I. WHATEVER QUESTION NAME

A quantum state is described by a vector $|\psi\rangle$ in the Hilbert space, whereas a measurement is described as a projection onto a complete orthogonal basis $\{|\phi_i\rangle\langle\phi_i|\}$ in the Hilbert space. For simplicity suppose the Hilbert space is of dimension 2. Let $\{|0\rangle, |1\rangle\}$ be some computational basis for the Hilbert space.

Here we will consider cloning, i.e. taking a state $|\psi\rangle$ and try to get two copies of $|\psi\rangle$.

Now, suppose you're given one copy of a quantum state $|\psi\rangle$ from some set of states $\Phi = \{|\phi_i\rangle\}$. Assume that we know what states are in Φ, but the state $|\psi\rangle$ is picked from Φ with a uniform probability, so we don't know which one $|\psi\rangle$ is.

A cloning machine is a transformation C that takes $|\psi\rangle$ and some blank state (say $|0\rangle$) and maps it to $C(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle$. For simplicity, we'll only consider linear processes, i.e. $C(\alpha_0|\psi_0\rangle + \alpha_1|\psi_1\rangle) = \alpha_0 C(|\psi_0\rangle) + \alpha_1 C(|\psi_1\rangle)$.

Problem 1 (2 marks)**.** *Show that if $\Phi = \{|0\rangle, |1\rangle\}$, then you can clone $|\psi\rangle$.*

Solution. *Just measure and prepare two copies of whatever state you got.*

Problem 2 (1 mark)**.** *Show that if all the vectors in Φ are mutually orthogonal, then you can clone $|\psi\rangle$.*

Solution. *The strategy from before works just fine.*

Problem 3 (4 marks)**.** *Show that when the states in Φ are not mutually orthogonal, then we can't always succeed in cloning the state.*

Solution. *Suppose process C clones the state $|\psi_0\rangle$ and $|\psi_1\rangle$ perfectly. Then $C^\dagger C = \mathbb{1}$ at least in the subspace spanned by $\{|\psi_0\rangle, |\psi_1\rangle\}$. Then $C|\psi_i\rangle|0\rangle = |\psi_i\rangle|\psi_i\rangle$. Then $\langle\psi_0|\psi_1\rangle = (\langle\psi_0|\langle 0|)(|\psi_1\rangle|0\rangle) = (\langle\psi_0|\langle 0|C^\dagger)(C|\psi_1\rangle|0\rangle) = \langle\psi_0|\psi_1\rangle^2$, which is true only iff $\langle\psi_0|\psi_1\rangle$ is either 0 or 1. Thus if there are two non-orthogonal states in Φ, you can't always succeed.*

Problem 4 (2 marks)**.** *If Φ contains more than 2 states, can we always clone the state?*

Solution. *Nope, cause you can't have more than 2 orthogonal states in a Hilbert space of dimension 2.*

We can actually do better than this. Let's try to give a success rate on how well can we clone something. We'll “consider measure and prepare“ strategies.

Suppose $\Phi = \{|\phi_0\rangle, |\phi_1\rangle\}, |\langle\phi_0|\phi_1\rangle| = \cos\theta \neq 0$, i.e. Φ contains only two states, but they are not orthogonal.

Problem 5 (3 marks)**.** *Suppose you do your measurement in some complete orthogonal basis $\{|\eta_0\rangle\langle\eta_0|, |\eta_1\rangle\langle\eta_1|\}$, where $|\eta_0\rangle = |\phi_0\rangle$. If we get outcome 0, then we prepare two copies of $|\phi_0\rangle$. Otherwise we prepare two copies of $|\phi_1\rangle$. What's the probability of success with this strategy?*

Solution. *Work it out, what's the probability of success if you get $|\phi_0\rangle$, etc. You should get $1 - \frac{1}{2}|\langle\phi_0|\phi_1\rangle|^2$.*

Problem 6 (4 marks)**.** *What if we vary $|\eta_0\rangle, |\eta_1\rangle$? What's the probability of success as a function of $|\eta_0\rangle$? Optimize over the choice of $|\eta_0\rangle, |\eta_1\rangle$. What's the optimal probability of success?*

Solution. *Suppose $|\phi_1\rangle = \cos\theta|\phi_0\rangle + \sin\theta\left|\phi_0^\perp\right\rangle$. Follow the same steps as before, you should get*

$$\frac{1}{2} + \frac{1}{2}\left((1-\cos\theta^2)|\langle\eta_0|\phi_0\rangle|^2 - \sin\theta^2|\left\langle\eta_0|\phi_0^\perp\right\rangle|^2\right)$$

Optimizing over $|\eta_0\rangle$, we get that the probability from before is the optimal one.

Problem 7 (2 marks)**.** *Does this mean we can't reliably distinguish any two states?*

Solution. *Yeah. Otherwise we can just measure and prepare, then we can clone things perfectly.*

Problem 8 (2 marks)**.** *Suppose instead of getting just one copy, we get n copies. Show that in the limit of large n, the probability of success goes to 1.*

Solution. *The optimal probability of success goes to 1 as $|\langle\phi_1|\phi_0\rangle| \to 0$. Now, instead of considering Φ, suppose we want to distinguish the states in $\Phi^{(n)} = \{|\phi_0\rangle^{\otimes n}, |\phi_1\rangle^{\otimes n}\}$. As $n \to \infty$, the overlap between the two states goes to 0, so you can do a measurement to distinguish them and prepare according to your measurement result.*