Preface J.H. Sikkema

After three hours of solving problems, you have finally arrived at the solutions. For both, we would like to thank the contributors; they did an excellent job at providing us with challenging questions (as you have noticed) and elegant answers. Also we would like to express our gratitude to our sponsors, without whom non of this would have been possible. We would especially like to thank Prof. Dr. Knoester for his efforts to provide us with a rare insight into the proceedings of one of our most acclaimed national research institutes. We would like to apologize to him for mispelling his name earlier and are grateful for his clemency in this matter. Finally our thanks go out to all contestants who were kind enough to make their way to Groningen and competed in a very sportsmanlike manner. We hope that you have enjoyed working on physics problems in this alternative way, and that Pion 2003 contributed to your enjoyment of the wonderful field of physics.

It was truly a pleasure to organize Pion 2003 and have you all in Groningen,

Jetze Sikkema,
Chairman,
Pion 2003

1. Blown away! E.A. Bergshoeff

If a tennis ball falls back on the boat, nothing happens, because the initial and final conditions are identical. These cases can be neglected, and we turn to those where the ball does in fact not fall back to the boat. Assume that you would throw a ball with momentum $+p$ in the direction of the wall (*i.e.* the sail). From conservation of momentum, it is known that you and the boat would gain a countering momentum of $-p$, where the minus sign tells us that it is directed away from the wall. The ball, with momentum $+p$, subsequently hits the wall and bounces back with momentum $-p$. Here, the minus sign is being introduced, because the direction of the momentum has been reversed. From conservation of momentum once again, it is known that the boat will receive a momentum $+2p$ in the forward direction. In total, the boat first gets a countering momentum $-p$ and subsequently a momentum contribution of $+2p$ – in total a gaining of momentum of $+p$. The boat will start moving!

2. Neutron Star N. Kalantar

The first term is the volume effect of a sphere of nuclei that are bound together by a short-ranged force. The larger the volume, the larger the binding energy. If the force had been long-ranged, the binding energy would have been proportional to the total number of pairs in the core, *i.e.* proportional to A^2. The short range of the transaction, on the other hand, translates into a closest neighbour interaction – giving a binding energy proportional to A. The second term is the contribution from surface effects. The third terms originates in the Coulomb repulsion of the protons. The fourth and final term arises from the asymmetry in the number of protons and neutrons in the core.
It is rather straightforward to calculate the gravitational energy inside a sphere with mass M. A short derivation is given by:

$$BE_{\text{grav}} = G \int \frac{1}{r}\ \rho \frac{4}{3}\pi r^3\ \rho 4\pi r^2 \mathrm{d}r,$$

with $\rho = \frac{M}{4/3\pi R^3}$. This totals to $\frac{3}{5}G\frac{M^2}{R}$ and it should be added to the binding energy. Together with the fact that $R = r_0 A^{1/3}$, we have (for a normal star):

$$BE(A,Z) = a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3} - a_a(A-2Z)^2 A^{-1} + \frac{3}{5}\frac{G}{r_0}M^2 A^{-1/3}.$$

For a neutron star, we have that $Z = 0$. Also, for a large sphere, one can assume the surface term to be much smaller than the volume term. The star is least bound if $BE(A,Z)$ is only slightly larger than zero. This gives us the following:

$$a_v A - a_a A + \frac{3}{5}G\left(\frac{M^2}{r_0}A^{-1/3}\right) = 0.$$

With $M = AM_n$ and the known coefficients, we get

$$\frac{3}{5}G(M_n^2/r_0)A^{2/3} = 7.4\ \text{ MeV}.$$

Combining all the data, we find for the minimal number of neutrons, the radius and the mass:

$$A \approx 5 \cdot 10^{55}$$

$$R \approx 4.3\ \text{ km}$$

$$M \approx 0.04 M_{\odot},$$

with the mass of the sun being $M_{\odot} = 1.99 \cdot 10^{30}$ kg. The radius of a neutron star is a factor 3 larger than the radius calculated here, corresponding to the total mass being a factor 27 larger. From the fact that this approach to the problem gives an approximation of the correct order of magnitude (taking into account the scale of the problem, with $A \approx 250$ to $A \approx 5 \cdot 10^{55}$), shows that the physics of neutron stars is indeed dominated by the gravitational and strong forces.

3. Solar Birth M. Spaans

(a) Take a line from the observer (p) to the center of the Sun (c), and a second one from p to an arbitrary point on the solar disc p'. The lines $p-c$ en $p-p'$ will now define an angle θ, and the flux associated with this angle is given by

$$F_\nu = \int I_\nu \cos\theta \mathrm{d}\Omega = I_\nu \int_0^{2\pi} \mathrm{d}\phi \int_0^\theta \sin\theta\cos\theta \mathrm{d}\theta,$$

for every frequency ν, an angle ϕ in the plane perpendicular to $p-c$, and with the intensity I_ν having been pulled through the integral, since the surface temperature of the Sun is constant. The angular size of the Sun in the sky can be put θ_h, where $\sin\theta_h = r_\odot/d$, d being the distance between the observer and the Sun. With this, we find:

$$F_\nu = \pi I_\nu(1-\cos^2\theta_h) = \pi I_\nu(r_\odot/d)^2.$$

It is also possible to note immediately that $\mathrm{d}\Omega = \mathrm{d}A/d^2$, with $\mathrm{d}A = \pi r_\odot^2$ being the surface area of the solar disc.

(b) We have a gravitational potential $\phi = GM(r)m_\mathrm{H}/r$, with $M(r)$ the mass inside a radius r and m_H the mass of a hydrogen atom. For an auto-gravitational system, the gas particles will move with sufficient kinetic energy to withstand the gravitational pressure. Because of this, we find that

$$\phi(r_\mathrm{vir}) = kT.$$

In the case of the Sun, we put $M(r_\mathrm{vir}) = M_\odot$, finding a radius of $7.1\cdot 10^{10}$ cm that is consistent with the observations given the approximation kT for the thermal energy.

(c) With the obtained result for r_vir, we find immediately that

$$E_\mathrm{bind} = \int_0^{r_\mathrm{vir}} 4\pi r^2\rho(r)\frac{GM(r)}{r}\mathrm{d}r,$$

since a volume element $\mathrm{d}V = 4\pi r^2\mathrm{d}r$ gives rise to a mass $\mathrm{d}m$ of $\mathrm{d}m = \rho(r)\mathrm{d}V$.

(d) Since T is a constant and $T\sim M(r)/r$, it follows that

$$M(r) = \int_0^r 4\pi r'^2\rho(r')\mathrm{d}r' \sim r.$$

We obtain that $\rho = \rho_0 r_\mathrm{vir}^2/r^2$, for a ball of a size given by r_vir.

(e) A single particle in the gas will experience an amount of work done on it to the size of

$$W = \int_{r_f}^{r_i} \frac{GM(r)m_{\mathrm{H}}}{r^2}\mathrm{d}r,$$

when it moves from a radius r_i to a radius r_f. Summed over all particles present, the total amount of work done comes to

$$W = \int_0^{r_{\mathrm{vir}}} 4\pi r_i^2 \rho(r_i) \int_{r_f}^{r_i} \frac{GM(r)}{r^2}\mathrm{d}r\mathrm{d}r_i.$$

We find that $E = W \sim E_{\mathrm{bind}}$.

(f) The time necessary to complete the contraction is given by the time of free fall t_{ff} of an average gas particle. The kinetic energy of a single atom lies in the order of

$$E_{\mathrm{kin}} = GM(r_{\mathrm{vir}})m_{\mathrm{H}}/r_{\mathrm{vir}} = 4\pi G\rho_0 r_{\mathrm{vir}}^2 m_{\mathrm{H}},$$

from which we can conclude that $4\pi G\rho_0 r_{\mathrm{vir}}^2 m_{\mathrm{H}} \approx \frac{1}{2} m_{\mathrm{H}} r_{\mathrm{vir}}^2 / t_{\mathrm{ff}}^2$ and accordingly

$$t_{\mathrm{ff}} \approx \frac{1}{\sqrt{8\pi G\rho_0}}.$$

4. Breaking the Waves H. Jordens

(a) Say that the direction of propagation in a certain point is at an angle α to the line perpendicular to the shore from that point, then we can conclude from the law of diffraction that $\sin(\alpha) = kv$. If $v = 0$, it follows that $\alpha = 0$; the waves will arrive in a direction perpendicular to the beach.

(b) The tangent to the trajectory of the wave will be given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\tan\alpha} = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha} = \frac{\sqrt{1-k^2v^2}}{kv} = \frac{\sqrt{1-a^2k^2y^2}}{aky} = \frac{\sqrt{1-c^2y^2}}{cy},$$

with a and c being constants. From this, we may conclude that

$$\int_0^y \frac{cy}{\sqrt{1-c^2y^2}}\mathrm{d}y = \int_0^x \mathrm{d}x$$

and therefore

$$\frac{1}{c}\left(1-\sqrt{1-c^2y^2}\right) = x.$$

This can be rewritten to give

$$\left(x-\frac{1}{c}\right)^2 + y^2 = \frac{1}{c^2},$$

the equation for a circle with its center at $(1/c, 0)$ and a radius $1/c$.

5. Wiring the Greek Way F.J. van Steenwijk

Consider the situation where a current I is fed into one vertex P while it is allowed to leave the structure at the remaining 19 vertices in equal portions $\frac{I}{19}$. In this situation we can, by symmetry, find the equipotential planes: if you take the line from P to its opposite to be vertical, the planes are the horizontal planes. If we let q_k denote the number of vertices in the k-th plane and n_k the number of resistors in parallel between the planes k and $k-1$ we get the following sequences of numbers:

$$\begin{aligned} \{q_0, \ldots, q_5\} &= \{1, 3, 6, 6, 3, 1\}, \\ \{n_1, \ldots, n_5\} &= \{3, 6, 6, 6, 3\}. \end{aligned}$$

Call the potential of the j-th plane (relative to P) V_j and the current flowing between plane $j-1$ and j, I_j. Then

$$V_i = R\sum_{j=1}^{i} \frac{I_j}{n_j} = RI\sum_{j=1}^{i} \frac{1}{n_j}\left(1 - \frac{1}{19}\sum_{k=1}^{j-1} q_k\right).$$

To find the equivalent resistor $R_e(i)$ between P and a point Q in the i-th plane we also consider the situation where the full current leaves at Q while it is fed into the circuit in 19 equal portions at the other vertices. We superpose this onto the former situation and the result is a current of $I + \frac{I}{19}$ going in at P and leaving at Q. This yields:

$$\left(I + \frac{I}{19}\right) R_e(i) = 2V_i.$$

Now all that remains to be done is substituting $i = 5$ for part (a) and $i = 1$ for part (b). The result is:

$$\begin{aligned} R_e(5) &= \frac{7}{6}R, \\ R_e(1) &= \frac{19}{30}R. \end{aligned}$$

Remark: the formulas in this solution can be used to calculate the equivalent resistor between any two vertices of any regular polyhedron.

6. Tooling Around in Space G. 't Hooft

1. (a) The trajectory followed by the tool is much like that of the spaceship, turned over a small angle around the z-axis at $t = 0$ (see figure). Even from a rough sketch, it can be seen that $x(t)$ and $z(t)$ won't differ from zero significantly, but $y(t)$ will follow a simple sine-curve, with the same period T.

   (b) By comparing the initial velocities, we find the amplitude of this sine-curve. In units of kilometers and hours:

   $$x(t) = 0, \quad y(t) = \frac{1}{\pi}\sin \pi t, \quad z(t) = 0$$

   (c) The coordinates (x, y, z) will be given by, respectively,

   $$(0, 1/\pi, 0), (0, 0, 0), (0, -1/\pi, 0) \text{ and } (0, 0, 0).$$

2. The shape of the orbit around Earth turns into an ellipsis (see figure). If the orbit of the spaceship has a fixed radius R, the half focal length of the ellipsis, c, will be in the order of a kilometer. From the figure, we note that the long axis, a, won't differ significantly from R. Since $b \approx a - c^2/2a$, to first order approximation, b will also still be equal to R, giving that the orbit is described sufficiently well by a slightly shifted circle. We conclude that c is too small to cause a notable difference between the circle and the ellipsis.

   (a) When $z > 0$, the tool will start to lag. Because of that, the x coordinate will differ from zero. During the second hour however, it will catch up again. Since $a \approx R$, the total period T of the tool will still be exactly 2 hours, making it a closed orbit. If this orbit is a circle or an ellipsis is not relevant at this point of the question.

   (b) It is easiest to use Kepler's second law (see the figure). The shaded area will take the same amount of time to sweep out as the other halve. From this, we conclude that the amplitude of the $x - axis$ should be twice the amplitude of the $z - axis$. Therefore, we have an ellipsis.

   $$\begin{aligned} z(t) &= (1/\pi)\sin(\pi t), \quad \text{giving} \\ x(t) &= (2/\pi)(\cos(\pi t) - 1), \\ y(t) &= 0. \end{aligned}$$

   (c) The coordinates (x, y, z) will be given by, respectively,

   $$(-2/\pi, 0, 1/\pi), (-4/\pi, 0, 0), (-2/\pi, 0, -1/\pi) \text{ en } (0, 0, 0).$$

3. (a) Now the orbit will resemble that of the third figure. The short and long axis are still essentially equal in size, but both are now larger than R. The period of revolution T will therefore be larger. After two hours, the tool will be lagging.

(b) We expect to see a spiralling motion in the negative x-direction (see figure). The orbit would have been identical to the orbit in part 2 (but with a different phase), if the spaceship had itself had been in an orbit with a slightly larger radius (*i.e.* $R \to R + z_0$, see below).

(c) Assume that

$$\begin{aligned} z &= z_0(1 - \cos(\pi t)), \\ x &= 2z_0 \sin(\pi t) - At, \\ y &= 0, \end{aligned}$$

where z_0 and A remain to be calculated. The long axis of the ellipsis (once again practically a circle) is given by $a = R + z_0$. Put $\delta R = z_0, T = T + \delta T$. From Kepler's first law, we derive:

$$\begin{aligned} T^2 &= C \cdot a^3, \\ 2T\delta T &= 3Ca^2\delta a, \\ \frac{2\delta T}{T} &= \frac{3\delta a}{a}, \\ v &= 2\pi R/T, \\ 2A &= v\delta T = \frac{3}{2}v\frac{T\delta R}{R} = 3\pi\delta R = 3\pi z_0. \end{aligned}$$

It is now possible to substitute the initial velocity:

$$1 = 2\pi z_0 - \frac{3}{2}\pi z_0,$$

which means that $z_0 = 2/\pi$. The formulae then become finally:

$$\begin{aligned} x(t) &= (4/\pi)sin(\pi t) - 3t; \\ y(t) &= 0; \\ z(t) &= (2/\pi)(1 - \cos(\pi t)). \end{aligned}$$

(d) After 2 hours, we get $x = -6$ km. Both y and z will at that point be 0 again. It is quite remarkable that, even if the initial velocity was 1 km h^{-1} in the positive x-direction, the object has, after 2 hours, managed to move 6 km *backwards*.

4. (a) With random initial velocity, we will find a linear superposition of the previous results. Noting that is already sufficient. The general formulae will be:

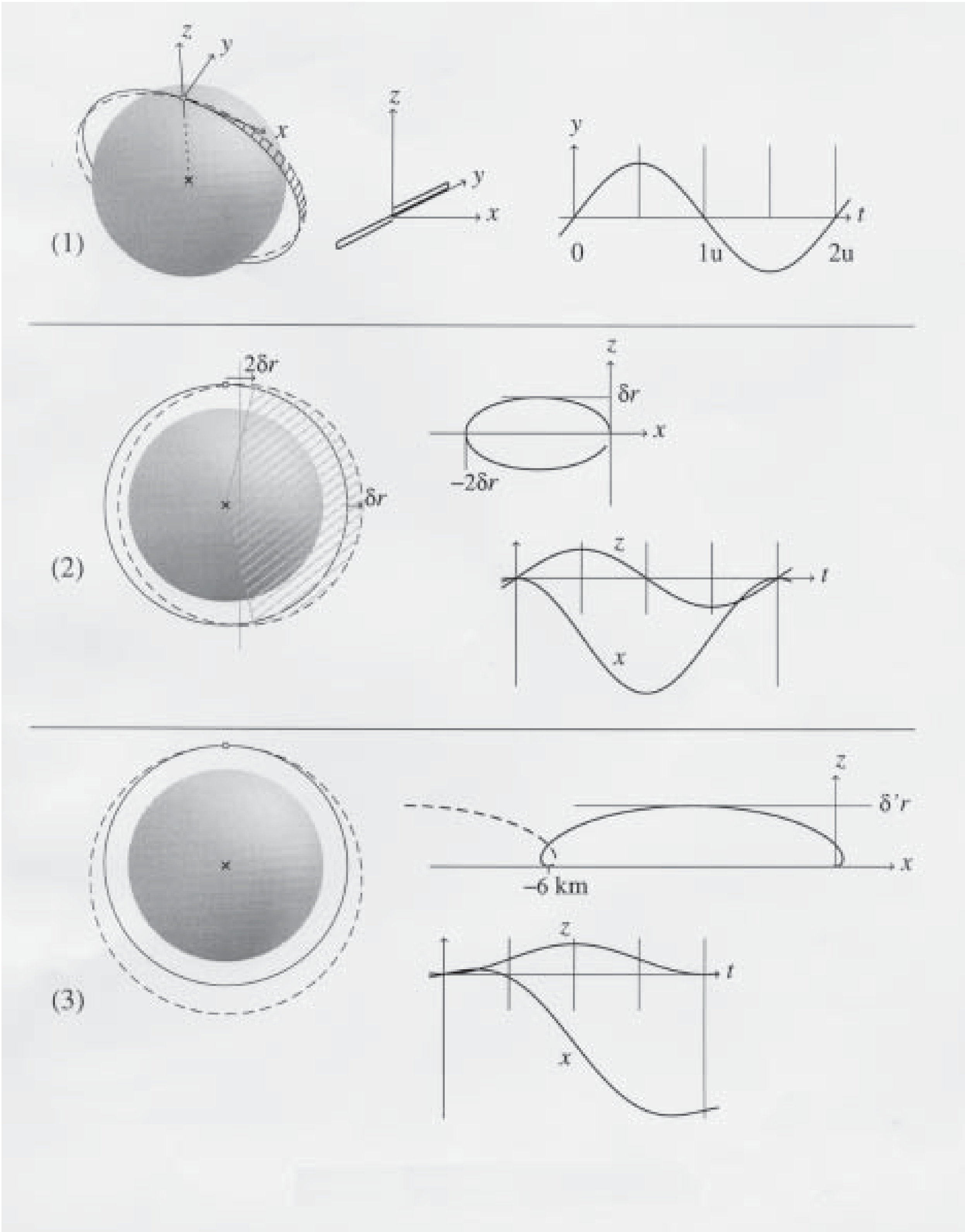
$$\begin{aligned} x(t) &= v_x(0)[(4/\pi)\sin(\pi t) - 3t] + (2v_z(0)/\pi)(\cos(\pi t) - 1); \\ y(t) &= (v_y(0)/\pi)\sin \pi t; \\ z(t) &= (2v_x(0)/\pi)(1 - \cos(\pi t)) + (v_z(0)/\pi)\sin \pi t. \end{aligned}$$

There are, by the way, other methods available to a theoretical physicist to arrive at these results; here, however, the intention was that Kepler's laws would be applied.

(b) The position of the tool after 2 hours will be solely dependent on the x-component:

$$x(t) = -3Tv_x(0) = -6v_x(0); \quad y(T) = z(T) = 0.$$

The tool will return to the astronaut after an hour then, but only if the x-component of the initial velocity is equal to zero.



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7. 日本大阪 O. Scholten

(a) Using conservation of momentum, we can write the following equations:

$$p_e - p_{uv} = p'_e \cos\theta_e + p_\gamma \cos\theta_\gamma$$
$$0 = p'_e \sin\theta_e + p_\gamma \sin\theta_\gamma$$
$$\sqrt{c^2 m_e^2 + p_e^2} + p_{uv} = \sqrt{c^2 m_e^2 + p_e'^2} + p_\gamma .$$

The first two equations can be squared and added, to give:

$$p_e'^2 = (p_e - p_{uv})^2 - 2p_\gamma (p_e - p_{uv}) \cos\theta_\gamma + p_\gamma^2 .$$

Combining this expression with the third of the three equations of the momenta finally gives:

$$p_\gamma (E_e - cp_e \cos\theta_\gamma) + cp_{uv}(1 + \cos\theta_\gamma) = p_{uv}(E_e + cp_e),$$

where $E_e = \sqrt{c^4 m_e^2 + c^2 p_e^2} \approx cp_e + \frac{c^4 m_e^2}{2cp_e}$, with $p_e = 8$ GeV/c and $p_{uv} = h\lambda_{uv} = 3.59$ eV.

(b) In the case of exact backscattering, the expression for the energy of the backscattered photons reduces to:

$$E_{\gamma,max} \approx cp_e \frac{1}{1 + \frac{m_e^2 c^2}{4p_e p_{uv}}} .$$

Numerically, this will give a value for the photon energy of about 2.4 GeV.

(c) This amounts to finding a solution to the equation:

$$1/10 \approx \frac{1 + \frac{m_e^2 c^2}{4p_e p_{uv}}}{\frac{p_e(1-\cos\theta_{1/10})}{2p_{uv}} + \frac{(1+\cos\theta_{1/10})}{2} + \frac{m_e^2 c^2}{4p_e p_{uv}}}$$

which is given by $1 - \cos\theta_{1/10} = 13.5 \cdot 10^{-9}$.

8. A Matter of Life and Death B. Hoenders

Starting from the Maxwell equations suitable for this problem:

$$\epsilon\frac{\partial E_x}{\partial t}+\sigma E_x=-\frac{\partial H_y}{\partial z},$$

$$\mu\frac{\partial H_x}{\partial t}=-\frac{\partial E_y}{\partial z},$$

we arrive at the following identity:

$$\left(\epsilon\frac{\partial E_x}{\partial t}+\sigma E_x\right)\mu H_y \mathrm{d}z+\mu\frac{\partial H_y}{\partial t}\epsilon E_z \mathrm{d}z=-\frac{\epsilon E_x^2+\mu H_y^2}{\partial z}\mathrm{d}z.$$

The right side of this equation gives the change in momentum (Poyntingvector), while the left side contains the "magnetic" and "electric" Lorentz forces, originating from the "magnetic" current $\epsilon\frac{\partial E_x}{\partial t}+\sigma E_x$ and the "electric" current $\mu\frac{\partial H_x}{\partial t}$ respectively.

9. The Magic Marble of Mystery W. K. Ma

(a) The linear motion of the center of mass is given by:

$$\begin{aligned} Ma &= -F_g \sin 30^\circ - \mu F_g \cos 30^\circ = -\frac{1}{2}Mg - \frac{1}{2}\sqrt{3}\mu Mg \\ a &= -\frac{1}{2}g(1+\sqrt{3}\mu) \\ v(t) &= v_0 - \frac{1}{2}g(1+\sqrt{3}\mu)t. \end{aligned}$$

The rotational motion around the center of mass follows from:

$$\begin{aligned} I\alpha &= \tau = F_f R = \frac{1}{2}\sqrt{3}\mu MgR \\ \alpha &= \frac{1}{2}\sqrt{3}\frac{\mu MgR}{\frac{2}{5}MR^2} = \frac{5}{4}\sqrt{3}\frac{\mu g}{R} \\ \omega(t) &= \frac{5}{4}\sqrt{3}\frac{\mu g}{R}t. \end{aligned}$$

For the rolling motion at time T, we have:

$$\begin{aligned} v(T) &= R\omega(T) \\ v_0 - \frac{1}{2}g(1+\sqrt{3}\mu)T &= \frac{5}{4}\sqrt{3}\mu g T \\ T &= \frac{v_0}{g}\frac{4}{2+7\sqrt{3}\mu}. \end{aligned}$$

Substituting this T:

$$v(T) = \frac{5}{4}\sqrt{3}\mu g T = v_0 \frac{5\sqrt{3}\mu}{2+7\sqrt{3}\mu}.$$

Note: angular momentum is not conserved, but one can use $\Delta L = \tau\delta t$. However, this gives an equation with two unknowns, $v(T)$ and T.

(b) Displacement of the center of mass before $t = T$ (integrate $v(t)$):

$$s(t) = v_0 t + \frac{1}{2}at^2 = v_0 t - \frac{1}{4}g(1+\sqrt{3}\mu)t^2.$$

Substitute $t = T$:

$$\begin{aligned} s(T) &= \frac{v_0^2}{g}\frac{4}{2+7\sqrt{3}\mu} - \frac{1}{4}\frac{v_0^2}{g}(1+\sqrt{3}\mu)\left(\frac{4}{2+7\sqrt{3}\mu}\right)^2 \\ &= \frac{v_0^2}{g}\frac{8+28\sqrt{3}\mu-(4+4\sqrt{3}\mu)}{(2+7\sqrt{3}\mu)^2} = \frac{v_0^2}{g}\frac{4+24\sqrt{3}\mu}{(2+7\sqrt{3}\mu)^2}. \end{aligned}$$

The change in height is then given by:

$$h(T) = s(T)\sin 30^\circ = \frac{v_0^2}{g}\frac{2+12\sqrt{3}\mu}{(2+7\sqrt{3}\mu)^2}.$$

(c) The heat produced equals the work done by friction, W_f. When the marble rolls without slipping, there is no relative displacement between the surfaces, hence friction does not do any work. For the period $t < T$, the work is found from conservation of energy:

$$\begin{aligned}
KE(T) &= \frac{1}{2}Mv(T)^2 + \frac{1}{2}I\omega(T)^2 = \frac{1}{2}Mv(T)^2 + \frac{1}{2}\cdot\frac{2}{5}MR^2\omega(T)^2 \\
&= \frac{7}{10}Mv(T)^2 = \frac{7}{10}Mv_0^2\left(\frac{5\sqrt{3}\mu}{2+7\sqrt{3}\mu}\right)^2 = \frac{1}{2}Mv_0^2\frac{105\mu^2}{(2+7\sqrt{3}\mu)^2} \\
PE(T) &= Mgh(T) = Mv_0^2\frac{2+12\sqrt{3}\mu}{(2+7\sqrt{3}\mu)^2} \\
KE(0) &= \frac{1}{2}Mv_0^2 \\
PE(0) &= 0 \\
W_f &= KE(0) + PE(0) - KE(T) - PE(T) \\
&= \frac{1}{2}Mv_0^2\left(1 - \frac{105\mu^2}{(2+7\sqrt{3}\mu)^2} - \frac{4+24\sqrt{3}\mu}{(2+7\sqrt{3}\mu)^2}\right) \\
&= \frac{1}{2}Mv_0^2\frac{(2+7\sqrt{3}\mu)^2 - 105\mu^2 - (4+24\sqrt{3}\mu)}{(2+7\sqrt{3}\mu)^2} = Mv_0^2\frac{4\sqrt{3}\mu + 21\mu^2}{(2+7\sqrt{3}\mu)^2}.
\end{aligned}$$

Note: because it rolls with slipping, one cannot apply $W = F_f \cdot s(T)$.

(d) The number of revolutions can be found from the total angle through which the marble has rotated. For $t < T$, integrate $\omega(t)$ to find:

$$\begin{aligned}
\theta(t) &= \frac{5}{8}\sqrt{3}\frac{\mu g}{R}t^2 \\
\theta(T) &= \frac{5}{8}\sqrt{3}\frac{\mu g}{R}\frac{v_0^2}{g^2}\left(\frac{4}{2+7\sqrt{3}\mu}\right)^2 = \frac{v_0^2}{Rg}\frac{10\sqrt{3}\mu}{(2+7\sqrt{3}\mu)^2}.
\end{aligned}$$

The number of revolutions is then:

$$\frac{\theta(T)}{2\pi} = \frac{\mu v_0^2}{\pi Rg}\frac{5\sqrt{3}}{(2+7\sqrt{3}\mu)^2}.$$

(f) For $t > T$, we set $t' = t - T$ and apply Newton's law:

$$\begin{aligned}
Ma &= -\frac{1}{2}Mg - F_f \\
I\alpha &= F_f R \\
\alpha &= \frac{a}{R} = \frac{-\frac{1}{2}g - \frac{F_f}{M}}{R} = \frac{-\frac{1}{2}g - \frac{I\alpha}{MR}}{R} \\
\alpha &= \frac{-\frac{1}{2}g}{R + \frac{I}{MR}} = -\frac{5}{14}\frac{g}{R} \\
\omega(t') &= \omega(T) - \alpha t' \\
\Delta\theta(t') &= \omega(T)t' - \frac{1}{2}\alpha t'^2.
\end{aligned}$$

The highest point is reached when $\omega(T')=0$, so $T'=\frac{\omega(T)}{\alpha}$. Then

$$\begin{aligned}\Delta\theta(T') &= \frac{1}{2}\frac{\omega(T)^2}{\alpha}\\ &= \frac{1}{2}v_0^2\left(\frac{5\sqrt{3}\mu}{2+7\sqrt{3}\mu}\right)^2\frac{14}{5}\frac{R}{g}\\ &= \frac{v_0^2}{Rg}\frac{105\mu^2}{(2+7\sqrt{3}\mu)^2}.\end{aligned}$$

One can check this using conservation of energy between the point when it first starts rolling without slipping, and the highest point:

$$\begin{aligned}Mg\Delta h &= KE(T)\\ \Delta h &= \frac{1}{2}\frac{v_0^2}{g}\frac{105\mu^2}{(2+7\sqrt{3}\mu)^2}\\ \Delta\theta &= \frac{s}{R}=\frac{\Delta h}{R\sin 30^\circ}=\frac{v_0^2}{Rg}\frac{105\mu^2}{(2+7\sqrt{3}\mu)^2}.\end{aligned}$$

Hence the total number of revolutions is:

$$\frac{\theta(T)+\Delta\theta}{2\pi} = \frac{\mu v_0^2}{2\pi Rg}\frac{105\mu+10\sqrt{3}}{(2+7\sqrt{3}\mu)^2}.$$

P.S. Doing the problem with a general angle β is not particularly insightful.

10. Logic for Spies L. P. Kok

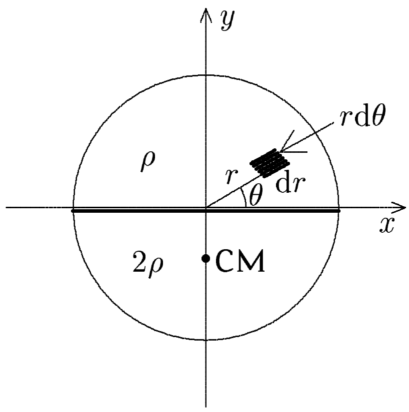
If we number the cards of spy A $1, \ldots, 6$, he can make the following statement to spy B: "I have either $1, \ldots, 6$ or $1, \ldots 4, 7, 8$ or $1, \ldots, 4, 9, 10$ or $1, \ldots, 3, 11, 12, 13$" and lots of permutations, all with the property that, each time, there are two cards not in the possession of spy A in the combination.
So, except for one combination given by spy A, there is always one card of spy B in the combinations. The result of this is that spy B knows which card spy C has, and spy B can just pass this information spy A.
The permutations are only necessary if spy C isn't allowed to know any card of spy A or B, and the combinations given are already sufficient to solve the problem.
This method of communication, where you don't need any encryption techniques, will not work if the number of cards spy A, B and C each have is not known.

11. A Rotating Disc

J. F. Schröder

The solution involves the following steps:

1. Determine the location of the center of mass and the mass of the disc.
2. Calculate the moment of inertia of the disc with respect to an axis through the center of mass.
3. Determine the center of mass velocity when the disc is rolling.
4. Determine the kinetic and potential energy.
5. Write down the Lagrangian.



We will now go through these steps one at a time.

1. The center of mass lies at the coordinate $(0, x_2)$. We can calculate x_2 from $x_2 = \frac{1}{M}\int y\mathrm{d}m$. This gives us

$$\begin{aligned}
x_2 &= \frac{1}{M}\Big[\int\limits_{\text{Lower part}} y2\rho\mathrm{d}x_1\mathrm{d}x_2 + \int\limits_{\text{Upper part}} y\rho\mathrm{d}x_1\mathrm{d}x_2\Big] \\
&= \frac{\rho}{M}\Big[\int\limits_{r=0}^{R}\int\limits_{\theta=\pi}^{2\pi} r\sin\theta r\mathrm{d}r\mathrm{d}\theta + \int\limits_{r=0}^{R}\int\limits_{\theta=0}^{\pi} 2r\sin\theta r\mathrm{d}r\mathrm{d}\theta\Big] \\
&= \frac{\rho}{M}\Big[\int\limits_{r=0}^{R} r^2\mathrm{d}r\int\limits_{\theta=\pi}^{2\pi}\sin\theta\mathrm{d}\theta + \int\limits_{r=0}^{R} 2r^2\mathrm{d}r\int\limits_{\theta=0}^{\pi}\sin\theta\mathrm{d}\theta\Big] \\
&= -\frac{2}{3}\frac{\rho R^3}{M}.
\end{aligned}$$

The mass of the disc is $M = \rho\frac{1}{2}\pi R^2 + 2\rho\frac{1}{2}\pi R^2 = \rho\frac{3}{2}\pi R^2$. It follows that $x_2 = -\frac{4}{9\pi}R$.

2. It's a hard job to find the moment of inertia with respect to the center of mass. That's why we first calculate the moment of inertia with respect to the center of the disc (w.r.t. the x_3 axis). After we have found this we use Steiner's theorem of parallel axes to find the moment of inertia.

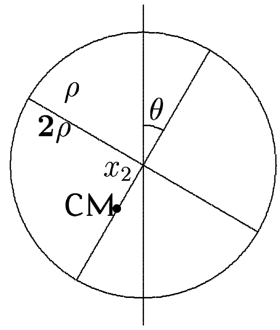
$$\begin{aligned}
I &= \int r^2\mathrm{d}m \\
I_3 &= \rho\Big[\int\limits_{r=0}^{R}\int\limits_{\theta=0}^{\pi} r^2 r\mathrm{d}r\mathrm{d}\theta + 2\int\limits_{r=0}^{R}\int\limits_{\theta=\pi}^{2\pi} 2r^2 r\mathrm{d}r\mathrm{d}\theta\Big] = \frac{3}{4}\rho R^4 = \frac{1}{2}MR^2
\end{aligned}$$

Using Steiner's theorem, we obtain

$$I_{\text{CM}} = I_3 - Mx_2^2 = \frac{1}{2}MR^2 - M\frac{16}{81\pi^2}R^2 = \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right].$$

3. We have that

$$\begin{aligned}
x_{\mathrm{CM}} &= R\theta - |x_2|\sin\theta \\
y_{\mathrm{CM}} &= R - |x_2|\cos\theta \\
\dot{x}_{\mathrm{CM}} &= R\dot{\theta} - |x_2|\dot{\theta}\cos\theta \\
\dot{y}_{\mathrm{CM}} &= |x_2|\dot{\theta}\sin\theta \\
v^2 &= \dot{x}_{\mathrm{CM}}^2 + \dot{y}_{\mathrm{CM}}^2 = R^2\dot{\theta}^2 + x_2^2\dot{\theta}^2 - 2R|x_2|\dot{\theta}^2\cos\theta = a^2\dot{\theta}^2.
\end{aligned}$$



Here

$$a = \sqrt{R^2 + x_2^2 - 2R|x_2|\cos\theta} = R\sqrt{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta},$$

where we used that $x_2 = -\frac{4}{9\pi}R$.

4. If we calculate the kinetic energy we get

$$T = T_{\text{translation}} + T_{\text{rotation}} = \frac{1}{2}Mv^2 + \frac{1}{2}I_{\mathrm{CM}}\dot{\theta}^2 = \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right].$$

The potential energy, with respect to the horizontal plane is

$$U = Mg(R - |x_2|\cos\theta) = \frac{1}{2}MgR\left[2 - \frac{8}{9\pi}\cos\theta\right].$$

5. The Lagrangian is given by

$$\begin{aligned}
L &= T - U \\
&= \frac{1}{2}MR\Big[R\dot{\theta}^2\left(\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right) - g\left(2 - \frac{8}{9\pi}\cos\theta\right)\Big] \\
&= \frac{3}{4}\rho\pi R^3\Big[R\dot{\theta}^2\left(\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right) - g\left(2 - \frac{8}{9\pi}\cos\theta\right)\Big].
\end{aligned}$$