cdots + m_{1,n}A_{1,n}$ where $A_{1,k} = (-1)^{1+k}M_{1,k}$.

Thus $d(\det)_M = \begin{pmatrix} A_{1,1}, & A_{1,2}, & \cdots, & A_{1,n}, & \cdots, & A_{n,n} \end{pmatrix}$, the critical points of $\mathrm{GL}_n(\mathbb{R})$ are the matrices whose $n-1$ minors are all 0 . The critical value is 0 .

Then $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ is a regular level set, thus a embedded submanifold with dimension $n^2 - 1$.

Propositon

Considering the smooth map $f : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{Sym}_n(\mathbb{R})$ (symmetric matrices) , $A \longmapsto A^T A$.

Thus $f(AC) = (AC)^T AC = C^T A^T AC = C^T f(A) C$, $f \circ r_C = l_{C^T} \circ r_C \circ f$.

Since the left transition $l_C : A \longmapsto CA$ and right transition $r_C : A \longmapsto AC$ are diffeomorphisms (as Lie groups) . One has $df_{AC} \circ d(r_C)_A = d(l_{C^T})_{A^T AC} \circ d(r_C)_{A^T A} \circ df_A$, $r(df_{AC}) = r(df_A)$.

By the Constant-rank Level Set Theorem, $\mathrm{O}(n) = f^{-1}(I)$ is an embedded submanifold.

$\mathrm{GL}_n(\mathbb{R})$ is open in $\mathbb{R}^{n\times n}$, then $\mathbf{T}_A \mathrm{GL}_n(\mathbb{R}) \cong \mathbf{T}_A \mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}$.

For the curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathrm{GL}_n(\mathbb{R})$, $0 \longmapsto A$ with velocity $X \in \mathbb{R}^{n\times n}$ at A , the differential df_A is given by

$$df_A(X) = \tfrac{d}{dt}(f \circ \gamma)(t)|_{t=0} = \tfrac{d}{dt}(\gamma^T(t)\gamma(t))|_{t=0} = (\gamma^T)'(0)\gamma(0) + \gamma^T(0)\gamma'(0) = X^T A + A^T X \ .$$

Take $B \in \mathrm{Sym}_n(\mathbb{R})$, $X = \frac{1}{2}(A^T)^{-1}B$, then $X^T A + A^T X = B$, df_A is surjective and $r(df_A) = \dim\ \mathrm{Sym}_n(\mathbb{R}) = \frac{n^2+n}{2}$, thus $\mathrm{O}(n) = f^{-1}(I)$ is an embedded submanifold with codimension $\frac{n^2+n}{2}$ and with dimension $\frac{n^2-n}{2}$.

Differential of $\det : \mathbf{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ at I

Take the curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathrm{GL}_n(\mathbb{R})$, $t \longmapsto e^{tX}$ with velocity $X \in \mathbf{T}_I \mathrm{GL}_n(\mathbb{R})$ at I , the differential $d(\det)_I$ is given by

$$d(\det)_I(X) = \tfrac{d}{dt}(\det \circ \gamma)(t)|_{t=0} = \tfrac{d}{dt}(\det(e^{tX}))|_{t=0} = \tfrac{d}{dt}(e^{t\cdot tr(X)})|_{t=0} = tr(X) \cdot e^{0\cdot tr(X)}) = tr(X) \ .$$

Inverse function theorem

For a smooth map $F : M \longrightarrow N$, if dF_p is invertible, then there are connected neighbourhood U of p and V of $F(p)$ such that $F|_U : U \longrightarrow V$ is a diffeomorphism ($F : M \longrightarrow N$ is a local diffeomorphism at p) .

F is a local diffeomorphism. $\Longleftrightarrow$ F is both an immersion and submersion at p . $\Longleftrightarrow$ dF_p is bijective.

Local section theorem

For any continuous map $p : M \longrightarrow N$:
a section of p is continuous right inverse of p that is $s : N \longrightarrow M$ such that $p \circ s = \mathbb{1}_N$,
a local section of p is continuous right inverse of $p|_U$ that is $s|_U : U \longrightarrow M$ such that $p|_U \circ s|_U = \mathbb{1}_U$.
(Thus s is injective, p is surjective.)

The smooth map $F : M \longrightarrow N$ is a submersion (or a topological submersion).
$\Longleftrightarrow$ every point $p \in M$ is in the image of a smooth local section (or a continuous local section) of π .

Proposition

(1) Suppose $\pi : M \longrightarrow N$ is a submersion, then π is an open map, and if it is surjective implies it is a quotient map.

(2) For smooth manifolds M , N and $p \in N$, $M \times \{p\}$ is an embedded submanifold of $M \times N$ that is diffeomorphic to M (called a slice of $M \times N$) .

(3) M is a smooth m-manifold without boundary, N is a smooth n-manifold with or without boundary, $U \subseteq N$ is an open set, $f : U \longrightarrow N$ is a smooth map, then :
the graph of f : $\Gamma(f) = \{(m, n) \mid m \in U, n = f(m)\}$ is an embedded m-submanifold of $M \times N$,
the global graph of f : $\Gamma(f) = \{(m, n) \mid m \in M, n = f(m)\}$ is a properly embedded m-submanifold of $M \times N$.

(4) The embedded submanifold $S \subseteq M$ is said to be properly embedded if the inclusion $S \longrightarrow M$ is a proper map.
One has S is properly embedded $\Longleftrightarrow$ S is a closed subset of M .
Moreover, the compact embedded submanifold is properly embedded since the compact subset in the Hausdorff space is closed.

9.4 Multilinear Algebra

Multilinear functions

If $f(v_1, \cdots, av_i + a'v_i', \cdots, v_k) = af(v_1, \cdots, v_i, \cdots, v_k) + a'f(v_1, \cdots, v_i', \cdots, v_k)$,
then $f : V_1 \times \cdots \times V_k \longrightarrow W$ is called a multilinear function, denote $f \in L(V_1, \cdots, V_k; W)$.

Tensor products in dual spaces

For $f^1 \in V_1^*$, $f^2 \in V_2^*$, define $f^1 \otimes f^2 : V_1 \times V_2 \longrightarrow \mathbb{R}$, $(v_1, v_2) \longmapsto f^1(v_1)f^2(v_2) \in \mathbb{R}$.
These tensor product form a space $L(V_1, V_2; \mathbb{R})$ denoted by $V_1^* \otimes V_2^*$.
Suppose $\omega_{(1)}^{i_1} = \{e_1^1, \cdots, e_1^{n_1}\}$ is the basis of V_1^* , $\omega_{(2)}^{i_2} = \{e_2^1, \cdots, e_2^{n_2}\}$ is the basis of V_2^* ,
then $\omega_{(1)}^{i_1} \otimes \omega_{(2)}^{i_2}$ is the basis of $V_1^* \otimes V_2^*$.

The tensor product of tensors

For $\alpha \in L(V_1, \cdots, V_n; \mathbb{R}) = V_1^* \otimes \cdots \otimes V_n^*$, $\beta \in L(W_1, \cdots, W_m; \mathbb{R}) = W_1^* \otimes \cdots \otimes W_m^*$,
$\alpha \otimes \beta : V_1 \times \cdots \times V_n \times W_1 \times \cdots \times W_m \longrightarrow \mathbb{R}$ is a element of $L(V_1, \cdots, V_n, W_1, \cdots, W_m; \mathbb{R})$.

Proposition

For $\alpha \in V_1^* \otimes \cdots \otimes V_n^*$, $\beta \in W_1^* \otimes \cdots \otimes W_m^*$, $\gamma \in U_1^* \otimes \cdots \otimes U_k^*$, one has :
(1) $(\alpha_1 + \alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta$, $\alpha \otimes (\beta_1 + \beta_2) = \alpha \otimes \beta_1 + \alpha \otimes \beta_2$.
(2) $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$.

The (p, q)-tensor space

Since $(V^*)^* \cong V$, denote $V_1 \otimes \cdots \otimes V_n = L(V_1^*, \cdots, V_n^*; \mathbb{R})$.
$v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_n}^{(n)}$ is the basis of $V_1 \otimes \cdots \otimes V_n$ where $v_{i_k}^{(k)} = \{e_1^{(k)}, \cdots, e_{n_k}^{(k)}\}$ is the basis of V_k .
$\omega_{(1)}^{i_1} \otimes \cdots \otimes \omega_{(n)}^{i_n}$ is the basis of $V_1^* \otimes \cdots \otimes V_n^*$ where $\omega_{(k)}^{i_k} = \{e_{(k)}^1, \cdots, e_{(k)}^{n_k}\}$ is the basis of V_k^* .

One can define $T^{(p,q)}(V) = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ with p copies of V and q copies of V^* .
The basis of $T^{(p,q)}(V)$ is $v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$.

Thus the element $T \in T^{(p,q)}(V)$ has the form : $T = T_{j_1, \cdots, j_q}^{i_1, \cdots, i_p} \cdot v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$.

The transform of bases in $T^{(p,q)}$

Suppose $v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$ is a basis of $T^{(p,q)}$, for another basis $v_{k_1} \otimes \cdots \otimes v_{k_p} \otimes \omega^{l_1} \otimes \cdots \otimes \omega^{l_q}$, if one has :

$$T = T_{j_1, \cdots, j_q}^{i_1, \cdots, i_p} \cdot v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q} = T_{l_1, \cdots, l_q}^{k_1, \cdots, k_p} \cdot v_{k_1} \otimes \cdots \otimes v_{k_p} \otimes \omega^{l_1} \otimes \cdots \otimes \omega^{l_q}$$

, then one has :

$$T_{l_1, \cdots, l_q}^{k_1, \cdots, k_p} = T_{j_1, \cdots, j_q}^{i_1, \cdots, i_p} \cdot A_{i_1}^{k_1} \cdots A_{i_p}^{k_p} B_{l_1}^{j_1} \cdots B_{l_q}^{j_q} \text{ where } A_{i_n}^{k_n} v_{k_n} = v_{i_n} \ , \ B_{l_n}^{j_n} \omega^{l_n} = \omega^{j_n} \ .$$

Symmetric and alternating tensors

For $f \in V^* \otimes \cdots \otimes V^*$, if one has $f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n) = f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_n)$, then f is a symmetric tensor, if one has $f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n) = -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_n)$, then f is an alternating tensor.

For the space $T^{(0,n)}(V)$, the symmetric tensors form a subspace denoted by $\sum^n(V)$.

For the space $T^{(0,n)}(V)$, the alternating tensors (exterior forms, multicovectors, n-covectors) form a subspace denoted by $\Lambda^n(V)$.

The symmetrization and alternation

Define the symmetrization of $\alpha \in T^{(0,n)}(V)$ by $\text{Sym}(\alpha) = \frac{1}{n!} \sum\limits_{\sigma \in S_n} \alpha \circ \sigma$ where σ is the permutation on the index of $(v_1, \cdots, v_n)$.

Define the alternation of $\alpha \in T^{(0,n)}(V)$ by $\text{Alt}(\alpha) = \frac{1}{n!} \sum\limits_{\sigma \in S_n} \text{sgn}(\sigma) \cdot \alpha \circ \sigma$ where σ is the permutation on the index of $(v_1, \cdots, v_n)$.

Proposition

(1) $\text{Sym}(\alpha)$ is symmetric, $\text{Alt}(\alpha)$ is alternating.

(2) $\text{Sym}(\alpha) = \alpha \Longleftrightarrow \alpha$ is symmetric. $\text{Alt}(\alpha) = \alpha \Longleftrightarrow \alpha$ is alternating.

(3) For $\alpha \in \sum^n(V)$, $\beta \in \sum^m(V)$, the symmetric product is $\alpha\beta = \text{Sym}(\alpha \otimes \beta) \in \sum^{n+m}(V)$.
For $\alpha \in \Lambda^n(V)$, $\beta \in \Lambda^m(V)$, the wedge product is $\alpha \wedge \beta = \frac{(n+m)!}{n!m!}\text{Alt}(\alpha \otimes \beta) \in \Lambda^{n+m}(V)$.
One has $\alpha\beta = \beta\alpha$, $\alpha \wedge \beta = (-1)^{nm}\beta \wedge \alpha$.

(4) If $\alpha, \beta \in V^* = T^{(0,1)}(V) = \sum^1(V)$, then $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.
If $\omega^1, \cdots, \omega^n \in V^* = T^{(0,1)}(V) = \Lambda^1(V)$, then $\omega^1 \wedge \cdots \wedge \omega^n(v_1, \cdots, v_n) = \det(M(\omega^i(v_j)))$.

(5) Every $(0,0)$-tensor (real number) is both symmetric and alternating.
Every $(0,1)$-tensor is both symmetric and alternating.

(6) For $\omega \in \Lambda^n(V)$, one has $\omega(Tv_1, \cdots, Tv_n) = \det(T)\omega(v_1, \cdots, v_n)$ where $T : V \longrightarrow V$ is a linear map.

(7) For $\alpha, \beta, \gamma \in \Lambda^n(V)$ and $k \in \mathbb{R}$, one has :
$(k_1\alpha_1 + k_2\alpha_2) \wedge \beta = k_1(\alpha_1 \wedge \beta) + k_2(\alpha_2 \wedge \beta)$, $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

(8) For $k > \dim\ V$, $\Lambda^k(V) = 0$.

Exterior differentiation

On smooth k-forms $\Omega^k(M)$, $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ is defined by $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg\ \alpha}\alpha \wedge d\beta$.

The interior multiplication

For $v \in V$, one can define a linear map $i_v : \Lambda^k(V) \longrightarrow \Lambda^{k-1}(V)$ by $i_v\omega(v_2, \cdots, v_k) = \omega(v, v_2, \cdots, v_k)$. For $v \in V$, $i_v \circ i_v = 0$. For $\alpha \in \Lambda^n(V)$, $\beta \in \Lambda^m(V)$, $i_v(\alpha \wedge \beta) = i_v\alpha \wedge \beta + (-1)^n \alpha \wedge i_v\beta$.

Proposition

(1) For covector $\alpha^1, \cdots, \alpha^k$, $i_v(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_2, \cdots, v_k) = (\alpha^1 \wedge \cdots \wedge \alpha^k)(v, v_2, \cdots, v_k)$.

$$
\begin{aligned}
(\alpha^1 \wedge \cdots \wedge \alpha^k)(v, v_2, \cdots, v_k) &= \det \begin{pmatrix} \alpha^1(v) & \alpha^1(v_2) & \cdots & \alpha^1(v_k) \\ \alpha^2(v) & \alpha^2(v_2) & \cdots & \alpha^2(v_k) \\ \vdots & \vdots & & \vdots \\ \alpha^k(v) & \alpha^k(v_2) & \cdots & \alpha^k(v_k) \end{pmatrix} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v)(\alpha^1 \wedge \cdots \wedge \hat{\alpha^i} \wedge \cdots \wedge \alpha^k)(v_2, \cdots, v_k)
\end{aligned}
$$

(2) $i_{fX}\omega = f i_X \omega$, $i_X(f\omega) = f i_X \omega$.

(3) The Lie derivative $\mathcal{L}_X : \Omega^k(M) \longrightarrow \Omega^k(M)$ is a derivation.
For $\alpha \in \Omega^n(M)$, $\beta \in \Omega^m(M)$, one has $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$.

(4) $\mathcal{L}_X$ commutes with exterior derivative d .

(5) Cartan homotopy formula : $\mathcal{L}_X = i_X \circ d + d \circ i_X$.

(6) Product formula : for $\omega \in \Omega^k(M)$, $Y_1, \cdots, Y_k \in \mathcal{F}(M)$, one has

$$\mathcal{L}_X(\omega(Y_1, \cdots, Y_k)) = (\mathcal{L}_X\omega)(Y_1, \cdots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \cdots, \mathcal{L}_X Y_i, \cdots, Y_k) ,$$

$$\mathcal{L}_X(f\omega) = \mathcal{L}_X f\omega + f\mathcal{L}_X\omega = Xf\omega + f\mathcal{L}_X\omega \text{ where } f : M \longrightarrow \mathbb{R} .$$

The pullback of a k-form ω on M

For a smooth map $F : M \longrightarrow N$ (not diffeomorphism necessarily) , the dual differential $F^* : \Lambda^k(N) \longrightarrow \Lambda^k(M)$ gives the pullback of k-form $\omega \in \Lambda^k(N)$ by

$$(F^*\omega)_p(v_1, \cdots, v_n) = \omega_{F(p)}(dF_p(v_1), \cdots, dF_p(v_n)) .$$

Proposition

For smooth map $F : M \longrightarrow N$, $\alpha, \beta \in \Omega^k(N)$, then $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$, $F^*(k\alpha) = kF^*\alpha$, $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$.

9.5 Lie Theory

Lie groups

A Lie group is a smooth manifold with smooth maps multiplication $m : G \times G \longrightarrow G$, $(g,h) \longmapsto gh$ and inverse $i : G \longrightarrow G$, $g \longmapsto g^{-1}$.

If G is a smooth manifold with smooth map $G \times G \longrightarrow G$, $(g,h) \longmapsto gh^{-1}$, then G is a Lie group.

For any element g of Lie group G , define maps left translation and right translation by $L_g : h \longmapsto gh$, $R_g : h \longmapsto hg$. They both smooth and actually diffeomorphisms of G by the definition of Lie group.

$(\mathbb{R}^n, +)$, $(\mathbb{R}^n_\times, \cdot)$, $(\mathrm{GL}_n(\mathbb{R}), \cdot)$, $\mathrm{T}^n = S^1 \times \cdots \times S^1$ are Lie groups.

Lie group homomorphisms

A Lie group homomorphism is a smooth map and also a group homomorphism.

If it is a diffeomorphism, then it is a Lie group isomorphism.

Every Lie group homomorphism has constant rank.

By the Global Rank Theorem, a bijective Lie group homomorphism is a Lie group isomorphism.

Lie subgroups

For two Lie groups H and G , if H is a subgroup of G and an immersed submanifold, then H is a Lie subgroup of G .

Proposition

(1) For a Lie group G , if H is a subgroup of G and an embedded submanifold, then H is a Lie subgorup of G (called embedded Lie subgroup) .

(2) For a Lie group G , if H is an open subgroup of G , then H is an embedded Lie subgroup and H is closed (not only open) , H is a union of connected components of G .

The Closed Subgroup Theorem

H is a Lie subgroup of G , then one has : H is closed in G . $\Longleftrightarrow$ H is an embedded Lie subgroup.

Proposition

(1) For an open neighbourhood U_e in a Lie group G containing $e \in G$, one has :
$< U_e >$ is an open subgroup of G (then it is also closed) .
If U_e is connected, then $< U_e >$ is a connected open subgroup of G .
If G is connected, then $< U_e >= G$.

(2) The connected component G_0 containing the identity e of the Lie group G is a normal subgroup of G , and also the only connected open subgroup of G .
Moreover, any connected component G_i is diffeomorphic to G_0 .

(3) Let $F : G \longrightarrow H$ be a Lie group homomorphism, then $\mathcal{K}er(F)$ is a properly embedded Lie subgroup with codimension $r(dF)$.

The Equivariant Rank Theorem

For a Lie group G , the map $F : M \longrightarrow N$ is equivariant with respect to a Lie group action if $F(g \cdot p) = g \cdot F(p)$, $F(p \cdot g) = F(p) \cdot g$ for $p \in M$, $g \in G$.

If this Lie group action is transitive on M , then the equivariant map F has the constant rank (thus the Global Theorem can be used) .

The dimension of some Lie groups

$\mathrm{GL}_n(\mathbb{R}) = \{A \mid \det(A) \neq 0\}$, dim $\mathrm{GL}_n(\mathbb{R}) = n^2$.

$\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ where $\det : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $A \longmapsto \det(A)$.
dim $\mathrm{SL}_n(\mathbb{R}) = n^2 - r(d(\det)) = n^2 - 1$.

$\mathrm{O}(n) = F^{-1}(I)$ where $F : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{Sym}_n(\mathbb{R})$, $A \longmapsto A^T A$.
dim $\mathrm{O}(n) = n^2 - r(dF) = n^2 - \frac{n^2+n}{2} = \frac{n^2-n}{2}$ (the Closed Subgroup Theorem) .

$\mathrm{SO}(n) = \det^{-1}(1)$ where $\det : \mathrm{O}(n) \longrightarrow \{1, -1\}$, $A \longmapsto \det(A)$.
dim $\mathrm{SO}(n) = \frac{n^2-n}{2} - r(d(\det)) = \frac{n^2-n}{2}$ (the Closed Subgroup Theorem) .

$\mathrm{U}(n) = F^{-1}(I)$ where $F : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{Hem}_n(\mathbb{C})$, $A \longmapsto (\overline{A})^T A$.
dim $\mathrm{U}(n) = 2n^2 - r(dF) = 2n^2 - \frac{n^2+n}{2} + \frac{n^2-n}{2} = n^2$ (the Closed Subgroup Theorem) .

$\mathrm{SU}(n) = \det^{-1}(1)$ where $\det : \mathrm{U}(n) \longrightarrow S^1 = \{z \mid |z| = 1\}$, $A \longmapsto \det(A)$.
dim $\mathrm{SU}(n) = n^2 - r(d(\det)) = n^2 - 1$ (the Closed Subgroup Theorem) .

Left-invariant vector fields

X is a vector field (not smooth necessarily) on Lie group G , for the left transition $l_g : G \longrightarrow G$, the pushforward $d(l_g)(X)$ is a well defined vector field on G .

If $d(l_g)_h(X_h) = X_{gh}$ for every $g, h \in G$, then X is a left-invariant vector field.

A left-invariant vector field X is determined by X_e since $X_g = d(l_g)_e(X_e)$.
Given a tangent vector $A_e \in \mathbf{T}_e G$, A_e generates a left-invariant vector field A by $A_g = d(l_g)_e(A_e)$.

Proposition

(1) Any left-invariant vector field on a Lie group is a smooth vector field.

(2) If X and Y are left-invariant vector field, then so is $[X,Y]=XY-YX$.

Lie brackets

For two smooth vector fields X and Y , the Lie bracket $[X,Y]=XY-YX$ is also a smooth vector, whose value is given by $[X,Y]_p=X_pY-Y_pX$.

For a smooth map $F:M\longrightarrow N$, the smooth vector fields X_1 , X_2 on M is F-related to the smooth vector fields Y_1 , Y_2 on N respectively, then $[X_1,Y_1]$ is F-related to $[X_2,Y_2]$.

If F is a diffeomorphism, with respect to the pushforward one has $F_*[X_1,X_2]=[F_*X_1,F_*X_2]$.

If S is an immersed submanifold of M , and the vector fields Y_1 , Y_2 on M are the vector fields on S , then so is $[Y_1,Y_2]$.

For two tangent vectors A_e , B_e at e of Lie group G , the generated left-invariant (smooth) vector fields are $<A_e>$ and $<B_e>$, one has $[<A_e>,<B_e>]=<[A_e,B_e]>$.

The coordinate formula of Lie bracket

For two smooth vector fields X and Y ,

$$[X,Y]=[\sum v^i\frac{\partial}{\partial x^i},\sum u^i\frac{\partial}{\partial x^i}]=\sum v^i\frac{\partial}{\partial x^i}(\sum u^j\frac{\partial}{\partial x^j})-\sum u^j\frac{\partial}{\partial x^j}(\sum v^i\frac{\partial}{\partial x^i})$$

$$=\sum v^i(\sum\frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial x^j}+\sum u^j\frac{\partial^2}{\partial x^i\partial x^j})-\sum u^j(\sum\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial x^i}+\sum v^i\frac{\partial^2}{\partial x^j\partial x^i})$$

$$=\sum v^i(\sum\frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial x^j})-\sum u^j(\sum\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial x^i})\ .$$

Bilinearity : $[aX+bY,Z]=a[X,Z]+b[Y,Z]$, $[Z,aX+bY]=a[Z,X]+b[Z,Y]$.

Antisymmetry : $[X,Y]=-[Y,X]$.

Jacobi identity : $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

For smooth function $f\in\mathbf{C}^\infty(M)$: $[fX,gY]=fg[X,Y]+(fXg)Y-(gYf)X$.

Intergral curves

If X is a vector field, $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ is a smooth curve on M such that the tangent vector X_p is the velocity of γ at p , then γ is called an integral curve of X .

Proposition

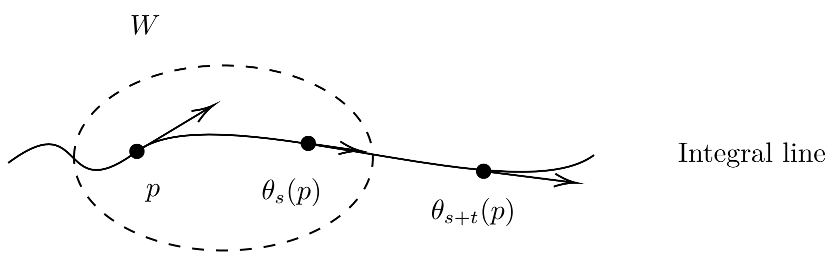
(1) If X is a smooth vector field, then the integral curve always exists.

(2) $F : M \longrightarrow N$ is smooth, X and Y are smooth vector fields on M and N . One has :
X and Y are F-related. $\Longleftrightarrow$ For each integral curve γ of X , $F \circ \gamma$ is an integral curve of Y .

Flows

For a smooth vector field X on M , there is a unique integral curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ with $\gamma(0) = p$. Then extend to all $t \in \mathbb{R}$, $\theta(p) : \mathbb{R} \longrightarrow M$ is a smooth curve on M .

Define a group ($\mathbb{R}$ as additive group) action $\theta : \mathbb{R} \times M \longrightarrow M$ by $(t, p) \longmapsto \theta_t(p)$.



A local flow of a point $p \in U$ is a smooth function $\theta : (-\epsilon, \epsilon) \times W \longrightarrow U$ where $p \in W \subseteq U$, such that $\theta_0(p) = p$, $\theta_t(\theta_s(p)) = \theta_{t+s}(p)$.

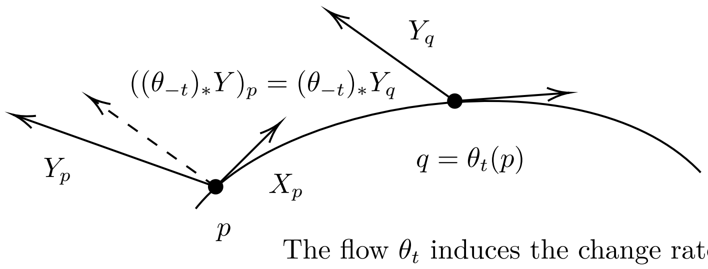
For each t the map $\theta_t : W \longrightarrow \theta_t(W)$ is a diffeomorphism.

For a smooth vector field, there is a unique integral curve γ with velocity X_p . Then one can define a globel flow generated by X such that

$$\theta_0(p) = p \ , \text{ the velocity } \frac{d}{dt}\theta(t, p) = X_{\theta_t(p)} \ .$$

Lie derivatives

$\theta_t : W \longrightarrow \theta_t(W)$ and $(\theta_t)^{-1} = \theta_{-t} : \theta_t(W) \longrightarrow W$ are both diffeomorphisms.



The flow θ_t induces the change rate of any vector field Y (not smooth necessarily) .

Define the Lie derivative of vector field Y with respect to X by

$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{(\theta_{-t})_* Y_q - Y_p}{t} = \lim_{t\to 0} \frac{((\theta_{-t})_* Y)_p - Y_p}{t} = \frac{d}{dt}((\theta_{-t})_* Y)_p \ .$$

Proposition

If X and Y are both smooth vector field, then $\mathcal{L}_X Y = [X, Y]$.

Lie derivatives of k-forms

Define the Lie derivative of k-form ω with respect to X by

$$\mathcal{L}_X \omega = \lim_{t\to 0} \frac{\theta_{-t}^* \omega_q - \omega_p}{t} = \lim_{t\to 0} \frac{(\theta_{-t}^* \omega)_p - \omega_p}{t} = \frac{d}{dt}(\theta_{-t}^* \omega)_p \ .$$

Proposition

If X is a smooth vector field, f is a smooth function (smooth covector) , then $\mathcal{L}_X f = Xf$.

9.6 The Riemannian Manifold

Riemannian metrics

In smooth local coordinate on M , a Riemannian metrics is a symmetric covariant 2-tensor field, written by $g = g_{ij} dx^i \otimes dx^j$ with $g_{ij} = g_{ji}$.

The Euclidean metric $\overline{g} = \delta^i_j dx^i dx^j$ on $\mathbb{R}^n$ is a Riemannian metric, and $\overline{g} = (dx^1)^2 + \cdots + (dx^n)^2$.

For the diffeomorphism $F : M \longrightarrow \mathbb{R}^n$, $g_{ij} = \frac{\partial}{\partial u} \cdots \frac{\partial}{\partial v}$ where (u, v) is the coordinate on M .

Proposition

(1) For a smooth map $F : M \longrightarrow N$ with a Riemannian metric g on N , one has :
$F^* g$ is a Riemannian metric on M . $\Longleftrightarrow$ F is a smooth immersion.

9.7 Vector Calculas

Differentials of tangent vectors

For smooth map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(x, y, z) \longmapsto x^2 y$ and smooth vector field $X = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}$, assume the coordinate on $\mathbb{R}^3$ is (x, y, z) , the coordinate on $\mathbb{R}$ is (t) .

$$X_{(1,1,0)} = \frac{\partial}{\partial x} + \frac{\partial}{\partial z} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \ , \ dF_{(1,1,0)} = \begin{pmatrix} 2xy & x^2 & 0 \end{pmatrix}_{(1,1,0)} = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \ .$$

$$dF_{(1,1,0)}(X_{(1,1,0)}) = \begin{pmatrix} 2 \end{pmatrix} = 2\frac{d}{dt} \ .$$

The pushforwards of smooth vector fields

For smooth map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(x, y, z) \longmapsto (x \cos y \sin z, x \sin y \sin z, x \cos z)$ and smooth vector field $X = \frac{\partial}{\partial y}$, assume the coordinate on $\mathbb{R}^3$ is (x, y, z) , the coordinate on second $\mathbb{R}^3$ is (u, v, w) .

$$F_* X = dF(X) = \begin{pmatrix} \cos y \sin z & -x \sin y \sin z & x \cos y \cos z \\ \sin y \sin z & x \cos y \sin z & x \sin y \cos z \\ \cos z & 0 & -x \sin z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -x \sin y \sin z \frac{\partial}{\partial u} + x \cos y \sin z \frac{\partial}{\partial v} \ .$$

The pullbacks of smooth covectors and smooth k-forms

For the smooth map $F : (0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $(r, \phi) \longmapsto (r \cos \phi, r \sin \phi)$ and smooth covector field $\omega = xdx + dy$ where $x(r, \phi) = r$, $y(r, \phi) = \phi$ are the coordinate functions.

$$F^*\omega = F^*(xdx + dy) = F^*(xdx) + F^*dy = (F^*x)(F^*dx) + F^*dy = x \circ F(F^*dx) + F^*dy$$

$$F^*dx = d(F^*x) = d(r \cos \phi) = \cos \phi dr - r \sin \phi d\phi \ , \ F^*dy = d(F^*y) = d(r \sin \phi) = \sin \phi dr + r \cos \phi d\phi \ .$$

$$F^*\omega = r \cos \phi(\cos \phi dr - r \sin \phi d\phi) + \sin \phi dr + r \cos \phi d\phi \ .$$

$$F^*(dx \wedge dy) = F^*dx \wedge F^*dy = (\cos \phi dr - r \sin \phi d\phi) \wedge (\sin \phi dr + r \cos \phi d\phi) \ .$$

Integrals of smooth k-forms

For a diffeomorphism $F : M \longrightarrow N$, and a k-form $\omega \in \Omega^k(N)$, one has

$$\int_N \omega = \int_M F^*\omega \ .$$

The Stokes Theorem

For the oriented n-dimensional manifold M and $\omega \in \Omega^{n-1}(M)$, one has

$$\int_{\partial M} \omega = \int_M d\omega \ .$$

9.8 Bundle Structures on Manifolds

Tangent Bundles

$$
\begin{tikzcd}
U \times \mathbb{R}^n \arrow[dr, "\pi_1"'] & {} \arrow[d, phantom, "\circlearrowright"] & p^{-1}(U) \subseteq \mathbf{T}M \arrow[ll, "\cong"'] \arrow[dl, "p"] \\
& U \subseteq M &
\end{tikzcd}
\qquad
\begin{tikzcd}
U_\lambda \times \mathbb{R}^n \arrow[dr, "\pi_1"'] & {} \arrow[d, phantom, "\circlearrowright"] & p^{-1}(U_\lambda) \subseteq \mathbf{T}M \arrow[ll, "\cong"'] \arrow[dl, "p"] \\
& U_\lambda \subseteq M &
\end{tikzcd}
$$

The tangent bundle $\mathbb{R}^n \longrightarrow \mathbf{T}M \xrightarrow{p} M$ is a vector bundle of rank $n = \dim\ M$ over M .

If the local trivialisation (homeomorphism) $h : p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ can be chosen to be a diffeomorphism, then it is a smooth tangent bundle. This local trivialisation is celled smooth local trivialisation.

Transition function between smooth local trivialisations

Suppose $\mathbb{R}^n \longrightarrow TM \xrightarrow{p} M$ is a smooth tangent bundle. For a smooth local trivialisation h_1 on $U \subseteq M$ and a smooth local trivialisation h_2 on $V \subseteq M$, there is a transition function such that this diagram commutes.

$$
\begin{tikzcd}
(U \cap V) \times \mathbb{R}^n \arrow[dr] & p^{-1}(U \cap V) \arrow[l, "h_1"'] \arrow[r, "h_2"] \arrow[d, "p"] & (U \cap V) \times \mathbb{R}^n \arrow[dl] \\
& U \cap V &
\end{tikzcd}
$$

$$
\begin{tikzcd}
U \times \mathbb{R}^n \arrow[dr] & p^{-1}(U) \arrow[l, "h_1"'] \arrow[d] \\
& U
\end{tikzcd}
\qquad
h_2 \circ h_1^{-1} : (p, v) \longmapsto (p, M_n v)
\qquad
\begin{tikzcd}
p^{-1}(V) \arrow[r, "h_2"] \arrow[d] & V \times \mathbb{R}^n \arrow[dl] \\
V &
\end{tikzcd}
$$

The function $M_n(p)$ (nonsingular matrix) of p is a transition function between h_1 and h_2 .