Problem 1:Warming up

Gauss' law: differential form

Assuming the electric potential is space can be described by the well-known Yukawa potential

$$V(r) = \frac{qe^{-r/a}}{4\pi\varepsilon_0 r}$$

(a) Find the electric field and charge distribution

(b) Is Gauss's law valid for this electric field? Is this electric field curl-less? Give a brief explanation.

(c) Explain briefly what does Yukawa potential represent.

Gauss' law: integral form

Consider a dielectric sphere of radius R, relative permittivity ϵ_r and a free charge density ρ with charge uniformly distributed over its volume.

(a) Derive an expression for the electric field $\mathbf{E}$ as a function of the distance $\mathbf{r}$ from the center of the sphere.

(b) Sketch $E(r)$ both inside and outside the sphere.

Dielectric media

Assuming a hollow **dielectric** sphere shell with a dielectric constant ε, the inner and outer radii are R_1 and R_2, respectively. The charge is uniformly distributed with a volume charge density ρ. Find

(a) The electric field in space $\mathbf{E}(\mathbf{r})$

(b) Volume bound charge density ρ_b (Hint: use $\rho_b = -\nabla \cdot \mathbf{P}$)

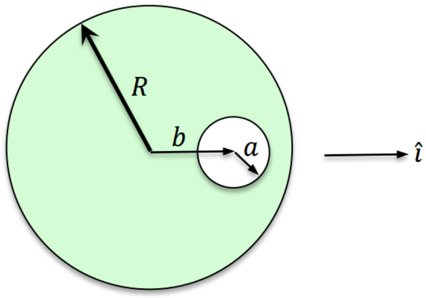
(c) Surface bound charge density σ_b on both inner and outer surfaces (Hint: $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}$)

(d) Prove that the total bound charge including both volume bound charge and surface bound charge is zero.

Problem 2: Principle of Superposition

For Electric Field

Consider an infinite insulating cylinder of radius R and charge density ρ with an off-centered cylindrical cavity of radius a as shown. Show that the electric field in the cavity is uniform.



For Magnetic Field

A hollow, infinitely long cylindrical conductor of radius R carries uniform current density. The conductor has an off-centered cylindrical cavity of radius a as shown above. Find the magnetic field inside the cavity.

Food for Thought

Redo the case for magnetic field by considering the cylinder with uniform charge density ρ moving with a velocity v in it's axis direction. (Hint: Use Edy)

Problem 3: Method of Images

Plate images

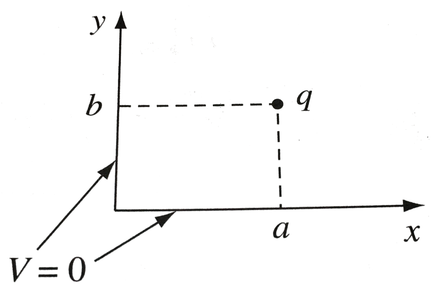
Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated . Set up the image configuration. Hints: what charges do you need and where should they be located?

(a) Calculate the potential in this region.

(b) What is the force on q?

(c) How much work did it take to bring q in from infinity? Find this quantity in two ways.

(d) Suppose the two planes don't meet at right-angle. Find the condition for the angle such that we can still apply Method of Images.



More Plate images

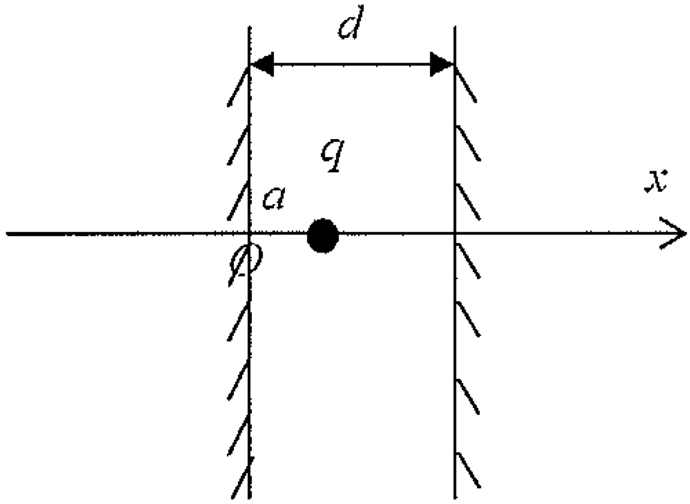
Two infinite vertical, grounded, conducting planes are separated at a distance d. A charge q is located at a distance a away from the left plane. The configuration of the planes and the charge is shown in the following figure. Assuming the charge q is put along the x-axis with the origin as the interception of x-axis and the left plane/

(a) Draw schematically the positions of mirror charges

(b) Find the potential at any point P(x,y,z) between the two planes.

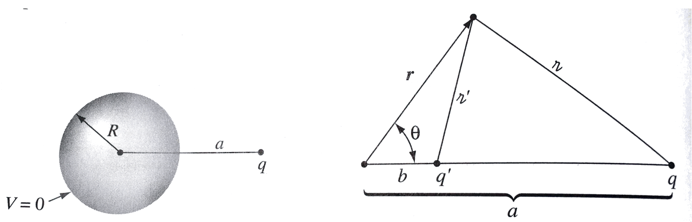
(c) Find the force exerted on q.

(d) How much work did it take to bring q in from infinity?



Sphere images

Consider a point charge q placed at a distance a from the centre of a conducting sphere of radius R. Find the (i) electric potential V outside the sphere, (ii) the image charge q' and (iii) the surface charge density σ on the sphere. (iv) Find the force of attraction between the point charge and the conducting sphere.

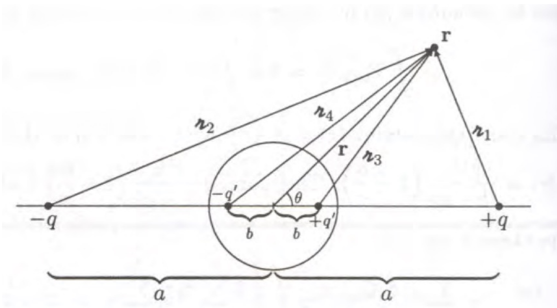


More Sphere images

To solve the example of a metal sphere of radius R placed in a uniform electric field, one can place a dipole inside the sphere and a dipole outside the sphere as shown in the diagram below. The solution was made of two terms – one due to the external field and one due to a dipole.

(a) Show that you obtain the same answer as before with this set up, $V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$, where $E_0 = -\frac{2q}{4\pi\varepsilon_0 a^2}$.

(b) What should q' and b be, and give a physical reason why $a \gg r$ is required in order for this method to give the same result as before.



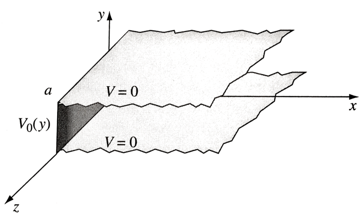
Food for Thought

What happens if we change the conducting sphere with a dielectric sphere of permittivity ε? Find the electric potential V outside the dielectric sphere of radius R with a point charge q placed at a distance a from the center.
What happens if we change the conducting plates with dielectric plates of permittivity ε? Find V for the case of a charge located between two dielectric plates.

Problem 4: Separation of Variables in Solving Poisson Equation

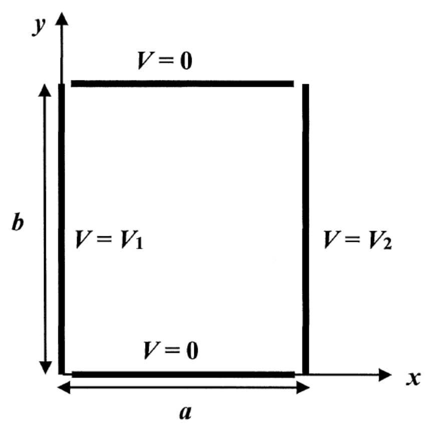
Cartesian Coordinates

Two infinite grounded metal plates lie parallel to the xz plane. One is at $y = 0$ and the other is at $y = a$. The left end at $x = 0$ is closed off by an insulating strip of infinite length, maintained at a specific potential $V_0(y)$. Find the potential inside the "slot".



More Cartesian Coordinates

The following figure shows the cross section through an infinitely long channel. Two sides are connected to ground potential and the other two sides are connected to two differential potentials. Find the electric potential inside the cross section. [Hint: You can use principle of superposition.]



Cylindrical Coordinates

(a) Derive the potential within the space between two (infinitely) long coaxial concentric cylinders $V(s, \phi, z)$ (in cylindrical coordinates), starting from Laplace's equation. The boundary conditions are that the inner cylinder (radius a) is at a potential of V_0 and the outer cylinder (radius b) is at 0 V.

(b) Hence, find the electric field $\vec{E}(s, \phi, z)$ within this region.

More Cylindrical Coordinates

Find the potential outside an infinitely long metal pipe of radius R, placed at right angles to an otherwise uniform electric field $\vec{E}_0$. Hence find the surface charge induced on the pipe.

Spherical Coordinates: Metal sphere in a uniform electric field

A metal sphere of radius R is placed in a uniform electric field pointing in the $z-$direction, $\vec{E} = E_0\hat{z}$. Find the electric potential and electric field in the region outside the sphere.

More Spherical Coordinates: Metal sphere in a uniform electric field with net charge Q

Derive the potential outside a charged metal sphere of charge Q, radius R, placed in a uniform electric field $\vec{E}_0$. Show that it is the sum of the result from previous question, plus a "point charge" $\frac{kQ}{r}$ term. Note: Set $V = 0$ on the xy plane, far from the sphere. Take care to explain each step carefully to demonstrate your understanding.

Even More Spherical Coordinates

The potential $V_0(\theta) = \alpha \sin^2 \theta$ is specified on the surface of a hollow sphere of radius R. Find the potential (i) inside the sphere, and (ii) outside the sphere. (iii) Find the surface charge density of the sphere.

Problem 5: Magnetic fields

Ampere's law

A long, straight solid cylinder of radius a, oriented with its axis in the z-direction, carries a current density $\vec{J} = J_0 \left(1 - \frac{s^2}{a^2}\right) \hat{z}$, where s is the perpendicular distance from the z-axis.

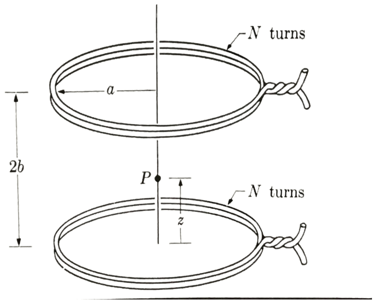
1. Find the total current passing through the entire cross section of the wire by summing up infinitesimal currents through cross-sectional rings of radius s and thickness ds centred at the axis.

2. Using Ampere's Law, derive expressions for the magnetic field $\vec{B}$ in the regions $s \leq a$ and $s > a$.

3. Hence, determine the location at which the magnetic field is a maximum.

Biot-Savart law

Using the Biot-Savart law, determine the magnetic field $\vec{B}$ on the axis of a pair of circular coils of wire with N turns each, radius a, separated by a distance $2b$ along their axis of symmetry. This set up is known as a Helmholtz coil. Take the z-axis to be along the axis of symmetry of the set up. Once you have determined $\vec{B}$, evaluate the first and second derivatives along the z-axis, $\frac{\partial B_z}{\partial z}$, $\frac{\partial^2 B_z}{\partial z^2}$. Determine the location and condition at which these derivatives vanish.



One can reverse the current in one coil. This set up is called the anti Helmholtz coil. This produce a zero magnetic field at the center of the set up. Calculate the Field gradient for the anti Helmholtz set up.

Magnetic cloaking

A long cylindrical shell (outer radius b, inner radius a, relative permeability μ) with a concentric superconducting shell (radius a) inside is oriented normal to a uniform magnetic induction field B_0. Show that the condition for no distortion of the external magnetic field is

$$\mu = \frac{a^2 + b^2}{b^2 - a^2}$$

Redo the calculation for spherical shells and find the condition is now

$$\mu = \frac{2b^3 + a^3}{2\left(b^3 - a^3\right)}$$

Problem 6: Force

Force on hemispheres

Find the net force that the southern hemisphere of a uniformly charged sphere exerts on the northern hemisphere. The sphere is of radius R and total charge Q.

Force on a dielectric medium

An uncharged, spherical, conducting shell of mass m floats with one quarter of its volume submerged in a liquid dielectric of dielectric constant K. To what potential must the sphere be charged to float half submerged? (Hint: assume the electric field of the half-submerged, charged shell to be purely radial. Can you think of why it should be from boundary conditions and symmetry?