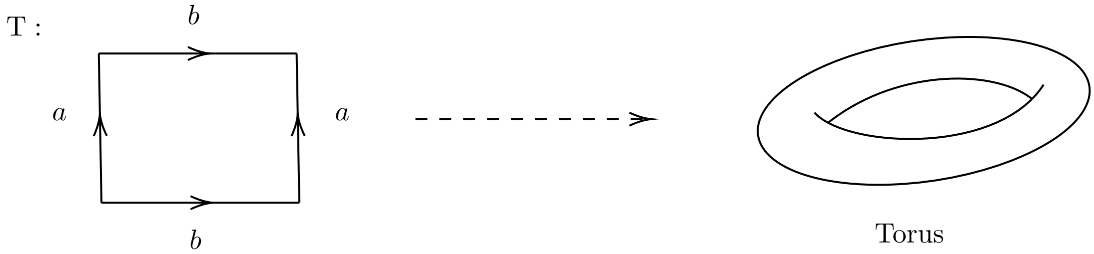
Chapter 4

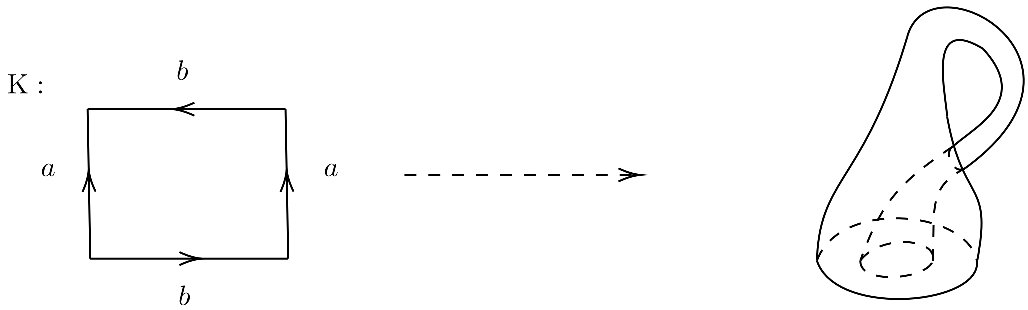
Homology Theory

4.1 Topological Spaces

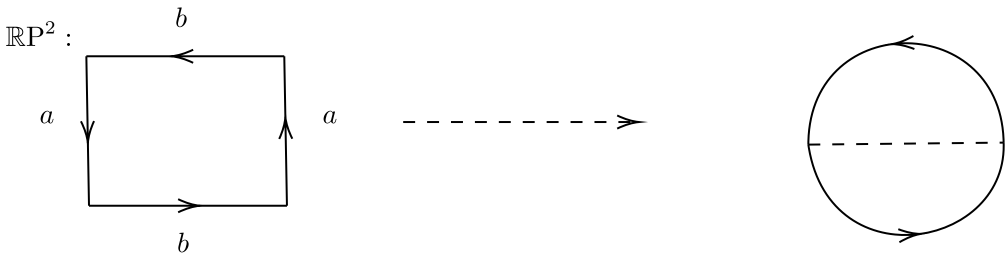
Surfaces



Torus



Klein Bottle



Projection Space

A topological space S is a surface.
$\Longleftrightarrow$ For any $x \in S$, there exists an open neighbourhood U_x of x and a homeomorphism $h_x : U_x \longrightarrow \mathbb{R}^2$.

A surface S is a closed surface.
$\Longleftrightarrow$ S is compact and connected, and it has no boundary ($\partial S = \emptyset$) .

A surface S is a open surface.
$\Longleftrightarrow$ S is noncompact and connected, and it has no boundary ($\partial S = \emptyset$) .

Two surfaces are homeomorphic.
$\Longleftrightarrow$ They are both orientable or non-orientable, and they have the same Euler characteristic.

Classification theorem of closed surfaces

Every closed surface is homeomorphic to one of the following :
(1) $S^2 : aa^{-1}$.
(2) $n\mathrm{T} = \mathrm{T}\#\cdots\#\mathrm{T}$ (n copies) : $a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$.
(3) $m\mathbb{R}\mathrm{P}^2 = \mathbb{R}\mathrm{P}^2\#\cdots\#\mathbb{R}\mathrm{P}^2$ (m copies) : $a_1^2\cdots a_m^2$.

Any two of them are not homeomorphic. By the way, $n\mathrm{T}$ is a quotient space of D^2 identified pairs of $4n$ edges, $m\mathbb{R}\mathrm{P}^2$ is a quotient space of D^2 identified pairs of $2m$ edges (for any group G , there is a 2-dimension cell complex X such that $\pi_1(X) = G$) .

Proposition

Let M be the Möbius bond, K be the Klein bottle.

(1) $\mathbb{R}\mathrm{P}^2 = \mathrm{M}\cup_f D^2$, $\mathrm{K} = \mathrm{M}\cup_f \mathrm{M}$, $f : \partial\mathrm{M} = S^1 \longrightarrow S^1$ (which is ∂D^2 or $\partial\mathrm{M}$) .

(2) $\mathrm{K} = \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$

(3) $\mathbb{R}\mathrm{P}^2\#\mathrm{T} \cong \mathbb{R}\mathrm{P}^2\#\mathrm{K}$

Hausdorff properties

For Hausdorff spaces X and Y :
(1) Every one point set $\{x\} \subseteq X$ is closed.
(2) Every subspace $X' \subseteq X$ is a Hausdorff space.
(3) Every product space $X \times Y$ is a Hausdorff space.

For a map $f : X \longrightarrow Y$ with Hausdorff Y :
(1) The graph of f , $G_f = \{(x, f(x)) \mid x \in X\}$ is a closed set of $X \times Y$.
(2) For another map $g : X \longrightarrow Y$ with Hausdorff Y , the set $Eq(f, g) = \{x \in X \mid f(x) = g(x)\}$ is a closed set of X .

Connectedness

A topological space X is connected.
$\Longleftrightarrow$ $X = U_1 \sqcup U_2$ such that $U_1, U_2 \neq \emptyset$ are disjoint and open.
$\Longleftrightarrow$ The only subsets of X which are both open and closed are $\emptyset$ and X .
$\Longleftrightarrow$ Any continuous map $X \longrightarrow \{0, 1\}$ is constant.

Proposition

(1) As a subspace of a topological space X , $B \subseteq X$ is both open and closed, then for every connected subspace $K \subseteq X$, either $B \cap K = \emptyset$, or $K \subseteq B$.

(2) Let $\{X_\alpha\}$ be a collection of connected subspaces of X , if for any $X_a, X_b \in \{X_\alpha\}$, $X_a \cap X_b \neq \emptyset$, then $Y = \bigcup_{\alpha} X_\alpha \subseteq X$ is connected.

(3) If X and Y are connected, then $X \times Y$ is connected.

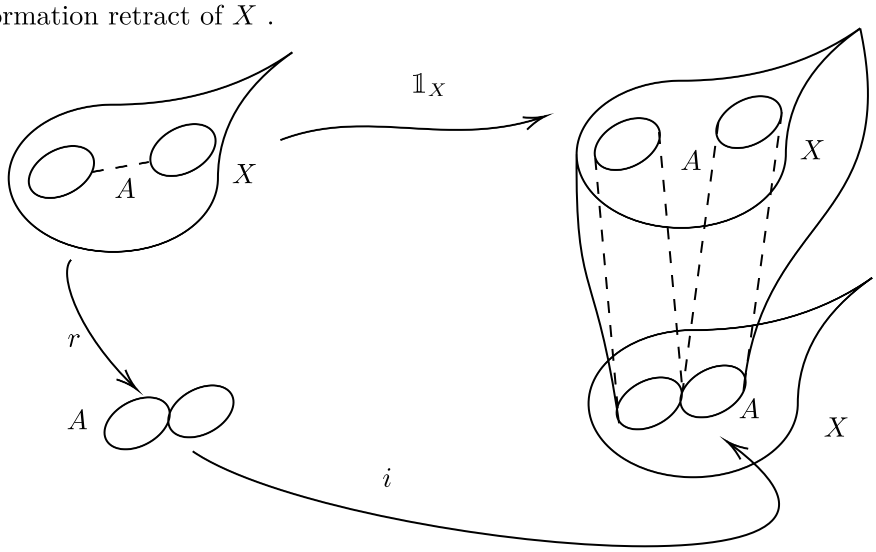
(4) Every connected components of X is closed set.

(5) X is path-connected. $\Longrightarrow$ X is connected.
If $X \subseteq \mathbb{R}^n$, then : X is path-connected. $\Longleftrightarrow$ X is connected.

(6) The Hausdorff property, compactness and connectedness (path-connecedness) are all topological properties.

Deformation retract

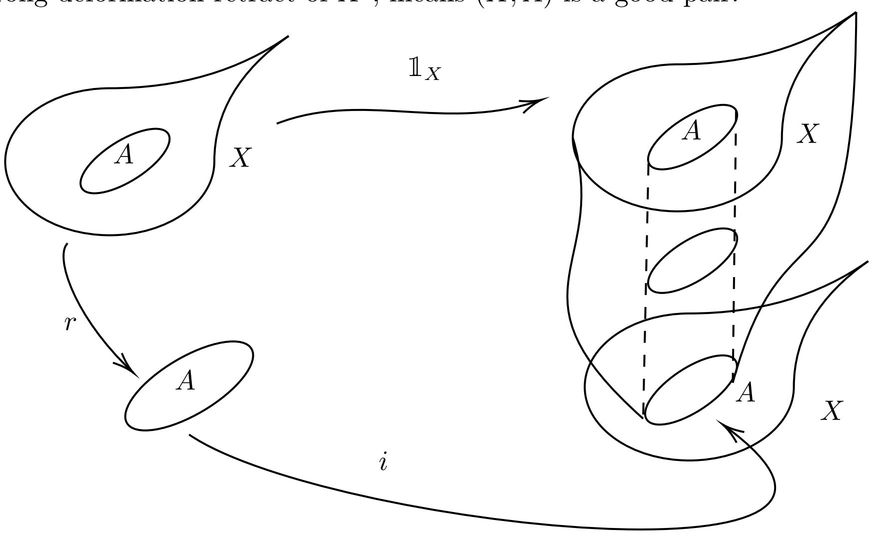
If for the inclusion $i : A \longrightarrow X$, there exists a retract $r : X \longrightarrow A$, such that $i \circ r \simeq \mathbb{1}_X$, then A is a deformation retract of X .



Easily, A is homotopy equivalent to X .

Strong deformation retract

If for the inclusion $i : A \longrightarrow X$, there exists a retract $r : X \longrightarrow A$, such that $i \circ r \simeq \mathbb{1}_X$ rel A , then A is a strong deformation retract of X , means (X, A) is a good pair.

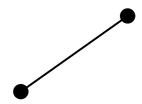


4.2 Chain Complexes

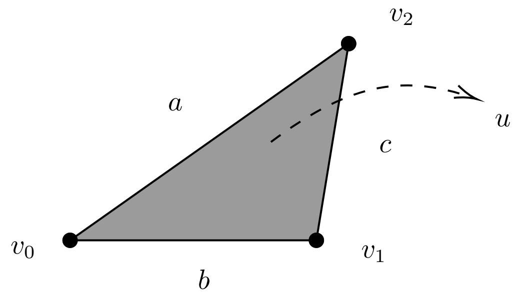
Simplical complex

In one simplex, a vertex only appears one time and every $n+1$ vertices certainly specify an n-simplex.

0-simplex



1-simplex



simplex complex

In a simplical complex X :

(1) For a simplex $\Delta^n = [v_0, \cdots, v_n] \in X$, its faces are also in X .

(2) For two simplices $[v_0, \cdots, v_n]$ and $[u_0, \cdots, u_n]$, their intersection is a face of them.

Simplical chain complex

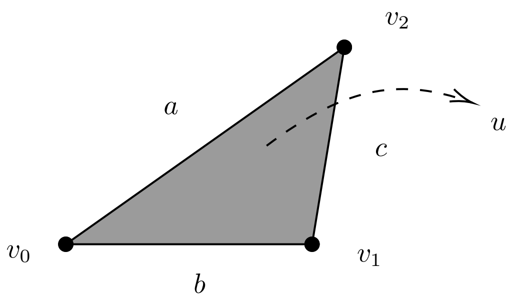
For a topological space X with simplical structure, one has a chain complex $C^{\Delta}_{\bullet}(X)$ called simplical chain complex :

$$\cdots \longrightarrow C^{\Delta}_{n+1}(X) \xrightarrow{\partial_{n+1}} C^{\Delta}_{n}(X) \xrightarrow{\partial_n} C^{\Delta}_{n-1}(X) \longrightarrow \cdots$$

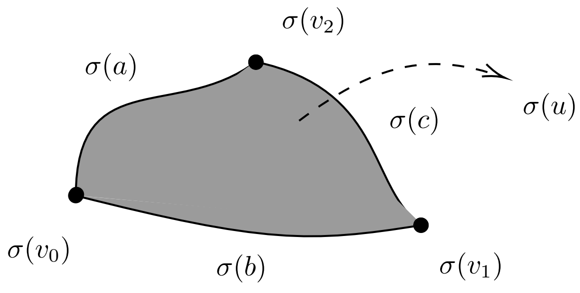
where $\partial_n : [v_0, \cdots, v_n] \longmapsto \sum_i (-1)^i [v_0, \cdots, \hat{v_i}, \cdots, v_n]$.

Singular complex

A singular n-simplex is a continuous map $\sigma : X \longrightarrow |X|$, $[v_0, \cdots, v_n] \longmapsto \sigma[v_0, \cdots, v_n] = (v_0, \cdots, v_n)$.



simplical complex X



singular complex $|X|$

Singular chain complex

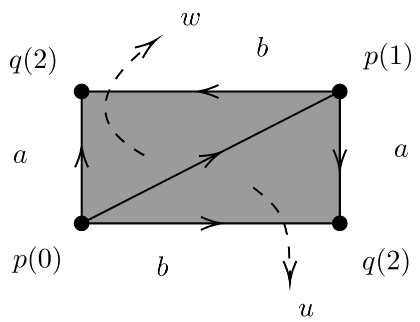
$C_n(X) = \{\sigma_n \mid \sigma_n : [v_0, \cdots, v_n] \longmapsto (v_0, \cdots, v_n) \in |X| \text{ is continuous}\}$ is a group. One has a chain complex $C_\bullet(X)$ called singular chain complex :

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots$$

where $\partial_n : \sigma_n \longmapsto \sigma_n \circ \partial_n = \sum_i (-1)^i \sigma_n|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}$.

Δ-complex (semi-simplical complex)

By identifying the maps in simplical set of different dimensions, one can define the Δ-complex. For a map $\sigma : \Delta^n \longrightarrow X$ in a Δ-complex $|X|$, a vertex could appear many times and every $n+1$ vertices do not necessarily specify an n-simplex.



The simplical set of $\mathbb{RP}^2$:

$\Delta_0 = \{p, q\}$ $\quad$ $\Delta_1 = \{a, b, c\}$ $\quad$ $\Delta_2 = \{u, w\}$

$\partial^0(a) = q$ $\quad$ $\partial^0(b) = q$ $\quad$ $\partial^0(c) = p$

$\partial^1(a) = p$ $\quad$ $\partial^1(b) = p$ $\quad$ $\partial^1(c) = p$

$\partial^0(u) = a$ $\quad$ $\partial^0(w) = b$

$\partial^1(u) = b$ $\quad$ $\partial^1(w) = a$

$\partial^2(u) = c$ $\quad$ $\partial^2(w) = c$

In a Δ-complex $|X|$:

(1) Every map $\sigma|_{\mathrm{Int}(\Delta)} \in |X|$ is injective.

(2) For a map $\sigma : \Delta^n \longrightarrow X$ in $|X|$, the restriction on every face of Δ^n is a map $\sigma' : \Delta^{n-1} \longrightarrow X$ in $|X|$.

(3) $A \subseteq |X|$ is open (closed) . $\Longleftrightarrow$ $\sigma_\alpha^{-1}(A)$ is open (closed) for each α .

(4) For a common face in some (of even different dimensions) simplices, the orientation should be same.

Δ-chain complex

For a simplex set $\{\Delta_0, \cdots, \Delta_n\}$ of X , the maps ∂ specify a Δ-complex inducing a chain complex $\Delta_\bullet(X)$ called Δ-chain complex :

$$\cdots \longrightarrow \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \longrightarrow \cdots$$

where ∂_n has been defined.

Cell complex (CW complex)

(1) Let X^0 be a discrete set of 0-cells.

(2) Let X^n be the n-skeleton formed from X^{n-1} by attaching n-cells e^n_α via maps $S^{n-1} \longrightarrow X^{n-1}$.
Thus $X^n = X^{n-1} \bigsqcup\limits_\alpha D^n_\alpha / (\varphi^n_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}) = X^{n-1} \bigsqcup\limits_\alpha e^n_\alpha$.

(3) Let $X = \bigcup\limits_n X^n (= X^n$ if n is finite) , then X is a cell complex.
X is given the weak topology : $A \subseteq X$ is open (closed) . $\Longleftrightarrow$ For every n , $A \cap X^n$ is open (closed) .

(4) Each cell e^n_α in the cell complex X induces a characteristic map $\widetilde{\varphi^n_\alpha} : D^n_\alpha \longrightarrow X$ which extends the attaching map $\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$. The characteristic map restricting on the interior of D^n_α is a homeomorphism $\varphi_\alpha(\mathrm{Int}(D^n_\alpha)) \cong e^n_\alpha$.

The Euler characteristic

The Euler characteristic of X is $\chi(X) = \sum\limits_n (-1)^n \#\{e^n_\alpha\}$ where e^n_α are the n-cells in X .

Cellular chain complex

$C^{cell}_n(X) = \{e^n_\alpha \mid e^n_\alpha \in X\}$ is a group. One has a chain complex $C^{cell}_\bullet(X)$ called cellular chain complex :

$$\cdots \longrightarrow C^{cell}_{n+1}(X) \xrightarrow{d_{n+1}} C^{cell}_n(X) \xrightarrow{d_n} C^{cell}_{n-1}(X) \longrightarrow \cdots \ .$$

The boundary map d_n is defined as the degree of composition

$$S^{n-1}_\alpha \xrightarrow{\text{attaching map}} X^{n-1} \xrightarrow{\text{delete } X^{n-1} \setminus \{e^{n-1}_\beta, e^0_\gamma\}} S^{n-1}$$

where $e^{n-1}_\beta, e^0_\gamma$ is all the cells that intersect with the closure $\overline{e^n_\alpha}$.

Homotopy equivalence in (Top) , $(\mathbf{Top})^2$ and (Comp)

$f, g : X \longrightarrow Y$ are homotopic, $h : f \simeq g$ is a homotopy.

$\Longleftrightarrow$ There is a continuous $h : X \times I \longrightarrow Y$,

$$\text{such that} \begin{cases} f = h(x,0) : X \times \{0\} \longrightarrow Y \\ g = h(x,1) : X \times \{1\} \longrightarrow Y \end{cases} .$$

$f, g : (X, X') \longrightarrow (Y, Y')$ are homotopic, $h : f \simeq g$ is a homotopy.

$\Longleftrightarrow$ There is a continuous $h : (X \times I, X' \times I) \longrightarrow (Y, Y')$,

$$\text{such that} \begin{cases} f = h(x,0) \colon (X \times \{0\}, X' \times \{0\}) \longrightarrow (Y, Y') \\ g = h(x,1) \colon (X \times \{1\}, X' \times \{1\}) \longrightarrow (Y, Y') \end{cases} .$$

$f_\#, g_\# : C_\bullet(X) \longrightarrow C_\bullet(Y)$ are chain homotopic, denote $f_\# \simeq g_\#$, h is a chain homotopy.

$\Longleftrightarrow$ There is a continuous $h_n : C_n(X) \longrightarrow C_{n+1}(Y)$,

such that $\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_\# - g_\#$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[dl, dashed, "h_n"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, dashed, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"'] & C_n(Y) \arrow[r, "\partial_n"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Proposition

(1) $f_\# : C_\bullet(X) \longrightarrow C_\bullet(Y)$ is a chain homotopy equivalence (the isomorphism in **Ho**(**Comp**)).

$\Longleftrightarrow$ There is a chain map $g_\# \colon C_\bullet(Y) \longrightarrow C_\bullet(X)$, such that $g_\# \circ f_\# \simeq \mathbb{1}_{C_\bullet(X)}$, $f_\# \circ g_\# \simeq \mathbb{1}_{C_\bullet(Y)}$.

(2) h is a contracting chain homotopy (of $(C_\bullet, \partial)$).

$\Longleftrightarrow$ There exist $h_n : C_n \longrightarrow C_{n+1}$, $\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \mathbb{1}_{C_n(X)}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[dl, dashed, "h_n"'] \arrow[d, shift right, "\mathbb{1}"'] \arrow[d, shift left, "0"] & C_{n-1}(X) \arrow[r] \arrow[dl, dashed, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"'] & C_n(X) \arrow[r, "\partial_n"'] & C_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

$\Longleftrightarrow$ There is a chain homotopic $\mathbb{1}_{C_\bullet(X)} \simeq 0_{C_\bullet(X)}$.

(3) $(C_\bullet, \partial)$ has a contracting homotopy.
$\Longleftrightarrow$ Chain complex $C_\bullet$ is chain null-homotopic.
$\Longrightarrow$ $(C_\bullet, \partial)$ is acyclic, that means $H_n(C_\bullet(X)) = 0$.

(4) $(C_\bullet, \partial)$ has a contracting homotopy and $C_\bullet$ is a free chain complex.
$\Longleftrightarrow$ $(C_\bullet, \partial)$ is acyclic, that means $H_n(C_\bullet(X)) = 0$.
$\Longrightarrow$ $H_n(f)$ is an isomorphism implies that $f_\#$ is a chain homotopy equivalence.

(5) the inclusion $i : X' \longrightarrow X$ induces an inclusion $i_\# : C_n(X') \longrightarrow C_n(X)$.

(6) $H_n : \mathbf{Ho}(\mathbf{Comp}) \longrightarrow (\mathbf{Ab})$ is an additive functor which means $H_n(f) + H_n(g) = H_n(f+g)$.

Reduced and relative homology groups

Define the relative chain by

$$C_\bullet(X, X') = C_\bullet(X)/C_\bullet(X') \ ,$$

then one has

$$H_n(C_\bullet(X)/C_\bullet(X')) = H_n(C_\bullet(X, X')) = H_n(X, X') \ .$$

Specially,

$$H_n(X) = H_n(X, \emptyset) \text{ for } n \geq 0 \ ,$$

$$H_n(X) = H_n(X, *) \text{ for } n \geq 1 \ ,$$

$$H_0(X) = H_0(X, *) \oplus \mathbb{Z}$$

One can define the reduced homology group by

$$\widetilde{H_n}(X) = H_n(X, *) \ .$$

Thus

$$H_n(X) = \widetilde{H_n}(X) = H_n(X, *) \text{ for } n \geq 1 \ ,$$

$$H_0(X) = \widetilde{H_0}(X) \oplus \mathbb{Z} = H_0(X, *) \oplus \mathbb{Z} \ .$$

If (U, X') is a good pair where U is an open neighbourhood such that $X' \subseteq U \subseteq X$, then one has

$$H_n(X, X') = H_n(X/X', *) = \widetilde{H_n}(X/X') \ .$$

Thus

$$H_n(X/X') = \begin{cases} H_0(X, X') \oplus \mathbb{Z} & n = 0 \\ H_n(X, X') & n > 0 \end{cases} .$$

When $n = 0$ and X is 0-connected, $H_0(X, *) = \widetilde{H_0}(X) = 0$.
If X is contractible, then $\widetilde{H_n}(X) = 0$ for all $n \geq 0$.

4.3 Tor and Ext

$\mathbf{Hom}_R(B,-)$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Ab})$

Objects : $A \xrightarrow{f} A' \dashrightarrow \mathrm{Hom}_R(B,A) \xrightarrow{\mathrm{Hom}_R(\mathbb{1},f)} \mathrm{Hom}_R(B,A')$.
Morphisms : For R-module homomorphism f , $\mathrm{Hom}_R(\mathbb{1},f)$ is a homomorphism of abelian groups.

$\mathbf{Ext}^n_R(B,-)$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Mod}_R)$

Consider the injective resolution of R-module A

$$0 \longrightarrow A \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_n \longrightarrow \cdots \ ,$$

take the chain complex

$$0 \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_n \longrightarrow \cdots \ ,$$

then take functor $\mathrm{Hom}_R(B,-)$

$$0 \xrightarrow{d_0} \mathrm{Hom}_R(B,J_0) \xrightarrow{d_1} \mathrm{Hom}_R(B,J_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(B,J_n) \xrightarrow{d_{n+1}} \mathrm{Hom}_R(B,J_{n+1}) \longrightarrow \cdots \ ,$$

one can define

$$\mathrm{Ext}^n_R(B,A) = \mathcal{K}er(d_{n+1})/\mathcal{I}m(d_n) \ .$$

$\mathbf{Hom}_R(-,B)$ functor (contravariant) : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Ab})$

Objects : $A \xrightarrow{f} A' \dashrightarrow \mathrm{Hom}_R(A,B) \xleftarrow{\mathrm{Hom}_R(\mathbb{1},f)} \mathrm{Hom}_R(A',B)$.
Morphisms : For R-module homomorphism f , $\mathrm{Hom}_R(f,\mathbb{1})$ is a homomorphism of abelian groups.

$\mathbf{Ext}^n_R(-,B)$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Mod}_R)$

Consider the projective resolution of R-module A

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \ ,$$

take the chain complex

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \ ,$$

then take contravariant functor $\mathrm{Hom}_R(-,B)$

$$0 \xrightarrow{d_0} \mathrm{Hom}_R(P_0,B) \xrightarrow{d_1} \mathrm{Hom}_R(P_1,B) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(P_n,B) \xrightarrow{d_{n+1}} \mathrm{Hom}_R(P_{n+1},B) \longrightarrow \cdots \ ,$$

one can define

$$\mathrm{Ext}^n_R(A,B) = \mathcal{K}er(d_{n+1})/\mathcal{I}m(d_n) \ .$$

$B\otimes_R$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Ab})$

Objects : $A \xrightarrow{f} A' \dashrightarrow B\otimes_R A \xrightarrow{\mathbb{1}\otimes_R f} B\otimes_R A'$.

Morphisms : For R-module homomorphism f , $\mathbb{1}\otimes_R f$ is a homomorphism of abelian groups.

$\mathbf{Tor}_n^R(B,-)$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Mod}_R)$

Consider the projective resolution of R-module A

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 ,$$

take the chain complex

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 ,$$

then take functor $B\otimes_R$

$$\cdots \longrightarrow B\otimes_R P_{n+1} \xrightarrow{\partial_{n+1}} B\otimes_R P_n \longrightarrow \cdots \longrightarrow B\otimes_R P_1 \xrightarrow{\partial_1} B\otimes_R P_0 \xrightarrow{\partial_0} 0 ,$$

one can define

$$\mathrm{Tor}_n^R(B,A) = \mathcal{K}er(\partial_n)/\mathcal{I}m(\partial_{n+1}) .$$

$\otimes_R B$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Ab})$

Objects : $A \xrightarrow{f} A' \dashrightarrow A\otimes_R B \xrightarrow{f\otimes_R \mathbb{1}} A'\otimes_R B$

Morphisms : For R-module homomorphism f , $f\otimes_R \mathbb{1}$ is a homomorphism of abelian groups.

$\mathbf{Tor}_n^R(-,B)$ functor : $(\mathbf{Mod}_R) \longrightarrow (\mathbf{Mod}_R)$

Consider the projective resolution of R-module A

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 ,$$

take the chain complex

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 ,$$

then take functor $\otimes_R B$

$$\cdots \longrightarrow P_{n+1}\otimes_R B \xrightarrow{\partial_{n+1}} P_n\otimes B \longrightarrow \cdots \longrightarrow P_1\otimes_R B \xrightarrow{\partial_1} P_0\otimes_R B \xrightarrow{\partial_0} 0 ,$$

one can define

$$\mathrm{Tor}_n^R(A,B) = \mathcal{K}er(\partial_n)/\mathcal{I}m(\partial_{n+1}) .$$

Proposition

$B\otimes_R$ and $B\otimes_R$ are right exact functor.

$\mathrm{Hom}_R(B,-)$ and $\mathrm{Hom}_R(-,B)$ are left exact sequence.

Proposition

(1) Both $B\otimes$ and $\otimes B$ are additive functors.

(2) If $f_\# : C_\bullet \longrightarrow C'_\bullet$ is a chain map (or chain homotopy equivalence) , then :

$\mathbb{1} \otimes_R f_\#$ and $f_\# \otimes_R \mathbb{1}$ are also chain maps (or chain homotopy equivalences) ,
$\mathrm{Hom}_R(\mathbb{1}, f)$ and $\mathrm{Hom}_R(f, \mathbb{1})$ are also chain maps (or chain homotopy equivalences) .

If $f_\#$ and $g_\#$ are chain homotopic, then :

$f_\# \otimes_R \mathbb{1}$ and $g_\# \otimes_R \mathbb{1}$ are chain homotopic,
$\mathbb{1} \otimes_R f_\#$ and $\mathbb{1} \otimes_R g_\#$ are also chain homotopic.
($\mathbb{1} \otimes_R \partial_{n+1} \circ \mathbb{1} \otimes_R h_n + \mathbb{1} \otimes_R h_{n-1} \circ \mathbb{1} \otimes_R \partial_n = \mathbb{1} \otimes_R f_\# - \mathbb{1} \otimes_R g_\#$)

(3) $0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$ is a short free resolution of abelian group A , then we have :
$H_0(C_\bullet) = A$, $H_0(C_\bullet \otimes B) = \mathrm{Tor}_0^{\mathbb{Z}}(A, B) = A \otimes B$, $H^0(\mathrm{Hom}(C_\bullet, B)) = \mathrm{Ext}^0_{\mathbb{Z}}(A, B) = \mathrm{Hom}(A, B)$.

(4) For two short free resolutions $C_\bullet$ and $C'_\bullet$ of abelian group A , we have $H_n(C_\bullet \otimes B) \cong H_n(C'_\bullet \otimes B)$.

Universal property of tensor products

A tensor product of A and B is an abelian group $T(A \times B)$ together with bilinaer map T satisfying the universal property :

$$\begin{tikzcd}
A \times B \arrow[rr, "T"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft" near start] & T(A \times B) \cong A \otimes B \arrow[dl, dashed, "h"] \\
& C &
\end{tikzcd}$$

For any bilinear map f mapping to any abelian group C , there exists a unique map $h : T(A \times B) \longrightarrow C$ such that the diagram commutes.

This is well defined up to isomorphism of abelian groups.

Proposition

As a $\mathbb{Z}$-module, if abelian group B is torsion-free, then $B\otimes$ and $\otimes B$ are all exact functors.

Proposition

(1) As $\mathbb{Z}$-modules, one has :

$$\mathrm{Hom}(\mathbb{Z}, A) \cong A \ , \ A \otimes \mathbb{Z} \cong \mathbb{Z} \otimes A \cong A \ .$$

$$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)} \ , \ \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(m,n)} \ .$$

$$\mathrm{Hom}(B \oplus B', A) \cong \mathrm{Hom}(B, A) \oplus \mathrm{Hom}(B', A) \ .$$

$$(\bigoplus_k A_k) \otimes B \cong \bigoplus_k (A_k \otimes B) \ , \ A \otimes (\bigoplus_k B_k) \cong \bigoplus_k (A \otimes B_k) \ .$$

(2) $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$, $\mathrm{Tor}^n_{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$.

(3) $\mathrm{Ext}^n_R(R/ < u >, B) = \begin{cases} \{b \mid ub = 0\} & n = 0 \\ B/uB & n = 1 \\ 0 & \text{else} \end{cases}$ where u is not a zero divisor, R is commutative.

$\mathrm{Tor}^R_n(R/ < u >, B) = \begin{cases} B/uB & n = 0 \\ \{b \mid ub = 0\} & n = 1 \\ 0 & \text{else} \end{cases}$ where u is not a zero divisor, R is commutative.

(4) $\mathrm{Ext}(F, B) = 0$ if F is free. $\mathrm{Ext}(B, D)$ if D is divisible.

$\mathrm{Tor}^R_n(A, B) = \mathrm{Tor}^R_n(\mathrm{T}(A), B)$, $\mathrm{Tor}^R_n(A, B) = \mathrm{Tor}^R_n(B, A)$.

(5) For a short exact sequence $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$, one has other exact sequences

$$0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A', B) \longrightarrow \mathrm{Tor}(A'', B) \longrightarrow A \otimes B \longrightarrow A' \otimes B \longrightarrow A'' \otimes B \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{Tor}(B, A) \longrightarrow \mathrm{Tor}(B, A') \longrightarrow \mathrm{Tor}(B, A'') \longrightarrow B \otimes A \longrightarrow B \otimes A' \longrightarrow B \otimes A'' \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{Hom}(A'', B) \longrightarrow \mathrm{Hom}(A', B) \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Ext}(A'', B) \longrightarrow \mathrm{Ext}(A', B) \longrightarrow \mathrm{Ext}(A, B) \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(B, A') \longrightarrow \mathrm{Hom}(B, A'') \longrightarrow \mathrm{Ext}(B, A) \longrightarrow \mathrm{Ext}(B, A') \longrightarrow \mathrm{Ext}(B, A'') \longrightarrow 0 \ .$$

(6)

$$\mathrm{Ext}^n_R(\bigoplus_k A_k, B) \cong \prod_k \mathrm{Ext}^n_R(A_k, B) \ , \ \mathrm{Ext}^n_R(A, \prod_k B_k) \cong \prod_k \mathrm{Ext}^n_R(A, B_k) \ .$$

$$\mathrm{Tor}^R_n(\bigoplus_k A_k, B) \cong \bigoplus_k \mathrm{Tor}^R_n(A_k, B) \ , \ \mathrm{Tor}^R_n(\underrightarrow{colim}_k A_k, B) \cong \underrightarrow{colim}_k \mathrm{Tor}^R_n(A_k, B)$$

4.4 Cochain Complexes

$C_n(-;A)$ **functor :** $\begin{cases} (\mathbf{Top}) \longrightarrow (\mathbf{Ab}) \\ (\mathbf{Top}^2) \longrightarrow (\mathbf{Ab}) \end{cases}$

Objects : $\begin{cases} X \xrightarrow{f} Y \dashrightarrow C_n(X) \otimes A \xrightarrow{f_n \otimes \mathbb{1}} C_n(Y) \otimes A \\ (X, X') \xrightarrow{f} (Y, Y') \dashrightarrow C_n(X, X') \otimes A \xrightarrow{f_n \otimes \mathbb{1}} C_n(Y, Y') \otimes A \end{cases}$

Morphisms :

(1) For continuous map f , $f_n \otimes \mathbb{1}$ is a homomorphism of Abelian groups.

(2) If f is a homotopy equivalence, then $f_n \otimes \mathbb{1}$ is an isomorhism.

(The arguements of $C_n^{\Delta}(-;A)$, $C_n^{cell}(-;A)$ and $\Delta_n(-;A)$ are the same.)

$H_n(-;A)$ **functor :** $\begin{cases} \mathbf{Ho}(\mathbf{Top}) \longrightarrow (\mathbf{Ab}) \\ \mathbf{Ho}(\mathbf{Top}^2) \longrightarrow (\mathbf{Ab}) \\ \mathbf{Ho}(\mathbf{Comp}) \longrightarrow (\mathbf{Ab}) \end{cases}$

Objects : $\begin{cases} X \xrightarrow{f} Y \dashrightarrow H_n(X;A) \xrightarrow{H_n(f)} H_n(Y;A) \\ (X, X') \xrightarrow{f} (Y, Y') \dashrightarrow H_n(X, X';A) \xrightarrow{H_n(f)} H_n(Y, Y';A) \\ C_\bullet \xrightarrow{f} D_\bullet \dashrightarrow H_n(C_\bullet \otimes A) \xrightarrow{H_n(f \otimes \mathbb{1})} H_n(D_\bullet \otimes A) \end{cases}$

Morphisms :

(1) For continuous maps $f \simeq g$, $H_n(f) = H_n(g)$ is a homomorphism of Abelian groups.

(2) If f is a homotopy equivalence, then $H_n(f)$ is an isomorhism.

(3) For chain maps $f \simeq g$, $H_n(f \otimes \mathbb{1}) = H_n(g \otimes \mathbb{1})$ is a homomorphism of Abelian groups.

(4) If f is a chain homotopy equivalence, then $H_n(f \otimes \mathbb{1})$ is an isomorhism.

Cochain groups

Define $C^n(X;A) = \mathrm{Hom}(C_n(X), A)$. Then $(C^\bullet(X;A), d)$ is a cochain complex.

$d_n = \mathrm{Hom}(\partial_n, \mathbb{1}) : \mathrm{Hom}(C_n(X), A) \longleftarrow \mathrm{Hom}(C_{n-1}(X), A)$ is called differential operator.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] & C_n(X) \arrow[r] & C_{n-1}(X) \arrow[r] & \cdots \arrow[r] & C_0(X) \arrow[r] \arrow[d, dashed, "\mathrm{Hom}(-,\mathbb{1})"'] & 0 \\
\cdots & C^{n+1}(X;A) \arrow[l] & C^n(X;A) \arrow[l] & C^{n-1}(X;A) \arrow[l] & \cdots \arrow[l] & C^0(X;A) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$d_n(\gamma) = \gamma \circ \partial_n$ where $\{\gamma : C_{n-1} \longrightarrow A\} \in C^{n-1} = \mathrm{Hom}(C_{n-1}, A)$, and we have $d_n \circ d_{n+1} = 0$.

Then define $H^n(X;A) = H^n(C^\bullet(X;A)) = \mathcal{K}er(d_{n+1})/\mathcal{I}m(d_n)$.

(Differentiating from $H_n(X) = \mathcal{K}er(\partial_n)/\mathcal{I}m(\partial_{n+1})$.)

$$H^n \textbf{ functor : } \begin{cases} \mathbf{Ho}(\mathbf{Top}) \longrightarrow (\mathbf{Ab}) \\ \mathbf{Ho}(\mathbf{Top}^2) \longrightarrow (\mathbf{Ab}) \\ \mathbf{Ho}(\mathbf{Comp}) \longrightarrow (\mathbf{Ab}) \end{cases}$$

$$\text{Objects : } \begin{cases} X \xrightarrow{f} Y \dashrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y) \\ (X, X') \xrightarrow{f} (Y, Y') \dashrightarrow H^n(X, X') \xrightarrow{H^n(f)} H^n(Y, Y') \\ C^\bullet \xrightarrow{f} D^\bullet \dashrightarrow H^n(C^\bullet) \xrightarrow{H^n(f)} H^n(D^\bullet) \end{cases}$$

Morphisms :

(1) For continuous maps $f \simeq g$, $H^n(f) = H^n(g)$ is a homomorphism of Abelian groups.

(2) If f is a homotopy equivalence, then $H^n(f)$ is an isomorphism.

(3) For cahin maps $f \simeq g$, $H^n(f) = H^n(g)$ is a homomorphism of Abelian groups.

(4) If f is a chain homotopy equivalence, then $H^n(f)$ is an isomorphism.

Proposition

(1) If A is a torsion-free Abelian group, then $H_n(X) \otimes A \cong H_n(X; A)$.

(2) Topological space X is said to be of finite type.

$\Longleftrightarrow$ $H_n(X)$ is finitely generated for each n .

$\Longrightarrow$ $T_n(X)$ is the torsion subgroup of $H_n(X)$, then $H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$.

(3) X is a topological space of finite type.

$\Longrightarrow$ There exists non-negative free chain complex $E_\bullet$ chain homotopy equivalent to $C_\bullet(X)$ such that E_n is finitely generated for each n .

$E_\bullet$ is non-negative free chain complex such that E_n is finitely generated for each n .

$\Longrightarrow$ For abelian group A one has $E^\bullet \otimes A = \mathrm{Hom}(E_\bullet, \mathbb{Z}) \otimes A \cong \mathrm{Hom}(E_\bullet, A)$.

Universal coefficients theorem

For a topological space X and an abelian group A (or a free chain complex $E_\bullet$) , there are two split exact sequences :

$$0 \longrightarrow H_n(X) \otimes A \xrightarrow{\omega} H_n(X;A) \longrightarrow \mathrm{Tor}(H_{n-1}(X),A) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H_n(E_\bullet) \otimes A \xrightarrow{\omega} H_n(E_\bullet;A) \longrightarrow \mathrm{Tor}(H_{n-1}(E_\bullet,A) \longrightarrow 0 \ ,$$

where $\omega : H_n(X) \otimes A \longrightarrow H_n(X;A)$, $\langle c \rangle \otimes a \longmapsto \langle c \otimes a \rangle$.

Thus one has

$$H_n(X;A) \cong (H_n(X) \otimes A) \oplus \mathrm{Tor}(H_{n-1}(X),A) \ .$$

If we use the contravariant functor $\mathrm{Hom}(-,A)$ instead of covariant functor $\otimes A$, there are dual version split exact sequences :

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X),A) \longrightarrow H^n(X;A) \xrightarrow{\zeta} \mathrm{Hom}(H_n(X),A) \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(E_\bullet),A) \longrightarrow H^n(\mathrm{Hom}(E_\bullet,A)) \xrightarrow{\zeta} \mathrm{Hom}(H_n(E_\bullet),A) \longrightarrow 0 \ ,$$

where $\zeta : H^n(X;A) \longrightarrow \mathrm{Hom}(H_n(X),A)$, $\langle \gamma \rangle \longmapsto \{\zeta\langle\gamma\rangle : \langle c \rangle \longmapsto \gamma(c)\}$.

Thus one has

$$H^n(X;A) \cong \mathrm{Hom}(H_n(X),A) \oplus \mathrm{Ext}(H_{n-1}(X),A) \ .$$

Cohomology universal coefficients theorem

For a topological space X of finite type and an abelian group A , take free chain complex $\mathrm{Hom}(E_\bullet,\mathbb{Z}) = E^\bullet$ in negative degrees and use the universal coefficients theorem in negative degrees :

$$0 \longrightarrow H^n(E^\bullet) \otimes A \xrightarrow{h} H^n(E^\bullet \otimes A) \longrightarrow \mathrm{Tor}(H^{n+1}(E^\bullet),A) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^n(\mathrm{Hom}(E_\bullet,\mathbb{Z})) \otimes A \xrightarrow{h} H^n(\mathrm{Hom}(E_\bullet,A)) \longrightarrow \mathrm{Tor}(H^{n+1}(\mathrm{Hom}(E_\bullet,\mathbb{Z}),A)) \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^n(\mathrm{Hom}(C_\bullet(X),\mathbb{Z})) \otimes A \xrightarrow{h} H^n(\mathrm{Hom}(C_\bullet(X),A)) \longrightarrow \mathrm{Tor}(H^{n+1}(\mathrm{Hom}(C_\bullet(X),\mathbb{Z})),A) \longrightarrow \ ,$$

and there is a split exact sequence (all the cohomology functors are isomorphic to H^n)

$$0 \longrightarrow H^n(X) \otimes A \xrightarrow{h} H^n(X;A) \longrightarrow \mathrm{Tor}(H^{n+1}(X),A) \longrightarrow 0 \ .$$

where $h : H^n(X) \otimes A \longrightarrow H^n(X;A)$, $\langle \gamma \rangle \otimes a \longmapsto \{\langle \gamma \cdot a \rangle : \langle c \rangle \longmapsto \langle \gamma(c) \rangle \cdot a\}$.

Thus one has

$$H^n(X;A) \cong (H^n(X) \otimes A) \oplus \mathrm{Tor}(H^{n+1}(X),A) \ .$$

Tensor product chain complexes

$(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two non-negative chain complexes, then the tensor product complex $(C \otimes C', \Delta)$ is non-negative with $\Delta \circ \Delta = 0$ defined by

$$(C \otimes C')_n = \bigoplus_{i+j=n} C_i \otimes C'_j \ ,$$

$$\Delta : c_i \otimes c'_j \longmapsto \partial c_i \otimes c'_j + (-1)^i \cdot \partial' c'_j \ .$$

If $f : C_\bullet \longrightarrow C'_\bullet$ and $g : D_\bullet \longrightarrow D'_\bullet$ are chain maps, then $f \otimes g : C \otimes C' \longrightarrow D \otimes D'$ is a chain map between two tensor product chain complexes, defined by $(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j$.

Moreover, if $f \simeq f'$ are chain homotopic, $g \simeq g'$ are chain homotopic, then $f \otimes g \simeq f' \otimes g'$ is chain homotopic.

Thus if $C_\bullet$ and $C'_\bullet$ are chain homotopy equivalent, $D_\bullet$ and $D'_\bullet$ are chain homotopy equivalent, then $f \otimes g : C \otimes C' \longrightarrow D \otimes D'$ is a chain homotopy equivalence.

Algebraic Künneth theorem

For two non-negative free chain complex $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$, there is a split exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \xrightarrow{\omega} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{k+l=n-1} \mathrm{Tor}(H_k(C_\bullet), H_l(D_\bullet)) \longrightarrow 0$$

where $\omega : \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \longrightarrow H_n(C_\bullet \otimes D_\bullet)$, $\langle c_i \rangle \otimes \langle d_j \rangle \longmapsto \langle c_i \otimes d_j \rangle$.

Thus one has

$$H_n(C_\bullet \otimes D_\bullet) \cong (\bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet)) \oplus (\bigoplus_{k+l=n-1} \mathrm{Tor}(H_k(C_\bullet), H_l(D_\bullet))) \ .$$

The Eilenberg-Zilber theorem

For two topological spaces X and Y , there is a natural chain homotopy equivalence $\Omega : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$ which is unique up to chain homotopy. Thus $H_n(C_\bullet(X \times Y)) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y))$. This Ω is called Eilenberg-Zilber morphism.

Künneth formula

For two topological spaces X and Y , there is a split exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\omega} H_n(X \times Y) \longrightarrow \bigoplus_{k+l=n-1} \mathrm{Tor}(H_k(X), H_l(Y)) \longrightarrow 0$$

where $\omega : \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y)$, $\langle c_i \rangle \otimes \langle d_j \rangle \longmapsto \langle c_i, d_j \rangle$.

Thus one has

$$H_n(X \times Y) \cong (\bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)) \oplus (\bigoplus_{k+l=n-1} \mathrm{Tor}(H_k(X), H_l(Y))) \ .$$

Cohomology Künneth formula

For two topological spaces X and Y of finite type, there are two split exact sequences

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(E_\bullet) \otimes H^j(F_\bullet) \xrightarrow{h} H^n(E_\bullet \otimes F_\bullet) \longrightarrow \bigoplus_{k+l=n+1} \mathrm{Tor}(H^k(E_\bullet), H^l(F_\bullet)) \longrightarrow 0 \ ,$$

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(C_\bullet(X)) \otimes H^j(C_\bullet(Y)) \xrightarrow{h} H^n(C_\bullet(X \times Y)) \longrightarrow \bigoplus_{k+l=n+1} \mathrm{Tor}(H^k(C_\bullet(X)), H^l(C_\bullet(Y))) \longrightarrow 0$$

where $E_\bullet$ is chain homotopy equivalent to $C_\bullet(X)$, $F_\bullet$ is chain homotopy equivalent to $C_\bullet(Y)$, then $E_\bullet \otimes F_\bullet$ is chain homotopy equivalent to the tensor product $C_\bullet(X) \otimes C_\bullet(Y)$ and $\Omega : C_\bullet(X) \otimes C_\bullet(Y) \longrightarrow C_\bullet(X \times Y)$ is a chain homotopy equivalence.

Thus one has

$$H^n(X \times Y) \cong (\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)) \oplus (\bigoplus_{k+l=n+1} \mathrm{Tor}(H^k(X), H^l(Y))) \ .$$

Cohomology Künneth formula for PID

For two topological spaces X and Y of finite type and a PID R , one has a split exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X;R) \otimes_R H^j(Y;R) \xrightarrow{\times} H^n(X \times Y;R) \longrightarrow \bigoplus_{k+l=n+1} \mathrm{Tor}_1^R(H^k(X;R), H^l(Y;R)) \longrightarrow 0 \ .$$

The Mayer-Vietoris sequence

For open subsets A, B of X such that $A \cap B \neq \emptyset$ and $A \cup B = X$, one has long exact sequences

$$\cdots H_{n+1}(X) \xrightarrow{\delta} H_n(A \cap B) \xrightarrow{H_n(i_1), H_n(i_2)} H_n(A) \oplus H_n(B) \xrightarrow{H_n(i_A) - H_n(i_B)} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \longrightarrow \cdots \ ,$$

$$\cdots H^{n-1}(A \cap B) \xrightarrow{\delta} H^n(X) \longrightarrow H^n(A) \oplus H^n(B) \longrightarrow H^n(A \cap B) \xrightarrow{\delta} H^{n+1}(X) \longrightarrow \cdots \ .$$

For the reduced version, one has

$$\cdots \widetilde{H}_{n+1}(X) \xrightarrow{\delta} \widetilde{H}_n(A \cap B) \xrightarrow{\widetilde{H}_n(i_1), \widetilde{H}_n(i_2)} \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \xrightarrow{\widetilde{H}_n(i_A) - \widetilde{H}_n(i_B)} \widetilde{H}_n(X) \xrightarrow{\delta} \widetilde{H}_{n-1}(A \cap B) \longrightarrow \cdots \ ,$$

$$\cdots \widetilde{H}^{n-1}(A \cap B) \xrightarrow{\delta} \widetilde{H}^n(X) \longrightarrow \widetilde{H}^n(A) \oplus H^n(B) \longrightarrow \widetilde{H}^n(A \cap B) \xrightarrow{\delta} \widetilde{H}^{n+1}(X) \longrightarrow \cdots \ .$$

4.5 Homology and Cohomology Rings

Graded rings

A graded ring is a ring $R = \bigoplus_n R_n$ together with Abelian subgroups R_n for $n \geq 0$ (the direct sum decomposition of R as an Abelian group, R_n is not a subring in general) such that $R_m \cdot R_n \subseteq R_{m+n}$.

R_0 is always a subring, and the other R_n is naturally a R_0-module since $R_0 \cdot R_n \subseteq R_n$.
If for $r_m \in R_m$, $r_n \in R_n$, one has $r_m \cdot r_n = (-1)^{mn} \cdot r_n \cdot r_m$, then R is a commutative graded ring.
For any ring R , it can be made into a graded ring by taking $R_0 = R$ and $R_n = 0$ for $n > 0$.

Graded ring homomorphisms

A graded ring homomorphism $\prod_n f_n : \bigoplus_n R_n \longrightarrow \bigoplus_n S_n$ is a ring homomorphism with group homomorphisms f_n such that $f_n(R_n) \subseteq S_n$ for $n \geq 0$.
For any element $x \in R$, on has a unique decomposition $x = \sum r_i$ where $r_i \in R_i$ since it is a direct sum decomposition. $x \in \bigoplus_n R_n$ is a degree d homogeneous element if $x = \sum r_i = r_d$ where $r_d \in R_d$, denote $\deg(x) = d$.

Thus the zero 0 is in every degree since every R_n is an Abelian subgroup, and the identity $\mathbb{1}$ (if it exists in ring R) is in degree 0 , since R_0 is a subring of R .

The ideal generated by the homogeneous elements $I =< r_{d1}, \cdots, r_{dn} >$ is a homogeneous ideal. For a homogeneous ideal I , let $I_n = I \cap R_n$ and $I = \bigoplus_n I_n$, then R_n/I_n is an Abelian group.
R/I is also a graded ring and one has $R/I = \bigoplus_n (R_n/I_n)$.
As R-module, one has $R_n/I_n = (R_n + I)/I$, thus $R/I = \bigoplus_n ((R_n + I)/I)$.

Graded modules

A graded module is a R-module M together with the direct sum decomposition of M such that $M = \bigoplus_n M_n$ and $R_m \cdot M_n \subseteq M_{m+n}$.
The graded module homomorphism between $A = \bigoplus_n A_n$ and $B = \bigoplus B_n$ is defined to be $\mathrm{Hom}(A, B) = \{f \mid f_n : A_n \longrightarrow B_n\}$ or $\mathrm{Hom}(A, B)_k = \{f \mid f_n : A_n \longrightarrow B_{n+k}\}$.
The tensor product of graded modules can be defined as $(A \otimes B)_n = \bigoplus_{p+q=n} (A_p, B_q)$.

Cross products

For $\{\alpha : C_m(X) \longrightarrow G_1\} \in C^m(X; G_1)$, $\{\beta : C_n(Y) \longrightarrow G_2\} \in C^m(Y; G_2)$, by the Eilenberg-Zilber theorem, one has the commutative diagram.

$$\begin{tikzcd}
C_\bullet(X \times Y) \arrow[r, dashed, "\Omega"] \arrow[d, "\alpha \times \beta"'] & C_\bullet(X) \otimes C_\bullet(Y) \arrow[d, "\alpha \otimes \beta"] \\
G_1 \otimes G_2 \arrow[r, dashed] & G_1 \otimes G_2
\end{tikzcd}$$

Then define

$$\alpha \times \beta = (\alpha \otimes \beta) \circ \Omega \ .$$

One has

$$d(\alpha \times \beta) = d\alpha \times \beta + (-1)^m \alpha \times d\beta \ .$$

By this property, define the cross product of the class $\langle\alpha\rangle \in H^m(X; G_1)$ and the class $\langle\beta\rangle \in H^n(Y; G_2)$ to be the class $\langle\alpha\rangle \times \langle\beta\rangle \in H^{m+n}(X \times Y; G_1 \otimes G_2)$.

Cup products

Consider the functor H^{m+n} with the map $\Delta : X \longrightarrow X \times X$, with the cross product of $H^m(X; G_1)$ and $H^n(X; G_2)$ one has the commutative diagram :

$$\begin{tikzcd}
H^m(X; G_1) \otimes H^n(X; G_2) \arrow[r, "\times"] \arrow[dr, "\smile"'] & H^{m+n}(X \times X; G_1 \otimes G_2) \arrow[d, "H^{m+n}(\Delta)"] \arrow[dr, phantom, "\circlearrowleft"] & X \times X \arrow[l, dashed, "H^{m+n}"'] \\
\arrow[ur, phantom, "\circlearrowleft"] & H^{m+n}(X; G_1 \otimes G_2) & X \arrow[l, dashed, "H^{m+n}"] \arrow[u, "\Delta"']
\end{tikzcd}$$

Then define the cup product of $\langle\alpha\rangle \in H^m(X; G_1)$ and $\langle\beta\rangle \in H^n(X; G_2)$ to be

$$\langle\alpha\rangle \smile \langle\beta\rangle = H^{m+n}(\Delta)(\langle\alpha\rangle \times \langle\beta\rangle) \ .$$

Proposition

(1) For $\langle x_1\rangle, \langle x_2\rangle \in H^m(X;G_1)$ and $\langle y_1\rangle, \langle y_2\rangle \in H^n(Y;G_2)$, one has :

$$(\langle x_1\rangle + \langle x_2\rangle) \times \langle y\rangle = \langle x_1\rangle \times \langle y\rangle + \langle x_2\rangle \times \langle y\rangle \ ,$$

$$\langle x\rangle \times (\langle y_1\rangle + \langle y_2\rangle) = \langle x\rangle \times \langle y_1\rangle + \langle x\rangle \times \langle y_2\rangle \ ,$$

$$(\langle x_1\rangle + \langle x_2\rangle) \smile \langle x\rangle = \langle x_1\rangle \smile \langle x\rangle + \langle x_2\rangle \smile \langle x\rangle \ ,$$

$$\langle x\rangle \smile (\langle x_1\rangle + \langle x_2\rangle) = \langle x\rangle \smile \langle x_1\rangle + \langle x\rangle \smile \langle x_2\rangle \ .$$

(2) For $H^m(f) : H^m(X;G_1) \longrightarrow H^m(X';G_1)$ and $H^n(g) : H^n(Y;G_2) \longrightarrow H^n(Y';G_2)$, one has

$$H^m(f)\langle x\rangle \times H^n(g)\langle y\rangle = H^{m+n}(f \times g)(\langle x\rangle \times \langle y\rangle) \ .$$

Which means this diagram commutes :

$$\begin{tikzcd}
H^m(X;G_1) \otimes H^n(Y;G_2) \arrow[r, dashed, "\times"] \arrow[d, "H^m(f) \otimes H^n(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{m+n}(X \times Y;G_1 \otimes G_2) \arrow[d, "H^{m+n}(f \times g)"] \\
H^m(X';G_2) \otimes H^n(Y';G_2) \arrow[r, dashed, "\times"'] & H^{m+n}(X' \times Y';G_1 \otimes G_2)
\end{tikzcd}$$

(3) For $H^m(f) : H^m(X;G_1) \longrightarrow H^m(X';G_1)$ and $H^n(f) : H^n(X;G_2) \longrightarrow H^n(X';G_2)$, one has

$$H^m(f)\langle x_1\rangle \smile H^n(g)\langle x_2\rangle = H^{m+n}(f \smile g)(\langle x_1\rangle \smile \langle x_2\rangle) \ .$$

Which means this diagram commutes :

$$\begin{tikzcd}
H^m(X;G_1) \otimes H^n(X;G_2) \arrow[r, dashed, "\smile"] \arrow[d, "H^m(f) \otimes H^n(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{m+n}(X;G_1 \otimes G_2) \arrow[d, "H^{m+n}(f \smile g)"] \\
H^m(X';G_2) \otimes H^n(X';G_2) \arrow[r, dashed, "\smile"'] & H^{m+n}(X';G_1 \otimes G_2)
\end{tikzcd}$$

Cup products on the chain level

Take $\alpha \in C^k(X;R)$ and $\beta \in C^l(X;R)$ where R is a ring, $\alpha : \Delta_k \longrightarrow R$, $\beta : \Delta_l \longrightarrow R$. The cup product $\alpha \smile \beta \in C^{k+l}(X;R)$ is given by

$$\alpha \smile \beta : (v_0, \cdots, v_{k+l}) \longmapsto \alpha(v_0, \cdots, v_k) \cdot \beta(v_k, \cdots, v_{k+l}) \ .$$

One has $d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^k \alpha \smile d\beta \in C^{k+l+1}(X;R)$.

Since $\sum\limits_i \alpha_i \smile \sum\limits_j \beta_j = \sum\limits_{i,j} \alpha_i \smile \beta_j$, define the cochain $\gamma : \Delta_0 = \{(v_0) \mid v_0 = * \in X\} \longrightarrow R$, $(v_0) \longmapsto \mathbb{1}$, then $C^*(X;R)$ becomes a graded ring (This ring structure does not restrict the homotopy axiom, and it is not graded commutative) .

$C^*(-;R)$ **functor :** $(\textbf{Top}) \longrightarrow (\textbf{GrRg})$

Objects : $X \xrightarrow{f} Y \dashrightarrow C^*(X;R) \xrightarrow{C^*(f)} C^*(Y;R)$.
Morphisms : For continuous map f , $C^*(f)$ is a graded ring homomorphism.

$H^*(-;R)$ **functor :** $\textbf{Ho}(\textbf{Top}) \longrightarrow (\textbf{GrRg})$

Objects : $X \xrightarrow{f} Y \dashrightarrow H^*(X;R) \xrightarrow{H^*(f)} H^*(Y;R)$. (graded commutative if R is commutative)
Morphisms :
(1) For continuous maps $f \simeq g$, $H^*(f) = H^*(g)$ is a graded ring homomorphism.
(2) If f is a homotopy equivalence, then $H^*(f)$ is a graded ring isomorphism.

Relative cup products

By the definition, one has relative cup products

$$H^m(X;G_1) \otimes H^n(X,A;G_2) \xrightarrow{\smile} H^{m+n}(X,A;G_1 \otimes G_2) \ ,$$

$$H^m(X,A;G_1) \otimes H^n(X;G_2) \xrightarrow{\smile} H^{m+n}(X,A;G_1 \otimes G_2) \ ,$$

$$H^m(X,A;G_1) \otimes H^n(X,A;G_2) \xrightarrow{\smile} H^{m+n}(X,A;G_1 \otimes G_2) \ ,$$

if $A, B \subseteq X$ are open subsets or subcomplexes, then one has :

$$H^m(X,A;G_1) \otimes H^n(X,B;G_2) \xrightarrow{\smile} H^{m+n}(X,A \cup B;G_1 \otimes G_2) \ .$$

Proposition

(1) $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/ < \alpha^{n+1} >$, $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ where $\deg(\alpha) = 1$.

(2) $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/ < \alpha^{n+1} >$, $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$ where $\deg(\alpha) = 2$.
$H^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/ < \alpha^{n+1} >$, $H^*(\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$ where $\deg(\alpha) = 4$.

(3) $H^*(\mathbb{R}\mathrm{P}^{2k};\mathbb{Z}) = \mathbb{Z}[\alpha]/ < 2\alpha, \alpha^{k+1} >$ where $\deg(\alpha) = 2$.
$H^*(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z}) = \mathbb{Z}[\alpha,\beta]/ < 2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta >$ where $\deg(\alpha) = 2$, $\deg(\beta) = 2k+1$.
$H^*(\mathbb{R}\mathrm{P}^{\infty};\mathbb{Z}) = \mathbb{Z}[\alpha]/ < 2\alpha >$ where $\deg(\alpha) = 2$.

Cross products for homology

For $(v_0, \cdots, v_m) \in C_m(X)$, $(w_0, \cdots, w_n) \in C_n(Y)$, by the barycentric subdivision, one can make $((v_0, \cdots, v_m), (w_0, \cdots, w_n))$ to be a singular $(m+n)$-simplex $(u_0, \cdots, u_{m+n})$. By the Eilenberg-Zilber theorem, one can define

$$(u_i) = (v_i) \times (w_i) = \Omega^{-1}((v_i) \otimes (w_i)) \ .$$

One has

$$\partial(u_0, \cdots, u_{m+n}) = \partial(v_0, \cdots, v_m) \times (w_i, \cdots, w_n) + (-1)^m (v_0, \cdots, v_m) \times \partial(w_i, \cdots, w_n) \ .$$

By this property, define the cross product of the class $\langle v\rangle \in H_m(X)$ and the class $\langle w\rangle \in H_n(Y)$ to be the class $\langle v\rangle \times \langle w\rangle \in H_{m+n}(X \times Y)$.

Pontryagin products

Consider the functor H_{m+n} with the H-space map $\mu : X \times X \longrightarrow X$, with the cross product of $H_m(X)$ and $H_n(X)$ one has the commutative diagram :

$$\begin{tikzcd}
H_m(X) \otimes H_n(X) \arrow[r, "\times"] \arrow[dr, "\text{Pontryagin}"'] & H_{m+n}(X \times X) \arrow[d, "H_{m+n}(\mu)"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & X \times X \arrow[l, dashed, "H_{m+n}"'] \arrow[d, "\mu"] \\
 & H_{m+n}(X) & X \arrow[l, dashed, "H_{m+n}"]
\end{tikzcd}$$

Then define the Pontryagin product of $\langle x\rangle \in H_m(X)$ and $\langle y\rangle \in H_n(X)$ to be

$$\langle x\rangle \cdot \langle y\rangle = H_{m+n}(\langle x\rangle \times \langle y\rangle) \ .$$

Cap products

For $c = (c_0, \cdots, c_{m+n}) \in C_{m+n}(X;R)$, $\{\alpha : \Delta_n \longrightarrow R\} \in C^n(X;R)$, define the cap product to be

$$\alpha \frown c = \alpha(c_0, \cdots, c_n) \cdot (c_n, \cdots, c_{m+n}) \in C_m(X;R) \ .$$

Proposition

For $\alpha \in C^n(X;R)$, $\beta \in C^p(X;R)$, $c \in C_{m+n}(X;R)$ and continuous map $f : X \longrightarrow Y$, $\gamma \in C^n(Y;R)$, one has :

$$\partial(\alpha \frown c) = (-1)^n(\alpha \frown \partial c - d\alpha \frown c) \ ,$$

$$\beta(\alpha \frown c) = (\alpha \smile \beta)(c) \text{ if } p = m \ ,$$

$$\beta \frown (\alpha \frown c) = (\alpha \smile \beta) \frown c \text{ if } p \leq m \ .$$

$$f_\#(f^\#(\gamma) \frown c) = \gamma \frown f_\#(c) \ .$$

Algebras

An R-module $M = \{\sum r_i m_j \mid r_i \in R \ , \ m_j \in M\}$ with product $m_j m_k$ is an R-algebra.
A ring R is always an R-algebra.

If $x_i x_j = -x_j x_i$, $x_i^2 = 0$, then this algebra is called an exterior algebra $\Lambda_R[x_1, x_2, \cdots]$.

If $(m!)x_i^m (n!)x_i^n = (m+n)!x_i^{m+n}$, then this algebra is called a divided power algebra $\Gamma_R[x_1, x_2, \cdots]$.

James reduced product

For a based space X , let $X^k = X \times \cdots \times X$ (k copies) , define

$$J(X) = \bigsqcup_k X^k / \sim \ \text{ where } (x_1, \cdots, x_i, \cdots, x_k) \sim (x_1, \cdots, \hat{x}_i, \cdots, x_k) \text{ if } x_i = * \ .$$

Notice that $J_m(X) = \{(x_1, \cdots, x_k) \mid k \leq m\}$ is a CW complex, thus $J(X) = \bigcup_m J_m(X)$ is a CW complex.

Proposition

(1) $H^*(J(S^{2n});\mathbb{Q}) = \mathbb{Q}[x]$ where $\deg(x) = 2n$.

(2) $H^*(J(S^{2n});\mathbb{Z}) = \Gamma[x]$ where $\deg(x) = 2n$.
$H^*(J(S^{2n+1});\mathbb{Z}) \cong H^*(S^n;\mathbb{Z}) \otimes H^*(J(S^{2n});\mathbb{Z}) = \Lambda[\alpha] \otimes \Gamma[\beta]$ where $\deg(\alpha) = n$, $\deg(\beta) = 2n$.

(3) $J(X)$ is an associative H-space.
Since $J(X)$ is a CW complex, the associative H-space is H-group.

4.6 The Orientability and Duality

Proposition

Let X be a topological manifold of dimension n , A be an Abelian group. For any point $X \in M$ one has

$$H_k(M, M \setminus \{x\}; A) = \begin{cases} A & k = n \\ 0 & \text{else} \end{cases} .$$

Local R-orientations

Let R be a commutative ring with identity, then for any point $x \in M$, $H_n(M, M \setminus \{x\}; R)$ is a free R-module with rank 1 . One has $H_n(M, M \setminus \{x\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; R) = R$.

A generator of $H_n(M, M \setminus \{x\}; R)$ is called a local R-orientation at x .

R-orientations

For a closed subset $K \subseteq M$, if there is a continuous function $f : K \longrightarrow R$ such that $f(k) \in R = H_n(M, M \setminus \{x\}; R)$ is a generator for each $k \in K$, then M is local orientable along K .

If $K = M$, then M is called R-orientable. M with an R-orientation f is called oriented.

Proposition

Let M be a compact connected manifold of dimension n .

If M is not orientable, then $H_n(M) = 0$.

If M is orientable, then $H_n(M) = \mathbb{Z}$, and for each $x \in M$, one has the isomorphism

$$H_n(M) \cong H_n(M, M \setminus \{x\}) .$$

Poincaré Duality

Let M be an oriented closed topological manifold of dimension n with fundamental class $\langle o_M \rangle \in H_n(M)$, then one has isomorphism

$$H^k(M; \mathbb{Z}) \longrightarrow H_{n-k}(M) \ , \ \langle c \rangle \longmapsto \langle c \rangle \frown \langle o_M \rangle \ .$$

4.7 Generalised Homology and Cohomology

$\mathcal{R}$ functor : $(\mathbf{Top})^2 \longrightarrow (\mathbf{Top})^2$

Define the functor by $\mathcal{R} : (X, X') \longmapsto (X', \emptyset)$ and the morphisms are well defined.
Then the connecting map δ is a natural transform $\delta : H_n \longrightarrow H_{n-1} \circ \mathcal{R}$.

Homology theory (Eilenberg-Steenrod Axioms)

A homology theory is a sequence $\mathcal{H}_n : (\mathbf{Top}^2) \longrightarrow (\mathbf{Ab})$ of functors for $n \geq 0$ and a sequence $\delta_n : \mathcal{H}_n \longrightarrow \mathcal{H}_{n-1} \circ \mathcal{R}$ of natural transforms for $n \geq 1$, satisfying these six axioms below.

(1) The homotopy axiom :
If $f, g : (X, X') \longrightarrow (Y, Y')$ are homotopic rel X' , then $\mathcal{H}_n(f) = \mathcal{H}_n(g)$ for $n \geq 0$.

(2) The exact sequence axiom :
For pair (X, X') with inclusions $(X', \emptyset) \longrightarrow (X, \emptyset)$ and $(X, \emptyset) \longrightarrow (X, X')$, there is a long exact sequence

$$\cdots \longrightarrow \mathcal{H}_n(X') \longrightarrow \mathcal{H}_n(X) \longrightarrow \mathcal{H}_n(X, X') \xrightarrow{\delta} \mathcal{H}_{n-1}(X') \longrightarrow \cdots ,$$

where $\mathcal{H}_n(X, \emptyset) = \mathcal{H}_n(X)$.

(3) The excision axiom :
For pair (X, X') with subset $U \subseteq X$ such that $\overline{U} \subseteq \mathrm{Int}(X')$, the inclusion $(X \setminus U, X' \setminus U) \longrightarrow (X, X')$ induces an isomorphism : $\mathcal{H}_n(X \setminus U, X' \setminus U) \cong \mathcal{H}_n(X, X')$ for $n \geq 0$.

(4) The dimension axiom :
$\mathcal{H}_0(*) = \mathbb{Z}$, $\mathcal{H}_n(*) = 0$ for $n \geq 1$.

(5) The additive axiom :
For a family (X_k, X'_k) of pairs, there is an isomorphism :

$$\bigoplus_k \mathcal{H}_n(X_k, X'_k) \cong \mathcal{H}_n(\bigsqcup_k X_k, \bigsqcup_k X'_k) \text{ for } n \geq 0 .$$

(6) The weak equivalence axiom:
If $f : (X, X') \longrightarrow (Y, Y')$ is a weak equivalence, then $\mathcal{H}_n(f) : \mathcal{H}_n(X, X') \longrightarrow \mathcal{H}_n(Y, Y')$ is an isomorphism for all $n \geq 0$.

Generalised homology theorem

A generalised homology theorem is a homology theorem satisfying these axioms except the dimension axiom such as topological K-theory and symplectic homology.

Baby uniqueness theorem

Let $(H_\bullet, \delta)$ and $(K_\bullet, \epsilon)$ satisfy the first four axioms (homotopy, exact sequence, excision and dimension) , suppose $\Phi_n : \mathcal{H}_n \longrightarrow \mathcal{K}_n$ and $\Phi'_n : \mathcal{H}_{n-1} \circ \mathcal{R} \longrightarrow \mathcal{K}_{n-1} \circ \mathcal{R}$ is a sequence of natural transformations such that $\epsilon_n \circ \Phi_n = \Phi'_n \circ \delta_n$ and $\Phi_0(pt) : \mathcal{H}_0(pt) \longrightarrow \mathcal{K}_0(pt)$ is an isomorphism.

Then there is an isomorphism $\Phi_n(X, X') : \mathcal{H}_n(X, X') \longrightarrow \mathcal{K}_n(X, X')$ for finite cell complex X and subcomplex X' .

Cohomology theory with coefficient A (The Eilenberg-Steenrod Axioms)

A cohomology theory with coefficient A is a sequence $\mathcal{H}^n : (\mathbf{Top}^2) \longrightarrow (\mathbf{Ab})$ of contravariant functors for $n \geq 0$ and a sequence $\delta_n : \mathcal{H}^n \circ \mathcal{R} \longrightarrow \mathcal{H}^{n+1}$ of natural transforms for $n \geq 0$, satisfying these six axioms below.

(1) The homotopy axiom :

If $f, g : (X, X') \longrightarrow (Y, Y')$ are homotopic rel X' , then $\mathcal{H}^n(f) = \mathcal{H}^n(g)$ for $n \geq 0$.

(2) The exact sequence axiom :

For pair (X, X') with inclusions $(X', \emptyset) \longrightarrow (X, \emptyset)$ and $(X, \emptyset) \longrightarrow (X, X')$, there is a long exact sequence

$$\cdots \longrightarrow \mathcal{H}^n(X'; A) \xrightarrow{\delta_n} \mathcal{H}^{n+1}(X, X'; A) \longrightarrow \mathcal{H}^{n+1}(X; A) \longrightarrow \mathcal{H}^{n+1}(X'; A) \longrightarrow \cdots \ ,$$

where $\mathcal{H}^n(X, \emptyset; A) = \mathcal{H}^n(X; A)$.

(3) The excision axiom :

For pair (X, X') with subset $U \subseteq X$ such that $\overline{U} \subseteq \mathrm{Int}(X')$, the inclusion $(X \setminus U, X' \setminus U) \longrightarrow (X, X')$ induces an isomorphism : $\mathcal{H}^n(X \setminus U, X' \setminus U; A) \cong \mathcal{H}^n(X, X'; A)$ for $n \geq 0$.

(4) The dimension axiom :

$\mathcal{H}^0(*) = A$, $\mathcal{H}^n(*) = 0$ for $n \geq 1$.

(5) The additive axiom :

For a family (X_k, X'_k) of pairs, there is an isomorphism :

$$\bigoplus_k \mathcal{H}^n(X_k, X'_k; A) \cong \mathcal{H}^n(\bigsqcup_k X_k, \bigsqcup_k X'_k; A) \text{ for } n \geq 0 \ .$$

(6) The weak equivalence axiom:

If $f : (X, X') \longrightarrow (Y, Y')$ is a weak equivalence, then $\mathcal{H}^n(f) : \mathcal{H}^n(X, X'; A) \longrightarrow \mathcal{H}^n(Y, Y'; A)$ is an isomorphism for all $n \geq 0$.

Generalised cohomology theorem

A generalised homology theorem is a homology theorem satisfying these axioms except the dimension axiom such as topological K-theory.