USEFUL EQUATIONS AND FORMULAE

Time independent perturbation theory

$$E_n^{(1)} = \langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle,$$

$$E_n^{(2)} = \sum_{k \neq n} \frac{|H'_{kn}|^2}{E_n^{(0)} - E_k^{(0)}}, \quad H'_{kn} \equiv \langle \psi_k^{(0)} | H' \psi_n^{(0)} \rangle$$

$$\det |H'_{nu,ns} - E_{nr}^{(1)} \delta_{us}| = 0, \quad (s, u = 1, 2, \ldots, \alpha), \quad H'_{nu,ns} \equiv \langle \psi_{nu}^{(0)} | H' \psi_{ns}^{(0)} \rangle$$

Variational Method

$$E[\psi] = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0.$$

Harmonic Oscillator

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \tfrac{1}{2} m \omega^2 x^2,$$

$$x = \left(\frac{\hbar}{2m\omega} \right)^{1/2} \left(a + a^\dagger \right), \quad p = -\mathrm{i} \left(\frac{\hbar m \omega}{2} \right)^{1/2} \left(a - a^\dagger \right), \quad [a, a^\dagger] = \mathbb{1}.$$

$$\hat{H} |n\rangle = \left(n + \tfrac{1}{2} \right) \hbar \omega |n\rangle, \quad n = 0, 1, 2, \ldots.$$

$$a |0\rangle = 0, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle.$$

$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x), \quad \alpha = \left(\frac{m\omega}{\hbar} \right)^{\frac{1}{2}}, \quad n = 0, 1, 2, \ldots$$

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2 - 1$$

Time-Dependent Perturbation Theory

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}_0|\psi\rangle, \quad |\psi(0)\rangle = \sum_n c_n(0)|n\rangle, \quad |\psi(t)\rangle = \sum_n c_n(0)e^{-iE_nt/\hbar}|n\rangle.$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left(\hat{H}_0 + \hat{H}'(t)\right)|\psi\rangle, \quad |\psi(0)\rangle = \sum_n c_n(0)|n\rangle, \quad |\psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle.$$

$$i\hbar\dot{c}_n(t) = \sum_m H'_{nm}(t)\,e^{i\omega_{nm}t}\,c_m(t), \quad H'_{nm}(t) = \langle n|H'(t)|m\rangle, \quad \omega_{nm} = \frac{E_n - E_m}{\hbar}.$$

(Fermi's Golden Rule)

$$\hat{H}_1(t) = \hat{H}_1\,e^{i\omega t}, \quad \omega = \frac{2\pi}{T}, \quad \Gamma_{i\to f} = \frac{P_{i\to f}}{T} = \frac{2\pi}{\hbar}\left|\langle f|\hat{H}_1|i\rangle\right|^2\,\delta(E_f - E_i - \hbar\omega).$$

Scattering

$$\psi = \psi_{\text{inc}} + \psi_{\text{sc}}, \quad \psi_{\text{inc}} = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \psi_{\text{sc}} = f(\theta,\phi)\frac{e^{ikr}}{r}, \quad \frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2.$$

First Born Approximation

$$f_k^B(\theta,\phi) = f(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})\,d^3\mathbf{r}.$$

Partial Wave Expansion

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)\,P_l(\cos\theta), \quad f_k(\theta,k) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta).$$

$$a_l(k) = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k}, \quad \sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l, \quad \sigma = \sum_{l=0}^{\infty}\sigma_l.$$

$$\alpha = -\lim_{k\to 0}\frac{\tan\delta_0}{k} \quad a_0(k) = \frac{1}{k\cot\delta_0 - ik}$$

$$P_0(\cos\theta) = 1, \quad \int_{-1}^{+1} d(\cos\theta)P_l(\cos\theta)P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}$$

Spherical polar co-ordinates

Laplacian $$\nabla^2 = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Hydrogenic wavefunctions

$$\psi_{1s} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_\mu}\right)^{3/2} e^{-Zr/a_\mu}, \quad a_\mu = \frac{(4\pi\epsilon_0)\hbar^2}{\mu e^2}$$

$$\psi_{2s} = \frac{1}{2\sqrt{2\pi}} \left(\frac{Z}{a_\mu}\right)^{3/2} \left(1 - \frac{Zr}{2a_\mu}\right) e^{-Zr/2a_\mu},$$

$$\psi_{2p_0} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_\mu}\right)^{3/2} \left(\frac{Zr}{a_\mu}\right) e^{-Zr/2a_\mu} \cos\theta,$$

$$\psi_{2p_{\pm1}} = \mp\frac{1}{8\sqrt{\pi}} \left(\frac{Z}{a_\mu}\right)^{3/2} \left(\frac{Zr}{a_\mu}\right) e^{-Zr/2a_\mu} \sin\theta e^{\pm i\phi},$$

Angular momentum matrices

$\underline{l = 1}$

$$L^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Relativistic Quantum Mechanics

(Klein-Gordan Equation)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi = 0.$$

(Dirac Equation)

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = \left(c\alpha \cdot \frac{\hbar}{\mathrm{i}}\nabla + \beta mc^2\right)\psi.$$

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad i = 1,2,3; \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(Pauli matrices)

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(metric tensors)

$$g^{\mu\nu} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = g_{\mu\nu}.$$