Dielectric Image Methods

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1 Problem

The method of images is most often employed in electrostatic examples with point or line charges in vacuum outside conducting planes, cylinders or spheres. Develop similar prescriptions for the electric scalar potential in examples where the conductor is a linear, isotropic dielectric medium with relative permittivity ϵ.

2 Solution

In a linear isotropic dielectric medium with relative permittivity ϵ, the electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$ are related by $\mathbf{D} = \epsilon\mathbf{E}$ (in Gaussian units). We assume that there are no free charges in/on the dielectric medium, such that $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$, and hence $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ within the dielectric medium. The net polarization charge density, if any, resides only on the surface of the dielectric medium.

Of course, $\boldsymbol{\nabla} \times \mathbf{E} = 0$ in static examples, so the electric field can be related to a scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla} V$. Thus, inside a linear, isotropic dielectric medium, the scalar potential obeys Laplace's equation, $\nabla^2 V = 0$, and the scalar potential can be represented by familiar Fourier series in rectangular, cylindrical and spherical coordinates (and 8 other coordinate systems as well).

If the interface between the dielectric medium and vacuum (or another dielectric medium) supports no free charge, then the normal component of the displacement field $\mathbf{D}$ and the tangential component of the electric field $\mathbf{E}$ are continuous across that interface. The latter condition is equivalent to the requirement that the potential V be continuous across the interface.

2.1 Point Charge Outside a Dielectric Half Space

We first consider the case of a dielectric medium with relative permittivity ϵ in the half space $z < 0$ with a point charge q at $(x, y, z) = (0, 0, a)$, where otherwise the region $z > 0$ is vacuum. The image method is to suppose that the potential in the region $z > 0$ is that due to the original point charge q at $(0, 0, a)$ plus an image charge q' at $(0, 0, -b)$, and that the potential in the region $z < 0$ is that due to the original point charge plus a point charge q'' at $(0, 0, c)$.

According to the suggested image method, the electric scalar potential at $(x, 0, z > 0)$ is

$$V(x, 0, z > 0) = \frac{q}{[x^2 + (z - a)^2]^{1/2}} + \frac{q'}{[x^2 + (z + b)^2]^{1/2}}\,, \tag{1}$$

and that at $(x, 0, z < 0)$ is

$$V(x, 0, z < 0) = \frac{q}{[x^2 + (z - a)^2]^{1/2}} + \frac{q''}{[x^2 + (z - c)^2]^{1/2}}, \tag{2}$$

Then, continuity of the potential V across the plane $z = 0$ requires that

$$b = c, \qquad \text{and} \qquad q'' = q'. \tag{3}$$

Continuity of D_z across the plane $z = 0$ requires that $E_z(x, 0, 0^+) = \epsilon E_z(x, 0, 0^+)$, *i.e.*,

$$\frac{\partial V(x, 0, 0^+)}{\partial z} = \epsilon \frac{\partial V(x, 0, 0^-)}{\partial z}, \tag{4}$$

$$\frac{qa}{[x^2 + a^2]^{3/2}} - \frac{q'b}{[x^2 + b^2]^{3/2}} = \epsilon \left(\frac{qa}{[x^2 + a^2]^{3/2}} + \frac{q'b}{[x^2 + b^2]^{3/2}} \right), \tag{5}$$

which implies that

$$a = b = c, \qquad \text{and} \qquad q'' = q' = -q \frac{\epsilon - 1}{\epsilon + 1}. \tag{6}$$

The potential and electric field in the region $z < 0$ are as if that region were vacuum and the original charge q were replaced by charge $q + q'' = 2q/(\epsilon + 1)$. In the region $z > 0$ the potential and field are due to the original charge q plus an image charge q''.

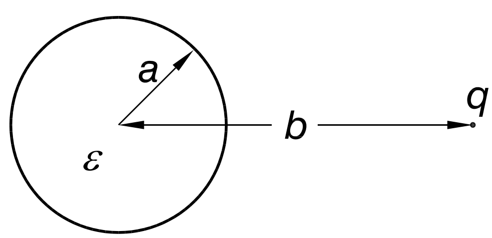
In the limit that $a = 0$ (such that charge q lies on the interface between the dielectric and vacuum) the electric field in vacuum is also as if the region $z < 0$ were vacuum but the charge were $2q/(\epsilon + 1)$.[1]

In the limit that $\epsilon \to \infty$ we obtain the image prescription for a point charge above a grounded, conducting plane; the potential above the plane is that due to the original charge plus an image charge $-q$ at $(0, 0, -a)$, and the potential below the plane is zero.

2.2 Line Charge and Dielectric Cylinder

This section follows secs. 404-406 of [1]. See [2] for an extension to case of a time-harmonic line charge.

We next consider the case of a dielectric cylinder of radius a and relative permittivity ϵ when a thin wire that carries charge q per unit length is located in vacuum at distance $b > a$ from the center of the cylinder.



[1]If the space $z > 0$ were also a dielectric of constant ϵ then the field everywhere is as if the charge q were actually q/ϵ, which effect is sometimes called "screening" with screening factor ϵ. The case of a charge on the interface between a dielectric half space and vacuum can also be described as if the charge is "screened", but with a screening factor of $(\epsilon + 1)/2$.

In this two dimensional problem we take the axis of the cylinder to be z axis, and take the position of the wire to be $(r, \theta) = (b, 0)$ in a cylindrical coordinate system. Then, the potential has the symmetry $V(r, -\theta) = V(r, \theta)$, so the Fourier expansion for the potential contains terms in $\cos n\theta$, but not $\sin n\theta$.

The potential due to the wire in the absence of the dielectric cylinder has the general form

$$V_{\rm wire}(r, \theta) = \begin{cases} a_0 + \sum_{n=1} a_n \left(\frac{r}{b}\right)^n \cos n\theta & (r < b), \\ a_0 + b_0 \ln \frac{r}{b} + \sum_{n=1} a_n \left(\frac{b}{r}\right)^n \cos n\theta & (r > b), \end{cases} \tag{7}$$

since the potential should not blow up at the origin, should be continuous at $r = b$, and can have a logarithmic divergence at infinity. For large r the electric field due to the wire is $\mathbf{E}_{\rm wire} = 2q\,\hat{\mathbf{r}}/r$, and the corresponding asymptotic potential is defined to be $V_{\rm wire} = -2q \ln r$. Thus,

$$a_0 = -2q \ln b, \qquad b_0 = -2q. \tag{8}$$

The remaining coefficients a_n are determined the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ by considering a Gaussian surface (of unit length in z) that surrounds the cylindrical shell (b, θ):

$$4\pi q_{\rm in} = \int \mathbf{E} \cdot d\mathbf{Area} = b \int d\theta\ (E_{r^+} - E_{r^-}). \tag{9}$$

From this we learn that

$$\begin{aligned} 4\pi q \delta(\theta) &= b(E_{r^+} - E_{r^-}) = b\left(-\frac{\partial V_{\rm wire}(b^+)}{\partial r} + \frac{\partial V_{\rm wire}(b^-)}{\partial r}\right) \\ &= 2q + 2\sum_n n a_n \cos n\theta. \end{aligned} \tag{10}$$

Multiplying eq. (10) by $\cos n\theta$ and integrating over θ we find

$$a_n = \frac{2q}{n}. \tag{11}$$

Then, the potential due to the wire can be written as

$$V_{\rm wire}(r, \theta) = \begin{cases} 2q[-\ln b + \sum_{n=1} \frac{1}{n} \left(\frac{r}{b}\right)^n \cos n\theta] & (r < b), \\ 2q[-\ln r + \sum_{n=1} \frac{1}{n} \left(\frac{b}{r}\right)^n \cos n\theta] & (r > b), \end{cases} \tag{12}$$

When the dielectric cylinder is present it supports a polarization charge density that results in an additional scalar potential which can be expanded in a Fourier series similar to eq. (7),

$$V_{\rm cylinder}(r, \theta) = \begin{cases} \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (r < a), \\ \sum_{n=1} A_n \left(\frac{a}{r}\right)^n \cos n\theta & (r > a), \end{cases} \tag{13}$$

noting that the dielectric cylinder has zero total charge, so its potential goes to zero at large r. The coefficients A_n can be evaluated by noting that the radial component of the total electric displacement field $\mathbf{D} = \epsilon\mathbf{E}$ is continuous across the boundary $r = a$,

$$\epsilon \frac{\partial V(a^-)}{\partial r} = \frac{\partial V(a^+)}{\partial r}\,, \tag{14}$$

$$\epsilon \sum_{n=1} \left[\frac{2q}{a} \left(\frac{a}{b} \right)^n + \frac{nA_n}{a} \right] \cos n\theta = \sum_{n=1} \left[\frac{2q}{a} \left(\frac{a}{b} \right)^n - \frac{nA_n}{a} \right] \cos n\theta, \qquad (15)$$

$$A_n = -\frac{2q}{n} \frac{\epsilon - 1}{\epsilon + 1} \left(\frac{a}{b} \right)^n, \qquad (16)$$

$$V_{\rm cylinder}(r,\theta) = \begin{cases} -2q\frac{\epsilon-1}{\epsilon+1} \sum_{n=1} \frac{1}{n} \left(\frac{r}{b}\right)^n \cos n\theta & (r < a), \\ -2q\frac{\epsilon-1}{\epsilon+1} \sum_{n=1} \frac{1}{n} \left(\frac{a^2/b}{r}\right)^n \cos n\theta & (r > a), \end{cases} \qquad (17)$$

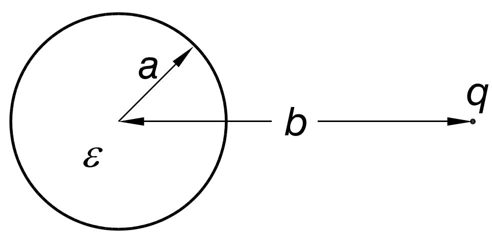
Outside the dielectric cylinder the potential is the same as if the cylinder were replaced by a wire at the origin of charge $q(\epsilon - 1)/(\epsilon + 1)$ per unit length, and an oppositely charged wire at $(r, \theta) = (a^2/b, 0)$. Inside the cylinder, the potential is, to within a constant, as if charge per unit length $-q(\epsilon - 1)/(\epsilon + 1)$ had been added to that of the original wire, bringing its effective linear charge density to $2q/(1 + \epsilon)$.

In the limit $\epsilon \to \infty$ we obtain the image prescription for a charged wire outside an neutral, conducting cylinder; the potential outside the cylinder is as if it were replaced by a wire of linear charge density $-q$ at $(r, \theta) = (a^2/b, 0)$ plus a wire of line charge density q along the z-axis, and the potential inside the cylinder is constant.

Another limit of interest is that the radius a of the dielectric cylinder goes to infinity while distance $d = b - a$ remains constant, such that the dielectric cylinder becomes a half space. For the potential outside the dielectric, the line charge is at depth $a - a^2/b \to d$ below the planar surface of the dielectric, as previously found in sec. 2.1.

2.3 Point Charge Outside a Dielectric Sphere

We finally consider the case of a dielectric sphere of radius a and relative permittivity ϵ when a point charge q per unit length is located in vacuum at distance $b > a$ from the center of the cylinder .



In this three dimensional problem we take the center of the sphere to be the origin, and take the position of the point charge to be $(r, \theta, \phi) = (b, 0, 0)$ in a spherical coordinate system. The geometry is azimuthally symmetric, so the potential does not depend on coordinate ϕ.

The potential due to the point charge q in the absence of the dielectric sphere has the general form

$$V_q(r,\theta,\phi.) = \begin{cases} \sum_{n=0} a_n \left(\frac{r}{b}\right)^n P_n(\cos\theta) & (r < b), \\ \sum_{n=0} a_n \left(\frac{b}{r}\right)^{n+1} P_n(\cos\theta) & (r > b), \end{cases} \qquad (18)$$

where the P_n are Legendre Polynomials of integer order. The coefficients a_n are determined the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ by considering a Gaussian surface that surrounds the spherical shell (b, θ, ϕ):

$$4\pi q_{\rm in} = \int \mathbf{E} \cdot d\mathbf{Area} = 2\pi b^2 \int_{-1}^{1} d\cos\theta\ (E_{r^+} - E_{r^-}). \tag{19}$$

From this we learn that

$$\begin{aligned} 2q\,\delta(\cos\theta) &= b^2(E_{r^+} - E_{r^-}) = b^2\left(-\frac{\partial V_q(b^+)}{\partial r} + \frac{\partial V_q(b^-)}{\partial r}\right) \\ &= b\sum_n (2n+1) a_n P_n(\cos\theta). \end{aligned} \tag{20}$$

Multiplying eq. (20) by $P_n(\cos\theta)$ and integrating over $\cos\theta$ we find

$$a_n = \frac{q}{b}. \tag{21}$$

Then, the potential due to the charge q can be written as

$$V_q(r,\theta) = \begin{cases} \frac{q}{b}\sum_{n=0}\left(\frac{r}{b}\right)^n P_n(\cos\theta) & (r < b), \\ \frac{q}{b}\sum_{n=0}\left(\frac{b}{r}\right)^{n+1} P_n(\cos\theta) = \frac{q}{r}\sum_{n=0}\left(\frac{b}{r}\right)^n P_n(\cos\theta) & (r > b), \end{cases} \tag{22}$$

When the dielectric sphere is present it supports a surface polarization charge density that results in an additional scalar potential which can be expanded in a Fourier series similar to eq. (18),

$$V_{\rm sphere}(r,\theta) = \begin{cases} \sum_{n=0} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) & (r < a), \\ \sum_{n=0} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) & (r > a). \end{cases} \tag{23}$$

The coefficients A_n can be evaluated by noting that the radial component of the total electric displacement field $\mathbf{D} = \epsilon\mathbf{E}$ is continuous across the boundary $r = a$,

$$\epsilon\frac{\partial V(a^-)}{\partial r} = \frac{\partial V(a^+)}{\partial r}, \tag{24}$$

$$\epsilon\sum_{n=0}\left[\frac{nq}{ab}\left(\frac{a}{b}\right)^n + \frac{nA_n}{a}\right] P_n(\cos\theta) = \sum_{n=0}\left[\frac{nq}{ab}\left(\frac{a}{b}\right)^n - \frac{(n+1)A_n}{a}\right] P_n(\cos\theta), \tag{25}$$

$$A_n = -\frac{q}{b}(\epsilon - 1)\frac{n}{n(\epsilon+1)+1}\left(\frac{a}{b}\right)^n, \tag{26}$$

$$V_{\rm sphere}(r,\theta) = \begin{cases} -\frac{q}{b}(\epsilon-1)\sum_{n=0}\frac{n}{n(\epsilon+1)+1}\left(\frac{r}{b}\right)^n P_n(\cos\theta) & (r < a), \\ -\frac{qa/b}{a^2/b}(\epsilon-1)\sum_{n=0}\frac{n}{n(\epsilon+1)+1}\left(\frac{a^2/b}{r}\right)^{n+1} P_n(\cos\theta) & (r > a), \end{cases} \tag{27}$$

The potential of the dielectric sphere does not quite have the form of the potential due to point charges. Hence, there is no general image method for the case of a point charge outside a dielectric sphere.

In the limit $\epsilon \to \infty$ we do obtain the image prescription for a point charge outside an neutral, conducting sphere; the potential outside the cylinder is as if it were replaced by a point charge $-qa/b$ at $(r, \theta, \phi) = (a^2/b, 0, 0)$ plus a charge qa/b at the origin, and the potential inside the sphere is constant.

3 Conducting Dielectric Media

In general a dielectric medium has a small electrical conductivity σ. Then if an electric field exists inside the medium, a free-current density flows according to

$$\mathbf{J}_{\rm free} = \sigma \mathbf{E} = \frac{\sigma \mathbf{D}}{\epsilon}, \tag{28}$$

where ϵ is the (relative) permittivity. Conservation of free charge can then be expressed as

$$\frac{\partial \rho_{\rm free}}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\rm free} = -\frac{\sigma \boldsymbol{\nabla} \cdot \mathbf{D}}{\epsilon} = -\frac{4\pi\sigma}{\epsilon}\rho_{\rm free}\,, \tag{29}$$

such that in the absence of an energy flow to maintain the electric field, the free charge distribution decays according to

$$\rho_{\rm free}(t) = \rho_0\, e^{-t/\tau}. \tag{30}$$

with time constant $\tau = \epsilon/4\pi\sigma$. For metals this time constant is so short that the above calculation should be modified to include wave motion [3], but for dielectric with low conductivity this approximation is reasonable.

If an external charge is brought near a conducting dielectric medium, the potentials and associated charge distributions found in sec. 2 apply only for times small compared to the relaxation time τ. At longer times the medium behaves like a static conductor, with zero internal electric field, for which the potentials and charge distributions are the familiar versions for good conductors. For example, glass has (relative) dielectric constant ≈ 4 and electrical conductivity $\approx 10^{-4}$ Gaussian units, and hence $\tau_{\rm glass} \approx 1$ hour. Rock has conductivity $\sigma \approx 10^6$ Gaussian units, and hence $\tau_{\rm rock} \approx 1\ \mu$s.

If the time structure of the external charges and currents is known, it is preferable to perform a full time-domain analysis. For a review, see [4]. For an incident wave of angular frequency ω the medium can be approximated as a good conductor if $\omega \ll 1/\tau$ and as a nonconductor if $\omega \gg 1/\tau$. For frequencies high enough that the medium is a "good conductor," the waves are attenuated as the pentrate into the medium over the skin depth, $d = c/\sqrt{2\pi\mu\omega\sigma}$, where μ if the relative permeability of the medium. When $\omega \approx 1/\tau$, $\delta \approx \sqrt{\epsilon/\mu}\, c/2\pi\sigma$, which provides an estimate of the maximum distance a transient field can penetrate into the medium.[2] For glass, this maximum depth is very large, but for rock is of order 1 m.

References

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[2] I.V. Lindell and K.I. Nikoskinen, *Two-dimensional image image method for time-harmonic line current in front of a material cylinder*, Elec. Eng. **81**, 357 (1999), http://puhep1.princeton.edu/~mcdonald/examples/EM/lindell_ee_81_357_99.pdf

[2] Compare eq. (7.70) of [5]. This argument seems to have disppeared from the 3rd edition.

[3] H.C. Ohanian, *On the approach to electro- and magneto-static equilibrium*, Am. J. Phys. **51**, 1020 (1983),
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