Uniform accelerations in special relativity

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In special relativity, an object moving in a fixed direction is said to be undergoing uniform acceleration if its 4-acceleration a^μ has a constant magnitude. In this problem, we will study some interesting phenomena associated with uniform acceleration motion in special relativity.

A. Uniformly accelerating point particle

For simplicity, we consider uniform acceleration motion with one space dimension. So the relevant Minkowski space-time is two-dimensional, with coordinates (t, x). Assume that a particle is moving with uniform acceleration $g \equiv \sqrt{a^\mu a_\mu}$. At time $t = 0$, it is at rest and located at $x = d(> 0)$.

(A1) Find the worldline of this particle, namely find the space and time coordinates of the particle in terms of the proper time as measure by the particle $(t(\tau), x(\tau))$.

(A2) Find d in terms of g such that all events on the worldline of the particle has a the same proper distance from the event $(0, 0)$. Denote the sought d as $d(g)$.

(A3) [Difficult] Now fix d as $d(g)$. We define the line of simultaneity of the particle as the line of simultaneity of the instantaneous inertial observer comoving with the particle. Prove that this line passes through $(0, 0)$ for all τ.

B. Uniformly accelerating rigid ruler

Suppose now a person carries a ruler in uniform acceleration g. At $t = 0$, the person is at rest and is located at $x = d = g^{-1}$. The ruler has a length L, and is very rigid, by which we mean that the proper length of the ruler as measured by an observer comoving with the ruler is fixed. At $t = 0$, the two ends of the ruler are located $x = d$ (end A) and $x = d + L$ (end B) respectively.

(B1) Is end B of the ruler in uniform acceleration motion? You can use the result of (A3) and justify your answer by considering the proper distance between events on the wordline of end B and the event $(0, 0)$ in the instaneous inertial frame comoving with end A.

(B2) Find the worldline of end B of the ruler.

(B3) Compare the accelerations of the two ends of the ruler.

Solution

(A1)[8'] Use proper time to parameterize the worldline,

$$a^\mu = (a^0, a^1) = \left(\frac{du^0}{d\tau}, \frac{du^1}{d\tau}\right) \tag{1}$$

One then has

$$-\left(\frac{du^0}{d\tau}\right)^2 + \left(\frac{du^1}{d\tau}\right)^2 = g^2. \tag{2}$$

Recall that 4-velocity is a unit vector

$$-(u^0)^2 + (u^1)^2 = -1. \tag{3}$$

Eliminating u^0, the equation for u^1 becomes

$$\frac{du^1}{d\tau} = \pm\sqrt{1 + (u^1)^2}. \tag{4}$$

We discard the case with a negative sign choice, since the particle is assumed to move in the positive x-direction. The solution to u^1 and x is thus

$$u^1 = \sinh g\tau, \quad x = g^{-1}\cosh g\tau + d - g^{-1}, \tag{5}$$

where the initial condition $u^1(\tau = 0) = 0$ has been used. Solving for u^0 and demanding proper time increases as t elapse forward yields

$$u^0 = \cosh g\tau, \quad t = g^{-1}\sinh g\tau \tag{6}$$

The wordline of the particle is thus

$$x^\mu = (g^{-1}\sinh g\tau, g^{-1}\cosh g\tau + d - g^{-1}), \tag{7}$$

with 4-velocity and 4-acceleration given by

$$u^\mu = (\cosh g\tau, \sinh g\tau), \qquad a^\mu = (g\sinh g\tau, g\cosh g\tau). \tag{8}$$

Clearly, $u^\mu u_\mu = -1$ and $a^\mu a_\mu = g^2$.

(A2)[2'] Note that an event on the worldline of the particle and the event $(0, 0)$ space-like separated. Suppose the event on the worldline is given by the proper time τ, the proper distance is then

$$\sqrt{-\Delta t^2 + \Delta x^2} = \sqrt{-(g^{-1}\sinh g\tau)^2 + (g^{-1}\cosh g\tau + d - g^{-1})^2}. \tag{9}$$

For this to be constant, one requires

$$d = g^{-1}. \tag{10}$$

(A3)[5'] If $d = g^{-1}$, the worldline of the particle is given by

$$x^\mu = (g^{-1} \sinh g\tau, g^{-1} \cosh g\tau), \tag{11}$$

which is a hyperbola in a t-x spacetime diagram

$$-t^2 + x^2 = g^{-2}. \tag{12}$$

The 4-velocity of the instantaneous inertial observer is given by the tangent of the worldline of the particle $u^\mu = (\cosh g\tau, \sinh g\tau)$. The tangent line to the worldline of the particle represents the time axis of the instantaneous inertial observer.

The instantaneous inertial observer's line of simultaneity and his time axis make the same angle with his light cone. Using this fact, one can determine the equation of the line of simultaneity of the instantaneous inertial observer and thus that of the particle. Note that the line of simultaneity makes up the spatial axis.

The angle made by the tangent line of the worldline of the particle and the x-axis, denoted by α, is

$$\tan \alpha = \frac{dt}{dx} = \frac{dt/d\tau}{dx/d\tau} = \frac{u^0}{u_1} = \coth g\tau. \tag{13}$$

The angle made by the line of simultaneity of the instantaneous observer is thus $\frac{\pi}{2} - \alpha$. The equation of the line of simultaneity of the instantaneous observer is thus

$$\begin{aligned} t - g^{-1} \sinh g\tau &= \tan\left(\frac{\pi}{2} - \alpha\right)\left(x - g^{-1} \cosh g\tau\right) \\ &= \tanh g\tau \left(x - g^{-1} \cosh g\tau\right) \qquad (14) \\ t &= \tanh(g\tau) x \qquad (15) \end{aligned}$$

Obviously, it passes through the origin for all τ.

(B1)[2'] End B of the ruler is in uniform acceleration motion.

Explanations of (B1) and (B2)[6'] We first note that the result of (A3) implies that from the perspective of the instantaneous inertial observer (which can be label by τ) comoving with end A, event E_O ($t = 0, x = 0$) occurs simultaneously with event E_A $x^\mu = (g^{-1}\sinh g\tau, g^{-1}\cosh g\tau)$. So this observer's reference frame, the time difference of the two events E_O and E_A is

$$\Delta t'_{OA} = 0. \tag{16}$$

The spatial difference between events E_O and E_A measured by this observer is

$$\Delta x'_{OA} = g^{-1}, \tag{17}$$

since $-\Delta t'^2 + \Delta x'^2 = g^{-2}$ is an invariant and should be satisfied.

At the same instant, the comoving inertial observer measures that end B is a distance L away from end A. Denote the event "the comoving inertial observer measures end B" as event B, then it is clear that

$$\Delta t'_{OB} = 0, \qquad \Delta x'_{OB} = g^{-1} + L. \tag{18}$$

Hence, in the perspective of the comoving observer,

$$-\Delta t'^2_{OB} + \Delta x'^2_{OB} = (g^{-1} + L)^2. \tag{19}$$

The lefthand side of the above equation is an invariant. This means that in the stationary frame, we have

$$-\Delta t^2_{OB} + \Delta x^2_{OB} = (g^{-1} + L)^2. \tag{20}$$

Since event O is has coordinates $(0, 0)$, the above equation reduces to

$$-t_B^2 + x_B^2 = (g^{-1} + L)^2. \tag{21}$$

We note that this equation is true regardless of the value of τ. Hence the worldline of end B is

$$-t^2 + x^2 = (g^{-1} + L)^2. \tag{22}$$

This is a hyperbola, indicating that end B is undergoing uniform acceleration.

(B3)[2'] The acceleration of end B can be found by comparing (22) with (12), and is given by

$$a_B = \frac{g}{1 + gL}. \tag{23}$$

Noting that

$$a_A = g, \tag{24}$$

we observe the interesting phenomenon that two ends of a uniformly accelerating rigid ruler has different accelerations in special relativity.