Problem 1: Some Methods in the Calculus of Variations

1. Find the curve $y(x)$ that passes through the endpoints $(0,0)$ and $(1,1)$ and minimizes the functional $I[y] = \int_0^1 [(dy/dx)^2 - y^2]dx$. What is the minimum value of the integral? Evaluate $I[y]$ for a straight line $y = x$ between the points $(0,0)$ and $(1,1)$.

2. You are mountain climbing on a conical peak described by the equation $z = -\sqrt{x^2 + y^2}$. There is a storm coming and you need to take refuge quickly. What is the equation of the shortest path to the refuge at position $(-1, 0, -1)$ if you are currently located at $(1, 0, -1)$.

3. A disk of radius R rolls without slipping inside the parabola $y = ax^2$. Find the equation of constraint. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

4. A flexible cable of a given length is suspended from two fixed points. Using the method of Lagrange multiplier, find the curve that will minimize the total gravitational potential energy of the cable.

5. Show that the Euler equation corresponding to the integral

$$J[y(x)] = \int_{x_1}^{x_2} f(y, y_x, y_{xx}, x)dx,$$

where $y_{xx} = d^2y/dx^2$, is given by

$$\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y_{xx}}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) + \left(\frac{\partial f}{\partial y}\right) = 0$$

Note: In order to obtain this equation, the variation as well its first derivative need to be set to zero at the end points.

Problem 2: Lagrangian and Hamiltonian Mechanics

1. A non-relativistic particle that is moving in an electromagnetic field described by the scalar potential ϕ and the vector potential $\mathbf{A}$ is governed by the Lagrangian
$$L = \frac{m\mathbf{v}^2}{2} + q(\mathbf{v} \cdot \mathbf{A}) - q\phi$$
where m and q are the mass and the charge of the particle.

   (a) Find the Hamiltonian of the system.

   (b) Show that the equation of motion of the particle is given by
$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$
   where $\mathbf{E}$ and the $\mathbf{B}$ are the electric and the magnetic fields given by
$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{dt}, \text{ and } \mathbf{B} = \nabla \times \mathbf{A}.$$

   Note: The scalar and the vector potential (ϕ and $\mathbf{A}$) are dependent on time as well as space. Also, since $\mathbf{A}$ depends on time as well as space, we have
$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}$$

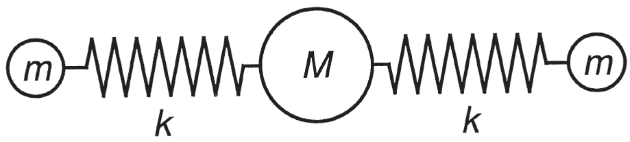
2. Find the equation of motion corresponding to the Lagrangian
$$L = e^{-(x^2+\dot{x}^2)} + 2\dot{x}e^{-x^2}\int_0^{\dot{x}} d\alpha e^{-\alpha^2}$$
   Obtain the energy integral for the system and also construct a simpler Lagrangian that will lead to the same equation of motion.

3. If a system has the Lagrangian
$$L = \frac{1}{2}G(q,t)\dot{q}^2 + F(q,t)\dot{q} - V(q,t)$$
   show that the corresponding Hamiltonian is given by
$$H = \frac{[p - F(q,t)]^2}{2G(q,t)} + V(q,t)$$
   where $p = G(q,t)\dot{q} + F(q,t)$.

Problem 3: Oscillation

1. Explain what is meant by a *normal mode* for an oscillating system.

2. A model for a water molecule consists of two masses m , each connected to a central mass M by a spring. The two springs are identical and each has a spring constant k. Consider the motion of this system when all the masses are constrained to lie along the same straight line, as shown in the figure below.

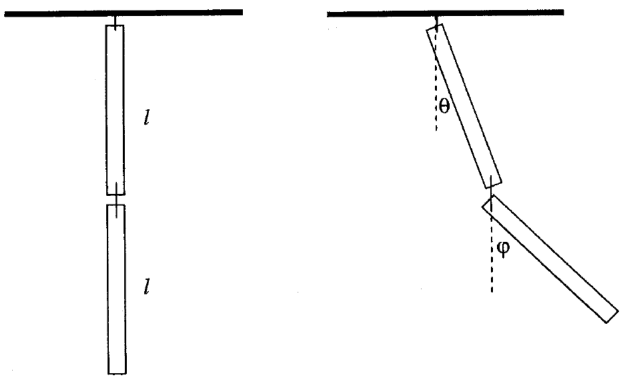
   

   Show that the normal mode (angular) frequencies are

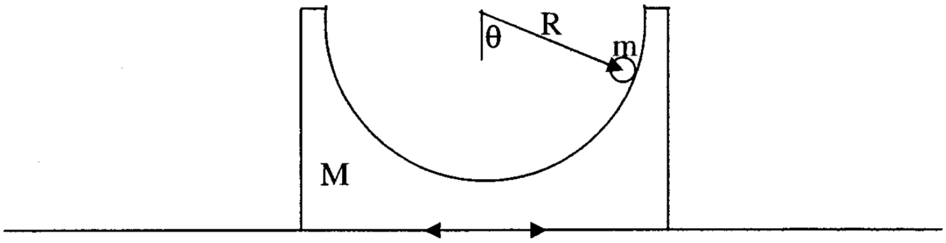
   $$0, \qquad (k/m)^{1/2}, \qquad \text{and} \qquad (k/m + 2k/M)^{1/2}.$$

   Find the displacements of the masses in each of the three normal modes.

3. Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal mode.

4. A particle of mass m is constrained to slide without friction on the surface of circular bowl of mass M. The particle remains in the vertical plane (xz-plane). The circular bowl has an inner radius R and is free to slide along the horizontal surface without friction. Find the frequency of the normal mode

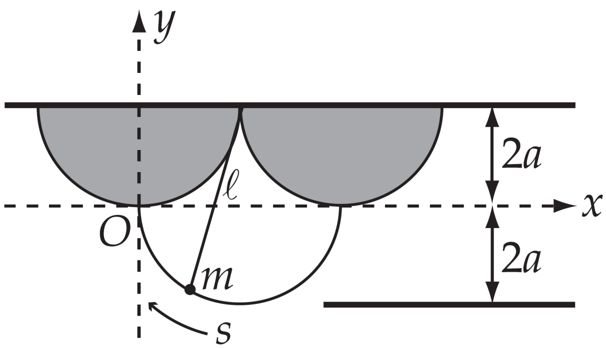


of this system for small oscillations around the equilibrium position at the bottom of the bowl. Describe the motion for this normal mode oscillation.

5. A pendulum is suspended from the cusp of a cycloid cut in rigid support (figure below). The path described by the pendulum bob is cycloidal and is given by

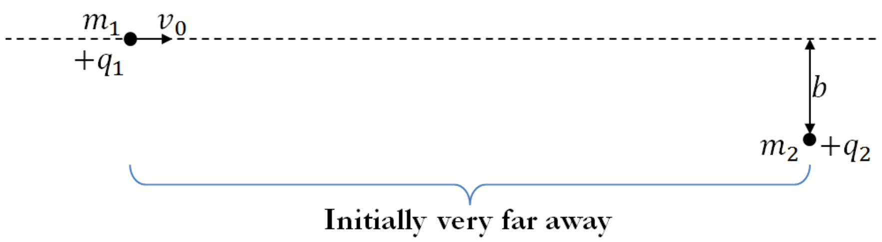
$$x = a(\phi - \sin\phi), y = a(\cos\phi - 1)$$

where the length of the pendulum is $l = 4a$, and where ϕ is the angle of rotation of the circle generating the cycloid. Show that the oscillations are exactly isochronous with a frequency of $\omega_0 = \sqrt{g/l}$, independent of the amplitude.



Problem 4: Central Potential

1. A particle of mass m_1 and electric charge $+q_1$ moves from infinity with initial speed v_0. Particle m_1 approaches the vicinity of a second particle of mass m_2 and electric charge $+q_2$ initially at rest. The perpendicular distance between m_2 and the initial path of m_1, sometimes called the impact parameter, is b.



   (a) Write down the appropriate expression for the effective potential $V(r)$ for this system, where r is the distance between m_1 and m_2.

   (b) What is the distance of closest approach between the two masses?

   (c) Determine the magnitude and direction of the final velocity of m_1.

   (d) Determine the magnitude and direction of the final velocity of m_2.

2. The interaction potential energy between two planets of mass m_1 and m_2 is given by $V(r) = -\frac{k}{r}$, where r denotes the separation between the two planets. At time $t = 0$, the velocity of mass m_1 and m_2 are equal to $\vec{v_0}$ and $-\vec{v}_0$ respectively, with $\vec{v_0}$ perpendicular to the line connecting the two planets. The initial separation of the two masses is r_0. What is the maximum value of v_0 such that r remains bounded? Assuming the condition is satisfied, at what time $T > 0$ does the distance between the two planets become r_0 again?

   Hint: try using substitution $r = \frac{a+b}{2} + \frac{a-b}{2}\sin(\alpha)$ for an integral of the form $\int_a^b \frac{f(r)}{\sqrt{(a-r)(r-b)}} dr$

3. A particle with mass m moves in a central force field given by

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3},$$

   where $k, \lambda > 0$. Show that the motion is a precessing ellipse. Consider the cases $\lambda < l^2/\mu$, $\lambda = l^2/\mu$, and $\lambda > l^2/\mu$

4. A planet of mass m moves in a central force field described by $F(r) = -\frac{k}{r^2}$. In this problem, we are going to re-derive the trajectory of the planet without using effective potential.

   (a) Show that a vector $\vec{A}$ defined by

   $$\vec{A} = \vec{p} \times \vec{L} - \frac{mk}{r}\vec{r},$$

   where $\vec{p}$ and $\vec{L}$ denote the momentum and the angular momentum of the planet respectively, is a constant of motion.

   (b) By considering $\vec{A} \cdot \vec{L}$ and $\vec{A} \cdot \vec{r}$, derive the trajectory of the planet. How do you interpret geometrically this vector?

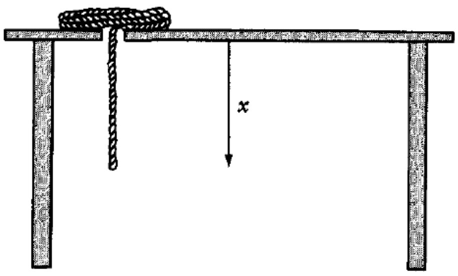
5. Consider a force law of the form

   $$F(r) = -\frac{k}{r^2} - \frac{k'}{r^4}$$

   Show that if $\rho^2 k > k'$, then a particle can move in a stable circular orbit at $r = \rho$

Problem 5: Dynamics of a system of particles

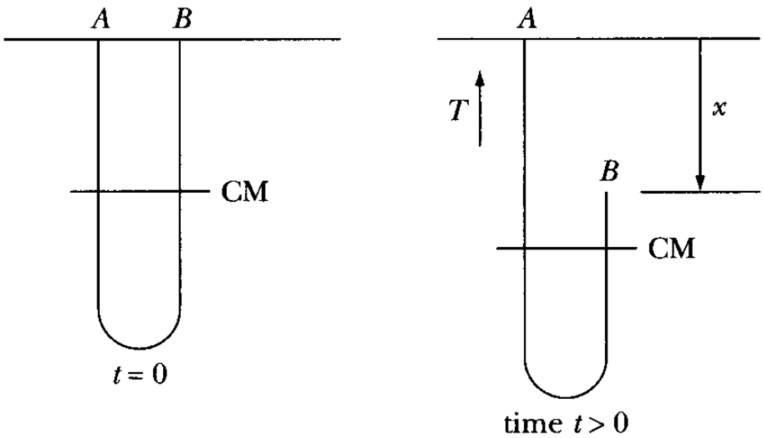
1. A smooth rope is place above a hole in a table. One end of the rope falls though the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity and acceleration of the rope as a function of the distance to the end of the rope x. Ignore all friction. The total length of the rope is L.



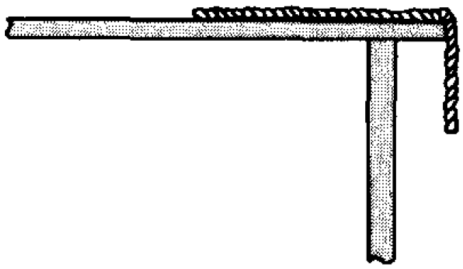
2. A chain of uniform linear mass density ρ, length b, and mass M ($\rho = M/b$) hangs as shown in the following figure. At time t=0, the ends A and B are adjacent, but end B is released. Find the tension in the chain at point A after end B has fallen a distance x by

   (a) assuming free fall, and

   (b) by using energy conservation.

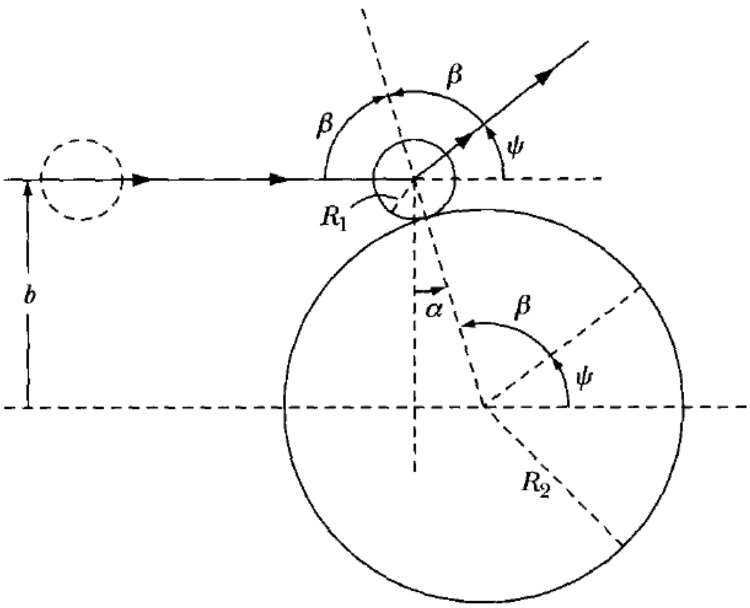


3. A flexible rope of length 1.0 m slides from a frictionless table top as shown in the following figure. The rope is initially released from rest with 30 cm hanging over the edge of the table. Find the time at which the left end of the rope reaches the edge of the table.



4. A billiard ball of initial velocity u_1 collides with another billiard ball (same mass) intinally at rest. The first ball moves off at $\psi = 45°$. For an elastic collision, what are the velocities of both balls after the collisions?

5. Consider molecules of radius R_1 moving toward the right with identical velocities scattering from dust particles of radius R_2 that are at rest. Consider both as hard spheres and find the differential and total scattering cross sections. Interpret this result.



Problem 6: Relativistic particles

Please do not set $\mathbf{c} = 1$!!!

1. The Lagrangian for a relativistic particle moving in a potential $U(r)$ is given by
$$L = -mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - U(r),$$
where m is the mass of the particle and c is a constant that denotes the speed of light.
   (a) Obtain the equation of motion of the relativistic particle.
   (b) What happens to the equation of motion when $|\mathbf{v}| \ll c$?
   (c) Find the Hamiltonian for this relativistic particle.

2. Consider a one-dimensional, relativistic harmonic oscillator for which the Lagrangian is
$$L = mc^2\left(1 - \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\right) - \frac{1}{2}kx^2$$
Obtain the Lagrange equation of motion and show that it can be integrated to yield
$$E = mc^2 + \frac{1}{2}ka^2$$
where a is the maximum excursion from equilibrium of the oscillating particle. Show that the period
$$\tau = 4\int_{x=0}^{x=a} dt$$
can be expressed as
$$\tau = \frac{2a}{\kappa c}\int_0^{\pi/2} \frac{1 + 2\kappa^2\cos^2\phi}{\sqrt{1 + \kappa^2\cos^2\phi}}d\phi$$
Expand the integrand in powers of $\kappa = (a/2)\sqrt{k/mc^2}$ and show that, to first order in κ,
$$\tau \approx \tau_0\left(1 + \frac{3}{16}\frac{ka^2}{mc^2}\right)$$
where τ_0 is the non-relativistic period for small oscillation, $2\pi\sqrt{m/k}$.

3. The relativistic charge particle Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - q\phi + q\vec{v}\cdot\vec{A}$$

Derive the relativistic equations of motion of the charge particle.