Cavity within a cylinder

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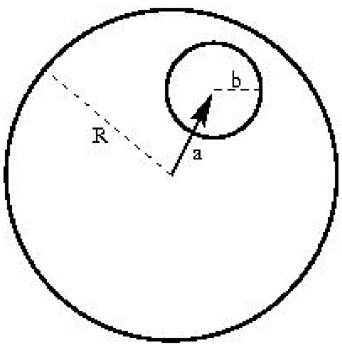
The problem:
There is a cylindrical cavity of a radius b inside an infinite cylinder. The cylinder's radius is R and it is charged with a uniform charge density ρ. The distance between the axis of the cavity and the axis of the cylinder is a.
Find the electric and the magnetic field as follows:

1. The electric field within the cavity.
2. By using the electric field, find the magnetic field inside the cavity. Consider the cylinder moving with a velocity v in it's axis direction.
3. Compare your result to direct method of finding the magnetic field (q.e_45_4_102).



The solution:
1. If we look carefully, and ignoring the cavity, then we notice that we have cylindrical symmetry. Therefore, we can use the Gauss' law in order to calculate the electric field which is produced by the electrical charges in the cylinder and in the cavity (we will consider them negative). According to the formula

$$Div\vec{E} = 4\pi k\rho \tag{1}$$

For the entire cylinder we will use a cylindrical Gaussian surface with a radius r and a height h. Because then the electric field will be perpendicular to the surface and equal at every point on it. So by using the Gauss' law we get

$$\oint_s \vec{E}d\vec{s} = \int_v Div\vec{E}dv = 4\pi k \int_v \rho dv \tag{2}$$

We know that the charge density is uniform and the surface area is $2\pi rh$ we get:

$$\begin{aligned} \oint_s \vec{E}\cdot d\vec{s} &= E_{r,cylinder}\oint_s ds = E_{r,cylinder}2\pi rh & (3)\\ 4\pi k\int_v \rho dv &= 4\pi k\rho\int_v dv = 4\pi k\rho\pi r^2 h & (4)\\ E_{r,cylinder} &= 2\pi k\rho r & (5)\end{aligned}$$

where $a-b \le r \le a+b$. And with a vector notation we get that the electric field produced by the cylinder alone a distance r from the cylinder axis:

$$\vec{E}_{r,cylinder} = 2\pi k\rho\vec{r} \tag{6}$$

From the same considerations we can calculate in the same manner the electric field which is produced by the cavity, because we assign every infinitesimal volume element a charge density $-\rho$ and say that it is uniform, in order to "create" the cavity in the first place. Hence the electric field from the cavity is:

$$\begin{aligned}
\oint_s \vec{E}d\vec{s} &= E_{r,cavity}\oint_s ds = E_{r,cavity}2\pi\left|\vec{r}-\vec{a}\right|h & (7)\\
4\pi k\int_v -\rho dv &= -4\pi k\rho\int_v dv = -4\pi k\rho\pi\left|\vec{r}-\vec{a}\right|^2 h & (8)\\
\vec{E}_{r,cavity} &= -2\pi k\rho(\vec{r}-\vec{a}) & (9)
\end{aligned}$$

By using the superposition idea we can find the total electric field inside the cavity.

$$\begin{aligned}
\vec{E}_{TOT} &= \vec{E}_{r,cylinder}+\vec{E}_{r,cavity} & (10)\\
&= 2\pi k\rho\vec{r}-2\pi k\rho(\vec{r}-\vec{a}) & (11)\\
&= 2\pi k\rho\vec{a} & (12)
\end{aligned}$$

We can see that the total electric field inside the cavity is constant everywhere inside the cavity and pointed along the vector $\vec{a}$.

2. It is given that the entire cylinder is moving along it's axis with a velocity v. Let us choose a cylindrical coordinates so that $\vec{v}=v\hat{z}$. We shall use two system frame that are moving relative to each other. The Lab system is not moving and we shall call it the S system. The cylinder is in the rest in another system which is moving with the velocity v, and we shall call it S'.
In S' the parallel electric field is zero, hence the parallel electric field in S is also zero, according to the equation:

$$E_z = E'_z = 0 \tag{13}$$

In addition, in S' there is no magnetic field at all, i.e $B'=0$. So in S the parallel magnetic field is also zero, i.e $B_z=0$. According to the transformation (from S to S'):

$$\vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{1}{c^2}\vec{V}\times\vec{E}) \tag{14}$$

And the opposite transformation (from S' to S):

$$\begin{aligned}
\vec{B}_\perp &= \gamma(\vec{B}'_\perp + \frac{1}{c^2}\vec{V}\times\vec{E}') & (15)\\
\vec{B}_\perp &= \gamma\frac{1}{c^2}\vec{V}\times\vec{E}' & (16)
\end{aligned}$$

Calculating the cross product of $\vec{V}\times\vec{E}'$

$$\vec{V}\times\vec{E}' = \begin{vmatrix} \hat{r} & \hat{\varphi} & \hat{z} \\ 0 & 0 & v \\ E_r & 0 & 0 \end{vmatrix} = E_r v\hat{\varphi} \tag{17}$$

Which mean that the direction of $\vec{B}_\perp$ is along the $\hat{\varphi}$ direction, i.e

$$\vec{B}_\perp = \gamma\frac{1}{c^2}\vec{V}\times\vec{E}' = \gamma\frac{1}{c^2}E_r v\hat{\varphi} = \gamma\frac{1}{c^2}2\pi k\rho a v\hat{\varphi} \tag{18}$$

By using the following relations we can arrive to a simpler solution

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{19}$$

$$k = \frac{1}{4\pi\varepsilon_0} \tag{20}$$

$$\vec{B}_\perp = \gamma \frac{1}{(\frac{1}{\sqrt{\mu_0\varepsilon_0}})^2} 2\pi (\frac{1}{4\pi\varepsilon_0}) \rho a v \hat{\varphi} = \mu_0 \frac{\gamma\rho}{2} a v \hat{\varphi} \tag{21}$$

3. In order to calculate directly the magnetic field we first have to understand that we have an infinite cylinder which is uniformly charged and the charge is moving. So we can analog this to an infinite wire, with a radius R, and a cavity parallel to it's axis at a distance a and with a radius b, which has a current. The current density is $\vec{J}_{cylinder} = J\hat{z}$. Due to the cylindrical symmetry we can use Ampere' law to calculate the magnetic field. First, let us conclude, by using the right hand rule, the direction of the magnetic field. If we shall put our thumb along the z axis (parallel to the current), we can see that the direction of the magnetic field is pointed along $\hat{\varphi}$. In addition we shall indicate the existence of the cavity by a current which is $\vec{J}_{cavity} = -\vec{J}_{cylinder} = -J\hat{z}$.
According to the Maxwell's equation

$$rot\vec{B} = 4\pi\kappa\vec{J} + \frac{\kappa}{k}\frac{\partial\vec{E}}{\partial t} \tag{22}$$

We know that $\frac{\partial\vec{E}}{\partial t} = 0$ because there is no change in the electric field and we have no EMF. So the Maxwell equation reduces to

$$rot\vec{B} = 4\pi\kappa\vec{J} \tag{23}$$

By using Stokes's theorem we get:

$$\oint_L \vec{B}\cdot d\vec{l} = \int_s rot\vec{B}d\vec{s} = 4\pi\kappa\int_s \vec{J}d\vec{s} = \mu_0\int_s \vec{J}\cdot d\vec{s} \tag{24}$$

If we choose a closed circular curved the magnetic field which is created by the cylinder or the cavity will be tanget to the curve and equal at every point. Also the current density is uniform so we can consider it as a constant, i.e we can take it out of the integral.

$$\oint_L \vec{B}d\vec{l} = B_\varphi \oint_L dl = B_\varphi 2\pi r \tag{25}$$

$$\mu_0\int_s \vec{J}d\vec{s} = \mu_0 J\int_s ds = \mu_0 J\pi r^2 \tag{26}$$

$$B_\varphi 2\pi r = \mu_0 J\pi r^2 \tag{27}$$

$$B_\varphi = \frac{\mu_0 J r}{2} \tag{28}$$

And in vector notation B_φ will be:

$$\vec{B}_{cylinder} = \frac{\mu_0 J r}{2}\hat{\varphi} = \frac{\mu_0 J}{2}\vec{r}\times\hat{z} \tag{29}$$

By using the same consideration we can find the magnetic field from the cavity:

$$\begin{aligned}
\oint_L \vec{B}d\vec{l} &= B_\varphi \oint_L dl = B_\varphi 2\pi \left|\vec{r}-\vec{a}\right| && (30)\\
\mu_0 \int_s \vec{J}d\vec{s} &= \mu_0 J \int_s ds = -\mu_0 J\pi \left|\vec{r}-\vec{a}\right|^2 && (31)\\
B_\varphi 2\pi \left|\vec{r}-\vec{a}\right| &= -\mu_0 J\pi \left|\vec{r}-\vec{a}\right|^2 && (32)\\
B_\varphi &= -\frac{\mu_0 J \left|\vec{r}-\vec{a}\right|}{2} && (33)
\end{aligned}$$

And in vector notation B_φ will be:

$$\vec{B}_{cavity} = -\frac{\mu_0 J \left|\vec{r}-\vec{a}\right|}{2}\hat{\varphi} = -\frac{\mu_0 J}{2}(\vec{r}-\vec{a}) \times \hat{z} \tag{34}$$

Using the superposition principle for both $\vec{B}_{cylinder}$ and $\vec{B}_{cavity}$ to find the total magnetic field in the cavity, we get:

$$\begin{aligned}
\vec{B}_{TOT} &= \vec{B}_{cylinder} + \vec{B}_{cavity} && (35)\\
&= \frac{\mu_0 J}{2}\vec{r} \times \hat{z} - \frac{\mu_0 J}{2}(\vec{r}-\vec{a}) \times \hat{z} && (36)\\
&= \frac{\mu_0 J}{2}\vec{a} \times \hat{z} = \frac{\mu_0 J a}{2}\hat{\varphi} && (37)
\end{aligned}$$

In the previus question we have found the magnetic field respect to the lab system.

$$\vec{B}_\perp = \mu_0 \frac{\gamma\rho}{2} a v \hat{\varphi} \tag{38}$$

In order to get the right answer for the magnetic field we have to "move" to the cylinder system, to the S' system. We can do this by presenting $\gamma\rho$ as $\rho^{'}$ because $\rho^{'} = \gamma\rho$. And if we look carefully and the dimensions of the product $\rho^{'}v$ we will get J'. Eventually the magnetic field will be also

$$\vec{B}^{'}_\perp = \mu_0 \frac{Ja}{2}\hat{\varphi} \tag{39}$$

So both answers are correct.