PLANCKS 2017
Preliminaries for Team Singapore

Exercise Booklet

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School of Physical and Mathematical Sciences
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11:30 am - 2:30 pm

Problem 1: Yukawa potential

A particle of mass m moves in a central force field given by the potential

$$V = -k\frac{e^{-\alpha r}}{r}$$

where k and α are positive constants. This potential is also called the **screened Coulomb potential** because it falls off with distance more rapidly than $1/r$ and hence approximates the electrostatic potential of the atomic nucleus in the vicinity of the nucleus by taking into account the partial "cancellation" or "screening" of the nuclear charge by the atomic electrons.

(a) Show that the kinetic energy of the particle, in spherical coordinates, is given by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and hence, obtain the Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + k\frac{e^{-\alpha r}}{r}$$

[2 mark]

(b) Write down the equations of motion for the particle using the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

[3 marks]

(c) What are the conditions for a possible circular orbit, given that the particle orbital radius is r_0 with angular velocity $\dot{\theta}$? [2 marks]

(d) Assuming the particle is in a possible circular orbit at radius $r = r_0$. If there is another particle that comes close to the particle for a short moment giving some perturbation to the circular motion. Find the equation that governs the time evolution of these perturbations. [6 marks]

(e) Find the condition for a **stable** circular orbit in which case the particle in **d)** will just oscillate around a fix orbit $r = r_0$. Find the frequency of this oscillation. [7 marks]

Problem 2: Electromagnetic Charges

Suppose you had an electric charge q_e and a "magnetic charge" (monopole) q_m, the fields produced by the charges are:

$$\mathbf{E} = \frac{q_e}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{\mathscr{r}}}$$

$$\mathbf{B} = \frac{\mu_0 q_m}{4\pi r_m^2}\hat{\boldsymbol{\mathscr{r}}}_{\mathbf{m}}$$

where $\boldsymbol{\mathscr{r}}$ and $\boldsymbol{\mathscr{r}}_m$ are the separation vectors from the location of the electric charge and magnetic charge, respectively, to $\mathbf{r}$, the location of interest.

Assume that the electric charge and the magnetic charge are located at the origin and $\mathbf{d} = d\hat{\mathbf{z}}$, respectively.

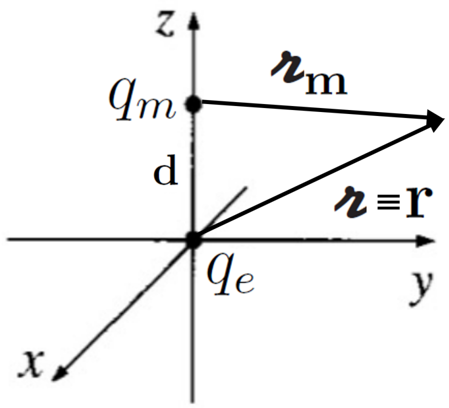

Figure 1: Schematic diagram showing electric charge q_e and magnetic charge q_m with their separation vector $\boldsymbol{\mathscr{r}} \equiv \mathbf{r}$ and $\boldsymbol{\mathscr{r}}_m$, respectively

(a) Find the momentum density of the fields. Hint: $\mathfrak{p} = \epsilon_0 \mathbf{E} \times \mathbf{B}$, $\boldsymbol{\mathscr{r}}_m = \boldsymbol{\mathscr{r}} - \mathbf{d}$ (form a triangle). [10 marks]

(b) Find the angular momentum density $\mathfrak{L} = \mathbf{r} \times \mathfrak{p}$ of the fields. [4 marks]

(c) Find the total angular momentum stored in the fields by integrating $\mathfrak{L}$ over whole space. Hint:

$$\int_0^{\pi}\int_0^{\infty} \frac{r\sin^3\theta}{(d^2 + r^2 - 2dr\cos\theta)^{3/2}}\,d\theta dr = \frac{2}{d}$$

[6 marks]

Problem 3: (No-)Cloning of Quantum States

A quantum state is described by a vector $|\psi\rangle$ in the Hilbert space, whereas a measurement is described as a projection onto a complete orthogonal basis $\{|\phi_i\rangle\langle\phi_i|\}$ in the Hilbert space. For simplicity suppose the Hilbert space is of dimension 2. Let $\{|0\rangle, |1\rangle\}$ be some computational basis for the Hilbert space.

Here we will consider cloning, i.e. taking a state $|\psi\rangle$ and try to get two copies of $|\psi\rangle$.

Now, suppose you're given one copy of a quantum state $|\psi\rangle$ from some set of states $\Phi = \{|\phi_i\rangle\}$. Assume that we know what states are in Φ, but the state $|\psi\rangle$ is picked from Φ with a uniform probability, so we don't know which one $|\psi\rangle$ is.

A cloning machine is a transformation C that takes $|\psi\rangle$ and some blank state (say $|0\rangle$) and maps it to $C(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle$. For simplicity, we'll only consider linear processes, i.e. $C(\alpha_0|\psi_0\rangle + \alpha_1|\psi_1\rangle) = \alpha_0 C(|\psi_0\rangle) + \alpha_1 C(|\psi_1\rangle)$.

(a) Show that if $\Phi = \{|0\rangle, |1\rangle\}$, then you can clone $|\psi\rangle$. [2 marks]

(b) Show that if all the vectors in Φ are mutually orthogonal, then you can clone $|\psi\rangle$. [1 mark]

(c) Show that when the states in Φ are not mutually orthogonal, then we can't always succeed in cloning the state. [4 marks]

(d) If Φ contains more than 2 states, can we always clone the state? [2 marks]

We can actually do better than this. Let's try to give a success rate on how well can we clone something. We'll "consider measure and prepare" strategies.
Suppose $\Phi = \{|\phi_0\rangle, |\phi_1\rangle\}, |\langle\phi_0||\phi_1\rangle| = \cos\theta \neq 0$, i.e. Φ contains only two states, but they are not orthogonal.

(e) Suppose you do your measurement in some complete orthogonal basis $\{|\eta_0\rangle\langle\eta_0|, |\eta_1\rangle\langle\eta_1|\}$, where $|\eta_0\rangle = |\phi_0\rangle$. If we get outcome 0, then we prepare two copies of $|\phi_0\rangle$. Otherwise we prepare two copies of $|\phi_1\rangle$. What's the probability of success with this strategy? [3 marks]

(f) What if we vary $|\eta_0\rangle, |\eta_1\rangle$? What's the probability of success as a function of $|\eta_0\rangle$? Optimize over the choice of $|\eta_0\rangle, |\eta_1\rangle$. What's the optimal probability of success? [4 marks]

(g) Does this mean we can't reliably distinguish any two states? [2 marks]

(h) Suppose instead of getting just one copy, we get N copies. Show that in the limit of large N, the probability of success goes to 1. [2 marks]

Problem 4: Hard Rods

A collection of N asymmetric molecules in two dimensions may be modeled as a gas of rods, each of length $2l$ and lying in a plane. A rod can move by translation of its center of mass and rotation about latter, as long as it does not encounter another rod. Without treating the hard-core interaction exactly, we can incorporate it approximately by assuming that the rotational motion of each rod is restricted (by the other rods) to an angle θ, which in turn introduces an excluded volume $\Omega(\theta)$ (associated with each rod). The value of θ is then calculated self consistently by maximizing the entropy at a given density $n = N/V$, where V is the total accessible area.

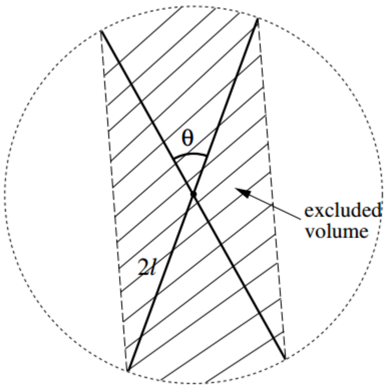

Figure 2: Schematic diagram of a rod

(a) Show that the total non-excluded volume available in the positional phase space of the system of N rods is approximately given by $(V - N\Omega/2)^N$. [Hint: Think of how to assemble the system.] [2 marks]

(b) Express the entropy in terms of N, n, $\Omega(\theta)$, and $A(\theta)$, the entropy associated to the rotational freedom of a single rod. (You may ignore the momentum contributions throughout, and consider the large N limit.) [2 marks]

(c) Extremizing the entropy as a function of θ, show that the density n can be expressed as

$$n = \frac{2A'}{\Omega A' + \Omega' A}$$

where Ω', A' denotes the derivatives with respect to θ. [3 marks]

(d) Express the excluded volume Ω in terms of θ and sketch n as a function of $\theta \in [0, \pi]$, assuming $A \propto \theta$ [6 marks]

(e) Describe the equilibrium state at high densities. Can you identify a phase transition as the density is decreased? Draw the corresponding critical density n_c on your sketch. What is the critical angle θ_c at the transition? You don't need to calculate θ_c explicitly, but give an (implicit) relation defining it. What value does θ adopt at $n < n_c$? [7 marks]

Problem 5: Uniform Accelerations in Special Relativity

In special relativity, an object moving in a fixed direction is said to be undergoing uniform acceleration if its 4-acceleration a^μ has a constant magnitude. In this problem, we will study some interesting phenomena associated with uniform acceleration motion in special relativity.

A. Uniformly accelerating point particle

For simplicity, we consider uniform acceleration motion with one space dimension. So the relevant Minkowski space-time is two-dimensional, with coordinates (t, x). Assume that a particle is moving with uniform acceleration $g \equiv \sqrt{a^\mu a_\mu}$. At time $t = 0$, it is at rest and located at $x = d(> 0)$.

A1 Find the world line of this particle, namely find the space and time coordinates of the particle in terms of the proper time as measure by the particle $(t(\tau), x(\tau))$. [4 marks]

A2 Find d in terms of g such that all events on the world line of the particle has the same proper distance from the event $(0, 0)$. Denote the sought d as $d(g)$. [2 marks]

A3 [Difficult] Now fix d as $d(g)$. We define the line of simultaneity of the particle as the line of simultaneity of the instantaneous inertial observer comoving with the particle. Prove that this line passes through $(0, 0)$ for all τ. [4 marks]

B. Uniformly accelerating rigid ruler

Suppose now a person carries a ruler in uniform acceleration g. At $t = 0$, the person is at rest and is located at $x = d = g^{-1}$. The ruler has a length L, and is very rigid, by which we mean that the proper length of the ruler as measured by an observer comoving with the ruler is fixed. At $t = 0$, the two ends of the ruler are located $x = d$ (end A) and $x = d + L$ (end B) respectively.

B1 Is end B of the ruler in uniform acceleration motion? You can use the result of (A3) and justify your answer by considering the proper distance between events on the word line of end B and the event $(0, 0)$ in the instantaneous inertial frame comoving with end A. [2 marks]

B2 Find the world line of end B of the ruler. [6 marks]

B3 Compare the accelerations of the two ends of the ruler. [2 marks]

Acknowledgment

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