Chapter 3

Algebraic Geometry

3.1 Affine Varieties

Polynomial rings in n variables

$K[x_1, \cdots, x_n]$ is the ring of polynomials in n variables.
For $f \in K[x_1, \cdots, x_n]$, the value of f at $a = (a_1, \cdots, a_n) \in K^n$ is $f(a) = f(a_1, \cdots, a_n)$.

Affine n-spaces

For an algebraically closed field K , denote the affine n-space over K by :

$$\mathbb{A}^n = \mathbb{A}^n_K = \{(a_1, \cdots, a_n) \mid a_i \in K\} .$$

As a set, the affine n-space over K is denoted by K^n .

Zero loci of polyniomials

For a subset $S \subseteq K[x_1, \cdots, x_n]$, the zero locus of S is defined by :

$$V(S) = \{x \mid x \in \mathbb{A}^n \ , \ f(x) = 0 \text{ for all } f \in S\}$$

$$V(f_1, \cdots, f_k) = V(\{f_1, \cdots, f_k\}) .$$

Trivially, $V(S)$ is a subset of $\mathbb{A}^n$, and this form of subsets of $\mathbb{A}^n$ are called affine varieties.

Proposition

These are all affine varieties :
(1) $\mathbb{A}^n = V(0)$, $\emptyset = V(1)$.
(2) Linear subspaces of $\mathbb{A}^n = K^n$.
(3) One point set $\{a\} = \{(a_1, \cdots, a_n)\} = V(x_1 - a_1, \cdots, x_n - a_n)$.
(4) Finite subsets of $\mathbb{A}^n$ (or K^n) like $\{a, b, c, d\}$.

Proposition

(1) For varieties $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, $X \times Y \subseteq \mathbb{A}^{m+n}$ is also a variety.

(2) $\{$Affine varieties in $\mathbb{A}^1\} = \{$Finite subsets in $\mathbb{A}^1\} \cup \{\mathbb{A}^1\}$.

(3) Finite unions and arbitrary intersections of affine varieties are still affine varieties.

The zero locus $V(I)$ of ideal I

For an ideal $I \lhd K[x_1, \cdots, x_n]$ (by the Hilbert's basis theorem, $I =< S >$) , define its zero locus to be

$$V(I) = V(< S >) = V(S) \ .$$

Thus any affine variety can be written as a zero locus of an ideal (or the generators set) .

The vanishing ideal $I(X)$ of subset $X \subseteq \mathbb{A}^n$

For a subset $X \subseteq \mathbb{A}^n$ (X need not be considered only finite) , define its ideal to be

$$I(X) = \{f \mid f \in K[x_1, \cdots, x_n] \ , \ f(x) = 0 \text{ for all } x \in X\} \ .$$

And this ideal $I(X)$ actrually is a radical ideal.

The Hilbert's Nullstellensatz (Theorem of the Zeros)

(1) For any affine variety $X \subseteq \mathbb{A}^n$, one has $V(I(X)) = X$.
(2) For any ideal $J \lhd K[x_1, \cdots, x_n]$, one has $I(V(J)) = \sqrt{J}$.

Coordinate rings

For an affine variety $X \subseteq \mathbb{A}^n$, a polynomial function on X is a map $f : X \longrightarrow K$, $x \longmapsto f(x)$ where $f \in K[x_a, \cdots, x_n]$.

Given the X , the ring of all the polynomial functions is the quotient ring $A(X) = K[x_1, \cdots, x_n]/I(X)$, called the coordinate ring of X .

The coordinate ring $A(X)$ is an automatic K-algebra.

The Relative Nullstellensatz

For a fixed affine variety $X \subseteq \mathbb{A}^n$, define :
the affine subvariety

$$V_X(S) = \{x \mid x \in X \ , \ f(x) = 0 \text{ for all } f \in S \subseteq A(X)\} \ ,$$

the ideal of subvariety Y in X

$$I_X(Y) = \{f \mid f \in A(X) \ , \ f(Y) = 0\} \ .$$

(1) For any affine subvariety $Y \subseteq X$, one has $V_X(I_X(Y)) = Y$.
(2) For any ideal $J \lhd A(X) = K[x_1, \cdots, x_n]/I(X)$, one has $I_X(V_X(J)) = \sqrt{J}$.
(3) For $A(Y) = K[x_1, \cdots, x_n]/I(Y)$, one has $A(Y) \cong A(X)/I_X(Y)$ for any subvariety Y in X .

Properties of $V(\cdot)$ and $I(\cdot)$

(1) For $S_1, S_2 \subseteq K[x_1, \cdots, x_n]$, one has :

$$V(S_1) \cup V(S_2) = V(S_1 S_2)$$

$$\text{where } S_1 S_2 = \{fg \mid f \in S_1 \ , \ g \in S_2\} \ ,$$

$$\bigcap_i V(S_i) = V(\bigcup_i S_i) \ .$$

(2) For $J_1, J_2 \lhd K[x_1, \cdots, x_n]$, one has :

$$V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$$

$$\text{since } \sqrt{J_1 J_2} = \sqrt{J_1 \cap J_2} \ ,$$

$$V(J_1) \cap V(J_2) = V(J_1 + J_2) \ .$$

(3) For $X_1, X_2 \subseteq \mathbb{A}^n$, one has :

$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V((X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)} \ ,$$

$$I(X_1 \cup X_2) = I(V(I(X_1)) \cup V(I(X_2))) = I(V(I(X_1) \cap I(X_2))) = I(X_1) \cap I(X_2)$$

$$\text{since} \sqrt{J_1 \cap J_2} = \sqrt{J_1} \cap \sqrt{J_2} \ .$$

(5) For a prime ideal $P \lhd A(X)$, $V(P)$ is a nonempty irreducible subvariety of X .
For a irreducible subvariety Y of X , $I(Y)$ is a prime ideal of $A(X) = K[x_1, \cdots, x_n]/I(X)$.

(6) For a minimal prime ideal $M \lhd A(X)$, $V(M)$ is an irreducible component of X .
For an irreducible component Y of X , $I(Y)$ is a minimal prime ideal of $A(X) = K[x_1, \cdots, x_n]/I(X)$.

Proposition

(1)

$$S_1 \subseteq S_2 \subseteq K[x_1, \cdots, x_n] \Longrightarrow V(S_2) \subseteq V(S_1) \subseteq \mathbb{A}^n \ .$$

(2)

$$X_2 \subseteq X_1 \subseteq \mathbb{A}^n \Longrightarrow I(X_1) \subseteq I(X_2) \subseteq K[x_1, \cdots, x_n] \ .$$

(3) The Weak Nullstellensatz : for an ideal $J \lhd K[x_1, \cdots, x_n]$, if $J \neq K[x_1, \cdots, x_n]$, then J has a 0 .

(4) For $J \lhd K[x_1, \cdots, x_n]$, one has $V(\sqrt{J}) = V(J)$.

Equations of varieties and vanishing ideals

$$V(f_1 \cdots , f_n)$$
$$= V(\{f_1\} \cup \cdots \cup \{f_n\})$$
$$= V(f_1) \cap \cdots \cap V(f_n)$$
$$= V(< f_1 >) \cap \cdots \cap V(< f_n >)$$
$$= V(< f_1 > + \cdots + < f_n >) \ .$$

Thus one has

$$< f_1, \cdots , f_n >=< f_1 > + \cdots + < f_n > \ .$$

$$V(gh) = V(\{g\}\{h\}) = V(g) \cup V(h)$$
$$= V(< g >) \cup V(< h >) = V(< g > \cap < h >) \ .$$

Thus one has

$$< gh >=< g > \cap < h > \ .$$

The $1:1$ **correspondences**

$$\{\text{Affine varieties } X \subseteq \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \{\text{Radical ideals } J \lhd K[x_1, \cdots , x_n]\}$$
$$\{\text{Affine subvarieties of } X \subseteq \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \{\text{Radical ideals } J \lhd A(X) = K[x_1, \cdots , x_n]/I(X)\}$$
$$\{\text{Nonempty irreducible subvarieties of } X \subseteq \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \{\text{Prime ideals } P \lhd A(X) = K[x_1, \cdots , x_n]/I(X)\}$$
$$\{\text{Irreducible components of } X \subseteq \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \{\text{Minimal prime ideals } N \lhd A(X) = K[x_1, \cdots , x_n]/I(X)\}$$
$$\{\text{Points } a = (a_1, \cdots , a_n) \in \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \{\text{Maximal ideals } < x_1 - a_1, \cdots , x_n - a_n > \lhd K[x_1, \cdots , x_n]\}$$

Products of varieties

For $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, let $I(X) \subseteq K[x_1, \cdots , x_n]$, $I(Y) \subseteq K[y_1, \cdots , y_m]$.
Denote $R = K[x_1, \cdots , x_n, y_1, \cdots , y_m] = A(\mathbb{A}^{m+n})$ since $I(\mathbb{A}^{m+n}) = 0$.

Define an ideal $I_{X \times Y} = I(X) \cdot R + I(Y) \cdot R \lhd R$.
Then one has $I(X \times Y) = I_{X \times Y}$ and the coordinate ring is given by :

$$A(X \times Y) = A(X) \otimes_K A(Y)$$

where K is algebraically closed.

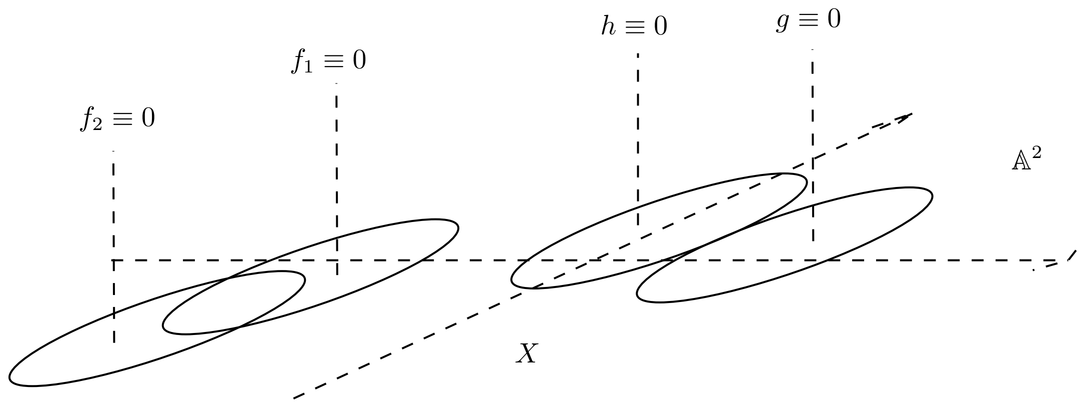
Between varieties

Considering the variety $X \subseteq \mathbb{A}^2$ shown below :

$< g > \cap < h >$ is not prime in $A(X)$. $\Longleftrightarrow V(g) \cup V(h)$ is reducible.

$< f_1 > \cap < h >$ is prime in $A(X)$. $\Longleftrightarrow V(f_1) \cup V(h)$ is irreducible.

$< f_1 >$ is minimal prime in $A(X)$. $\Longleftrightarrow V(f_1)$ is an irreducible component



3.2 The Zariski Topology

The Topology on affine varieties

For an affine variety $X \subseteq \mathbb{A}^n$, define the Zariski topology on X whose closed sets are the affine subvarieties of X . The Zariski topology agrees with the subspace topology while fixed Y , $X \subseteq Y \subseteq \mathbb{A}^n$.

Proposition

(1) The Zariski topology on $\mathbb{A}^1$ is the cofinite topology and the closed sets are the finite sets and $\mathbb{A}^1$.

(2) The product topology of the Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is not a Zariski topology.

Irreducible spaces

$X = X_1 \cup X_2$ where $X_1, X_2 \subsetneqq X$ are closed	$X \neq X_1 \cup X_2$ where $X_1, X_2 \subsetneqq X$ are closed
reducible	irreducible
$\Uparrow$	$\Downarrow$
disconnected	connected
$X = X_1 \sqcup X_2$ where $X_1, X_2 \subsetneqq X$ are closed	$X \neq X_1 \sqcup X_2$ where $X_1, X_2 \subsetneqq X$ are closed

Proposition

(1) Let X be irreducible, any nonempty open subsets $U_1, U_2 \subseteq X$ have nonempty intersection $U_1 \cap U_2$ (The open set is big in irreducible X) .

(2) Let X be irreducible, any nonempty open subset $U \subseteq X$ is dense.

(3) For a disconnected affine variety $X = X_1 \sqcup X_2$ where $X_1, X_2 \subsetneqq X$ and $X_1 \cap X_2 = \emptyset$, one has

$$A(X) \cong A(X_1) \times A(X_2) \ .$$

By the Chinese reminder theorem, one has $A(X) \cong \prod_i A(X_i)$ where X_i is the connected component of X .

(4) A nonempty affine variety X is irreducible. $\Longleftrightarrow$ $A(X)$ is a domain.

Noetherian spaces

If there is no infinite strictly decreasing chain of closed subsets of topological space X like

$$X \supsetneqq X_0 \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots \ ,$$

then X is called Noetherian.

Proposition

(1) Any affine variety is a Noetherian space.

(2) Given the subspace topology, the subspace of a Noetherian space is also Noethrian.

The irreducible decomposition

Every Noetherian space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of nonempty irreducible closed subsets, called the irreducible decomposition of X . If $X_i \nsubseteq X_j$ for $i \neq j$, then $X_1, \cdots, X_r$ are unique up to permutation, called the irreducible components of X .

The primary decomposition

For an affine variety $X \subseteq \mathbb{A}^1$, consider the primary decomposition of $I(X) \subseteq K[x_1, \cdots, x_n]$:

$$I(X) = Q_1 \cap \cdots \cap Q_n \subseteq K[x_1, \cdots, x_n]$$

where Q_i is prime ideal. Then take $P_i = \sqrt{Q_i}$, one has

$$I(X) = \sqrt{I(X)} = \sqrt{Q_1 \cap \cdots \cap Q_n} = \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_n} = P_1 \cap \cdots \cap P_n \ ,$$

$$X = V(I(X)) = V(P_1) \cap \cdots \cap V(P_n) \ .$$

If M_i is the minimal prime ideal, then there is an irreducible copmosition of X :

$$X = V(M_1) \cup \cdots \cup V(M_n) \ .$$

The dimension of topological spaces

Define the dimension of X to be the supremum of n in chains where $X_i \subseteq X_n$ is irreducible closed subset :

$$\emptyset \subsetneqq X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n \subseteq X \ ,$$

denoted by $\dim X \in \mathbb{N} \cup \{\infty\}$.

Proposition

(1) Since X_i is irreducible and $X_i \subsetneqq X_{i+1}$, one has $\dim X_i < \dim X_{i+1}$. Thus for a chain with finite length n , $\dim X = \dim X_n = n$.

(2) For a Noetherian space, the dimension might be infinite. For example, $X = \mathbb{N}$ with closed sets $\{\emptyset, \mathbb{N}\} \cup \{\{1, 2, \cdots, n\} \mid n \in \mathbb{N}\}$ is Noetherian and $\dim X = \infty$.

The codimension of topological spaces

Define the codimension of the nonempty irreducible closed subset Y to be the supremum of n in chains where $Y_i \subseteq X$ is irreducible closed subset :

$$Y \subseteq Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_n \subseteq X \ ,$$

denoted by $\mathrm{codim}_X Y$.

The Krull dimension of rings

Define the Krull dimension of R to be the supremum of n in chains where $P_i \lhd R$ is peime ideal :

$$R \supseteq P_0 \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_n \ .$$

Define the height of prime ideal $P \lhd R$ to be the supremum of m in chains where $P_i \lhd R$ is prime ideal :

$$P = P_0 \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_m \ ,$$

denoted by $\mathrm{ht}(P)$.

The dimension of affine varieties

For affine variety $X \subseteq \mathbb{A}^n$:

(1) The dimension $\dim X$ is equal to the Krull dimension $\dim A(X)$.

(2) The codimension $\mathrm{codim}_X Y$ is equal to the height $\mathrm{ht}(I_X(Y))$.

Proposition

(1) For nonempty irreducible affine varieties, dimensions and codimensions are always finite.

(2) For nonempty irreducible affine varieties X, Y , one has : $\dim(X \times Y) = \dim X + \dim Y$.

(3) For nonempty irreducible affine varieties $Y \subseteq X$, one has : $\dim X = \dim Y + \mathrm{codim}_X Y$.

(4) If $f \in A(X)$ is nonzero, then every irreducible component of $V(f)$ has codimension 1 in X and dimension $\dim X - 1$.

Pure dimensional spaces

A Noetherian topological space X is said to be of pure dimension n if every irreducible component of X has dimension n .

An affine variety is called :

(1) a curve if it is of pure dimension 1 ,

(2) a surface if it is of pure dimension 2 ,

(3) a hypersurface in a pure dimensional affine variety Y if it is an affine subvariety of Y of pure dimension $\dim Y - 1$.

Regular functions

For an open subset $U \subseteq X$ of affine variety X , define a map $\varphi : U \longrightarrow K$. If for any point $a \in U$, there is an open neighbourhood $U_a \subseteq U$ and functions $f, g \in A(X)$ such that $\begin{cases} f(x) \neq 0 \\ \varphi(x) = \frac{g(x)}{f(x)} \end{cases}$ on U_a , then φ is called a regular function on U .

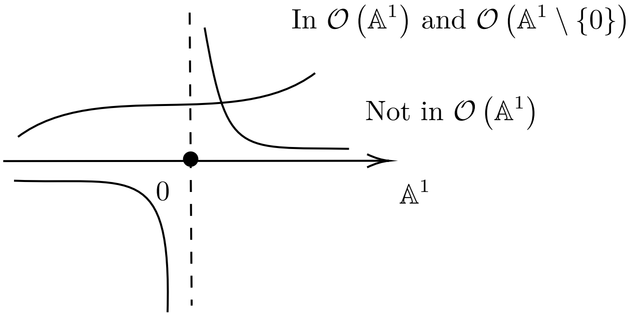
The regular functions on U is denoted by $\mathcal{O}_X(U)$ and it is a K-algebra.

The zero locus of regular functions

The zero loci $V(\varphi) = \{x \mid x \in U \ , \ \varphi(x) = 0\}$ of a $\varphi \in \mathcal{O}_X(U)$ is closed.

Restriction maps

For open subsets $U \subseteq V \subseteq X$ of affine variety, there is a well defined restriction map $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$, $\{\varphi : V \longrightarrow K\} \longmapsto \{\varphi|_U : U \longrightarrow K\}$. In general, this is not surjective.



The Identity Theorem for Regular Functions

Let $U \subseteq V$ be nonempty open subsets of an irreducible affine variety X , if $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ with $\varphi_1 \equiv \varphi_2$ on U , then $\varphi_1 \equiv \varphi_2$ on whole V (restriction map $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$ is injective) .

Distinguished open subsets

For an affine variety X and a polynomial function $f \in A(X)$, one can define the distinguished open subset of f in X by $D(f) = X \setminus V(f) = \{x \mid f(x) \neq 0\}$.

Proposition

(1)

$$D(f) \cap D(g) = D(fg) \text{ for } f, g \in A(X) \ .$$

$$V_X(f) \cup V_X(g) = V_X(fg) \text{ for } f, g \in A(X) \ .$$

(2) Finite intersections of distinguished open subsets are again distinguished open subsets.

(3) Any open subset is a union of distinguished open subsets, one has :

$$U = X \setminus V(f_1, \cdots, f_k) = X \setminus (V(f_1) \cap \cdots \cap V(f_k)) = D(f_1) \cup \cdots \cup D(f_k) \ .$$

The generalised partition of unity

Given an affine variety X , assume that

$$D(f) = \bigcup_i D(f_i)$$

where $f_i \in A(X)$. Then one has

$$f^n = \sum_i f_i \cdot g_i$$

where $n \in \mathbb{N}$, $g_i \in A(X)$.

Take $f = 1$, then $X = \bigcup_i D(f_i) \Longrightarrow 1 = \sum_i f_i \cdot g_i$.

Regular functions on distinguished open subsets

Let X be an affine variety, $f \in A(X)$. Then

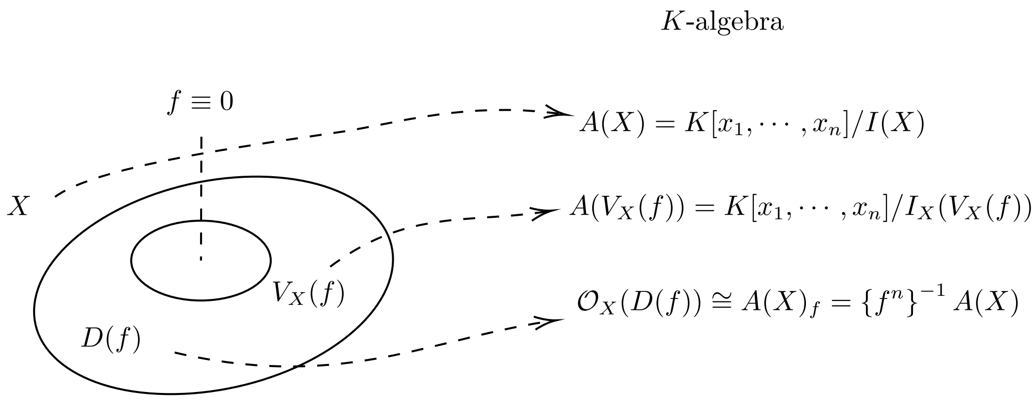
$$\mathcal{O}_X(D(f)) = \{\frac{g}{f^n} \mid f, g \in A(X) \ , \ n \in \mathbb{N}\} \ .$$

(1) Take $f = 1$, then $\mathcal{O}_X(X) = A(X)$, means the regular function on whole X are exactly the polynomial functions.

(2) More generally, $\mathcal{O}_X(D(f))$ can be considered as the localization $A(X)_f$ of ring $A(X)$ at multiplicatively closed $S = \{f^n \mid f \in A(X) \ , \ n \in \mathbb{N}\}$.
There is a K-algebra isomorphism $\mathcal{O}_X(D(f)) \cong S^{-1}A(X) = A(X)_f$.

(2) The regular function on a distinguished open subset is always globally the quotient of two polynomial functions.

K-algebra of sets



Extending regular functions

For open set $U = \mathbb{A}^2 \setminus \{(0,0)\} = D(x_1) \cup D(x_2)$, the regular function $\varphi \in \mathcal{O}_{\mathbb{A}^2}(D(x_1) \cup D(x_2))$, by the restriction one has $\varphi = \begin{cases} \frac{f}{x_1^m} & x \in D(x_1) \\ \frac{g}{x_2^n} & x \in D(x_2) \end{cases}$ where $f, g \in A(\mathbb{A}^2) = K[x_1, x_2]$.

Without loss of generality, $x_1 \nmid f$, $x_2 \nmid g$, $m = n + d$. Restricting on $D(x_1) \cap D(x_2)$ one has

$$\frac{f}{x_1^{n+d}} - \frac{g}{x_2^n} = 0 \in \mathcal{O}_{\mathbb{A}^2}(D(x_1) \cap D(x_2)) = \mathcal{O}_{\mathbb{A}^2}(D(x_1 x_2)) \cong K[x_1, x_2]_{x_1 x_2} \ .$$

Thus one has

$$x_2^d(f - x_1^{n+d} \cdot g) = 0 \in K[x_1, x_2]$$

Since $K[x_1, x_2]$ is an integral domain and a UFD , $m = n = 0$, $f = g$, $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{(0,0)\}) = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$.

3.3 Sheaves

Presheaves and sheaves

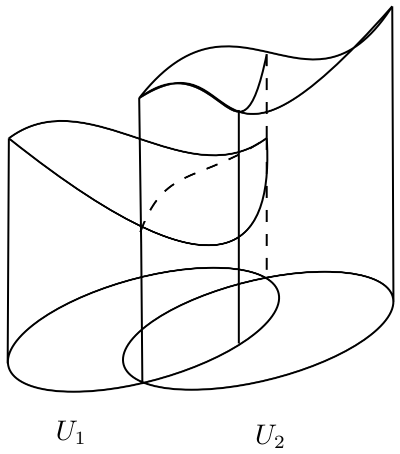
A presheaf $\mathcal{F}$ on a topological space X satisfies :

(1) For every open $U \subseteq X$, $\mathcal{F}(U)$ is a ring , $\mathcal{F}(\emptyset) = 0$.

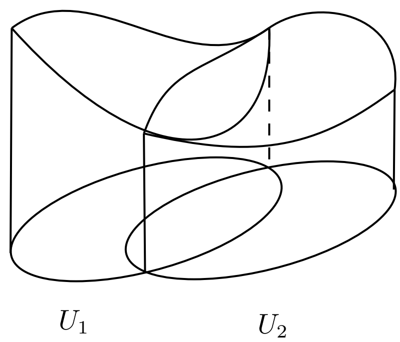
(2) For every inclusion $V \subseteq U$ of open sets of X , $P_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$, $\varphi \longmapsto \varphi|_V$ is a ring homomorphism called the restriction map such that :

$$P_{U,U} = \mathbb{1}_U \ , \ P_{V,U} \circ P_{W,V} = P_{W,U} \text{ for any inclusion } U \subseteq V \subseteq W$$

(The element in $\mathcal{F}(U)$ is called the section of $\mathcal{F}$ over U) .



Presheaf



Sheaf

A presheaf $\mathcal{F}$ is called a sheaf if it satisfies the gluing property :

For any open cover $\{U_i\}$ of an open $U \subseteq X$, if the section $\varphi_i \in \mathcal{F}(U_i)$ satisfies $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then the gluing section $\varphi \in \mathcal{F}(U)$ with $\varphi|_{U_i} = \varphi_i$ is unique.

The sheaf of rational functions

The ring $\mathcal{O}_X(U_i)$ of regular functions on open subsets $U_i \subseteq X$, together with the restriction maps and the identity theorem, form a sheaf $\mathcal{O}_X$ of K-algebras on X .

The restriction on presheaf or sheaf

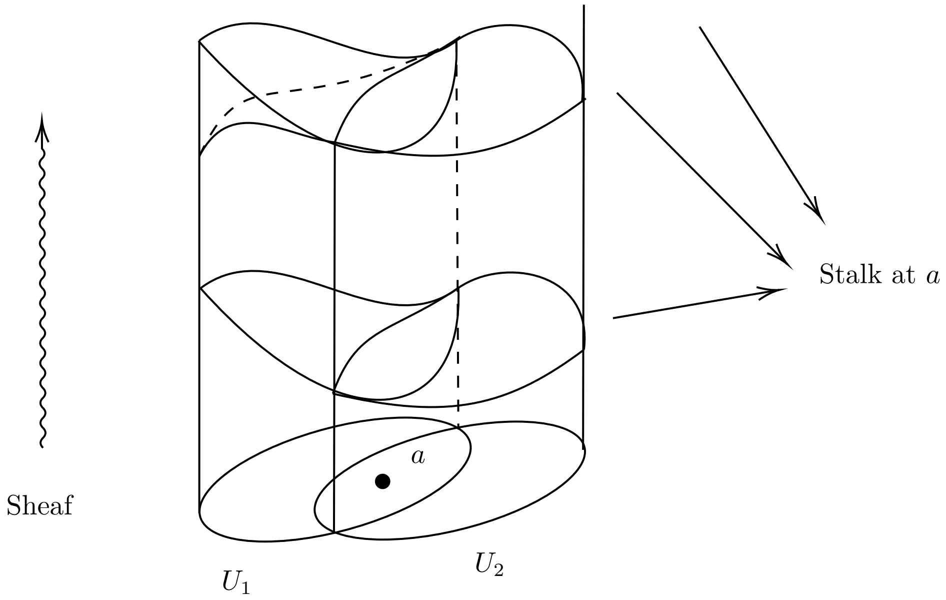
For a presheaf (or sheaf) $\mathcal{F}$ on X , the restriction of $\mathcal{F}$ on open $W \subseteq X$ is given by $\mathcal{F}|_W(U_i) = \mathcal{F}(U_i)$ for $U_i \subseteq W \subseteq X$.

Stalks

For a presheaf $\mathcal{F}$ on topological space X and a point $a \in X$, define the stalk of $\mathcal{F}$ at a to be

$$\mathcal{F}_a = \{(U, \varphi) \mid U \subseteq X \text{ is open containing } a\ ,\ \varphi \in \mathcal{F}(U)\}/\sim$$

where $(U, \varphi) \sim (U', \varphi')$ if there is an open $V \subseteq U \cap U'$ such that $\varphi|_V \equiv \varphi'|_V$ (The elements of $\mathcal{F}_a$ is called the germs of $\mathcal{F}$ at a) .



3.4 Morphisms

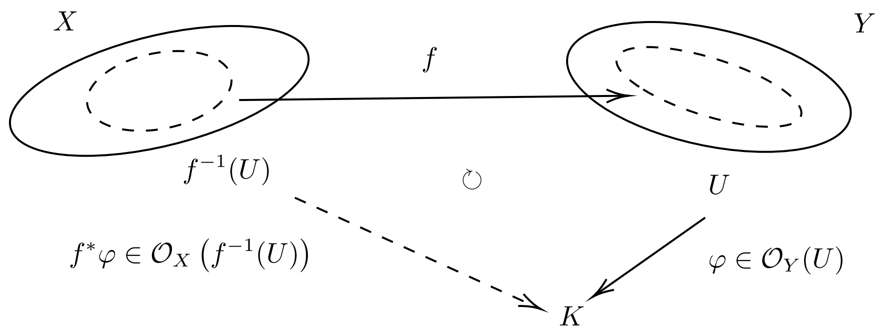
Ringed spaces

A ringed space is a topological space X with a sheaf of rings on X , denoted by $(X, \mathcal{O}_X)$ or just X ($\mathcal{O}_X$ denotes the sheaf on X) . For an open set $U \subseteq X$, $(U, \mathcal{O}_X|_U)$ is also a ringed space.

If X is an affine variety, then always take $\mathcal{O}_X$ to be the regular function.
Every sheaf of rings is assumed to be a sheaf of K-valued functions before the section of schemes.

Morphisms of ringed spaces

A map (topological space level) $f : X \longrightarrow Y$ of ringed spaces is called a morphism if it is continuous and for any open set $U \subseteq Y$ one has :



and the pullback $f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$, $\varphi \longmapsto f^*\varphi = \varphi \circ f$ is a K-algebra homomorphism.

Proposition

For a map $f : X \longrightarrow Y$ of ringed space, if one has an open cover $\{U_i\}$ of X such that every restriction $f|_{U_i} : U_i \longrightarrow Y$ is a morphism, then f is a morphism (glue together) .

Morphisms of affine varieties

Let U be an open subset of affine variety X , for another affine variety $Y \subseteq \mathbb{A}^n$, the morphism $f : U \longrightarrow Y$ must haev form

$$f = (f_1, \cdots, f_n) : U \longrightarrow Y \ , \ x \longmapsto (f_1(x), \cdots, f_n(x)) \text{ where } f_i \in \mathcal{O}_X(U) \ .$$

Proposition

There is a $1:1$ correspondence :

$$\{\text{morphisms } f : X \longrightarrow Y\} \overset{1:1}{\longleftrightarrow} \{K\text{-algebra homomorphisms } f^* : A(Y) \longrightarrow A(X)\} \ .$$

Proposition

(1) $X = V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$ and $\mathbb{A}^1 \subseteq \mathbb{A}^3$ are isomorphic.

$$A(X) \longrightarrow A(\mathbb{A}^1)$$

$$K[x,y,z]/ < y - x^2, z - x^3 > \longrightarrow K[t]$$

$$f(x,y,z) \longmapsto f(t,t^2,t^3) = f'(t) \ .$$

(2) The morphism $f : \mathbb{A}^1 \longrightarrow X = V(x^2 - y^3) \subseteq \mathbb{A}^2$, $t \longmapsto (t^3, t^2)$ is bijective but not isomorphism. The pullback

$$f^* : A(X) \longrightarrow A(\mathbb{A}^1)$$

$$K[x,y]/ < x^2 - y^3 > \longrightarrow K[t]$$

$$f(x,y) \longmapsto f(t^3, t^2)$$

is not a K-algebra isomorphism since $t \in K[t]$ is not in the image.

(3) For affine varieties $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $X \times Y \subseteq \mathbb{A}^{m+n}$ is affine variety, $A(X \times Y) = A(X) \otimes_K A(Y)$.
The map $\pi_X : X \times Y \longrightarrow X$, $(x_1, \cdots, x_n, y_1, \cdots, y_m) \longmapsto (x_1, \cdots, x_n)$ is a morphism.
The map $\pi_Y : X \times Y \longrightarrow Y$, $(x_1, \cdots, x_n, y_1, \cdots, y_m) \longmapsto (y_1, \cdots, y_m)$ is a morphism.

The Isomorphism Theorem

For a finitely generated K-algebra R , take generators $b_1, \cdots, b_n \in R$, define a surjective K-algebra homomorphism

$$g : K[x_1, \cdots, x_n] \longrightarrow R \ , \ f(x_1, \cdots, x_n) \longmapsto f(b_1, \cdots, b_n) \ .$$

Then $R \cong K[x_1, \cdots, x_n]/\mathcal{K}er(g) = K[x_1, \cdots, x_n]/J$, if R is reduced, then J is a radical ideal, $X = V(J) \subseteq \mathbb{A}^n$ is a affine variety with $A(X) = R$.

Abstract affine varieties

A ringed space $(X', \mathcal{O}_{X'})$ isomorphic to the ringed space $(X, \mathcal{O}_X)$ where $X \subseteq \mathbb{A}^n$ is called an abstract variety

Distinguished open subsets as affine varieties

For X be an affine variety and $f \in A(X)$, then the distinguished open subset $D(f)$ is an affine variety with coordinate ring $A(X)_f$.

But not all open subsets are affine variety since the infinite union of affine varieties can be not an affine variety.

Embedded affine variety : $V(J)$ which is closed in the Zariski topology.

Abstract affine variety : distinguished open subset $D(f)$ and the finite union of them.

3.5 Varieties

Prevarieties

A prevariety is a ringed space X has a finite open cover $\{U_i\}$ where U_i is affine variety. The sheaf on X is the regular functions $\mathcal{O}_X$. Morphisms of prevarieties are morphisms of ringed spaces.

If an open subset of a prevariety is also an affine variety, then it is called an affine open set.

Proposition

(1) Any affine variety is a prevariety.

(2) Any open subset in an affine variety is a prevariety.

(3) Any open subset in a prevariety is a prevariety.

Gluing prevarieties

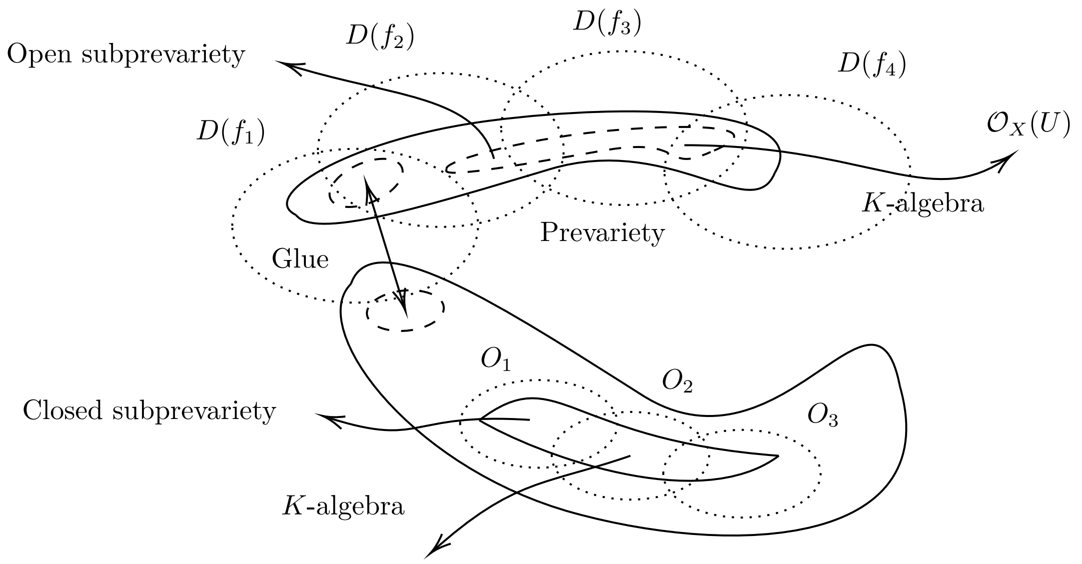
For open subsets $U_{12} \subseteq X_1$, $U_{21} \subseteq X_2$ of prevarieties, if there is an isomorphism $U_{12} \longrightarrow U_{21}$, then one has a prevariety $X = X_1 \sqcup X_2 / u \sim f(u)$. The ringed space structure is given by $\mathcal{O}_{X_i} = \mathcal{O}_X|_{X_i}$.

Similarly, one can glue finite prevarieties together $X = \bigsqcup\limits_i U_i / \sim$.

Open and closed subprevarieties

Let $X = \bigcup\limits_i X_i$ be a prevariety where X_i is an open affine variety, the open set $U \subseteq X$ is a prevariety given by $U = \bigcup\limits_i (X_i \cap U)$. U is called an open subprevariety of X .

For a closed subset $V \subseteq X$, the sheaf $\mathcal{O}_V(U)$ can not just take $\mathcal{O}_X(U)$ since the open subset $U \subseteq V$ is not open in X necessarily. The sheaf is given by the restriction on sheaves of an open cover of V .



$$\mathcal{O}_X(V) = \{f : V \longrightarrow K \mid f|_{O_i \cap V} \equiv g|_{O_i \cap V} \text{ for some } g \in \mathcal{O}_X(O_i)\}$$

Proposition

The product of prevarieties X , Y is a pushout but not $X \times Y$ necessarily.

Separated prevarieties

A prevariety X is called a variety or separated prevariety if the diagonal $\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$ is closed (an affine variety) in $X \times X$.

Proposition

(1) Any affine variety is a variety.

(2) For varieties X and Y , the product $X \times Y$ is a variety.

(3) Open or closed subprevariety of a variety is a variety, called open or closed subvariety.

Curves and surfaces

A variety of pure dimension 1 or 2 is called a curve or surface. For a pure dimension variety X , a closed subvariety Y of X of codimension 1 is called a hypersurface in Y .

Proposition

For morphisms $f, g : X \longrightarrow Y$ from prevariety X and varoety Y , one have closed sets

$$G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y \ ,$$

$$Eq(f,g) = \{x \mid f(x) = g(x)\} \subseteq X \ .$$

Constructible sets

For a topological space X , a subset $S \subseteq X$ is called locally closed if $S = U \cap V$ where $U \subseteq X$ is open and $V \subseteq X$ is closed.

A subset K is called consrtuctible if $K = \bigcup_{i=1}^{n} S_i$ is the union of finite locally closed sets.

Chevallay's theorem

For morphism $f : X \longrightarrow Y$ of prevarieties and $K \subseteq X$ constructible, the image $f(K) \subseteq Y$ is constructible.

3.6 Projective varieties

Projective spaces

The projective n-space $\mathbb{P}^n$ is defined as

$$\mathbb{P}^n = \mathbb{P}^n_K = \{L \mid L \subseteq K^{n+1} \text{ is the linear subspace of dimension } 1\} \ .$$

The element is denoted by $L = (x_0 : x_1 : \cdots : x_n)$.

$\mathbb{A}^n \subseteq \mathbb{P}^n$ is called the affine part of $\mathbb{P}^n$, and the points at infinity is $\mathbb{P}^{n-1}$, $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$.

Proposition

(1) Denote $U_i = \{(x_0, \cdots, x_n) \mid x_i \neq 0\}$, then $\mathbb{P}^n = \bigcup\limits_{i=1}^{n} U_i$. And for every i one has $\mathbb{A}^n \cong U_i$ and $U_i \cap U_j \longrightarrow U_j \cap U_i$ is an isomorphism of affine varieties, thus by gluing $\mathbb{P}^n$ is a prevariety.

(2) The projective space $\mathbb{P}^n_{\mathbb{C}}$ on $\mathbb{C}$ is compact.

Homogeneous ideals

A homogeneous ideal is generated by homogeneous polynomials (not necessarily of same degree) .

Proposition

(1) The homogeneous ideal is prime.

(2) For a graded ring R and ideal $J \lhd R$ one has :
J is homogeneous. $\iff$ for any $f = \sum\limits_{d \in \mathbb{N}} f^{(d)}$, one has $f^{(d)} \in J$.

(3) For homogeneous ideals $J_1, J_2 \lhd R$, these are also homogeneous ideals : $J_1 + J_2$, $J_1 \cap J_2$, $\sqrt{J_1}$, $J_1 J_2$ (not an ideal necessariliy if not prime) .

(4) For a homogeneous ideal $J \lhd R$ of graded ring R , one has a graded ring $R/J = \bigoplus\limits_{d \in \mathbb{N}} R_d/(J \cap R_d)$.

Projective varieties

For projective space $\mathbb{P}^n$, define the zero lucus of a set $S \subseteq K[x_0, x_1, \cdots, x_n]$ of homogeneous ideals or the homogeneous ideal $J \lhd K[x_0, x_1, \cdots, x_n]$ to be the projective variety

$$V_p(S) = \{x \mid x \in \mathbb{P}^n \ , \ f(x) = 0 \text{ for all } f \in S\} \ , \ V_p(J) = \{x \mid x \in \mathbb{P}^n \ , \ f(x) = 0 \text{ for all } f \in J\} \ ,$$

one has $V_p(S) = V_p(< S >)$. For any subset $X \subseteq \mathbb{P}^n$, define its vanishing ideals to be the homogeneous ideal

$$I_p(X) = \{f \mid f \in K[x_0, x_1, \cdots, x_n] \text{ is homogeneous} \ , \ f(x) = 0 \text{ on } X\} \ .$$

Proposition

(1) $\emptyset = V_p(1)$, $\mathbb{P}^n = V_p(0)$ are projective variety.

(2) $\{a\} \in \mathbb{P}^n$ is a projective variety.

(3) For homogeneous $f_1, \cdots, f_n \in K[x_0, \cdots, x_n]$, $V_p(f_1, \cdots, f_n)$ is called the linear subspace of $\mathbb{P}^n$.

Cones

An affine variety $X \subseteq \mathbb{A}^{n+1}$ is called a cone if $0 \in X$ and $kx \in X$ for any $x \in X$, $k \in K$.

For a cone $X \subseteq \mathbb{A}^{n+1}$, $\mathbb{P}(X) = \pi(X \setminus \{0\}) = \{(x_0 : \cdots : x_n) \mid (x_0, \cdots, x_n) \in X\} \subseteq \mathbb{P}^n$ is called the projectivization of X (it is a projective variety) , where $\pi : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$, $(x_0, \cdots, x_n) \longmapsto (x_0 : \cdots : x_n)$.
For a projective variety $X \subseteq \mathbb{P}^n$, $\text{Cone}(X) = \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0, \cdots, x_n) \mid (x_0 : \cdots : x_n) \in \mathbb{P}^n\} \subseteq \mathbb{A}^{n+1}$ is called the cone over X .

Proposition

For nonconstant homogeneous polynomials S , $V(S) \subseteq \mathbb{A}^n$ is a cone.
For cone $X \subseteq \mathbb{A}^{n+1}$, $I(X) \lhd K[x_0, \cdots, x_n]$ is a homogeneous ideal.

There is a $1:1$ correspondence

$$\{\text{Cones } X \subseteq \mathbb{A}^{n+1}\} \overset{1:1}{\longleftrightarrow} \{\text{Projective varieties } \mathbb{P}(X) \subseteq \mathbb{P}^n\} .$$

The irrelevant ideal

The ideal $Ir = < x_0, \cdots, x_n > \lhd K[x_0, \cdots, x_n]$ is called irrelevant ideal, it is radical and homogeneous.

The Projective Nullstellensatz

(1) For any projective variety $X \subseteq \mathbb{P}^n$, one has $V_p(I_p(X)) = X$.
(2) For any homogeneous ideal $J \lhd K[x_0, \cdots, x_n]$ with $\sqrt{J} \neq Ir$, one has $I_p(V_p(J)) = \sqrt{J}$.

Proposition

$V_p(x_0, \cdots, x_n) = \emptyset$, $\text{Cone}(\emptyset) = \{0\}$, $I(\{0\}) = Ir = < x_0, \cdots, x_n >$.

Properties of $V_p(\cdot)$ and $I_p(\cdot)$

(1) For sets $S_1, S_2 \subseteq K[x_0, \cdots, x_n]$ of homogeneous polynomials, one has :

$$V_P(S_1) \cup V_p(S_2) = V_p(S_1 S_2) \ ,$$

$$\bigcap_i V_p(S_i) = V_p(\bigcup_i S_i) \ .$$

(2) For homogeneous ideals $J_1, J_2 \lhd K[x_0, \cdots, x_n]$, one has :

$$V_p(J_1) \cup V_p(J_2) = V_p(J_1 J_2) = V_p(J_1 \cap J_2) \ ,$$

$$V_p(J_1) \cap V_p(J_2) = V_p(J_1 + J_2) \ .$$

(3) For projective varieties $X_1, X_2 \subseteq \mathbb{P}^n$, one has :

$$I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)} \ ,$$

$$I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2) \ .$$

Notice that $\sqrt{I_p(X_1) + I_p(X_2)} \neq Ir$ since one has : $X_1 \cap X_2 = \emptyset$. $\Longleftrightarrow$ $\sqrt{I_p(X_1) + I_p(X_2)} = Ir$.

Homogeneous coordinate rings

For projective variety $X \subseteq \mathbb{P}^n$, define the homogeneous coordinate ring to be $S(X) = K[x_0, \cdots, x_n]/I_p(X)$, since $I_p(X)$ is a homogeneous ideal, $S(X)$ is a graded ring.

The homogenization and dehomogenization

For homogeneous polynomial $f \in K[x_0, \cdots, x_n]$, one can make it to be a polynomial polynomial $f^{de} \in K[x_1, \cdots, x_n]$ by taking $x_0 = 1$.

For homogeneous ideal $J \lhd K[x_0, \cdots, x_n]$, one can make it to be an ideal $J^{de} = \{f^{de} \mid f \in J\} \lhd K[x_1, \cdots, x_n]$.

For polynomial $f = \sum\limits_{k_1, \cdots, k_n \in \mathbb{N}} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_{k_n}^n$ of degree d in $K[x_1, \cdots, x_n]$, one can make it to be a homogeneous polynomial of degree d by taking

$$f^{ho} = x_0^d \cdot f(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}) = \sum_{k_1, \cdots, k_n \in \mathbb{N}} a_{k_1 \cdots k_n} x_0^{d-k_1-\cdots-k_n} x_1^{k_1} \cdots x_{k_n}^n \ .$$

For an ideal $J \lhd K[x_1, \cdots, x_n]$, one can make it to be a homogeneous ideal $J^{ho} = \{f^{ho} \mid \text{nonzero } f \in J\} \lhd K[x_0, \cdots, x_n]$.

Proposition

(1) $\mathbb{A}^n$ is open in $\mathbb{P}^n$.

(2) $\mathbb{P}^n$ is irreducible of dimension n .

(3) $\mathbb{A}^n$ is irreducible of dimesion n .

(4) Affine variety X is irreducible. $\Longleftrightarrow$ $A(X)$ is an integral domain.

(5) Projective variety X is irreducible. $\Longleftrightarrow$ $S(X)$ is an integral domain.

Projective closures

For affine variety $X = V(J) \subseteq \mathbb{A}^n$ m define its projective closure to be $\overline{X} = V_p(J^{ho}) \subseteq \mathbb{P}^n$.
If $J =< f >$ is a nonzero principal ideal, then $\overline{X} = V_p(f^{ho}) \subseteq \mathbb{P}^n$.

Projective regular functions

For an open subset $U \subseteq X$ of projective variety X , define a map $\varphi : U \longrightarrow K$. If for any point $a \in U$,

there is an open neighbourhood $U_a \subseteq U$ and functions $f, g \in S(X)$ both of degree d such that $\begin{cases} f(x) \neq 0 \\ \varphi(x) = \frac{g(x)}{f(x)} \end{cases}$ on U_a , then φ is called a regular function on U .

The regular functions on U is denoted by $\mathcal{O}_X(U)$ and it is a K-algebra.

Proposition

(1) For a projective variety $X \subseteq \mathbb{P}^n$, $(X, \mathcal{O}_X)$ is a ringed space.

(2) Since projective variety $X \subseteq \mathbb{P}^n$ is closed, it is a closed subprevariety with sheaf $\mathcal{O}_X$.

(3) For projective variety $X \subseteq \mathbb{P}^n$, $U_i = \{(x_0 : \cdots : x_n) \mid (x_0 : \cdots : x_n) \in X \ , \ x_i \neq 0\}$ is an affine variety (isomorphic to an affine variety in $\mathbb{A}^n$) .

Morphisms of projective varieties

Let $U = X \setminus V_p(f_0, \cdots, f_m)$ be an open subset of projective variety X (f_i is homogeneous) , then one has morphism

$$f : U \longrightarrow \mathbb{P}^m \ , \ (x_0 : \cdots : x_n) \longmapsto (f_0(x_0, \cdots, x_n) : \cdots : f_m(x_0, \cdots, x_n)) \ .$$

The Segre embedding

Consider the map $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$, $([x_i],[y_j]) \longmapsto ([z_{ij} = x_i y_j])$.

The image $X = f(\mathbb{P}^n \times \mathbb{P}^m) = V_p(z_{ik}z_{jl} - z_{ij}z_{kl} \mid 0 \leq i,k \leq n \ , \ 0 \leq j,l \leq m) \subseteq \mathbb{P}^{(n+1)(m+1)-1}$ is a projective variety, and $f : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow X$ is an isomorphism.

Proposition

(1) The prevariety $\mathbb{P}^n$ is a variety.

(2) The map $\pi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ is a closed map.

(3) The map $\pi : \mathbb{P}^n \times X \longrightarrow X$ is closed for any variety X .

Complete varieties

A variety X is complete if the map $\pi : X \times Y \longrightarrow Y$ is closed for any variety Y .

The Veronese embedding

For $n, d \in \mathbb{N}$, $\{f_0, \cdots, f_N\} \subseteq K[x_0, \cdots, x_n]$ is the set of homogeneous polynomials of degree d where $N = C^n_{n+d} - 1$. Consider the map $F : \mathbb{P}^n \longrightarrow \mathbb{P}^n$, $(x_0 : \cdots : x_n) \longmapsto (f_0(x_0, \cdots, x_n), \cdots, f_n(x_0, \cdots, x_n))$.

The image $X = F(\mathbb{P}^n)$ is a projective variety, and $F : \mathbb{P}^n \longrightarrow X$ is an isomorphism.

Grassmannians

$Gr(k,n) = \{L \mid L \subseteq K^n$ is the linear subspace of dimension $k\}$ is the Grassmannian of k-planes in K^n .

3.7 Smooth varieties

Rational maps

Let X, Y be irreducible varieties, a rational map $f : X \overset{r}{\longrightarrow} Y$ is a morphism $f : U \longrightarrow Y$ where $\emptyset \neq U \subseteq X$ is open.

$$\mathrm{RatMap}(X, Y) = \{f \mid f : U \longrightarrow Y \ , \ U \subseteq X \text{ is open}\}/\sim$$

$f_1 \sim f_2$. $\Longleftrightarrow f_1|_V \equiv f_2|_V$ for some $V \subseteq U_1 \cap U_2$.

Birational maps

Let X, Y be irreducible varieties, a rational map $f : X \overset{r}{\longrightarrow} Y$ is called dominant if the image $f(U)$ is dense in Y . Then one has $f^{-1}(V) \neq \emptyset$ for any open $V \subseteq Y$, thus one can compose f with $g : Y \overset{r}{\longrightarrow} Z$ to get $g \circ f : X \overset{r}{\longrightarrow} Z$.

A rational map $f : X \overset{r}{\longrightarrow} Y$, if there exists a rational map $f : Y \overset{r}{\longrightarrow} X$ such that $f \circ g = \mathbb{1}_V$ and $g \circ f = \mathbb{1}_V$ for open $V \subseteq Y$ and $U \subseteq X$, then f or g is called birational, varieties X and Y are birational.

Proposition

Varieties X and Y are birational. $\Longleftrightarrow$ There are open subsets $U \subseteq X$, $V \subseteq Y$ such that $U \cong V$.

Rational functions

For an irreducible variety X , rational map $f : X \longrightarrow \mathbb{A}^1 = K$ is called a rational function. The set of rational functions is denoted by $K(X) = \{f \mid f : X \overset{r}{\longrightarrow} \mathbb{A}^1$, called the function field of X .

Proposition

(1) $K(X)$ is a stalk of $\mathcal{O}_X$ at X .

(2) For an open subvariety $U \subseteq X$ one has bijection $K(U) \longrightarrow K(X)$.

Blow-ups

For affine variety $X \subseteq \mathbb{A}^n$ and $f_0, \cdots, f_k \in A(X)$, consider the morphism $f : U = X \setminus V_X(f_0, \cdots, f_k) \longrightarrow \mathbb{P}^k$, $(x_1, \cdots, x_n) \longmapsto (f_0(x_1, \cdots, x_n) : \cdots : f_k(f_0, \cdots, f_k))$ and the subset $G_f = \{(u, f(u)) \mid u \in U\}$ closed in $U \times \mathbb{P}^k$ and open in $X \times \mathbb{P}^k$, one can define the blow-up of X to be

$$\pi : \widetilde{X} = \overline{G_f} \longrightarrow X$$

and denoted by $\widetilde{X} = \overline{G_f} = Bl_{f_0, \cdots, f_k}(X)$, the blow-up of X at $f_0, \cdots, f_k$,

$$Bl_{f_0, \cdots, f_k}(X) \subseteq \{(x, y) \mid x \in X \ , \ y \in \mathbb{P}^k \ , \ y_i f_j(x) = y_j f_i(x) \text{ for } i, j \in \{0, \cdots, k\}\} \ .$$

Proposition

(1) For affine variety $X \subseteq \mathbb{A}^n$ and $J < f_0, \cdots, f_k > \lhd A(X)$, define the blow-up of X at J to be $Bl_J(X) = Bl_{f_0,\cdots,f_k}(X)$.

(2) For affine variety $X \subseteq \mathbb{A}^n$ and a closed subvariety $Y \subseteq X$, define the blow-up of X at Y to be $Bl_Y(X) = Bl_{I_X(Y)}(X)$.

Tangent cones

For variety X and $a \in X$, $\pi : Bl_a(X) \longrightarrow X$ is the blow-up of X at a .
Define $C_a(X) = \text{Cone}(\pi^{-1}(a))$ to be the tangent cone of X at a .

Tangent spaces

For a variety X , define the tangent space of X at $a \in X$ to be

$$\mathbf{T}_a X = V(f^{(1)} \mid f \in I(X) \ , \ f = \sum_{d \in \mathbb{N}} f^{(d)} \ , \ f^{(0)} = 0) \ .$$

If $I(X) =< S >$, then $\mathbf{T}_a X = V(f^{(1)} \mid f \in S)$.

Proposition

For an affine variety $X \subseteq \mathbb{A}^n$, $a = 0 \in X$, $I(a) = I_X(a) =< x_1, \cdots, x_n > \lhd A(X)$, one has

$$I(a)/I(a)^2 \cong \text{Hom}(\mathbf{T}_a X, K) \ .$$

Singular varieties

If $C_a X = \mathbf{T}_a X$, then a is called a smooth, regular or nonsingular point.
Otherwise, a is called a singular point.

If X has a singular point, then X is called singular.
Otherwise, X is called smooth, regular or nonsingular.

Affine Jacobi criterion

For an affine variety $X \subseteq \mathbb{A}^n$, $I(X) =< f_1, \cdots, f_k >$, X is smooth at $a \in X$ if and only if

$$J = (\frac{\partial f_i}{\partial x_j}(a))_{i,j} \in M_{r \times n}(K) \text{ has rank } r(J) = n - \text{codim}_X\{a\} \ .$$

$$\text{In general, } r(j) \geq n - \text{codim}_X\{a\} = \dim \ C_a(X) \ .$$

Resolution of singularities