PLANCKS 2018
Preliminaries for Team Singapore

Exercise Booklet
Round 1

Division of Physics and Applied Physics
School of Physical and Mathematical Sciences
Nanyang Technological University

Submission deadline: 2nd January 2018

Mathematics

(2 marks each, **20 marks in total**)

1. Find the Taylor expansion of the following function by any methods at $x = 1$ up to the **forth order**:

$$\frac{1}{x-3}$$

2. Evaluate the following integrals:

   (a)

$$\int (5^x - 25^x)(1 + 5^x)^{100} \, dx$$

   (b)

$$\int (\ln x)^2 \, dx$$

3. Evaluate the following integral:

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{a^2 + b^2 - 2ab\cos\theta}} d\theta$$

   where a and b are positive real numbers and $a > b$.

4. Evaluate the following integrals:

   (a)

$$\int_{-2}^{2} (2x+3)\delta(2-3x)dx$$

   (b)

$$\int_{-\infty}^{\infty} f(x)\delta(x^2 - \alpha^2)dx$$

   where $f(x)$ is a real-valued continuous function on $(-\infty, \infty)$ and α is a positive real number.

5. Solve the following initial-value problem:

$$\frac{dy}{dx} = 1 + y + x + xy \qquad y(0) = 1$$

6. Use power series to solve the initial-value problem

$$y'' + (x-1)y' + y = 0 \qquad y(1) = 0 \qquad y'(1) = 1$$

7. Find the general solution to the following differential equation:

$$y'' - y = xe^{-x}$$

8. How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 12$$

   where x_1, x_2, and x_3 are nonnegative integers and

(a) x_2 can take any nonnegative values.

(b) $x_2 \leq 5$.

Hint: You do not have to find the solutions.

9. Find eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

10. Using contour integration, evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos(\beta x)}{x^2 + 4} dx$$

where $\beta \in \mathbb{R}$.

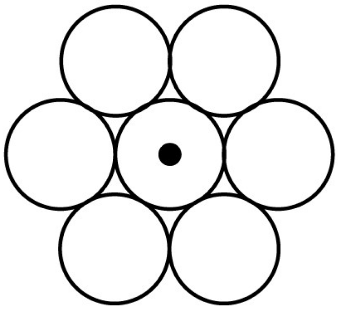
Physics

Short Questions

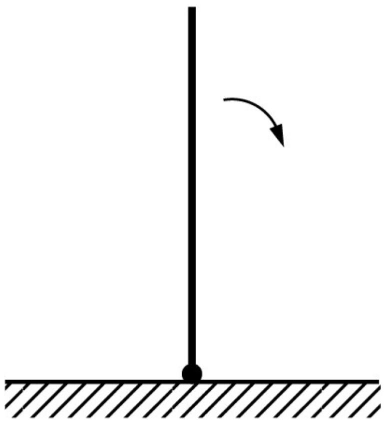
(2 marks each, **40 marks in total**. The solution to each question should be **no longer than half an A4 page** but you can write as long as you want.)

1. A simple pendulum of length l is suspended from the ceiling of an elevator that is accelerating upward with constant acceleration a. For small oscillations, what is the period T of the pendulum?

2. Seven pennies are arranged in a hexagonal, planar pattern so as to touch each neighbor, as shown in the figure below. Each penny is a uniform disk or radius r and mass m. What is the moment of inertia of the system of seven pennies about an axis that passes through the center of the central penny and is normal to the plane of the pennies?



3. A thin rod of mass M and length L is positioned vertically above an anchored frictionless pivot point, as shown below, and then allowed to fall to the ground. With what speed does the free end of the rod strike the ground?

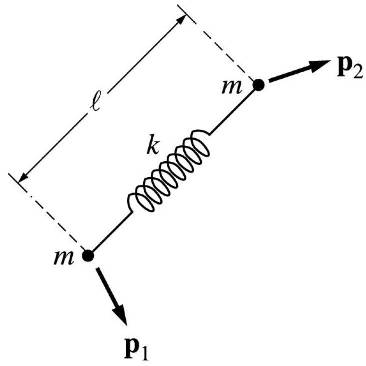


4. The Lagrangian for a mechanical system is

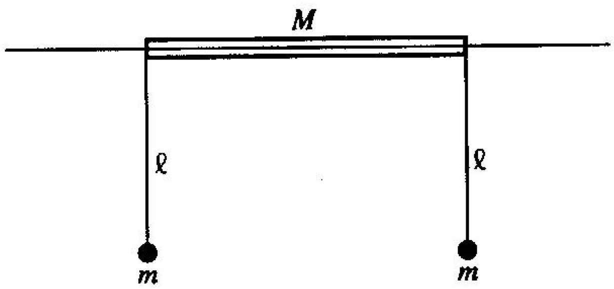
$$L = a\dot{q}^2 + bq^4$$

where q is a generalized coordinate and a and b are constants. What is the equation of motion for this system? What is the Hamiltonian of this system?

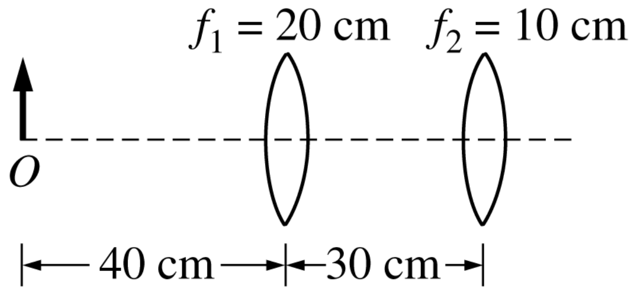
5. Two small equal masses m are connected by an ideal massless spring that has equilibrium length l_0, and force constant k, as shown in the figure above. The system is free to move without friction in the plance of the page. If $\mathbf{p_1}$ and $\mathbf{p_2}$ represents the magnitudes of the momenta of the two masses, what is the Hamiltonian for this system?



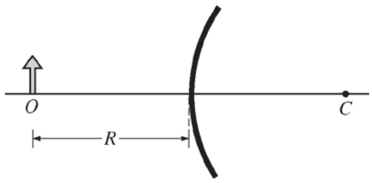
6. A cylindrical tube of mass M can slide on a horizontal wire. Two identical pendulums, each of mass m and length l, hang from the ends of the tube, as shown below. For small oscillations of the pendulums in the plane of the paper, what are the eigenfrequencies of the normal modes of oscillation of this system?



7. An object is located 40 centimeters from the first of two thin converging lenses of focal lengths 20 centimeters and 10 centimeters, respectively, as shown in the figure below. The lenses are separated by 30 centimeters. Where is the final image formed by the two-lens system is located?



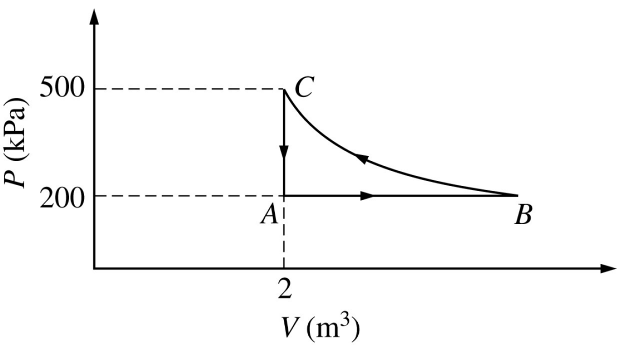
8. The figure below shows an object O placed at a distance R to the left of a convex spherical mirror that has a radius of curvature R (same as the distance R above). Point C is the center of curvature of the mirror. Where is the image formed by the mirror?



9. During a typhoon, a 1200-Hz warning siren on the town hall sounds. The wind is blowing at 55 m/s in a direction from the siren toward a person 1 km away. With what frequency does the sound wave reach the person? (The speed of sound in air is 330 m/s.)

10. A distant galaxy is observed to have its hydrogen-β line shifted to a wavelength of 580 nm, away from the laboratory value of 434 nm. What is the velocity of recession of the distant galaxy? (Note: $\frac{580}{434} \approx \frac{4}{3}$.)

11. A constant amount of an ideal gas undergoes the cyclic process ABCA in the PV diagram shown below. The path BC is isothermal. What is the work done by the gas during one complete cycle, beginning at A?

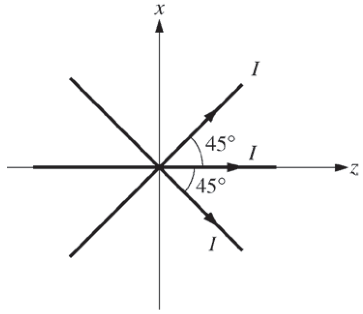


12. Consider 1 mole of a real gas that obeys the van der Waals equation of state

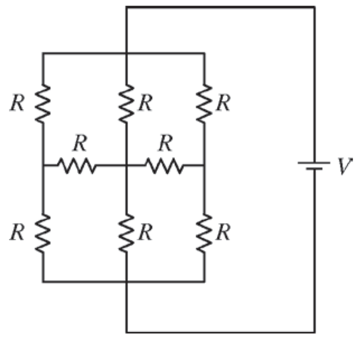
$$\left(P + \frac{a}{V^2}\right)(V - b) = RT.$$

If the gas undergoes an isothermal expansion at temperature T_0 from volume V_1 to volume V_2, what is the work done in this process?

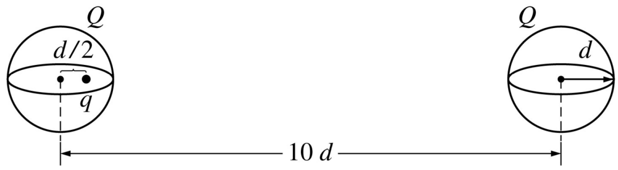
13. Three long, straight wires in the xz-plane, each carrying current I, cross at the origin of the coordinates, as shown in the figure below. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ denote the unit vectors in the x-, y-, and z-directions, respectively. What is the magnetic field along the x-axis ($y = z = 0$) as a function of x?



14. The circuit shown in the figure below consists of eight resistors, each with resistant R, and a battery with terminal voltage V and negligible internal resistance. What is the current flowing through the battery?



15. Two spherical, non-conducting, and very thin shells of uniformly distributed positive charge Q and radius d are located a distance $10d$ from each other. A positive point charge charge q is placed inside one of the shells at a distance $d/2$ from the center, on the line connecting the two shells, as shown in the figure below. What is the net force on the point charge q?



16. The normalized ground state wave function of hydrogen is $\psi_{100} = \frac{2}{(4\pi)^{1/2} a_0^{3/2}} e^{-r/a_0}$, where a_0 is the Bohr radius. What is the most likely distance that the electron is from the nucleus?

17. X rays of wavelength $\lambda = 0.250$ nm are incident on the face of a crystal at angle θ, measured from the crystal surface. The smallest angle that yields an intense reflected beam is $\theta = 14.5°$. What is the value of the interplanar spacing d? (Note: $\sin 14.5° \approx 1/4$.)

18. A uniform thin film of soapy water with refraction index $n = 1.33$ is viewed in air via reflected light. The film appears dark for long wavelengths and first appears bright for $\lambda = 540$ nm. What is the next shorter wavelength at which the film will appear bright on reflection?

19. An electron has total energy equal to four times its rest energy. What is the momentum of the electron?

20. A meter stick with a speed of 0.8c moves past an observer. In the observer's reference frame, how long does it take the stick to pass the observer?

Long Questions

(8 marks each, 40 marks in total)

1. A ball of mass m is projected vertically upwards with initial speed u in a resisting medium that produces a retardation force of magnitude kv^2, where v is the ball's speed. Show that when the ball returns to its initial position, its final speed w satisfies

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{k}{mg}.$$

What has happened to the missing energy?

2. A thin, conducting, spherical shell of radius R is charged uniformly with charge Q. Without using Gauss's law (by direct integration), find the potential at an arbitrary point

   (a) Inside the shell, and

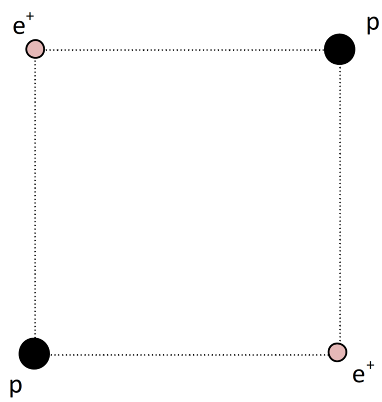
   (b) Outside the shell.

   Then compare with results from using Gauss's law. **Hint:** Only one integration is needed in the entire question. Just play around with the integration limits.

3. Two positrons and two protons are arranged in a square of side $a = 1$ cm, as illustrated in the figure below. The particles can be considered as classical point mass, moving in each other's electric fields. Gravity can be ignored. Mass of proton $M_p = 1.67 \times 10^{-27}$ kg; mass of positron $M_e = 9.11 \times 10^{-31}$ kg. (hint: mass of proton = 1836 × mass of positron). Initially the particles are held in these positions, but all four particles released at the same time.

   (a) What will their respective speeds be when they are of significant distance apart?

   (b) What are the speeds for (a) if the protons are replaced with positrons? Compare the speed of positron in (a) and comment.



4. Using the expression of hydrogenic wave function for the state 100, calculate the expectation values:

   (a) $\langle \hat{\mathbf{p}} \rangle$, (vector momentum)

   (b) $\langle p \rangle$, where in this case p is the magnitude of the momentum vector.

The ground state wave function for hydrogen atom is

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}.$$

The momentum operator is given as

$$\hat{\mathbf{p}} = -i\hbar\vec{\nabla} = -i\hbar\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)$$

where $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ are the unit vectors in spherical coordinates.

For the second case, rewriting $\psi_{100}(\vec{r})$ to $\psi_{100}(\vec{p})$ would be helpful:

$$\psi_{100}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \psi_{100}(\vec{r})\ r^2 \sin\theta\ dr\ d\theta\ d\phi.$$

5. The energy levels of a quantum mechanical rigid rotator (a model for the rotational degrees of freedom of a diatomic molecule) with moment of inertia I are

$$E_j = \frac{\hbar^2}{2I} j(j+1), \qquad j = 0, 1, 2, 3, \cdots$$

Each level is $(2j+1)$-fold degenerate.

(a) Write down the general expression for the partition function of this rotator.

(b) Show that at high temperatures, the expression for (a) can be approximated by an integral. What is the range of temperatures for which this is valid? In this limit, perform the integral to obtain the partition function, the internal energy, and the heat capacity.

(c) Find low temperature approximations to these quantities. For what range of temperatures are your expressions valid?

(d) Using Euler's Summation formula

$$\sum_{v=0}^{\infty} \varphi(v) = \int_0^{\infty} \varphi(x)\ dx + \frac{1}{2}\varphi(0) - \frac{1}{12}\varphi'(0) + \frac{1}{720}\varphi'''(0) + \cdots$$

where $\varphi'(0) \equiv \frac{d\varphi}{dx}|x=0$, and $\varphi'''(0) \equiv \frac{d^3\varphi}{dx^3}|x=0$, re-derive the high-temperature heat capacity to the next-higher order term compared to (b). Hence show that the heat capacity approaches its limiting high-temperature value from *above*.

(e) From your results in (b), (c), and (d), sketch a graph of the temperature dependence of the heat capacity.