Chapter 5

Homotopy Theory

5.1 Homotopy Groups

The table of homotopy groups of S^n (Toda 1962) :

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

From the table

(1) $\pi_i(S^n) = 0$ for all $i < n$ (cellular approximation theorem) .

(2) $\pi_1(S^1) = \mathbb{Z}$, $\pi_i(S^1) = 0$ for all $i > 1$.

(3) $\pi_i(S^{2n}) \cong \pi_{i-1}(S^{2n-1}) \oplus \pi_i(S^{4n-1})$ for all n and i (James's theorem) .

(4) The first non-zero homotopy group of X is isomorphic to the first non-zero homology group (the Hurewicz theorem) .

(5) For all $i > n$, $\pi_i(S^n)$ is a finite group except for $\pi_{4k-1}(S^{2k})$.
For all k , one has $\pi_{4k-1}(S^{2k}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ for a prime p
(Serre's theorem 1950) .

(6) The groups $\pi_{n+k}(S^n)$ with each fixed k has the stability property, it eventually become independent of n when n becomes large enough
(Freudenthal suspension theorem 1940) .

Homotopy sets

For $n \geq 0$, the n-th homotopy set

$$\pi_n(X) = \pi_n(X, *) = [f \mid f : (S^n, p) \longrightarrow (X, *)] = [S^n, X] = [f \mid f : (I^n, \partial I^n) \longrightarrow (X, *)] \ .$$

is a group when $n \geq 1$ and an abelian group when $n \geq 2$.

Peterson complex

For $n \geq 2$ and finitely generated abelian group $G = F \oplus T$, define the 1-connected Peterson complex $P^n(G)$ such that

$$\widetilde{H^i}(P^n(G); \mathbb{Z}) = \begin{cases} G & i = n \\ 0 & \text{else} \end{cases} \ .$$

Proposition

For $G = F \oplus T$, one has $\widetilde{H_i}(P^n(G)) = \begin{cases} T & i = n-1 \\ F & i = n \\ 0 & \text{else} \end{cases}$. Then $P^n(F \oplus T) \simeq M(F, n) \vee M(T, n-1)$ has unique homotopy type.

Proof:

By the universal coefficient theorem $H^n(X; A) = \mathrm{Hom}(H_n(X), A) \oplus \mathrm{Ext}(H_{n-1}(X), A)$ one has

$$\mathrm{Hom}(H_{n-1}(P^n(G)), \mathbb{Z}) = 0 \ , \ \mathrm{Ext}(H_n(P^n(G)), \mathbb{Z}) = 0 \ . \implies H_n(P^n(G)) \text{ is free.}$$

$$F \oplus T \cong \mathrm{Ext}(H_{n-1}(P^n(G)), \mathbb{Z}) \oplus \mathrm{Hom}(H_n(P^n(G)), \mathbb{Z}) \cong H_{n-1}(P^n(G)) \oplus H_n(P^n(G)) \ .$$

Proposition

(1) $\Sigma P^n(G) = P^{n+1}(G)$.

(2) $P^n(\mathbb{Z}) = S^n$, $P^n(\mathbb{Z}_k) = S^{n-1} \cup_k e^n$ where $k : S^n \longrightarrow S^n$ is of degree k with mapping cone $P^n(\mathbb{Z}_k)$.

Homotopy sets with coefficients

For $n = 1$ and $F = \bigoplus_\alpha \mathbb{Z}$ a finitely generated free abelian group,

$$\pi_1(X; F) = \pi_1(X, *; F) = [P^1(F), X] = [\bigvee_\alpha S^1, X] = \bigoplus_\alpha \pi_1(X) \ .$$

For $n \geq 2$ and $G = F \oplus T$ a finitely generated abelian group,

$$\pi_n(X; G) = \pi_n(X, *; G) = [P^n(G), X] \ .$$

Proposition

(1) $P^n(G \oplus G') \simeq P^n(G) \vee P^n(G')$, then one has

$$\pi_n(X; G \oplus G') = [P^n(G) \vee P^n(G'), X] = [P^n(G), X] \oplus [P^n(G'), X] = \pi_n(X; G) \oplus \pi_n(X; G') \ .$$

(2) If X is an H-group, the multiplication $\mu : X \times X \longrightarrow X$ define a group structure on $[Y, X]$.
If Y is an H-cogroup, the comultiplication $\nu : Y \longrightarrow Y \vee Y$ define a group structure on $[Y, X]$.

(3) If X is an H-space and Y is an H-cospace, then the set $[Y, X]$ is an abelian group where two structures in (2) are same.

(4) $\pi_n(X; G)$ is a group for $n \geq 3$ and abelian group for $n \geq 4$.

Relative homotopy sets

For $n \geq 1$, the n-th relative homotopy set

$$\pi_n(X, A) = \pi_n(X, A, *) = [f \mid f : (B^n, S^{n-1}, p) \longrightarrow (X, A, *)] = [(B^n, S^{n-1}), (X, A)] \ .$$

is a group when $n \geq 2$ and an abelian group when $n \geq 3$.

Relative homotopy sets with coefficients

For $n \geq 3$ and $G = F \oplus T$ a finitely generated abelian group,

$$\pi_n(X, A; G) = [(CP^{n-1}(G), P^{n-1}(G)), (X, A)]$$

is a group for $n \geq 4$ and abelian group $n \geq 5$.

Proposition

(1) $\pi_n(X, *, *) = [(B^n, S^{n-1}), (X, *)] = [(B^n/S^{n-1}, p), (X, *)] \cong [S^n, X] = \pi_n(X, *) = \pi_n(X)$.
$\pi_n(X, *; G) = \pi_n(X; G)$.

(2) And there is a well-defined map $\delta : \pi_{n+1}(X, A) \longrightarrow \pi_n(A, *)$ given by

$$[(B^{n+1}, S^n), (X, A)] \longrightarrow [(B^{n+1}, S^n), (X, A)]|_{S^n} = [(S^n, S^n), (A, A)] \cong [S^n, A] \ .$$

(3) There are exact sequences in $(\mathbf{Set}_*)$:

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{\pi_n(i)} \pi_n(X) \xrightarrow{\pi_n(j)} \pi_n(X, A) \xrightarrow{\delta} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(X) \ ,$$

$$\cdots \longrightarrow \pi_n(A; G) \longrightarrow \pi_n(X; G) \longrightarrow \pi_n(X, A; G) \xrightarrow{\delta} \pi_{n-1}(A; G) \longrightarrow \cdots \longrightarrow \pi_2(X; G) \ .$$

Proposition

(1) $X' \subseteq X$ is the path-component containing p . $\Longrightarrow \pi_n(X', p) \cong \pi_n(X, p)$.
There is a path from p_1 to p_2 in X . $\Longrightarrow \pi_n(X, p_1) \cong \pi_n(X, p_2)$.

(2) $f : X \longrightarrow Y$ is a homotopy equivalence.
$\Longleftrightarrow X \simeq Y$ (the equivalent relation in $\mathbf{Ho}(\mathbf{Top}_*)$) .
$\Longrightarrow \pi_n(f) : \pi_n(X, p) \cong \pi_n(Y, f(p))$.

(3) X is contractible.
$\Longleftrightarrow \mathbb{1}_X \simeq c$, $c : X \longrightarrow * \in X$.
$\Longleftrightarrow \mathbb{1}_X$ is a nullhomotopy.
$\Longleftrightarrow \pi_n(X) = \pi_n(X, *) = 0$.

(4) $\pi_1(X \times Y) \cong \pi_1(X) \oplus \pi_1(Y)$ if X and Y are path-connected.

(5) For $n \geq 2$ one has $\pi_i(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z}_2 & i = 1 \\ \pi_i(S^n) & i \geq 2 \end{cases}$.

(6) $\pi_1(SO(2)) = \mathbb{Z}$, $\pi_1(SO(3)) = \mathbb{Z}_2$.

(7) $p : C \longrightarrow X$ is a covering space, then $\pi_n(X) \cong \pi_n(C)$ for all $n \geq 2$.

(8) For a H-cospace X , $\pi_1(X)$ is a free group.

Classification theorem for covering spaces

For the covering space $p : E \longrightarrow X$, there is a one-to-one correspondence between all the connected covering spaces and the conjugacy classes of the subgroups of $\pi_1(X)$.

Moreover, for the covering space $p : E \longrightarrow X$, there is a one-to-one correspondence between all the connected covering spaces and the actual subgroups of $\pi_1(X)$.

π_0 functor : $(\mathbf{Top}) \longrightarrow (\mathbf{Sets})$

Objects : $X \xrightarrow{f} Y \dashrightarrow \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y)$
Morphisms :
(1) For continuous map f , $\pi_0(f)$ is a function on sets.
(2) If f keeps the number of path components, then $\pi_0(f)$ is bijective.

π_1 functor : $\mathbf{Ho}(\mathbf{Top}_*) \longrightarrow (\mathbf{Gp})$

Objects : $(X,x) \xrightarrow{f} (Y,y) \dashrightarrow \pi_1(X,x) \xrightarrow{\pi_1(f)} \pi_1(Y,y)$

Morphisms :

(1) For continuous maps $f \simeq g$, $\pi_1(f) = \pi_1(g)$ is a homomorphism of groups.

(2) If f is a homotopy equivalence, then $\pi_1(f)$ is an isomorphism.

π_n functor ($n \geq 2$) : $\mathbf{Ho}(\mathbf{Top}_*) \longrightarrow (\mathbf{Ab})$

Objects : $(X,x) \xrightarrow{f} (Y,y) \dashrightarrow \pi_n(X,x) \xrightarrow{\pi_n(f)} \pi_n(Y,y)$

Morphisms :

(1) For continuous maps $f \simeq g$, $\pi_n(f) = \pi_n(g)$ is a homomorphism of Abelian groups.

(2) If f is a homotopy equivalence, then $\pi_1(f)$ is an isomorphism.

n-connected spaces

A topological space X is (-1)-connected.

$\Longleftrightarrow X \neq \emptyset$.

A topological space X is 0-connected.

$\Longleftrightarrow X \neq \emptyset$ is path-connected.

$\Longleftrightarrow \pi_0(X) = 0$.

A topological space X is n-connected.

$\Longleftrightarrow \pi_1(X) = \cdots = \pi_n(X) = 0$ and $\pi_0(X)$ is a set with only one path-component ($\pi_0(X) = 0$) .

$\Longleftrightarrow$ Every $f : S^i \longrightarrow X$ is homotopic to a constant map $c : S^i \longrightarrow p \in X$ for all $i \leq n$.

$\Longleftrightarrow$ Every $f : S^i \longrightarrow X$ extends to a map $D^{i+1} \longrightarrow X$ for all $i \leq n$.

One has :

$$(-1)\text{-connected} \Longleftarrow \text{connected} \Longleftarrow 0\text{-connected} = \text{path-connected} \Longleftarrow$$

$$1\text{-connected} = \text{simply connected} \Longleftarrow 2\text{-connected} \Longleftarrow \cdots n\text{-connected} \Longleftarrow \cdots \ .$$

n-connected maps

A continuous map $f : X \longrightarrow Y$ is n-connected (or n-equivalence) .

$\Longleftrightarrow \pi_i(f) : \pi_i(X) \longrightarrow \pi_i(Y)$ is an isomorphism for $i \leq n-1$.

$\Longleftrightarrow$ The homotopy fibre M_f is an $(n-1)$-connected space.

$\Longleftrightarrow$ The mapping cone C_f is n-connected for 1-connected X .

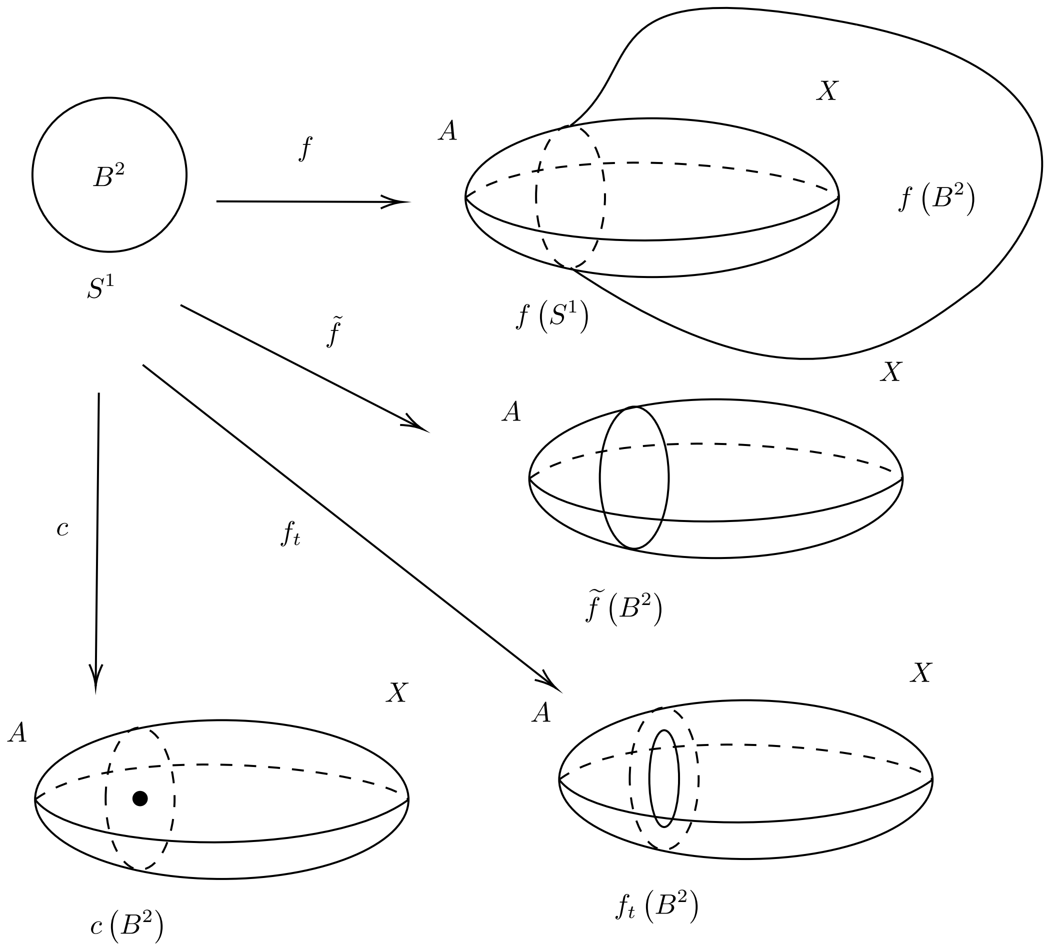
$\Longrightarrow$ The pair (I_f, X) is n-connected, C_f is n-connected.

The 0-connected map is also called connected map.

The 1-connected map is also called simply connected map.

n-connected pairs

The pair (X, A) is n-connected.
$\Longleftrightarrow$ The inclusion $A \longrightarrow X$ is an n-connected map.
$\Longleftrightarrow$ $\pi_1(X, A) = \cdots = \pi_n(X, A) = 0$ ($\pi_0(X, A)$ is not well defined) .
$\Longleftrightarrow$ Every $f : (B^n, S^{n-1}) \longrightarrow (X, A)$ is homotopic to a map $B^n \longrightarrow A$ rel S^{n-1} .
$\Longleftrightarrow$ Every $f : (B^n, S^{n-1}) \longrightarrow (X, A)$ is homotopic through such maps to map $\widetilde{f} : B^2 \longrightarrow A$.
$\Longleftrightarrow$ Every $f : (B^n, S^{n-1}) \longrightarrow (X, A)$ is homotopic through such maps to the constant map c .



Proposition

(1) (CX, X) is $(n+1)$-connected. $\Longleftrightarrow$ X is n-connected.

(2) If all the cells in $X \setminus A$ have dimension greater than n , then the CW pair (X, A) is n-connected.

(3) For n-connected X , ΩX is $(n-1)$-connected, ΣX is $(n+1)$-connected.

Proposition

(1) For an n-connected map $f : X \longrightarrow Y$ of 0-connected CW complexes, one has

$$H_i(f) : H_i(X) \longrightarrow H_i(Y) \text{ is } \begin{cases} \text{injective} & i \leq n-1 \\ \text{surjective} & i \leq n \end{cases} ,$$

$$H^i(f;G) : H^i(Y;G) \longrightarrow H^i(X;G) \text{ is } \begin{cases} \text{surjective} & i \leq n-1 \\ \text{injective} & i \leq n \end{cases} .$$

(2) For an n-connected map $f : X \longrightarrow Y$,

$$[CW, X] \longrightarrow [CW, Y] \text{ is } \begin{cases} \text{injective} & \dim CW \leq n-1 \\ \text{surjective} & \dim CW \leq n \end{cases} .$$

(3) For an n-connected inclusion $A \longrightarrow CW$ (n-connected pair) ,

$$[CW, Y] \longrightarrow [A, Y] \text{ is } \begin{cases} \text{surjective} & \text{if } \pi_i(Y) = 0 \text{ for } i \geq n \\ \text{injective} & \text{if } \pi_i(Y) = 0 \text{ for } i \geq n+1 \end{cases} .$$

(4) If f, g are n-connected, then $g \circ f$ is also n-connected.

(5) If $g \circ f$ is n-connected, then one has :
f is $(n-1)$-connected. $\Longrightarrow$ g is n-connected.
g is $(n+1)$-connected. $\Longrightarrow$ f is n-connected.

Whitehead's first theorem

The weak homotopy equivalence between connected CW complexes is a homotopy equivalence.

Whitehead's second theorem

An n-connected map between 0-connected CW complexes is a homological n-equivalence.
A weak homotopy equivalence between 0-connected CW complexes is a homological equivalence.

Proposition

A weak homotopy equivalence $f : X \longrightarrow Y$ induces isomorphisms

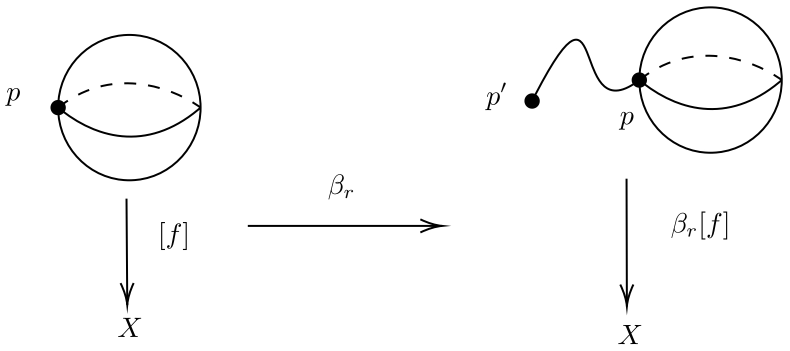
$$H_n(f) : H_n(X;G) \longrightarrow H_n(Y;G) ,$$

$$H^n(f) : H^n(X;G) \longrightarrow H^n(Y;G) ,$$

$$[CW, X] \longrightarrow [CW, Y] ,$$

for all n , coefficients G and CW complexes.

Group action of π_1 on π_n



$$\pi_n(X, x_0) \xrightarrow{\quad\beta_r\quad} \pi_n(X, x_1)$$

Take $[l \mid l : [0,1] \longrightarrow X \ , \ l(0) = l(1) = x_0 \ , \ l(\frac{1}{2}) = x_1] \in \pi_1(X, x_0)$.
Take a path $\gamma : [0,1] \longrightarrow X$, $\gamma(0) = l(0) = x_0$, $\gamma(1) = l(\frac{1}{2}) = x_1$.

Consider $\pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$, $[l] \cdot [f] = [\gamma \circ f \circ \gamma^{-1}]$, this gives a group action of π_1 on π_1 changing the base point.

For $n \geq 2$, the group action $\pi_1(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$ is given by $[l] \cdot [f] = \beta_\gamma[f]$, this gives a action of π_1 on π_n changing the base point.

For an $[l] \in \pi_1(X, x_0)$, there is a induced homomorphism $\beta_r : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$, since $\pi_1(X, x_0) \cong \pi_n(X, x_1)$, thus $\beta_r \in \mathrm{Aut}(\pi_n(X))$ called the action of π_1 on π_n ($n > 1$) . If π_1 acts trivially on π_n , the space X is called n-simple, and X is simple means it is n-simple for all n .

This action makes the Abelian group $\pi_n(X, x_0)$ a $\mathbb{Z}[\pi_1]$-module ($\mathbb{Z}[\pi_1] = \{\sum_i n_i r_i \mid n_i \in \mathbb{Z}, r_i \in \pi_1\}$) by

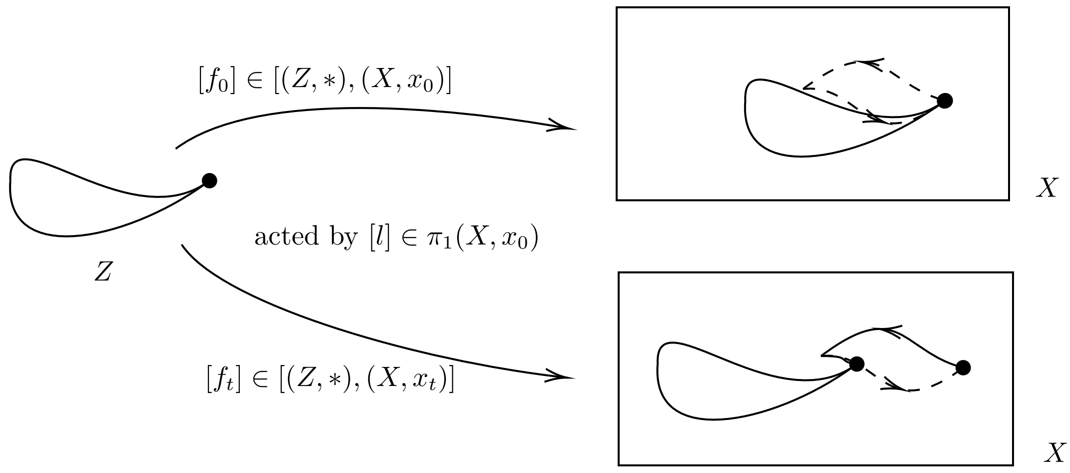
$$(r_1 + r_2) \cdot f = \beta_{r_1}(f) + \beta_{r_2}(f) \ ,$$

$$r \cdot (f + g) = \beta_r(f) + \beta_r(g) \ ,$$

$$r_1 \cdot (r_2 \cdot f) = r_1 r_2 \cdot f = \beta_{r_1} \circ \beta_{r_2}(f) \ ,$$

$$0 \cdot f = \beta_0(f) = \mathbb{1}(f) = f \ .$$

Group action of π_1 on homotopy classes



Take $[l \mid l : [0,1] \longrightarrow X \ , \ l(0) = l(1) = x_0 \ , \ l(\frac{1}{2}) = x_1] \in \pi_1(X,x_0)$.
Take a path $\gamma : [0,1] \longrightarrow X$, $\gamma(0) = l(0) = x_0$, $\gamma(1) = l(\frac{1}{2}) = x_1$.

Consider the right group action $[Z,X]_* \times \pi_1(X,x_0) \longrightarrow [Z,X]_*$, $[f_0] \cdot [l] = \beta_\gamma([f_0])$, this gives an action of π_1 on $[Z,X]_*$.

Proposition

(1) If Z is a CW complex and X is 0-connected, then the natural map $[Z,X]_* \longrightarrow [Z,X]$ induces a bijection from orbit set $[Z,X]_*/\pi_1(X,x_0)$ to $[Z,X]$.

(2) If X is 1-connected, then $[Z,X] \cong [Z,X]_*$.

(3) If X is an H-space, then the group action of π_1 on $[Z,X]_*$ is trivial.

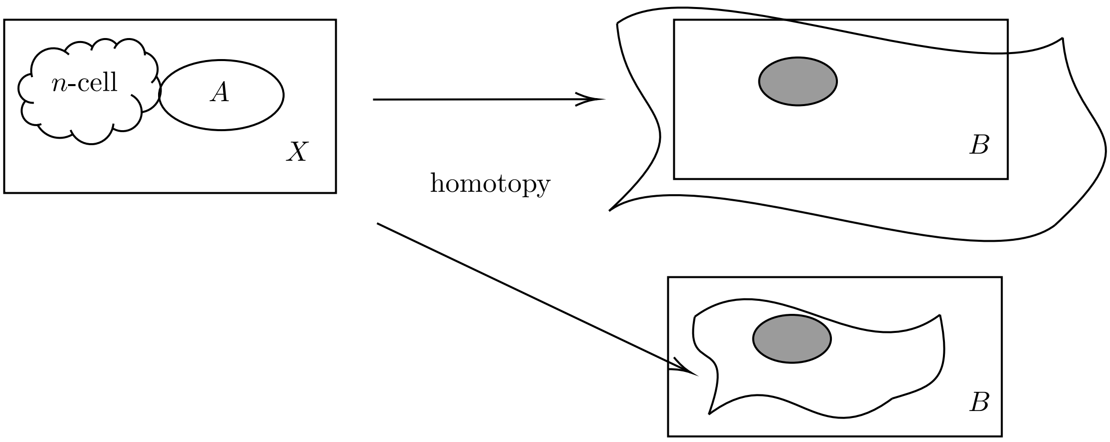
(4) One can construct a finite CW complex with π_n not finitely generated as a $\mathbb{Z}[\pi_1]$-module, for $n \geq 2$.

5.2 Techniques on Homotopy

Compression lemma

Let (X, A) be a CW pair, (Y, B) be any pair with $B \neq \emptyset$.

If $X \setminus A$ has cells of dimension n and $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$, then every map $f : (X, A) \longrightarrow (Y, B)$ is homotopic to a map $X \longrightarrow B$ rel A .



Extension lemma

Let (X, A) be a CW pair, Y be path-connected.

If $X \setminus A$ has cells of dimension n and $\pi_{n-1}(Y, y_0) = 0$ for all $y_0 \in Y$, then $f : A \longrightarrow Y$ can be extended to a map $X \longrightarrow Y$.

Cellular approximation theorem

Such a map $f : X \longrightarrow Y$ satisfying $f(X^n) \subseteq Y^n$ for all n is called a cellular map.

Every map $f : X \longrightarrow Y$ between CW complexes is homotopic to a cellular map. If $f : X \longrightarrow Y$ is cellular on the subcompex $A \subseteq X$, then the homotopy may be taken to be a stationary on A .

Relative cellular approximation theorem

Every map $f : (X, A) \longrightarrow (Y, B)$ of CW pairs can be deformed through maps $(X, A) \longrightarrow (Y, B)$ to a cellular map. This follows by first deforming $A \longrightarrow B$ to be cellular, then extending this to a homotopy of f on X (by the homotopy extension property) , and deforming $X \longrightarrow Y$ to be cellular.

The homotopy of f can be taken to be stationary on any subcomplex of X where f is already cellular.

Proposition

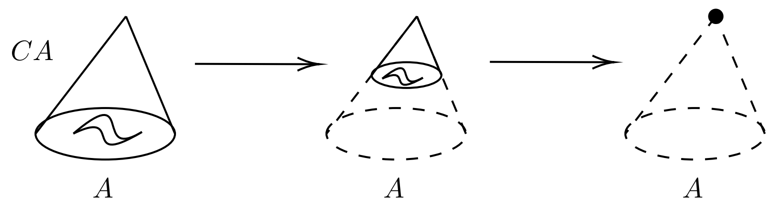
$\pi_n(S^n) \cong \mathbb{Z}$ is generated by the identity map $\mathbb{1}_{S^n}$ for all $n \geq 1$. The degree map $\pi_n(S^n) \longrightarrow \mathbb{Z}$ is an isomorphism.

Proposition

Let X be a CW complex with subcomplexes A and B such that $X = A \cup B$, $A \cap B \neq \emptyset$ is connected.
If $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected, then the induced map :
$\pi_i(A, A \cap B) \longrightarrow \pi_i(X, B)$ is an isomorphism for $i \leq m + n - 1$ and a surjection for $i = m + n$.

For a r-connected CW pair (X, A) with s-connected A , $(s+1)$-connected (CA, A) .
Then $(X, X \cap CA) = (X, A)$ is r-connected, $(CA, X \cap CA) = (CA, A)$ is $(s+1)$-connected, then :
$\pi_i(X, A) \longrightarrow \pi_i(X \cup CA, CA)$ is an isomorphism for $i \leq r + s$ and a surjection for $i = r + s + 1$.

$$
\begin{tikzcd}
 & \pi_{i-1}(CA) = 0 & & \\
\pi_i(X, A) \arrow[r] & \pi_i(X \cup CA, CA) \arrow[u] \arrow[r] & \pi_i(X \cup CA / CA) & = \pi_i(X/A) \\
\pi_i(CA) = 0 \arrow[r] & \pi_i(X \cup CA) \arrow[u] \arrow[ur, "\text{homotopy equivalence}"'] & &
\end{tikzcd}
$$



CW approximation theorem

For every space X , there is a CW complex and a weak homotopy equivalence $f : CW \longrightarrow X$. This such a map is called a CW approximation to X .

CW models

Suppose (X, A) is a pair with a CW complex A , an n-connected CW model for (X, A) is an n-connected CW pair (CW, A) and a map $f : CW \longrightarrow X$ satisfying $f|_A = \mathbb{1}$ such that $\pi_i(f) : \pi_i(CW) \longrightarrow \pi_i(X)$ is an isomorphism for $i \geq n+1$ and an injection for $i = n$ (thus isomorphism for all i) .

Constructing n-connected CW model

For every pair (X, A) with nonempty CW complex A , there exists an n-connected CW model $f : (CW, A) \longrightarrow (X, A)$ for all $n \geq 0$ and CW can be obtained form A by attaching cells of dimension greater than n (if $n = 0$, X is path-connected, $A = *$, this map f is a CW approximation) .

One can construct 0-connected (CW, A) to be the union of subcomplexes $A = CW_0 \subseteq CW_1 \subseteq CW_2 \subseteq \cdots$, $CW = \bigcup_n CW_n$ (for the n-connected, construct $A = CW_n \subseteq CW_{n+1} \subseteq CW_{n+2} \subseteq \cdots$) where CW_n is obtained by attaching n-cells to CW_{n-1} .

For every cellular map $[S^k, CW_k]_*$ generating the kernel of $\pi_k(f) : \pi_k(CW_k) \longrightarrow \pi_k(X)$, attach e^{k+1} to CW_k by this map to get $CW_k \bigsqcup_\alpha e^{k+1}_\alpha$, then $\pi_k(f)$ is injective.

For every cellular map $[S^{k+1}, X]_*$ generating $\pi_{k+1}(X)$, wedge with sphere S^{k+1}_β to $CW_k \bigsqcup_\alpha e^{k+1}_\alpha$ by this map to get CW_{k+1} , then $\pi_{k+1}(f)$ is surjective.

Then we get CW_{k+1} from CW_k such that $\pi_{k+1}(f)$: is surjective , $\pi_k(f)$ is injective.

$$
\begin{tikzcd}
 & & \text{surjective on } \pi_1 \text{ to } X \\
A = CW_0 \arrow[r, bend right, "e^1_\alpha"'] & CW_0 \sqcup e^1_\alpha \arrow[r, bend left] \arrow[d, dashed] \arrow[ur, dashed] & CW_1 = CW_0 \sqcup e^1_\alpha \vee S^1_\beta \\
 & \text{injective on } \pi_0 \text{ to } X &
\end{tikzcd}
$$

If the induction begins with $k = 0$, $CW_0 = A$, then we get a 0-connected CW model (CW, A) such that $\pi_i(f) : \pi_i(CW) \longrightarrow \pi_i(X)$ is an isomorphism for $i \geq 1$ and an injection for $i = 0$, then one can choose the A such that π_0 is automatically surjective.

Relative CW approximation

construct by the five lemma.

$$
\begin{tikzcd}
\pi_n(CW_0) \arrow[r] \arrow[d] & \pi_n(CW) \arrow[r] \arrow[d] & \pi_n(CW, CW_0) \arrow[r] \arrow[d] & \pi_{n-1}(CW_0) \arrow[r] \arrow[d] & \pi_{n-1}(CW) \arrow[d] \\
\pi_n(X_0) \arrow[r] & \pi_n(X) \arrow[r] & \pi_n(X, X_0) \arrow[r] & \pi_{n-1}(X_0) \arrow[r] & \pi_{n-1}(X)
\end{tikzcd}
$$

Proposition

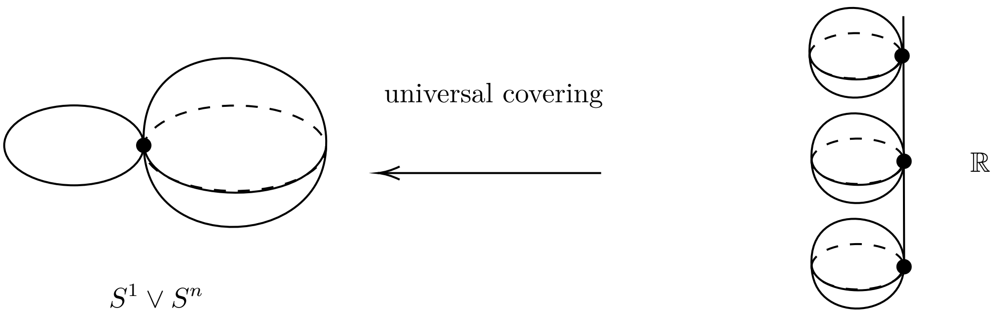
For an n-connected CW model $f:(CW,A)\longrightarrow(X,A)$ and an $(n+k)$-connected model $f':(CW',A')\longrightarrow(X',A')$ with map $g:(X,A)\longrightarrow(X',A')$, there is a unique $h:CW\longrightarrow CW'$, $h|_A=g$ up to homotopy rel A such that this diagram commutes up to homotopy.

$$\begin{tikzcd} CW \arrow[r, "f"] \arrow[d, dashed, "h"'] & X \arrow[d, "g"] \\ CW' \arrow[r, "f'"'] & X' \end{tikzcd}$$

Proposition

(1) The universal covering of $S^1\vee S^n$ is homotopy equivalent to $\bigvee\limits_{\alpha} S^n_\alpha$.

One has the fact that a finite CW complex need not have finitely generated homotopy groups.



$S^1\vee S^n$

(2) If the action of π_1 on all π_n is trivial, then one has :

the homotogy groups are finitely generated. $\Longleftrightarrow$ the homology groups are finitely generated. (Serre)

Plus construction theorem (Levin)

For every 0-connected CW complex, there is a 1-connected CW complex obtained by attaching 2-cells and 3-cells to CW , $CW \subseteq CW^+ = CW \sqcup e^2_\alpha \sqcup e^3_\beta$ such that the inclusion $i : CW \longrightarrow CW^+$ induces isomorphism $H_i(i) : H_i(CW) \longrightarrow H_i(CW^+)$ for $i > 1$.

Attach e^2 by the cellular map generating $\langle S^1, CW\rangle$, then $\pi_1(CW \sqcup e^2_\alpha) = 0$. $p : CW \sqcup e^2_\alpha \longrightarrow CW \sqcup e^2_\alpha/CW = \bigvee\limits_\alpha S^2$ is a collapsing. $\pi_2(\bigvee\limits_\alpha S^2)$ is free as well as its subgroup $\mathcal{I}m(\pi_2(p)) = \bigoplus\limits_\beta \mathbb{Z}$.

Take a section δ such that $\pi_2(p) \circ \delta = \mathbb{1}$.

Attach e^3 by the preimage $(\pi_2(p))^{-1}(p_i) = \langle S^2, CW \sqcup e^2_\alpha\rangle$, $CW^+ = CW \sqcup e^2_\alpha \sqcup e^3_\beta$.

By the Hurewicz Theorem : $H_3(CW^+, CW \sqcup e^2_\alpha) = \bigoplus\limits_\beta \mathbb{Z}$. $\pi_2(p) \circ \delta$ is injective.

$$\begin{tikzcd}
\pi_3\left(CW^+, CW \sqcup e^2_\alpha\right) \arrow[d, equal] & \pi_2\left(CW \sqcup e^2_\alpha\right) \arrow[d, equal] & & \\
H_3\left(CW^+, CW \sqcup e^2_\alpha\right) \arrow[r, "\delta"'] \arrow[d, equal] & H_2\left(CW \sqcup e^2_\alpha\right) \arrow[r, "\pi_2(p)"'] & H_2\left(\bigvee\limits_\alpha S^2\right) \arrow[r, equal] & \bigoplus\limits_\alpha \mathbb{Z} \\
\bigoplus\limits_\beta \mathbb{Z} & & &
\end{tikzcd}$$

Then there is a long exact sequence of triple $(CW^+, CW \sqcup e^2_\alpha, CW)$:

$$\begin{tikzcd}
H_3\left(CW \sqcup e^2_\alpha, CW\right) \arrow[r] \arrow[d, equal] & H_3\left(CW^+, CW\right) \arrow[r] & H_3\left(CW^+, CW \sqcup e^2_\alpha\right) \arrow[r, "\pi_2(p)\circ\delta"] & H_2\left(CW \sqcup e^2_\alpha, CW\right) \\
0 & & &
\end{tikzcd}$$

By the exactness, $H_3(CW^+, CW) = 0$ and $H_i(CW^+, CW) = 0$ for $i > 3$ by the dimensional resean.

Then there is a long exact sequence of pair (CW^+, CW) :

$$\cdots \longrightarrow H_{n+1}(CW^+, CW) \longrightarrow H_n(CW) \longrightarrow H_n(CW^+) \longrightarrow H_n(CW^+, CW) \longrightarrow \cdots$$

$$0 \longrightarrow H_2(CW) \longrightarrow H_2(CW^+) \longrightarrow H_2(CW^+, CW)$$

Then one has $H_i(CW^+) \cong H_i(CW)$ for $i > 2$ and an injection $H_2(CW) \longrightarrow H_2(CW^+)$.

$$\begin{tikzcd}
CW^+ \arrow[r, "j"] & CW^+/CW & & H_2(CW^+) \arrow[r, "j"] & H_2(CW^+, CW) \\
CW \sqcup e^2_\alpha \arrow[u, "i"] \arrow[r, "p"'] \arrow[ur, phantom, "\circlearrowright"] & CW \sqcup e^2_\alpha/CW \arrow[u, "f"'] \arrow[r, phantom, "\xrightarrow{\quad H_2 \quad}"] & & H_2\left(CW \sqcup e^2_\alpha\right) \arrow[u, "H_2(i)"] \arrow[r, "\pi_2(p)"'] \arrow[ur, phantom, "\circlearrowright"] & H_2\left(CW \sqcup e^2_\alpha, CW\right) \arrow[u, "H_2(f)"']
\end{tikzcd}$$

$H_2(i)$ is surjective by the exactness since $H_2(\bigvee\limits_\beta S^3) = 0$, $\pi_2(p) \circ H_2(f) = 0$ by the exactness since $\mathcal{K}er(H_2(f)) = \mathcal{I}m(\pi_2(p) \circ \delta)$, then $j = 0$, $H_2(CW) \longrightarrow H_2(CW^+)$ is surjective thus isomorphism.

5.3 Topics with Homology

Moore spaces

For $n \geq 2$ and an abelian group G , define the 1-connected Moore space $M(G,n)$ such that

$$\widetilde{H}_i(M(G,n)) = \begin{cases} G & i = n \\ 0 & \text{else} \end{cases} .$$

Eilenberg-MacLane spaces

For $n \geq 1$ and an abelian group G define the $n-1$-connected Eilenberg-MacLane sapce $K(G,n)$ such that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n \\ 0 & \text{else} \end{cases} .$$

Proposition

(1) $\Sigma M(G,n) = M(G,n+1)$, $\Omega K(G,n+1) = K(G,n)$.

(2) The Hurewicz map $h_n : \pi_n(M(G,n)) \longrightarrow H_n(M(G,n))$ is an isomorphism.

(3) The abelian group homomorphism $\varphi : G \longrightarrow H$ induces

$$f : M(G,n) \longrightarrow M(H,n) \text{ such that } H_n(f) = \varphi : H_n(M(G,n)) \longrightarrow H_n(M(H,n)) ,$$

$$g : K(G,n) \longrightarrow K(H,n) \text{ such that } \pi_n(g) = \varphi : \pi_n(K(G,n)) \longrightarrow \pi_n(K(H,n)) .$$

(4) $\pi_n(X;G) = [M(G,n),X]$ ($n \geq 2$) , $H^n(X;G) = [X, K(G,n)]$ where G is an abelian group.

The Hopf classification theorem

For a CW complex X with $\dim X \leq n$, one has $[X, S^n] \cong H^n(X;\mathbb{Z})$.

The Hurewicz theorem

For $n \geq 2$:

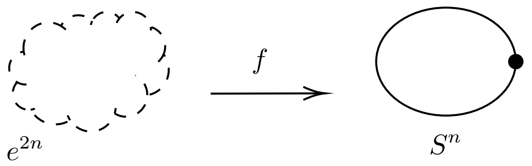
If X is $(n-1)$-connected, $\pi_1(X) = \cdots = \pi_{n-1}(X) = 0$,
then $\widetilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.

If (X,A) is $(n-1)$-connected with nonempty A is 1-connected, $\pi_1(X,A) = \cdots = \pi_{n-1}(X,A) = 0$,
then $H_i(X,A) = 0$ for $i \leq n-1$ and $\pi_n(X,A) = H_n(X,A)$.

The Hopf invariant

For a map $f : S^m \longrightarrow S^n$ with $m \geq n$, construct a CW complex CW_f by attaching an $(m+1)$-cell on S^n . Then the homotopy type of CW_f depends only on the homotopy class of f .

When $m = 2n-1$, choose generators $\alpha \in H^n(CW_f)$, $\beta \in H^{2n}(CW_f)$, then the cohomology ring $H^*(CW_f)$ is determined by $\alpha \smile \alpha = H(f)\beta$ where $H(f)$ is called Hopf invariant of f .



$$H^n(CW_f) = H^{2n}(CW_f) = \mathbb{Z}$$

Proposition

(1) If f is constant, then $H(f) = 0$.

(2) If n is odd for $f : S^{2n-1} \longrightarrow S^n$, then $H(f) = 0$.

(3) For even n , the map $f : S^{2n-1} \longrightarrow S^n$ of Hopf invariant 2 always exists.
For even n , the map $f : S^{2n-1} \longrightarrow S^n$ of Hopf invariant 1 only exists when $n = 2, 4, 8$. (Adams 1960)

(4) $\mathbb{R}^n$ is a division algebra only for $n = 1, 2, 4, 8$.

(5) S^n is an H–space only for $n = 0, 1, 3, 7$.

(6) S^n has n linearly independent tangent vector fields only for $n = 0, 1, 3, 7$.

(7) The only fiber bundles $S^p \longrightarrow S^q \longrightarrow S^r$ occur when $(p, q, r) = (0, 1, 1), (1, 3, 2), (3, 7, 4), (7, 15, 8)$.

(8) The Hopf invariant $H : [S^{2n-1}, S^n]_* \longrightarrow \mathbb{Z}$ is a homomorphism.

(9) $[S^{2n-1}, S^n]_*$ contains a $\mathbb{Z}$ direct summand when n is even.

Whitehead products

For $f \in [S^m, X]_*$, $g \in [S^n, X]_*$, define the Whitehead product $[f, g] \in [S^{m+n-1}, X]_*$ to be the composition

$$S^{m+n-1} \longrightarrow S^m \vee S^n \xrightarrow{f \vee g} X$$

where $S^{m+n-1} \longrightarrow S^m \vee S^n$ is the attaching map of attaching e^{m+n} to $S^m \times S^n$.

Minimal cell structure

For a 1-connected CW complex X with finitely generated $H_n(X)$, one has a CW complex M and a cellular homotopy equivalence $f : M \longrightarrow X$ such that the cell in M is either :

(a) a generator n-cell mapped by f to a generator in $H_n(X)$,

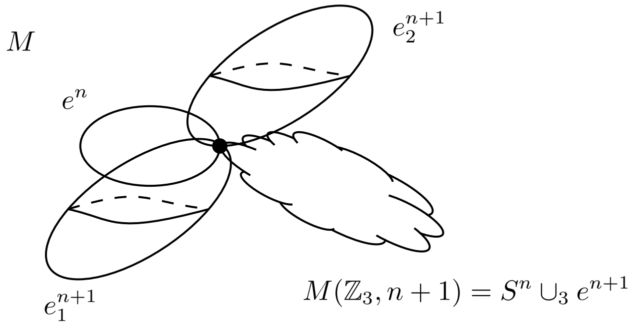
(b) a relator $(n+1)$-cell with boundary a multiple of one torsion generator cell.

$$H_{n-1}(X) = 0$$

$$H_n(X) = \mathbb{Z} \oplus \mathbb{Z}_3$$

$$H_{n+1}(X) = \mathbb{Z} \oplus \mathbb{Z}$$

$$H_{n+2}(X) = 0$$



Proposition

A map f between 1-connected CW complexes is a homotopy equivalence if $H_n(f)$ is an isomorphism for each n .

Proposition

For a space X homotopy equivalent to a 1-connected CW complex with only nontrivial reduced homology $H_2(X) = \bigoplus_k \mathbb{Z}$, $H_4(X) = \mathbb{Z}$, its homotopy type is determined by the cohomology ring $H^*(X)$. In particular the homotopy type of any 1-connected closed 4-manifold X is determined by the cohomology ring $H^*(X)$.

Stable splitting of spaces

For connected CW complexes X and Y , one has

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \ .$$

For James reduced product $J(X)$ of CW complex X one has

$$\Sigma J(X) \simeq \bigvee_n \Sigma(X^{\wedge n}) \text{ where } X^{\wedge n} = X \wedge \cdots \wedge X \text{ (} n \text{ copies) ,}$$

$$\Sigma J(S^n) \simeq S^{n+1} \vee S^{2n+1} \vee \cdots \ ,$$

$$\Sigma K(\mathbb{Z}_{p^n}, 1) \simeq X_1 \vee \cdots \vee X_{p-1}$$

where X_i is a CW complex such that $\widetilde{H_k}(X_i) = \begin{cases} \text{nonzero} & k \equiv 2i (mod\ 2p-2) \\ 0 & \text{else} \end{cases}$.

Loop-suspension spaces

For every connected CW complex X one has a weak homotopy equivalence $J(X) \longrightarrow \Omega\Sigma X$.

The Freudenthal suspension theorem

For an n-connected CW complex X , the suspension $\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ is an isomorphism for $i \leq 2n$ and surjection for $i = 2n+1$.

EHP sequences

There is an exact sequence

$$\pi_{3n-2}(S^n) \xrightarrow{\Sigma} \pi_{3n-1}(S^{n+1}) \longrightarrow \pi_{3n-2}(S^{2n}) \longrightarrow \pi_{3n-3}(S^n) \xrightarrow{\Sigma} \pi_{3n-2}(S^{n+1}) \longrightarrow \cdots .$$

Proposition

(1) $\pi_n(S^n) = \mathbb{Z}$ is generated by $\mathbb{1}_{S^n}$.

(2) $\pi_{n+1}(S^n) = \mathbb{Z}_2$ is generated by Hopf map $h : S^3 \longrightarrow S^2$ or $\Sigma^k h : S^{3+k} \longrightarrow S^{2+k}$.

(3) $\pi_i(S^{2n})$ in the EHP sequence are stable homotopy groups since $i = 3n{-}2, 3n-3, \cdots \leq 4n-2$.

(4) In the EHP sequence, table homotopy groups $\pi_i(S^{2n})$ are measuring the lack of stability of the groups $\pi_i(S^n)$ in the range $2n{-}1 \leq i \leq 3n{-}2$ called metastable range.

Excisive triads

pair $(X; A, B)$ is called an excisive triad if $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

For two excisive triads $(X; A, B)$ and $(Y; C, D)$, the map $f : (X; A, B) \longrightarrow (Y; C, D)$ satisfies $f : X \longrightarrow Y$, $f(A) \subseteq C$, $f(B) \subseteq D$.

Excision as homotopy

For a map $f : (X; A, B) \longrightarrow (Y; C, D)$ of excisive triads, one has (let $\pi_0(A, A \cap B) = \pi_0(A)/\pi_0(A \cap B)$) :

$$\begin{cases} \pi_i(A, A\cap B) \longrightarrow \pi_i(C, C\cap D) \text{ is an isomorphism for } i \leq n-1 \text{ and surjective for } i = n \ . \\ \pi_i(B, A\cap B) \longrightarrow \pi_i(D, C\cap D) \text{ is an isomorphism for } i \leq n-1 \text{ and surjective for } i = n \ . \end{cases}$$

$$\Longrightarrow \begin{cases} \pi_i(X, A) \longrightarrow \pi_i(Y, C) \text{ is bijective for } i \leq n-1 \text{ and surjective for } i = n \ . \\ \pi_i(X, B) \longrightarrow \pi_i(Y, D) \text{ is bijective for } i \leq n-1 \text{ and surjective for } i = n \ . \end{cases}$$

By the excision axiom, one has $H_i(X, A) \cong H^i(B, A\cap B)$, $H_i(Y, C) \cong H^i(D, C\cap D)$.

Gluing weak homotopy equivalences

For open covers $\{U_i\}$ of X , $\{V_i\}$ of Y , if $f : X \longrightarrow Y$ such that each $f(U_i) \subseteq V_i$ is a homotopy equivalence, then $f : U_1 \cap \cdots \cap U_k$ and $f : X \longrightarrow Y$ are also homotopy equivalences.

Quasi-fibrations

The map $p : E \longrightarrow X$ with 0-connected X is a quasi-fibration if any following one is satisfied :

(1) $\pi_i(E, p^{-1}(x)) \cong \pi_i(X, x)$ for all $x \in X$.

(2) Exist open sets V_1 , V_2 such that $X = V_1 \cup V_2$, $p^{-1}(V_1) \longrightarrow V_1$, $p^{-1}(V_2) \longrightarrow V_2$, $p^{-1}(V_1 \cap V_2) \longrightarrow V_1 \cap V_2$ are quasi-fibrations.

(3) $X = \underleftarrow{lim}(X_1 \xrightarrow{i} X_2 \xrightarrow{i} X_3 \longrightarrow \cdots) = \bigcup\limits_n X_n$ such that $p^{-1}(X_n) \longrightarrow X_n$ is quasi-fibration for each n .

(4) There are retract $r : E \longrightarrow E' \subseteq E$ and covering retract $r' : X \longrightarrow X' \subseteq X$ such that $E' \longrightarrow X'$ is a quasi-fibration and $r : p^{-1}(x) \longrightarrow p^{-1}(r'(x))$ is a weak homotopy equivalence.

Infinite symmetric product

For a based space X , let $X^k = X \times \cdots \times X$ (k copies) , define

$$SP(X) = \bigsqcup_k X^k / \sim \text{ where } (x_1, \cdots, x_k) \sim \text{ permutations of } (x_1, \cdots, x_k) \text{ .}$$

$SP_n(X)$ is a CW complex and $SP_n(X) \longrightarrow SP_{n+1}(X)$, $(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_n, *)$ is an inclusion.

Proposition

(1) $f \simeq g : X \longrightarrow Y$ induces $SP_n(f) = SP_n(g) : SP_n(X) \longrightarrow SP_n(Y)$.
For connected CW complexes $X \simeq Y$, one has $SP(X) \simeq SP(Y)$.

(3) $SP(S^2) = \mathbb{C}\mathrm{P}^\infty$. $SP_2(S^n) \cong C_f$ where $f : \Sigma^n \mathbb{R}\mathrm{P}^n \longrightarrow S^n$.

The Dold-Thom theorem

For cofibre sequence $A \xrightarrow{f} X \longrightarrow C_f$, $SP(X) \longrightarrow SP(C_f)$ is a Quasi-fibration with fibre $SP(A)$.
Functors $\widetilde{H_i}(-;\mathbb{Z})$ and $\pi_i(SP(-))$ coincides on the category of connected CW complexes.

Proposition

(1) $SP(X)$ is an commutative associative H-space.

(2) For connected CW complex X , $SP(X)$ is 0-connected, has weak homotopy type of $\prod\limits_n K(H_n(X), n)$.

5.4 On Fibre bundles

Fibre bundles

A surjective continuous map $p : E \longrightarrow X$ is a fibre bundle over X with the fibre F .

$\Longleftrightarrow$ $\forall\ x \in X$, $\exists$ open U containing x such that $p^{-1}(U) \cong U \times F$ and the first diagram commutes.

Thus $E_x = p^{-1}(x) \cong \{x\} \times F$.

$\Longleftrightarrow$ $\forall$ open cover $\{U_\lambda\}$ of X , one has $p^{-1}(U_\lambda) \cong U_\lambda \times F$ for all λ and the second diagram commutes.

$\{U_\lambda\}$ is called a trivialising cover of X .

$$\begin{tikzcd}
U \times F \arrow[dr, "\pi_1"'] & & p^{-1}(U) \subseteq E \arrow[ll, "\cong"'] \arrow[dl, "p"] \\
& U \subseteq X \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
U_\lambda \times F \arrow[dr, "\pi_1"'] & & p^{-1}(U_\lambda) \subseteq E \arrow[ll, "\cong"'] \arrow[dl, "p"] \\
& U_\lambda \subseteq X \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

Homeomorphism $p^{-1}(U) \longrightarrow U \times F$ is called local trivialisation of the fibre bundle $F \longrightarrow E \xrightarrow{p} X$.

p is an open map since π_1 is also an open map.

Proposition

(1) Fibre bundle $F \longrightarrow X \times F \xrightarrow{p} X$ is called trivial bundle.
Fibre bundle $\mathbb{R}^n \longrightarrow E \xrightarrow{p} X$ is called real vector bundle.
Fibre bundle $B^n \longrightarrow E \xrightarrow{p} X$ is called disk bundle.
Fibre bundle $S^n \longrightarrow E \xrightarrow{p} X$ is called sphere bundle.

(2) $\mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{p} S^1$ is a fibre bundle given by $p : \mathbb{R} \longrightarrow S^1$, $\theta \longmapsto e^{2\pi i\theta}$.
($p : \mathbb{R} \longrightarrow S^1$ is also a covering space.)

(3) For a continuous map $p : C \longrightarrow X$, if for every $x \in X$, there exists an open U containing x such that $p^{-1}(U)$ is a disjoint union of open sets of C which are homeomorphic to each other, then $p : C \longrightarrow X$ is a covering space.
($p^{-1}(U)$ need not be nonempty, so p need not be surjective.)

(4) For fibre bundle $F \longrightarrow E \xrightarrow{p} X$, if F carries the discrete topology, then $p : E \longrightarrow X$ is a covering space.
But a covering space is not a fibre bundle necessarily, since if X is not path-connected, $p^{-1}(U_x)$ and $p^{-1}(U_y)$ is not homeomorphic to the same F respectively.

(5) For a fibre bundle $F \longrightarrow E \xrightarrow{p} X$ with $A \subseteq X$, if (X, A) is n-connected, then $(E, p^{-1}(A))$ is also n-connected.

Hopf fibrations

There are some natrually continuous maps (which actually are fibre bundles) :

$$S^{(n+1)-1} = \{(x_0, \cdots, x_n) \mid x_i \in \mathbb{R} \ , \ \sum_i ||x_i|| = 1\} \longrightarrow \mathbb{R}\mathrm{P}^n \ ,$$

$$S^{2(n+1)-1} = \{(x_0, \cdots, x_n) \mid x_i \in \mathbb{C} \ , \ \sum_i ||x_i|| = 1\} \longrightarrow \mathbb{C}\mathrm{P}^n \ ,$$

$$S^{4(n+1)-1} = \{(x_0, \cdots, x_n) \mid x_i \in \mathbb{H} \ , \ \sum_i ||x_i|| = 1\} \longrightarrow \mathbb{H}\mathrm{P}^n \ .$$

Thus there are the fibre bundles called Hopf fibrations :

$$S^{1-1} \longrightarrow S^{(n+1)-1} \xrightarrow{p} \mathbb{R}\mathrm{P}^n \ ,$$

$$S^{2-1} \longrightarrow S^{2(n+1)-1} \xrightarrow{p} \mathbb{C}\mathrm{P}^n \ ,$$

$$S^{4-1} \longrightarrow S^{4(n+1)-1} \xrightarrow{p} \mathbb{H}\mathrm{P}^n \ .$$

Moreover we have $\mathbb{C}\mathrm{P}^n \cong S^{2n}$ thus the second Hopf fibration can be written by :

$$S^1 \longrightarrow S^{2n+1} \xrightarrow{p} S^{2n} \ .$$

The Bott periodicity theorem

There are three fiber bundles :

$$O(n-1) \longrightarrow O(n) \xrightarrow{p} S^{n-1} \ ,$$

$$U(n-1) \longrightarrow U(n) \xrightarrow{p} S^{2n-1} \ ,$$

$$Sp(n-1) \longrightarrow Sp(n) \xrightarrow{p} S^{4n-1} \ ,$$

where p is an evaluation of an orthogonal, unitary, or symplectic transformation on a fixed unit vector.

Since $O(n-1) \longrightarrow O(n)$ induces isomorphism on π_i for $i \leq n-3$, $\pi_i(O(n))$ is independent of n if n is large enough. Then one has the periodicity of homotopy groups.

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(O(n))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(U(n))$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$\pi_i(Sp(n))$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Pullback bundles

For a fibre bundle $F \longrightarrow E \xrightarrow{p} X$ and a continuous map $f : Y \longrightarrow X$, the pullback $E_f = E \times Y / \sim$ makes $F \longrightarrow E_f \xrightarrow{p'} Y$ a fibre bundle called pullback bundle.

$$\begin{tikzcd}
F \arrow[r] & E \arrow[r, "p"] \arrow[dr, phantom, "\circlearrowright"] & X \\
F \arrow[r] & E_f \arrow[u] \arrow[r] & Y \arrow[u, "f"']
\end{tikzcd}$$

Fibre bundle pairs

For two fibre bundles $F \longrightarrow E \xrightarrow{p} X$ and $F' \longrightarrow E' \xrightarrow{p'} X$ such that $F' \subseteq F$, $E' \subseteq E$ and $p' = p|_{E'}$, $(F, F') \longrightarrow (E, E') \xrightarrow{p} X$ is called a fibre bundle pair.

Cohomology extension of fibres

For a fibre bundle $F \longrightarrow E \xrightarrow{p} X$ and commutative ring R , define the cohomology extension of fibre to be the homomorphism $\xi : H^n(F; R) \longrightarrow H^n(E; R)$ such that

$$H^n(F; R) \xrightarrow{\xi} H^n(E; R) \xrightarrow{H^n(i_x)} H^n(E_x; R) \cong H^n(F; R)$$

is an isomorphism for $n \geq 0$. Equivalently, there exist $\{\langle c_j \rangle\} \subseteq H^n(E; R)$ such that $\{H^n(i_x)\langle c_j \rangle\}$ is a basis of $H^n(F; R)$.

For a fibre bundle pair $(F, F') \longrightarrow (E, E') \xrightarrow{p} X$ and commutative ring R , define the cohomology extension of fibre to be the homomorphism $\xi : H^n(F, F'; R) \longrightarrow H^n(E, E'; R)$ such that

$$H^n(F, F'; R) \xrightarrow{\xi} H^n(E, E'; R) \xrightarrow{H^n(i_x)} H^n(E_x, p^{-1}(x) \cap E'; R) \cong H^n(F, F'; R)$$

is an isomorphism for $n \geq 0$.

The Leray-Hirsch theorem

For a fibre bundle $F \longrightarrow E \xrightarrow{p} X$ such that $H^n(F; R)$ is a finitely generated free R-module for $n \geq 0$ and the cohomology extension exists, one has an isomorphism

$$H^*(X; R) \otimes_R H^*(F; R) \longrightarrow H^*(E; R) \ , \ \langle x \rangle \otimes \langle f \rangle \longmapsto H^*(p)\langle x \rangle \smile \xi\langle f \rangle \ .$$

This isomorphism need not be a ring homomorphism.

The relative Leray-Hirsch theorem

For a fibre bundle pair $(F, F') \longrightarrow (E, E') \xrightarrow{p} X$ such that $H^n(F, F'; R)$ is a finitely generated free R-module for $n \geq 0$ and the cohomology extension exists, one has an isomorphism

$$H^*(X; R) \otimes_R H^*(F, F'; R) \longrightarrow H^*(E, E'; R) \ , \ \langle x \rangle \otimes \langle f \rangle \longmapsto H^*(p)\langle x \rangle \smile \xi\langle f \rangle \ .$$

This isomorphism need not be a ring homomorphism.

Proposition

For CW complexes X and Y the cross product

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is a graded ring isomorphism if $H^n(Y;R)$ is a finitely generated free R-module for all n .

For CW pairs (X,A) and (Y,B) the cross product

$$H^*(X,A;R) \otimes_R H^*(Y,B;R) \longrightarrow H^*(X \times Y, A \times Y \cup B \times X;R)$$

is a graded ring isomorphism if $H^n(Y,B;R)$ is a finitely generated free R-module for all n .

Notice that $A \otimes_R B = A \otimes B$ if $R = \mathbb{Z}_m, \mathbb{Z}$ or $\mathbb{Q}$.

Proposition

(1) By the Leray-Hirsch theorem, one has

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, x_3, \cdots, x_{2n-1}] \text{ where } \deg(x_i) = i \ ,$$

$$H^*(SU(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_5, \cdots, x_{2n-1}] \text{ where } \deg(x_i) = i \ ,$$

$$H^*(Sp(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_7, \cdots, x_{4n-1}] \text{ where } \deg(x_i) = i \ .$$

(2) Denote $G_n(\mathbb{R}^\infty)$ as Grassmann manifold of n-dimensional vector subspaces of $\mathbb{R}^\infty$, by the Leray-Hirsch theorem, one has

$$H^*(G_n(\mathbb{R}^\infty);\mathbb{Z}_2) = \mathbb{Z}_2[x_1, \cdots, x_n] \text{ where } \deg(x_i) = i \ ,$$

$$H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[x_2, x_4, \cdots, x_{2n}] \text{ where } \deg(x_i) = i \ ,$$

$$H^*(G_n(\mathbb{H}^\infty);\mathbb{Z}) = \mathbb{Z}[x_4, x_8, \cdots, x_{4n}] \text{ where } \deg(x_i) = i \ .$$

Thom classes

For the fibre bundle pair $(B^n, S^{n-1}) \longrightarrow (E, E') \xrightarrow{p} X$, a Thom class for p is a cohomology class $\langle t\rangle \in H^n(E, E'; R)$ such that $H^n(i_x)\langle t\rangle \in H^n(E_x, E'_x; R) = H^n(B^n, S^{n-1}; R) = R$ is a generator of R .

The Thom isomorphism theorem

For a fibre bundle pair $(B^n, S^{n-1}) \longrightarrow (E, E') \xrightarrow{p} X$, if it has a Thom class $\langle t\rangle$, then one has isomorphsim

$$H^i(X; R) \longrightarrow H^{i+n}(E, E'; R)\ ,\ \langle x\rangle \longmapsto H^i(p)\langle x\rangle \smile \langle t\rangle \text{ for } i \geq 0$$

and $H^i(E, E'; R) = 0$ for $i \leq n-1$.

Proposition

(1) For fibre bundle pair $(B^n, S^{n-1}) \longrightarrow (E, E') \xrightarrow{p} X$, the Thom class exists for $R = \mathbb{Z}_2$ and it is unique.

(2) Disk bundles $(B^n, S^{n-1}) \longrightarrow (E, E') \xrightarrow{p} X$ is orientable. $\Longleftrightarrow$ It has a Thom classes with $\mathbb{Z}$ coefficients.

Gysin sequences

X is 1-connected or take cohomology with $\mathbb{Z}_2$.
$\Longrightarrow$ The fibre bundle $S^{n-1} \longrightarrow E \xrightarrow{p} X$ is orientable.
$\Longrightarrow$ There is an exact sequence

$$\cdots \longrightarrow H^{i-n}(X; R) \xrightarrow{\smile\langle e\rangle} H^i(X; R) \xrightarrow{H^i(p)} H^i(E; R) \longrightarrow H^{i-n+1}(X; R) \longrightarrow \cdots$$

where $\langle e\rangle \in H^n(X; R)$ is a Euler class.
Since $H^i(X; R) = 0$ for negative i , one has $H^i(X; R) \cong H^i(E; R)$ for $i \leq n-2$.

5.5 Steenrod Algebra

Bockstein homomorphisms

For an exact sequence of abelian groups $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, take the covariant exact functor $\mathrm{Hom}(C_n(X), -)$ since $C_n(X)$ is free, one has exact sequence

$$0 \longrightarrow C^n(X;A) \longrightarrow C^n(X;B) \longrightarrow C^n(X;C) \longrightarrow 0 \ .$$

Then one has long exact sequence

$$\cdots \longrightarrow H^n(X;A) \longrightarrow H^n(X;B) \longrightarrow H^n(X;C) \xrightarrow{\beta} H^{n+1}(X;A) \longrightarrow \cdots$$

where $\beta : H^n(X;C) \longrightarrow H^{n+1}(X;C)$ is called a Bockstein homomorphism.

Cohomology operators

A natrual transform $\Theta : \mathcal{H}^m(-;G) \longrightarrow \mathcal{H}^n(-;H)$ is called a cohomology operator.

$$\{\Theta \mid \Theta : \mathcal{H}^m(-;G) \longrightarrow \mathcal{H}^n(-;H)\} \cong H^n(K(G,m);H)$$

is given by $\Theta \longmapsto \Theta\langle\alpha\rangle$ where $\langle\alpha\rangle \in H^m(K(G,m);G)$ is a generator.

Steenrod squares

The Steenrod square $Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$ for $i \geq 0$ is a cohomology operator satisfying :

(1)

$$Sq^i(H^n(f)\langle\alpha\rangle) = H^{n+i}(f)(Sq^i\langle\alpha\rangle) \text{ where } f : X \longrightarrow Y \ .$$

(2)

$$Sq^i\langle\alpha+\beta\rangle = Sq^i\langle\alpha\rangle + Sq^i\langle\beta\rangle \ .$$

(3)

$$\text{The Cartan formula : } Sq^i\langle\alpha \smile \beta\rangle = \sum_j Sq^{i-j}\langle\alpha\rangle \smile Sq^j\langle\beta\rangle \ .$$

(4)

$$Sq^i\sigma = \sigma Sq^i \text{ where } \sigma : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+1}(\Sigma X;\mathbb{Z}_2) \text{ is an isomorphism .}$$

(5)

$$Sq^i\langle\alpha\rangle = \alpha^2 \text{ for } i = \deg(\alpha) \ , \ Sq^i\langle\alpha\rangle = 0 \text{ for } i \geq \deg(\alpha)+1 \ .$$

(6)

$$Sq^0 = \mathbb{1} \ .$$

$$Sq^1 = \beta \text{ is the } \mathbb{Z}_2 \text{ Bockstein homomorphism given by}$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

$$\cdots \longrightarrow H^n(X;\mathbb{Z}_2) \longrightarrow H^n(X;\mathbb{Z}_4) \longrightarrow H^n(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}_2) \longrightarrow \cdots \ .$$

Stable homotopy groups $\pi_{n+1}(S^n)$, $\pi_{n+3}(S^n)$ and $\pi_{n+7}(S^n)$ are nontrivial

If $f : S^{2k-1} \longrightarrow S^k$ has Hopf invariant 1 , then $[f] \in \pi_{n+k-1}(S^n)$ is nontrivial.
Proof :

For $f : S^{2n-1} \longrightarrow S^n$ with Hopf invariant $H(f) = 1$, take $CW_f = S^n \cup_f e^{2n}$, one has

$$Sq^n : H^n(CW_f;\mathbb{Z}_2) \longrightarrow H^{2n}(CW_f;\mathbb{Z}_2)$$

is nontrivial given by $\alpha \smile \alpha = \beta \in H^{2n}(CW_f;\mathbb{Z}_2)$, so is

$$Sq^n : H^{n+k}(\Sigma^k CW_f;\mathbb{Z}_2) \longrightarrow H^{2n+k}(\Sigma^k CW_f;\mathbb{Z}_2) \ .$$

If $[\Sigma^k f] = 0$ then one has $\Sigma^k CW_f \simeq S^{n+k}$ then one has contradiction.

$$\begin{tikzcd}
H^{n+k}\left(S^{n+k};\mathbb{Z}_2\right) \arrow[r, "\cong"] \arrow[d, "Sq^n"'] \arrow[dr, phantom, "\circlearrowright"] & H^{n+k}\left(\Sigma^k CW_f;\mathbb{Z}_2\right) \arrow[d, "Sq^n"] \\
0 = H^{2n+k}\left(S^{n+k};\mathbb{Z}_2\right) \arrow[r, "\cong"'] & H^{2n+k}\left(\Sigma^k CW_f;\mathbb{Z}_2\right)
\end{tikzcd}$$

Steenrod powers

The Steenrod power $P^i : H^n(X;\mathbb{Z}_p) \longrightarrow H^{n+2i(p-1)}(X;\mathbb{Z}_p)$ for p an odd prime is a cohomology operator satisfying :

(1)
$$P^i(H^n(f)\langle\alpha\rangle) = H^{n+i}(f)(P^i\langle\alpha\rangle) \text{ where } f : X \longrightarrow Y \ .$$

(2)
$$P^i\langle\alpha+\beta\rangle = P^i\langle\alpha\rangle + P^i\langle\beta\rangle \ .$$

(3)
$$\text{The Cartan formula : } P^i\langle\alpha \smile \beta\rangle = \sum_j P^{i-j}\langle\alpha\rangle \smile P^j\langle\beta\rangle \ .$$

(4)
$$P^i\sigma = \sigma P^i \text{ where } \sigma : H^n(X;\mathbb{Z}_p) \longrightarrow H^{n+1}(\Sigma X;\mathbb{Z}_p) \text{ is an isomorphism} \ .$$

(5)
$$P^i\langle\alpha\rangle = \alpha^p \text{ for } 2i = \deg(\alpha) \ , \ P^i\langle\alpha\rangle = 0 \text{ for } 2i \geq \deg(\alpha)+1 \ .$$
$$\text{Notice that } \alpha^2 = 0 \text{ for } \deg(\alpha) \text{ is odd.}$$

(6)
$$P^0 = \mathbb{1} \ .$$

Total Steenrod squares and powers

$Sq = \sum\limits_{i=0}^{\infty} Sq^i$ and $P = \sum\limits_{i=0}^{\infty} P^i$ are graded ring homomorphisms since $Sq\langle \alpha \smile \beta \rangle = Sq\langle \alpha \rangle \smile Sq\langle \beta \rangle$,

$P\langle \alpha \smile \beta \rangle = P\langle \alpha \rangle \smile P\langle \beta \rangle$ by the Cartan formula.

Proposition

$$Sq^i\langle \alpha^n \rangle = \binom{n}{i} \alpha^{n+i} \text{ for } \alpha \in H^1(X;\mathbb{Z}_2) \ .$$

$$P^i\langle \alpha^n \rangle = \binom{n}{i} \alpha^{n+i(p-1)} \text{ for } \alpha \in H^2(X;\mathbb{Z}_2) \ .$$

By these one has :

$$Sq\langle \alpha \rangle = \alpha + \alpha^2 = \alpha \smile (1+\alpha) \ , \ \alpha \in H^1(X;\mathbb{Z}_2) \ ,$$

$$Sq\langle \alpha^n \rangle = \sum_{i=0}^{\infty} \binom{n}{i} \alpha^{n+i} = \alpha^n \smile (1+\alpha)^n \ , \ \alpha \in H^1(X;\mathbb{Z}_2) \ ,$$

$$P\langle \alpha \rangle = \alpha + \alpha^p = \alpha \smile (1+\alpha^{p-1}) \ , \ \alpha \in H^2(X;\mathbb{Z}_p) \ ,$$

$$P\langle \alpha^n \rangle = \sum_{i=0}^{\infty} \binom{n}{i} \alpha^{n+i(p-1)} = \alpha^n \smile (1+\alpha^{p-1})^n \ , \ \alpha \in H^2(X;\mathbb{Z}_p) \ .$$

By the Pascal triangle :

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & \cdot & & 1 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & \cdot & & \cdot & & \cdot & & 1 & \\
1 & & 1 & & \cdot & & \cdot & & 1 & & 1
\end{array}
\qquad\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & \cdot & & \cdot & & 1 & & \\
 & 1 & & 1 & & \cdot & & 1 & & 1 & \\
1 & & 2 & & 1 & & 1 & & 2 & & 1
\end{array}$$

mod 2 mod 3

For $\alpha \in H^1(X;\mathbb{Z}_2)$:

$$Sq\langle \alpha^2 \rangle = \alpha^2 + \alpha^4 \ ,$$

$$Sq\langle \alpha^3 \rangle = \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \ .$$

For $\alpha \in H^2(X;\mathbb{Z}_3)$:

$$P\langle \alpha^2 \rangle = \alpha^2 + 2\alpha^4 + \alpha^6 \ ,$$

$$P\langle \alpha^3 \rangle = \alpha^3 + \alpha^9 \ .$$

Adem relations

$$Sq^a Sq^b = \sum_{j=0}^{\infty} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \text{ if } a \leq 2b-1 \ ,$$

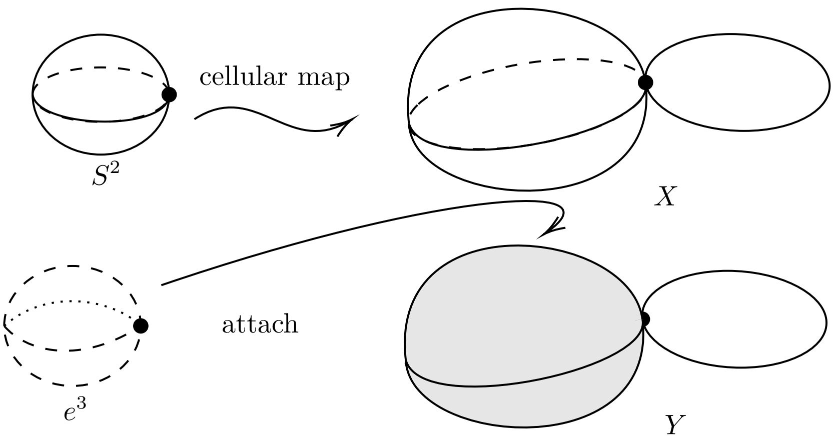
$$P^a P^b = \sum_{j=0}^{\infty} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j \text{ if } a \leq pb-1 \ ,$$

$$P^a \beta P^b = \sum_{j=0}^{\infty} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j - \sum_{j=0}^{\infty} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \text{ if } a \leq pb \ .$$

5.6 Homotopy Decomposition

The Postnikov Tower

For a connected CW complex X , one can construct a sequence $\widetilde{X_n}$ such that $\pi_i(X) \cong \pi_i(\widetilde{X_n})$ for $i \leq n$ and $\pi_i(\widetilde{X_n}) = 0$ for $i > n$.



For every generator of $\langle S^{n+1}, X \rangle$ in $\pi_{n+1}(X)$, by the cellular approximation theorem one can make it to be cellular. If we attach a e^{n+2} to X by this cellular map, then one has :

(1) $\pi_i(Y) \cong \pi_i(X)$ for $i \leq n$ since the i-skeletons of Y and X are the same, by the cellular approximation theorem, one can make $\langle S^i, Y \rangle$ homotopic to $\langle S^i, X \rangle$.

(2) $\pi_{n+1}(Y) = 0$ since the generators in X are nullhomotopy in Y .

Let $X = Y$ and repeat this process, one can get a CW complex $\widetilde{X_n}$ such that $\pi_k(i) : \pi_k(X) \longrightarrow \pi_k(\widetilde{X_n})$ is an isomorphism for $k \leq n$ and $\pi_k(\widetilde{X_n}) = 0$ for $k > n$.

$$\begin{tikzcd}
 & \vdots & \\
 & X_n \arrow[d, dotted] & \\
 & X_3 \arrow[d] & \varprojlim_n X_n \arrow[uul, dashed] \\
 & X_2 \arrow[d] & \\
X \arrow[r] \arrow[ur, phantom, "\circlearrowright"] \arrow[ur] \arrow[uur] \arrow[uuur] & X_1 &
\end{tikzcd}$$

(1) By the extension lemma :

$X \longrightarrow \widetilde{X_1}$ can be extended to a map $\widetilde{X_2} \longrightarrow \widetilde{X_1}$.

$$\begin{tikzcd}
 & \widetilde{X_2} \arrow[d, dashed] & (\ X \sqcup e^4_\beta \sqcup \cdots \text{ retracts to } X_2\ ) \\
X \arrow[ur] \arrow[r] & \widetilde{X_1} & (\ X \sqcup e^3_\alpha \sqcup e^4_\beta \sqcup \cdots \text{ retracts to } X_1\ )
\end{tikzcd}$$

(2) Map $\widetilde{X_n} \longrightarrow X_{n-1}$ factors as $\widetilde{X_n} \longrightarrow X_n \longrightarrow X_{n-1}$.

(any map can be turned into a fibration up to homotopy)

Proposition

(1) $(\widetilde{X_n}, X)$ is an $(n+1)$-connected CW model for (CX, X) .

(2) The unique map $X \longrightarrow \varprojlim_n X_n$ is a weak homotopy equivalence, X is a CW approximation to $\varprojlim_n X_n$ since $\pi_k(X) \longrightarrow \pi_k(\varprojlim_n X_n) \longrightarrow \varprojlim_n \pi_k(X_n)$ is an isomorphism for n sufficiently large.

Principal fibrations

A fibration $p : E \longrightarrow B$ with fibre F is called equivalent to a principal fibration if there is a homotopy equivalence $E \longrightarrow M_k$ where $k : B \longrightarrow K$ such that the diagram commutes.

$$\begin{tikzcd}
F \arrow[r] \arrow[d, dashed, "\text{weq}"'] & E \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, equal] & \\
\Omega K \arrow[r] & M_k \arrow[r, "p'"'] & B \arrow[r, "k"'] & K
\end{tikzcd}$$

Thus one must have a weak homotopy equivalence $F \longrightarrow \Omega K$.

The induced fibration $p' : M_k \longrightarrow B$ is called the principal fibration induced by $p : E \longrightarrow B$.

Proposition

A connected CW complex X has a Postnikov tower of principal fibrations.
$\Longleftrightarrow \pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n \geq 2$.

Any 1-conected CW complex X has a Postnikov tower of principal fibrations.

k-invariants

If the fibration $p : X_{n+1} \longrightarrow X_n$ is principal in the Postnikov tower, then one has an induced fibration $k_n : X_n \longrightarrow K(\pi_{n+1}(X), n+2)$ with fibre M_{k_n} (homotopy equivalent to X_{n+1}) .

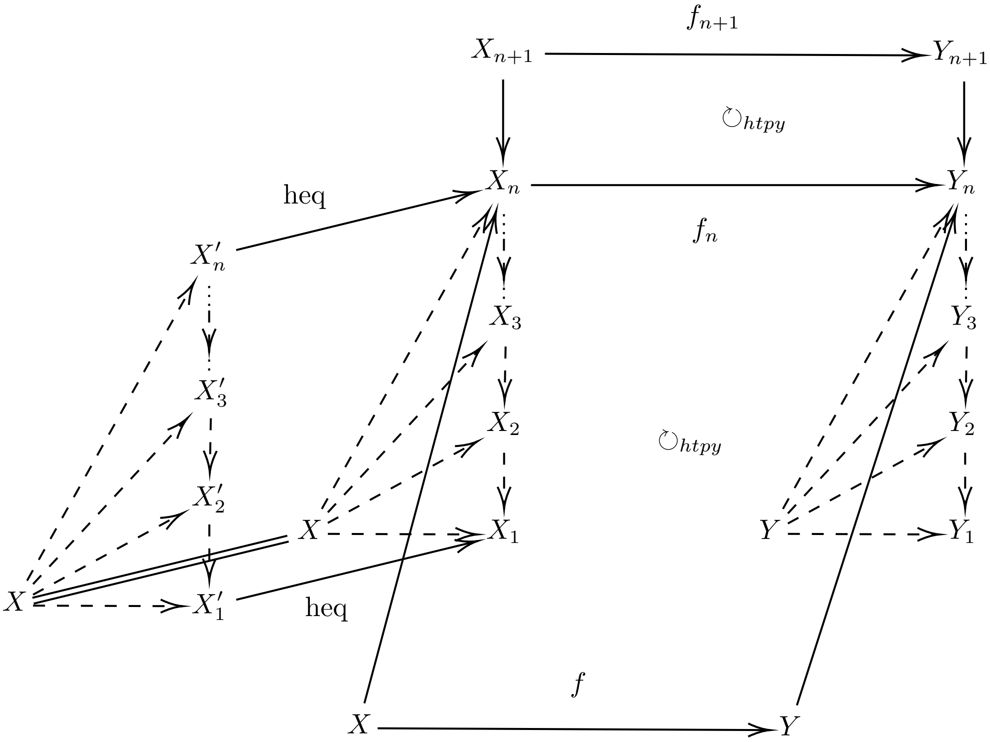
$$
\begin{tikzcd}
 & & \vdots & \\
 & & X_n \arrow[d, dotted] & \\
 & & X_3 \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
K(\pi_3(X),4) & & X_2 \arrow[ll, "k_2"'] \arrow[d] & \\
 & X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_1 &
\end{tikzcd}
$$

Thus there is a fibre sequence $K(\pi_n(X), n) \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow K(\pi_n, n+1)$.

$k_n : X_n \longrightarrow K(\pi_{n+1}(X), n+2)$ is a class in $H^{n+2}(X_n; \pi_{n+1}(X))$ called the n-th k-invariant (Postnikov invariant) of X (By the Brown representability theorem, $H^{n+2}(X_n; \pi_{n+1}(X)) \cong [X_n, K(\pi_{n+1}(X), n+2)]_*$) .

Functoriality of Postnikov towers

Consider the category of the tower-like diagrams, the object is the Postnikov tower $\mathcal{P}(X)$ of space X , the morphism is $f \prod_n f_n$ where $f : X \longrightarrow Y$, $f_n : X_n \longrightarrow Y_n$ (assume that all are 1-connected) .



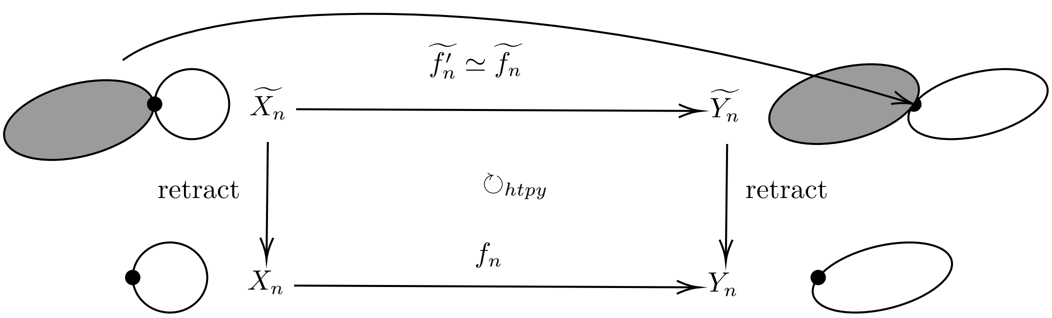
Proposition

For CW pairs (X, A) where cells in $X \setminus A$ have dimension $k \geq n+2$, then there is an induced map $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$.

If $\pi_m(Y) = 0$ for $n \geq n+2$, then $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$ is injective.

If $\pi_m(Y) = 0$ for $n \geq n+1$, then $\langle A, Y \rangle \longrightarrow \langle X, Y \rangle$ is surjective.

Consider the inclusion $i_n : X \longrightarrow \widetilde{X_n}$, then there is a unique $[\widetilde{f_n}]$ such that $[i'_n \circ f] \longmapsto [\widetilde{f_n}]$ where $i'_n : Y \longrightarrow \widetilde{Y_n}$, $\widetilde{f} : \widetilde{X_n} \longrightarrow \widetilde{Y_n}$, thus $f_n : X_n \longrightarrow Y_n$ is well defined.



Proposition

If f is a homotopy equivalence, then f_n is a homotopy equivalence.
For the homotopy inverse g , one has

$$f_n \circ g_n \simeq (fg)_n \simeq \mathbb{1}_{Y_n} \ , \ g_n \circ f_n \simeq (gf)_n \simeq \mathbb{1}_{X_n} \ .$$

Take $f = \mathbb{1}_X$, $Y = X$, then for two section X_n and X'_n they are homotopy equivalent.

Commutativity with k-invariant

For two Postnikov towers of X , one has X_n and X'_n are homotopy equivalent, then one has the diagram commutes.

$$\begin{tikzcd}
X_n \arrow[rr, "\text{heq}"] \arrow[dr] & \circlearrowleft_{htpy} & X'_n \arrow[dl] \\
& K(\pi_{n+1}(X), n+2) &
\end{tikzcd}$$

Thus $H^{n+2}(X_n; \pi_{n+1}(X)) \cong \langle X_n, K(\pi_{n+1}(X), n+2)\rangle = \langle X'_n, K(\pi_{n+1}(X), n+2)\rangle \cong H^{n+2}(X'_n; \pi_{n+1}(X))$.

The Whitehead tower

For a connected CW complex X , one has the commutative diagram such that $W_n \longrightarrow W_{n-1}$ is a fibration with fibre $K(\pi_n(X), n-1)$ for each n

$$\begin{tikzcd}
& \vdots \arrow[d, dotted, no head] & \\
& W_n \arrow[d, dotted] \arrow[dddl] & \\
& W_3 \arrow[d] \arrow[ddl] & K(\pi_3(X), 2) \arrow[l] \\
& W_2 \arrow[d] \arrow[dl] & \\
X & W_1 \arrow[l] &
\end{tikzcd}$$

where $\pi_k(W_n) \longrightarrow \pi_k(X)$ is an isomorphism for $k \geq n+1$ and $\pi_k(W_n) = 0$ for $k \leq n$.

The Postnikov tower of spectra

The Universal Coefficient Theorem for Homotopy

Define the homotopy group with coefficient $\pi_n(X;G) = \langle M(G,n), X\rangle$, for $n \geq 2$ there is an exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(G, \pi_{n+1}(X)) \longrightarrow \pi_n(X;G) \longrightarrow \mathrm{Hom}(G, \pi_n(X)) \longrightarrow 0 \ .$$

The Moore tower

If X is 1-connected, then X has a Moore tower (commutative diagram) of principal cofibrations.

$$\begin{tikzcd}
 & & \vdots & \\
 & & X_n \arrow[ddddl] & \\
 & & \vdots \arrow[u] & \\
 & & X_3 \arrow[u] \arrow[r] \arrow[ddl] & M(H_3(X),3) \\
M(H_3(X),2) \arrow[rr, "k'_2"] & & X_2 \arrow[u] \arrow[dl] & \\
 & X & X_1 \arrow[l] \arrow[u] &
\end{tikzcd}$$

$i_n : X_n \longrightarrow X_{n+1}$ is a principal cofibration inducing the cofibration $k'_n : M(H_{n+1}(X), n) \longrightarrow X_n$ with cofibre $C_{k'_n}$ (homotopy equivalent to X_3) .

Thus there is a cofibre sequence $M(H_{n+1}(X), n) \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow M(H_{n+1}(X), n+1)$.

The Moore tower has no factoriality

Take an 1-connected $X = M(\mathbb{Z}_2, n) \vee S^{n+1}$, take $X_n = M(\mathbb{Z}_2, n)$, $X_{n+1} = M(\mathbb{Z}_2, n) \vee S^{n+1} = X$. By the universal coefficient theorem one has $\langle M(\mathbb{Z}_2, n), S^{n+1}\rangle = \pi_n(X;\mathbb{Z}_2) \cong \mathrm{Ext}^1_{\mathbb{Z}_2}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$ since $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = 0$. Thus there is a nonconstant map $q : M(\mathbb{Z}_2, n) \longrightarrow S^{n+1}$. Consider $f = i_2 \circ \mathrm{fold} \circ (q \vee \mathbb{1}) : X \longrightarrow X$.

$$\begin{tikzcd}
 & M(\mathbb{Z}_2,n) \vee S^{n+1} \arrow[ddl, equal] & & M(\mathbb{Z}_2,n) \vee S^{n+1} \arrow[ddr, equal] & \\
 & M(\mathbb{Z}_2,n) \arrow[u] \arrow[dl] \arrow[rr, dashed, "f_n"] & & M(\mathbb{Z}_2,n) \arrow[u] \arrow[dr] & \\
M(\mathbb{Z}_2,n) \vee S^{n+1} \arrow[rrrr, bend right, "f = i_2 \circ \mathrm{fold} \circ (q \vee \mathbb{1})"'] & \vdots \arrow[u] & & \vdots \arrow[u] & M(\mathbb{Z}_2,n) \vee S^{n+1}
\end{tikzcd}$$

If $f \circ i_1 \simeq i_1 \circ f_n$, then $q = (q \vee c) \circ i_2 \circ q = (q \vee c) \circ f \circ i_1 \simeq (q \vee c) \circ i_1 \circ f_n = c$ makes a contradiction.

5.7 Spectral Sequences

Cohomology spectral sequences

A graded differential ring (algebra) is a graded ring R with a map $d : R \longrightarrow R$ such that $d \circ d = 0$ and satisfies the Leibniz rule $d(x_m \cdot y_n) = d(x_m) \cdot (y_n) + (-1)^{m+n} x_m \cdot d(y_n)$ for $x_m \in R_m$, $y_n \in R_n$.

A filtration of an R-module M is a sequence $0 \subseteq \cdots \subseteq F_{-1}M \subseteq F_0 M \subseteq F_1 M \subseteq \cdots \subseteq M = \bigcup\limits_n F_n M$, and M is not even to be graded. If M is a filtered module, then take $G_n = F_n M / F_{n-1} M$, $G = \bigoplus\limits_n G_n$ is a graded module.

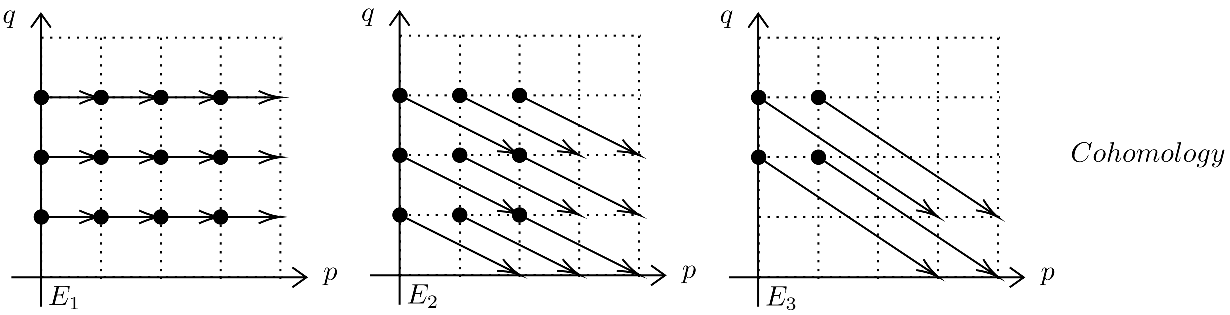
If R is a field, then we can get $F_1 M$ from the G_0 and G_1 since the exact sequence $0 \longrightarrow F_0 M \longrightarrow F_1 M \longrightarrow F_1 M / F_0 M \longrightarrow 0$ is unique, and we can use $G = \bigoplus\limits_n G_n$ to approximate $M = \bigoplus\limits_n F_n M$ and the same holds if G_n is a free R-module for each n .

A cohomological spectral sequence is a sequence of graded R-modules $E_r^{p,q}$ together with the differentials $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$, the bidegree is $(r, -r+1)$ and the total degree is 1 . In this sequence, the next page is $E_{r+1}^{p,q} = H^d(E_r^{p,q}) = \mathcal{K}er(d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}) / \mathcal{I}m(d_r : E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})$. The infinite page is $E_\infty^{p,q} = \underrightarrow{Colim}_r E_r^{p,q}$.

We say the spectral sequence $E_r^{p,q}$ converges to the graded R-module $M = \bigoplus\limits_n M_n$ denoted by $E_r^{p,q} \Longrightarrow M^{p+q}$, if for each (p,q) there exists a r_0 such that the differentials $d_r = 0$ for all $r \geq r_0$, and for the filtration of M , one has an isomorphism $0 \subseteq \cdots \subseteq M = \bigcup\limits_n F_n M$, $E_\infty^{p,n-p} = \underrightarrow{Colim}_r E_r^{p,n-p} \cong G_n = F_n M / F_{n-1} M$ for each n .

A homological spectral sequence is a sequence of graded R-modules $E_{p,q}^r$ together with the differentials $d^r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r$ such that $d^r \circ d^r = 0$, the bidegree is $(-r, r-1)$ and the total degree is -1 . In this sequence, the next page is $E_{p,q}^{r+1} = H_d(E_{p,q}^r) = \mathcal{K}er(d^r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r) / \mathcal{I}m(d^r : E_{p+r,q-r+1}^r \longrightarrow E_{p,q}^r)$. The infinite page is $E_{p,q}^\infty = \underrightarrow{Colim}_r E_{p,q}^r$.

First quadrant spectral sequences



Exact couples

Let X be a CW-complex, $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X = \bigcup\limits_n X_n$ is the cellular filtration where X_n is the n-skeleton, each $i_p : X_p \longrightarrow X_{p+1}$ induces a long exact sequence $\cdots \longrightarrow H_n(X_p) \longrightarrow H_n(X_{p+1}) \longrightarrow H_n(X_{p+1}, X_p) \xrightarrow{\delta} H_{n-1}(X_p) \longrightarrow \cdots$ in homology.

This is a homology spectral sequence where the first page is $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$ and $d^1 : E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \longrightarrow E^1_{p-1,q} = H_{p+q-1}(X_{p-1}, X_{p-2})$, E^r_{p+q} converges to $H_{p+q}(X)$, $E^r_{p+q} \Longrightarrow H_{p+q}(X)$

$$\begin{tikzcd}
H_n(X_{p-1}) \arrow[d] & & & \\
H_n(X_p) \arrow[r] \arrow[d] & H_n(X_p, X_{p-1}) \arrow[r, "\delta"] \arrow[rr, bend left, "d_1"] & H_{n-1}(X_{p-1}) \arrow[r] \arrow[d] & H_{n-1}(X_{p-1}, X_{p-2}) \\
H_n(X_{p+1}) \arrow[r] \arrow[d] & H_n(X_{p+1}, X_p) \arrow[r, "\delta"] \arrow[rr, bend left] & H_{n-1}(X_p) \arrow[r] \arrow[d] & H_{n-1}(X_p, X_{p-1}) \\
H_n(X_{p+2}) \arrow[r] \arrow[d, dashed] & H_n(X_{p+2}, X_{p+1}) \arrow[r, "\delta"] \arrow[rr, bend left] & H_{n-1}(X_{p+1}) \arrow[r] \arrow[d, dashed] & H_{n-1}(X_{p+1}, X_p) \\
H_n(X) & & H_{n-1}(X) &
\end{tikzcd}$$

$E^2_{p,q} = \mathcal{K}er(d^1 : H_{p+q}(X_p, X_{p-1}) \longrightarrow H_{p+q-1}(X_{p-1}, X_{p-2}))/\mathcal{I}m(d^1 : H_{p+q+1}(X_{p+1}, X_p) \longrightarrow H_{p+q}(X_p, X_{p-1}))$
$d^2 : E^2_{p,q} < H_{p+q}(X_p, X_{p-1}) \longrightarrow E^2_{p-2,q+1} < H_{p+q-1}(X_{p-2}, X_{p-3})$

In this homology spectral sequence, $H_{p+q}(X_p, X_{p-1}) = \widetilde{H_{p+q}}(X_p/X_{p-1}) = \widetilde{H_{p+q}}(\bigvee\limits_k S^p) = \begin{cases} \bigoplus\limits_k \mathbb{Z} & q = 0 \\ 0 & q \neq 0 \end{cases}$

The Leray-Serre spectral sequence

$p : E \longrightarrow B$ is a (weak) fibration with a connected fibre F where B is path-connected, for an Abelian group A , there is a first quadrant homology spectral sequence and a first quadrant cohomology spectral sequence.

$E^2_{p,q} = H_p(B; H_q(F; A))$, $E^r_{p,q} \Longrightarrow H_{p+q}(E; A)$, $H_q(F; A)$ is the local coefficient.
$E_2^{p,q} = H^p(B; H^q(F; A))$, $E_r^{p,q} \Longrightarrow H^{p+q}(E; A)$, $H^q(F; A)$ is the local coefficient.

If A is a commutative ring, then this cohomology spectral sequence has an algebraic structure.

Let B be a finite CW-complex, there is a filtration $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots B$, then lift to the total space there is another filtration $p^{-1}(B_0) \subseteq p^{-1}(B_1) \subseteq \cdots \subseteq p^{-1}(B_n) \subseteq \cdots E$.

$E^1_{p,q} = H_{p+q}(p^{-1}(B_p), p^{-1}(B_{p-1}))$, $E^r_{p,q} \Longrightarrow H_{p+q}(E)$

$E^1_{p,q} = H_{p+q}(p^{-1}(B_p), p^{-1}(B_{p-1})) = \widetilde{H_{p+q}}(p^{-1}(B_p)/p^{-1}(B_{p-1}))$

Since $p^{-1}(B_p)/p^{-1}(B_{p-1})$ is homotopy equivalent to $\bigvee\limits_k S^p \times F$, $E^1_{p,q} = \widetilde{H_{p+q}}(\bigvee\limits_k S^p \times F)$.

By the Künneth formula $H_n(X \times Y) \cong (\bigoplus\limits_{i+j=n} H_i(X) \otimes H_j(Y)) \oplus (\bigoplus\limits_{k+l=n-1} \mathrm{Tor}(H_k(X), H_l(Y)))$,

$H_{p+q}(\bigvee\limits_k S^p \times F) = (\bigoplus\limits_{i+j=p+q} H_i(\bigvee\limits_k S^p) \otimes H_j(F)) \oplus (\bigoplus\limits_{k+l=p+q-1} \mathrm{Tor}(H_k(\bigvee\limits_k S^p), H_l(F)))$

$= H_p(\bigvee\limits_k S^p) \otimes H_q(F) = \bigoplus\limits_k \mathbb{Z} \otimes H_q(F) \cong C^{cell}_p(B) \otimes H_q(F) = E^1_{p,q}$

By the definition, $E^2_{p,q} = H_d(E^1_{p,q}) = H^{cell}_p(B; H_q(F)) = H_p(B; H_q(F))$

Computations by Serre spectral sequence

The loop-path fibration $p : PX \longrightarrow X$ with fibre ΩX induced a Serre spectral sequence with $E_2^{p,q} = H^p(X; H^q(\Omega X))$ and $E_r^{p,q} \Longrightarrow H^{p+q}(PX) \cong H^{p+q}(\{x_0\})$.