PLANCKS 2018
Preliminaries for Team Singapore

Exercise Booklet
Round 2

Division of Physics and Applied Physics
School of Physical and Mathematical Sciences
Nanyang Technological University

19nd January 2018
3:30 pm - 7:30 pm
SPMS-TR+1

Part I: Short questions

Please select and complete **only two questions** in this part. If more questions are attempted, the two highest scores of all will be taken.

Charged Particle in a Dipole Field

[Kelvin Horia] [20 Marks]

An electric dipole with dipole moment $\vec{p} = p\hat{z}$ is situated at the origin. A negative charged particle Q ($Q < 0$) is placed on the z-axis, and its initial position is given by $z = z_0 > 0$. The charge is given an initial velocity v_0 in the direction perpendicular to the z-axis (see Fig. 1).

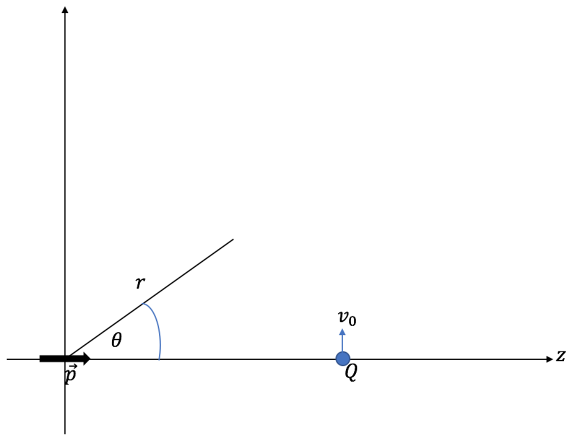

Figure 1: The electric dipole is located at the origin, and the negative charge Q is placed on the z-axis with its velocity.

In this problem, you are going to use cylindrical coordinate $(r; \theta)$ to denote the position of Q.

1. Derive the equation of motion of $r(t)$, and $\theta(t)$. You might want to start with the Lagrangian of the system. You are not required to solve the differential equations that you obtain. **Hint:** The electric potential created by the dipole is given as following

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}.$$

2. Show that $\frac{d}{dt}(r\dot{r})$ is a constant. What does this constant depend on?

3. It turns out that there is a maximum speed v_0 such that if the initial speed is larger than v_0, the charged particle will run away to infinity. Find the value of v_0.

4. If the initial speed is $v < v_0$, how long does it take for the charged particle to collide with the dipole?

Falling Magnet

[Kelvin Horia] [20 Marks]

In spherical coordinate, the magnetic field produced by a magnetic dipole, located at the origin, and pointing along the positive z-axis is given by,

$$\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right),$$

where r is the distance from the dipole, and θ is the angle between z-axis and $\vec{r}$.

1. A circular ring with radius R_0 is placed on the xy-plane, with its center located at the origin. The circular ring has resistance R. If a magnetic dipole with magnetic moment $\vec{m} = m\hat{z}$ is moving along the z-axis (see Fig. 2), what is the induced current flowing in the ring as a function of position and velocity of the dipole? You may assume that the dipole does not change its orientation.

2. An infinitely long hollow cylinder with resistivity ρ, radius R_0, and thickness $\delta \ll R_0$, is placed such that its cylindrical axis coincides with z-axis. A magnetic dipole with magnetic moment m is dropped at some point along the z-axis, and falls due to its weight. Its orientation is always parallel to z-axis (see Fig. 3). After sufficiently long time, the dipole will reach its terminal velocity. Find the terminal velocity of the dipole.

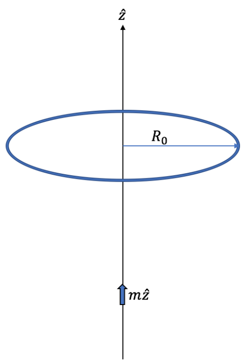

Figure 2: A magnetic dipole at a distance z away from the ring and is moving towards it.

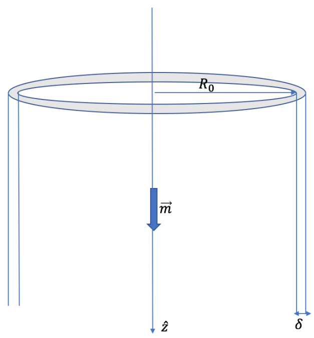

Figure 3: A magnetic dipole flying through an infinitely long hollow cylinder.

Hint: Depending on your approach, the following information **might or might not** be useful:

1. $$\int_{-\infty}^{+\infty} \frac{x^2}{\left(x^2+R^2\right)^5}\, dx = \frac{5\pi}{128R^7}$$

2. The force acting on a magnetic dipole due to external magnetic field $\vec{B}$ is given by $\vec{F} = -\nabla(\vec{m}\cdot\vec{B})$.

3. In spherical coordinates

$$
\begin{aligned}
\hat{r} &= \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}\,,\\
\hat{\theta} &= \cos\theta\cos\varphi\hat{x} + \cos\theta\sin\varphi\hat{y} - \sin\theta\hat{z}\,,\\
\hat{\varphi} &= -\sin\varphi\hat{x} + \cos\varphi\hat{y}\,.
\end{aligned}
$$

4. In cylindrical coordinates

$$
\begin{aligned}
\hat{\rho} &= \cos\theta\hat{x} + \sin\theta\hat{y}\,,\\
\hat{\varphi} &= -\sin\theta\hat{x} + \cos\theta\hat{y}\,,\\
\hat{z} &= \hat{z}\,.
\end{aligned}
$$

Half Integer Spherical Harmonics Are Not Allowed

[20 Marks]

We have seen that the commutation relations for angular momentum operators allow $l = 0, 1/2, 1, 3/2, 2, 5/2, \cdots$. However, closer inspection shows that it is not possible to formulate the quantum mechanics of orbital angular momentum consistently in the Hilbert spaces with half-integer orbital angular momentum.

1. Assume that $l = 1/2$ is allowed. Then spherical harmonics for $l = 1/2$ should exist. Find the spherical harmonic for $l = m = 1/2$ (as we found the Y_{lm} for integer l) by using the equations:

$$\begin{aligned} L_+ \left|l = \frac{1}{2}, m = \frac{1}{2}\right\rangle &= 0, \\ L_z \left|l = \frac{1}{2}, m = \frac{1}{2}\right\rangle &= \frac{\hbar}{2} \left|l = \frac{1}{2}, m = \frac{1}{2}\right\rangle . \end{aligned}$$

   Your answer should be of the form:

$$Y_{\frac{1}{2},\frac{1}{2}} = N(\sin\theta)^a e^{ib\varphi}.$$

   Find a, b, and show that $Y_{\frac{1}{2},\frac{1}{2}}$ is normalizable.

   Note: You may find these useful:

$$\begin{aligned} L_z &= \frac{\hbar}{i}\frac{\partial}{\partial\phi}, \\ L_\pm &= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right). \end{aligned}$$

2. Repeat part 1. for $l = \frac{1}{2}$, $m = -\frac{1}{2}$.

3. Now show that the operator L_- applied to $Y_{\frac{1}{2},\frac{1}{2}}$ does not give $Y_{-\frac{1}{2},\frac{1}{2}}$ [Since the only ingredients that went into the derivation of the relation $L_- |l, m\rangle \propto |l, m-1\rangle$ were the commutation relations of the $\hat{L}_k$ and the fact that they are Hermitian, we are forced to conclude that we cannot implement orbital angular momentum in terms of Hermitian operators on the Hilbert (sub)space spanned by $l = \frac{1}{2}$, $m = \pm\frac{1}{2}$.]

4. Suppose you were to construct the spherical harmonic for $l = 3/2$, $m = 3/2$ in the manner of part 1. You would be led to the result,

$$Y_{3/2,3/2} = c(\sin\theta)^{3/2}e^{3i\varphi/2},$$

   which satisfies $L_+Y_{3/2,3/2} = 0$. Generate $Y_{3/2,-3/2}$ by repeated use of L_-. Show that the state you obtained does not satisfy $L_-Y_{3/2,-3/2} = 0$ (and is not even normalizable [not required in this test]).

1D-lattice vibrations

[Dr. Chen Yu] [20 Marks]

In solids, atoms form periodic arrays. The motion of these atoms is modelled as vibrations about their equilibrium positions in the array. In this question, we explore vibrations of atoms in 1D solids and their thermodynamic properties.

Here 1D solids are modelled as a chain of $N+1$ identical atoms with the same mass m, separated by identical springs with the same spring constant k. Then atoms are aligned in the x-axis, and have an equilibrium separation a. When the atoms are vibrating, the s-th atom displaces from its equilibrium position by u_s. We study the equations that u_s must satisfy and their solutions.

1 [5 marks] For the s-th atom (for $s = 2, 3, ..., N-1$), find its equation of motion.

2 [5 marks] We will get a large number of equations if N is large. We seek solutions of the following form

$$u_s(t) = ue^{iqsa-i\omega t}. \tag{1}$$

Notice that sa represents the equilibrium position of the s-th atom x_s. When $a \to 0$ with Na fixed, the above solution represents a plane-wave. (1) contains two parameters ω and q (apart from u representing the magnitude), find their relation. This relation is known as dispersion relation.

3 [5 marks] We have not finished solving the problem yet. What are the possible values of the parameters in the above solution? Since the dispersion relation fixes ω in terms of q, we need to know what possible values that q can take. The answer to this question lies in the boundary condition.

We shall impose Born-Von Karman boundary condition. More specifically, we identify the fist atom with the last ($N+1$-th) atom, so

$$u_s(t) = u_{s+N}(t) \quad \text{for all } s = 1, 2, ..., N$$

In doing so, we have effectively defined our system on a circle. Find all physically distinct values that q can take, and count the total number.

4 [5 marks] In view of the dispersion relation, the solution (1) is uniquely labelled by q, whose possible values have also been found. Thus, each q represents a vibrating (harmonic) modes of the 1D solids. In classical theory, the magnitude of vibration and thus the energy it carries is continuous. In quantum theory, each harmonic mode can be treated as a harmonic oscillator with frequency ω, and thus the energy it carries is quantized as

$$E_l = n_l \hbar \omega_l,$$

where a zero-point energy $\frac{1}{2}\hbar\omega_l$ has been omitted. The quanta of each harmonic mode is called a phonon. Phonons are bosons which, at finite temperature, obey Plank's distribution given by

$$f = \frac{1}{e^{\hbar\omega_l/k_B T} - 1}.$$

Suppose the system has two atoms ($N = 2$). Find the heat capacity of this model.

Spin-Orbit Coupling

[20 Marks]

The semi-classical derivation of the spin orbit coupling in atoms is based on a coordinate transformation into the rest frame of the valence electron. In this picture, an orbiting (singly charged) ion creates a field

$$\vec{B} = -(\vec{v} \times \vec{E})/c^2$$

felt by the valence electron. In this B-field, the magnetic momentum μ of the electron, coupled to the electron spin s via

$$\vec{\mu}_s = -\frac{g_s \mu_B}{\hbar} \vec{S},$$

where μ_B is the Bohr magneton, g_s is the electron spin g-factor, and $\vec{S}$ is the vector spin of the electron. This gives rise to an energy contribution which is known as spin-orbit interaction energy $E_{SO} = \vec{\mu} \cdot \vec{B}$. A factor 0.5 needs to be added here due to the incorrect assumption of a linear motion of the electron in the transformation (Thomas-Factor).

Important quantities:

$$\mu_B = \frac{e\hbar}{2m_e}[J/T]\,, \qquad \varepsilon_0 = 8.85 \cdot 10^{-12}[F/m]\,,$$

$$g_s = 2\,, \qquad \hbar = \frac{h}{2\pi} = 1.0546 \cdot 10^{-34}[J.s]\,,$$

$$m_e = 9.10939 \cdot 10^{-31}[kg]\,. \qquad e = 1.6022 \cdot 10^{-19}[C]\,.$$

1. Write E_{SO} as a function of the angular momentum $\vec{L}$ and the spin $\vec{S}$ of the valence electron. Introduce the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ and try to find an expression for E_{SO} where the operators can be replaced by their corresponding quantum numbers j, l, and s. The radius of the n$^{\text{th}}$ orbit can be estimated via Bohr's ansatz $r_n = \frac{a_0 n^2}{z}$ with $a_0 = 0.529\mathring{A}$ and $z = 11$.

   Hint: You may find this expression useful: $\vec{L} \cdot \vec{S} = \frac{\vec{J}^2 - \vec{L}^2 - \vec{S}^2}{2}$, where $\vec{J}$, $\vec{L}$, and $\vec{S}$ are the respective total, orbital, and spin angular momentum operator.

2. With this formula, calculate the energy difference between the D1 and the D2 line of sodium in cm^{-1}. This splitting is a consequence of the SO-coupling in an electronically excited state of sodium where the 3p-orbital is singly occupied.

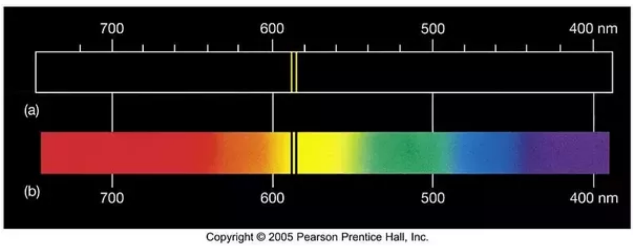

Figure 4: The absorption and emission spectrum of Sodium vapor show clearly the D-lines.

Part II: Long Questions

Please select and complete **only two questions** in this part. If more questions are attempted, the two highest scores of all will be taken.

Gravitational wave emission from a rotating binary

[Erickson Tjoa - University of Waterloo] [30 Marks]

Comment: in view of Nobel Prize in Physics 2017 for the detection of gravitational waves due to binary black hole coalescence, we shall consider a simplified problem of gravitational radiation where fully general relativistic treatment is not required. Instead, we work with Newtonian approximation.

Consider a binary system of two nearly point-like black holes of masses M, in orbit around each other with approximately constant separation $a \gg GM/c^2$. This orbit can therefore be adequately described by Newton's gravitational equation.

1. Choose coordinates in which the orbit lies in the (x, y) plane. Write down the nonvanishing components of the **mass quadrupole tensor** as a function of time, defined by
$$Q_{ij}(t) = \int d^3x\rho(x,t)\left(3r_ir_j - \delta_{ij}\vec{r}\cdot\vec{r}\right) .$$
where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.
Hint: tensor is not new, since the reader may recall a similar tensor in analytical mechanics called the *moment of inertia tensor* I_{ij}. Here it is essentially a 3×3 matrix.

2. Work out the total power emitted from the binary as *gravitational waves*, using the quadrupole formula
$$P_{\mathrm{GW}} = \frac{G}{45c^5}\left(\frac{\partial^3 Q_{ij}}{\partial t^3}\frac{\partial^3 Q^{ij}}{\partial t^3} - \frac{1}{3}\left|\frac{\partial^3 Q^i{}_i}{\partial t^3}\right|\right)$$
where we use Einstein summation convention i.e. repeated indices mean summation (e.g. $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3$ and $A_{ij}B^{ik} = A_{1j}B^{1k} + ... + A_{3j}B^{3k}$). Also in Newtonian approximation, $Q_{ij} = Q^{ij} = Q^i{}_j$. Your expression should not contain ω explicitly.

3. Consider two black holes of five solar masses $M = 5M_\odot$ in circular orbit of separation a, where $M_\odot = 2 \times 10^{30}$ kg. What is the maximum initial separation a from which the two black holes will merge in less than 10^{10} years?

4. Suggest a reason why the power emitted as gravitational waves is dominated by quadrupole moment. **Hint:** what is the analog of dipole for gravitational field?

5. In similar spirit to part (4.) and using the quadrupole formula, provide an argument for **Birkhoff theorem**: namely, a spherically symmetric 'pulsating' star does *not* emit gravitational waves. Pulsating here means the surface increases and decreases in size in time in spherically symmetric manner. This shows that not every oscillatory motion generates gravitational waves. In fact, the theorem has remarkable implication that such a time-dependent pulsating mass can generate *time-independent* external gravitational field! **Hint:** you do not need to explicitly calculate anything to provide this argument.

Optics and Special Relativity

[Erickson Tjoa - University of Waterloo] [30 Marks]

In special relativity, speed of light in inertial frames has a special role as compared to other velocities since it is the only velocity that is invariant under Lorentz transformation. We will see the implication of this on optical phenomena, some of which are both interesting and perhaps surprising. For simplicity we model 'light wave' as a scalar field ϕ satisfying wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0\,.$$

The general solution to this wave equation are plane waves, i.e. $\phi \propto \exp ik_\mu x^\mu$ where k^μ is the 4-wavevector given by $k^\mu = (\omega/c, \mathbf{k})$ and as usual ω is angular frequency. As a reminder, Lorentz transformation along x-direction is represented by the following:

$$\begin{aligned} t' &= \gamma\left(t \pm \frac{vx}{c^2}\right)\,, \\ x' &= \gamma\left(x \pm vt\right)\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned}$$

where the $\pm$ depends on the direction of the Lorentz boost and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

1. Using the dispersion relation of light, show that $k^\mu k_\mu = 0$ i.e. the 4-wavevector is null, as we would expect.

2. Consider an inertial observer O' moving at constant velocity v in the direction of the wave (i.e. in the direction of $\mathbf{k}$). Show that the frequency of the light wave as measured by this observer is *redshifted* by computing the ratio ω'/ω. This is known as *relativistic Doppler shift.*

3. Compute the classical Doppler redshift in non-relativistic case when observer moves towards a stationary light source at speed $v \ll c$. By comparing with part (b), show that the relativistic contribution of the Doppler effect is of order $O(v^2/c^2)$, i.e. Doppler effect of order $O(v/c)$ is the 'non-relativistic' part.

4. Show that there is still Doppler shift even if the observer is moving in the direction *orthogonal* (transverse) to $\mathbf{k}$ by computing ω'/ω in this case. Argue whether this is a relativistic or non-relativistic effect.

5. In classical optics, light rays bouncing off a mirror (at rest) is reflected without change in frequency and the incident angle is equal to reflected angle. Now consider another inertial frame where light ray bounces off the mirror *receding away* at velocity $-v$ in the x-direction. Show that in this frame, the incident angle α and the reflected angle β

are given by

$$\tan\alpha = \frac{\sqrt{1-\frac{v^2}{c^2}}\sin\alpha_0}{\left(\cos\alpha_0 + \frac{v}{c}\right)},$$
$$\tan\beta = \frac{\sqrt{1-\frac{v^2}{c^2}}\sin\beta_0}{\left(\cos\beta_0 - \frac{v}{c}\right)},$$

where $\alpha_0 = \beta_0$ are the angles in the mirror's rest frame.

6. Consider air-water interface where water has refractive index n. Show that n is not Lorentz invariant by calculating n', the refractive index of the water moving at velocity v as measured by an observer at rest. If instead we require that n is Lorentz invariant quantity i.e. it is still defined by n for the moving fluid, what quantity would have to change and what is the interpretation? (Fun fact: the possibility of light travelling very slowly due to refractive index of a medium allows optical equivalent of sonic boom, more famously known as *Cherenkov radiation.*)

The tippe top

[30 Marks]

The classical dime-store model tippe top (Fig. 5) consists of a section of a sphere upon whose planar surface is mounted on a short rod. The sportsman spinning this device will note that this perverse top refuses to sit on its elongated stem. In the process of inverting itself, the center of mass of the object is raised. Also, the direction of rotation reverses with respect to the body axes as the top turns over.

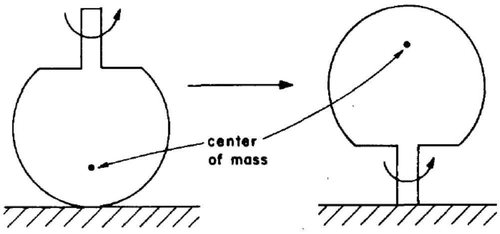

Figure 5: A tippe top inverting.

The tippe top's motion constitutes the sort of phenomenon abundant in physics, for which a simple physical analysis reveals the underlying principles, yet for which a detailed and rigorous solution (which may require the use of computing machines) is necessary to confirm the analysis.

1. It was shown by Pliskin, Braams, and Hugenholtz that the frictional interaction of the top with the table surface plays a decisive role in the inversion of the top. Explain in brief why must friction be important in this process?

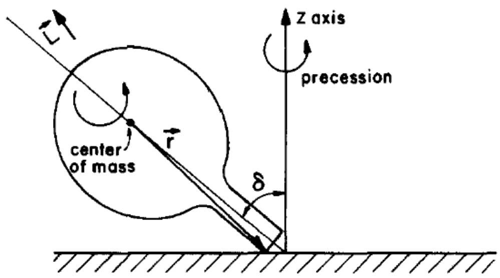

Figure 6: A conventional top.

Before proceeding with a discussion of the mechanics of the tippe top, it is instructive to briefly examine the influence of friction on the rising of a conventional top. Consider a top, consisting of a symmetrical mass, mounted on an axle of finite radius, set spinning about its axis of symmetry on a frictionless table an an angle δ to the vertical (Fig. 6).

2. The angular momentum $\vec{L}$ is not quite parallel to the axis of the top. Why is it the case?

3. Now, we introduce a small coefficient of friction for the table. What direction and where should friction be depicted in the diagram (Fig. 6)?

4. Briefly explain how this friction can help the top correct its motion?

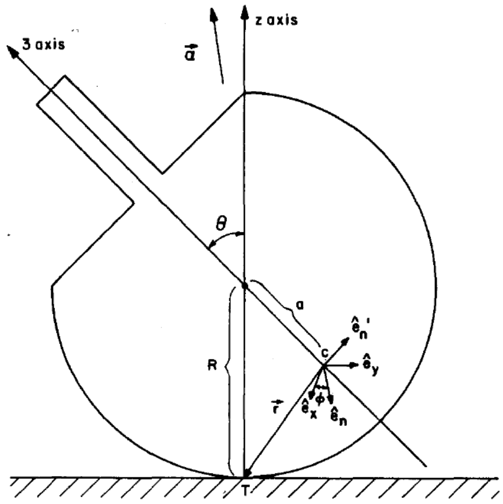

Figure 7: The tippe top with all axis definitions needed for this analysis.

This sort of analysis, however, will not suffice to account for the behavior of the tippe top. For one thing, this analysis depends on the fact that $\vec{L}$ points predominantly along the symmetry axis of the top, whereas the angular momentum of the tippe top points predominantly along the positive z-axis during the entire inversion process. Moreover, a correct analysis of the tippe top must take into account the change in the vector $\vec{r}$ (pointing from the CM of the top to the contact point) as a function of the angle of inclination. The only restriction relating to the mass distribution inside the top is that it be symmetric about the shaft and that the center of mass be epicentric as indicated in Fig. 7. The $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ coordinate system is fixed in the top's body with $\hat{e}_3$ parallel to the stem. The $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ system is fixed in the laboratory frame with $\hat{e}_z$ pointing up, perpendicular to the table surface. We may now proceed to define the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate system via the following equations:

$$\hat{e}_n = \frac{1}{|\hat{e}_z \times \hat{e}_3|}\hat{e}_z \times \hat{e}_3,$$
$$\hat{e}_{n'} = \hat{e}_3 \times \hat{e}_n.$$

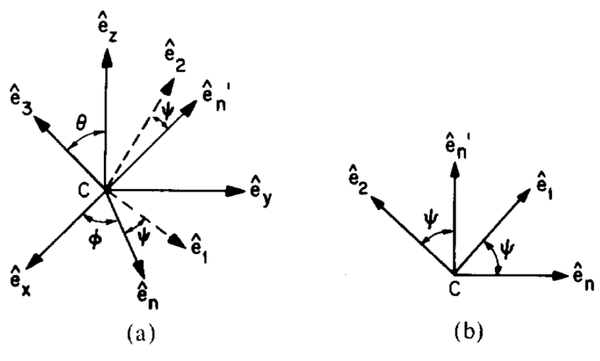

Figure 8: (a) Coordinate axes. (b) Coordinates axes as viewed from the direction of $\hat{e}_3$.

$\hat{e}_n$, $\hat{e}_{n'}$ remain fixed in the plane in which the vectors $\hat{e}_1$, $\hat{e}_2$ rotate (Fig. 8).

5. Utilizing the Euler angles θ, ϕ, ψ as defined in the figures, write down the transformation equations relating the $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$, and the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate systems.

6. The angular velocity of $\vec{\omega}$ of the top as measured in the laboratory frame is given as:

$$\vec{\omega} = \dot{\theta}\hat{e}_n + \dot{\phi}\hat{e}_z + \dot{\psi}\hat{e}_3.$$

   Express $\vec{\omega}$ in terms of $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinates.

7. Similarly, we also define the angular velocity $\vec{\alpha}$ of the $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinate system:

$$\vec{\alpha} = \dot{\theta}\hat{e}_n + \dot{\phi}\hat{e}_z.$$

   Express $\vec{\alpha}$ in terms of $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinates.

Any observer will readily confirm $\dot{\phi} \gg \dot{\theta}$, for the top rapidly precesses as it slowly turns over. Hence, $\vec{\alpha}$ points almost totally along the positive z-axis. Also, $\dot{\phi}$, the precessional velocity, is substantially larger in magnitude than $|\dot{\psi}|$. The fact that $\dot{\phi} \gg |\dot{\psi}|$ is equivalent to stating that L_z remains the dominant component of angular momentum throughout.

8. Explain if the value of $|\dot{\psi}|$ in the tippe top should be bigger or smaller compared to the conventional top.

Equipped with this terminology, we may now present a simple, semiquantitative physical argument that explains why the top turns over on its rounded head. The frictional force applied at point T (in Fig. 7) is label $\vec{F}_f$ and has magnitude of $|\vec{F}_f|$.

9. Write down an expression for the torque caused by the frictional force, $\vec{N}_f$ in terms of $\hat{e}_n$, $\hat{e}_{n'}$, $\hat{e}_3$ coordinates.

10. Analyze the motion of the top for the following cases:

    (a) $\theta < \cos^{-1}(a/R)$,

    (b) $\cos^{-1}(a/R) < \theta < \pi/2$,

    (c) $\theta > \pi/2$.

    Hint: It is useful to notice the trend of $\alpha_3/\alpha_{n'}$ which is a function of θ, where $\alpha_{3,m'}$ are the 3, and n'-component of α. Also, $N_3 \approx I_3\dot{\alpha}_3$, $N_{n'} \approx I_{n'}\dot{\alpha}_{n'}$, and we can assume $I_3 \approx I_{n'}$.

If you did part 10 correctly, you should see that the top will turn until its stem touches the table surface. Then, the main frictional force will be between the stem and the table surface. Based on the analysis of the conventional top, the tippe top will continue turning until the center of mass is upright.

Nerfing the nerve cell equation

[Suryadi Suryadi] [30 Marks]

The brain consists of individual cells known as neurons (or nerve cells) that communicate with each other using both electric and chemical signals. Here, we attempt a derivation of a differential equation describing a neuron's electrical behavior.

The electrical property of a neuron results from ionic movement through its cell membrane, characterized by its potential $V_m(x, t)$ that may vary both in space and time. Below is a simple illustration of a neuron membrane with its essential components:

- Phospholipid bilayer: the main component of the membrane with an insulating layer inside covered by two hydrophilic layers, which serves to prevent entry of any "unauthorized" substance.
- Ion channels: small openings in the membrane, created by proteins, that consumes energy to pump specific ions in/out of the cell to create concentration gradients. This gradient modulates the membrane potential, which facilitates neuron signaling. Each channel is typical described with a parameter known as reversal potential E_i, the potential at which there is no net movement of a particular ion species i.

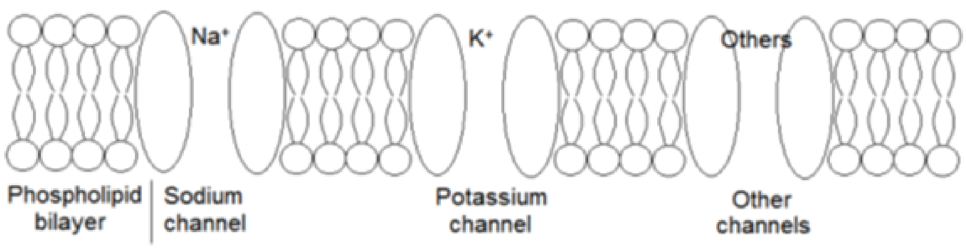

Figure 9: A part of cell boundary shows a lipid layer and channels for Sodium, Potassium, and other ions

In the Hodgkin–Huxley picture, three different ion currents contribute to the voltage signal of the neuron, that is, a sodium current, a potassium current, and a leak current that consists mainly of Cl^- ions. The flow of these ions through the cell membrane is controlled by their respective voltage-dependent ion channels. The leak current also takes care of other channel types which are not described in particular. The lipid membrane is considered an insulator that acts as a capacitor with constant capacitance, C_m. The channel proteins are considered being resistors described by conductances g_i (i being the index for a particular channel and ion, e.g. Na, K, and O (for other channels), and conductance can be thought of inverse of resistance). Upon change of voltage, V_m, two currents can be observed—a capacitive current charging the capacitor and ohmic currents through the protein. The total current through the membrane, I_m, is the sum of these two currents.

1. Given some incoming current I_m through a membrane patch, use a Kirchhoff's rule to derive a differential equation governing the intracellular Voltage, Vm. You may assume that the circuit element variables are constant, and that the entire system is Ohmic.

2. For this model, what is the steady state potential for the neuron membrane?

So far, we've only looked at the neuron body itself, ignoring all forms of signal propagation. Neuron signals travel via long projections known as axons (for outgoing signals) and dendrites (for incoming signals), which are nevertheless made up of cell membrane, hence we expect some form of ion leakage to be present. To account for this, we consider a cylindrical cable of radius a, which we use to model the axons and dendrites. For this question, suppose there is some current $I_a(x,t)$ propagating through the cable in the axial x-direction towards the neuron. You may assume that the intracellular resistance per unit length of the cable r_i is constant.

3. Derive an equation for the membrane potential of the cable changes by ∂V after a distance ∂x given that there is some current $I_a(x,t)$ propagating through the cell membrane.

4. Suppose the leakage mentioned above is uniform over the circumference of the cable cross-section, leading to an average outgoing (radial) current through the membrane of $\mathbf{i_m}$ **per unit of cable length**. Assuming conservation of charge, express i_m in terms of the current $I_a(x,t)$ going along the cell through a certain cross-section, and then in terms of the membrane potential V.

5. Your equation at (4) is described per unit length of cable, which is not uniform for each neuron. To generalize our result, express that equation using current $I(x,t)$ passing locally through a small patch of membrane and the resistivity inside the cell ρ_i.

6. Relate your answer with that in part (1) to obtain a final equation.

7. If the signal propagates at a constant speed v independent of the membrane potential, can the equation be simplified further?

1-D Fokker Planck Equation

[30 Marks]

In statistical mechanics, the Fokker–Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as in Brownian motion. The equation can be generalized to other observables as well.

The 1-dimensional Fokker-Planck equation has the form of

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[A(x)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x)P(x,t)\right], \tag{2}$$

where $A(x)$ and $D(x)$ are the drift and the diffusion coefficient, respectively, and $P(x,t)$ is the probability density of a random variable X_t.

In this first part, we will work with the 1-D Fokker-Planck equation with constant diffusion (also called as Smoluchowski equation), which is given by:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[V(x)P(x,t)\right] + D.\frac{\partial^2}{\partial x^2}\left[P(x,t)\right],$$

where $V(x)$ is the potential (also the drift coefficient in this case) and D the diffusion coefficient. This equation can also be written in the form of a continuity equation:

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}J(x,t),$$

where the probability current is $J(x,t) = \left[D.\frac{\partial}{\partial x} - V(x)\right]P(x,t)$.

1. First, for a stationary probability distribution $\left(\frac{\partial}{\partial t}P(x,t) = 0\right)$, J must be constant. Assume first that the probability current vanishes somewhere, this implies $J(x) = 0\ \forall x$. Find that in this case, the stationary probability distribution $P(x,t)$ has the form $P(x) = Ne^{-\Phi(x)}$, and derive $\Phi(x)$.

2. *Boundary condition of the Fokker-Planck equation:* Show that if the probability distribution J vanishes at the boundaries at $x = x_{min}$ and $x = x_{max}$, it follows that $\int_{x_{min}}^{x_{max}} P(x,t)dx = const.$ Hence, the probability distribution is constant with time (no particle can escape or can be absorbed, i.e. we have reflecting boundaries).

3. Next assume the case where the probability current J is constant but *non-zero*. Derive the stationary probability distribution $P(x)$ for this case. Assume that the problem is solved on a finite interval $J(a) = J(b) = J$. What do the boundary conditions now imply? Explain why in this situation (despite having a non-zero current J) the distribution $P(x)$ can still be stationary.

In this second part, we will show that the 1-D Fokker-Planck equation (Eqn. 2) can be formally made equivalent to a 1-D Schrödinger equation. This requires several transformations:

4. First, that you can always transform the 1-D FP equation with a drift coefficient $A(x)$ and a diffusion coefficient $D(x)$ into a form where $D(x) = D$ is a constant. Show that the transformation at accomplishes this is given by $y = y(x) = \int^x \sqrt{D/D(x')}dx'$. Find the transformed drift and diffusion coefficients. Show that you can thus treat in general the probability distribution equation of $P(x,t)$ in the form:

$$L_{FP}P(x,t) = \left[\frac{\partial}{\partial x}f'(x) + D \cdot \frac{\partial^2}{\partial x^2}\right] P(x,t) = \frac{\partial}{\partial t}P(x,t),$$

where L_{FP} is an operator as defined in the equation, $f(x) = -\int^x A(x')dx'$, and D is constant. **Note:** Please be careful about the order of applying operators.

5. Show that one can write the Fokker Planck Operator $L_{FP} = \frac{\partial}{\partial x}f'(x) + D \cdot \frac{\partial^2}{\partial x^2}$ in the FP equation $L_{FP}P(x,t) = \frac{\partial}{\partial t}P(x,t)$ in the form: $L_{FP} = \frac{\partial}{\partial x}D \cdot e^{-\Phi(x)} \cdot \frac{\partial}{\partial x}e^{\Phi(x)}$, where $\Phi(x)$ is the previously calculated function in part 1.

6. The Fokker-Planck operator L_{FP} is obviously not Hermitian (it acts differently on a function on the right and left). However, the operator $e^{\Phi(x)}L_{FP}$ is Hermitian, i.e. that its action to the right and to the left is identical[1]. The same is also true for the operator $L = e^{\Phi(x)/2}L_{FP}e^{-\Phi(x)/2}$. Show that L has the same form as the single particle Hamiltonian operator in quantum mechanics, i.e. $L = D\frac{\partial^2}{\partial x^2} - V(x)$, where $V(x) = \frac{1}{4}[f'(x)]^2/D - \frac{1}{2}f''(x)$, and where $\Phi(x) = f(x)/D$.

[1] i.e. for the boundary condition that $S(x_{min}) = S(x_{max}) = 0$, a Hermitian operator $\hat{L}$ implies $\int_{x_{min}}^{x_{max}} W_1(x)\hat{L}W_2(x)dx = \int_{x_{min}}^{x_{max}} W_2(x)\hat{L}W_1(x)dx$. Here, the functions $W_{1,2}(x)$ are two different solutions of the Fokker-Planck equation.

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