1) a) $x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

$$\Rightarrow \dot{x} = \dot{r} \sin\theta \cos\varphi + \dot{\theta} r \cos\theta \cos\varphi - \dot{\varphi} r \sin\theta \sin\varphi$$

$$\dot{y} = \dot{r} \sin\theta \sin\varphi + \dot{\theta} r \cos\theta \sin\varphi + \dot{\varphi} r \overset{\sin}{\cancel{\cos}}\theta \cos\varphi$$

$$\dot{z} = \dot{r} \cos\theta - \dot{\theta} r \sin\theta$$

$$\Rightarrow T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2\right) \quad [1]$$

We can choose the coordinate system depending on the initial condition to get $\dot{\varphi} = 0$.

$$\Rightarrow T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad [+]$$

$$\Rightarrow L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + k \frac{e^{-\alpha r}}{r} \quad [1]$$

b) $$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$\Rightarrow m r \dot{\theta}^2 - k\alpha \frac{e^{-\alpha r}}{r} - \frac{k e^{-\alpha r}}{r^2} = m\ddot{r} \quad [2]$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \Rightarrow \frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0$$

$$\Rightarrow m r^2 \dot{\theta} = \text{const} = L_{ang}. \quad [1]$$

c) For circular motion, $\dot{r} = 0 \Rightarrow \ddot{r} = 0$, $r = r_0$ [1]

$$\Rightarrow m r_0 \dot{\theta}^2 = \frac{k\alpha e^{-\alpha r_0}}{r_0} + \frac{k e^{-\alpha r_0}}{r_0^2} \quad [1]$$

Note: $L = m r_0^2 \dot{\theta} = \text{const} \Rightarrow \dot{\theta} = \frac{L}{m r_0^2}$

$$\Rightarrow \frac{L^2}{m r_0^3} = k\alpha \frac{e^{-\alpha r_0}}{r_0} + \frac{k e^{-\alpha r_0}}{r_0^2}$$

d) $$m\ddot{r} = m r^2 \dot{\theta}^2 - k\alpha \frac{e^{-\alpha r}}{r} - k\frac{e^{-\alpha r}}{r^2}$$

Let $r = r_0 + \delta(r)$ [1]

$$\Rightarrow m\ddot{\delta}(r) = \cancel{m}\frac{L^2}{m(r_0+\delta r)^3} - k\alpha \frac{e^{-\alpha(r_0+\delta r)}}{r_0+\delta r} - k\frac{e^{-\alpha(r_0+\delta r)}}{(r_0+\delta r)^2}$$

$$\simeq \frac{L^2}{m r_0^3}\left(1+\frac{\delta r}{r_0}\right)^{-3} - k\alpha \frac{e^{-\alpha r_0}(1-\alpha\delta r)}{r_0\left(1+\frac{\delta r}{r_0}\right)} - k\frac{e^{-\alpha r_0}(1-\alpha \delta r)}{r_0^2\left(1+\frac{\delta r}{r_0}\right)^2}$$

[1] [1] [1]

$$\simeq \frac{L^2}{m r_0^3}\left(1-3\frac{\delta r}{r_0}\right) - \frac{k\alpha e^{-\alpha r_0}}{r_0}\left(1-\alpha\delta r - \frac{\delta r}{r_0}\right) - \frac{k e^{-\alpha r_0}}{r_0^2}(1-\alpha\delta r)\left(1-2\frac{\delta r}{r_0}\right)$$

$$\simeq \underbrace{\frac{L^2}{m r_0^3} - \frac{k\alpha e^{-\alpha r_0}}{r_0} - \frac{k e^{-\alpha r_0}}{r_0^2}}_{=0} - 3\frac{L^2}{m r_0^4}\delta r + \frac{k\alpha^2 \delta r}{r_0}e^{-\alpha r_0} + \frac{k\alpha \delta r e^{-\alpha r_0}}{r_0^2}$$

By circular orbit assumption!

$$+ \frac{k\alpha \delta r e^{-\alpha r_0}}{r_0^2} + 2\frac{k \delta r e^{-\alpha r_0}}{r_0^3}$$

$$= -\delta r \underbrace{\left[\frac{3L^2}{m r_0^4} - \frac{k\alpha^2 e^{-\alpha r_0}}{r_0} - \frac{2k\alpha e^{-\alpha r_0}}{r_0^2} - \frac{2k\cancel{\alpha} e^{-\alpha r_0}}{r_0^3}\right]}_{\text{const}} \quad [2]$$

d) For stable circular orbit, the equation in c) needs to be in the form :

$$\delta \ddot{r} = -\omega^2 \delta r \quad [1], \quad \omega \in \mathbb{R}$$

$$\rightarrow \delta r = A \cos(\omega t) + B \sin(\omega t) \rightarrow \text{bounded oscillation}$$

Hence, we need $\dfrac{3L^2}{m r_0^4} - \dfrac{k\alpha^2 e^{-\alpha r_0}}{r_0} - \dfrac{2k\alpha e^{-\alpha r_0}}{r_0^2} - \dfrac{2k e^{-\alpha r_0}}{r_0^3} \geq 0$ [1]

$$= \frac{3}{r_0}\left(\frac{k\alpha e^{-\alpha r_0}}{r_0} + \frac{k e^{-\alpha r_0}}{r_0^2}\right) - \frac{k\alpha^2 e^{-\alpha r_0}}{r_0} - \frac{2k\alpha e^{-\alpha r_0}}{r_0^2} - \frac{2k e^{-\alpha r_0}}{r_0^3}$$

$$= \frac{3k\alpha e^{-\alpha r_0}}{r_0^2} + \frac{3k e^{-\alpha r_0}}{r_0^3} - \frac{k\alpha^2 e^{-\alpha r_0}}{r_0} - \frac{2k\alpha e^{-\alpha r_0}}{r_0^2} - \frac{2k e^{-\alpha r_0}}{r_0^3}$$

Derivation [2]

$$= \frac{k\alpha e^{-\alpha r_0}}{r_0^2} + \frac{k e^{-\alpha r_0}}{r_0^3} - \frac{k\alpha^2 e^{-\alpha r_0}}{r_0} \geq 0$$

$\Rightarrow \alpha r_0 + 1 - \alpha^2 r_0^2 \geq 0$ [1]

Let $x = \alpha r_0 \Rightarrow x^2 - x - 1 \leq 0$

$$\Rightarrow x \leq \frac{1+\sqrt{5}}{2}$$

x		$\frac{1-\sqrt{5}}{2}$		$\frac{1+\sqrt{5}}{2}$	
x^2-x-1	+	0	−	0	+

$\Rightarrow r_0 \leq \dfrac{1+\sqrt{5}}{2\alpha}$ [1] $\omega = \ldots$ [1]

② $\vec{E} = \dfrac{q_e}{4\pi\varepsilon_0 r^2}\hat{r}$ $\qquad \vec{p} = \varepsilon_0 \vec{E}\times\vec{B}$

$$= \frac{\cancel{\varepsilon_0}\, q_e \mu_0 q_m}{(4\pi)^2 \cancel{\varepsilon_0}\, r^2 r_m^2}\, \hat{r}\times\hat{r}_m$$

$\vec{B} = \dfrac{\mu_0 q_m}{4\pi r_m^2}\hat{r}_m$

$$= \frac{\mu_0 q_e q_m}{(4\pi)^2}\,\frac{\vec{r}\times\vec{r}_m}{r^3 r_m^3} \quad [2]$$

Note: $\vec{r} = \vec{r}$

$\vec{r}_m = \vec{r} - \vec{d} = \vec{r} - d\hat{z}.$

$|r_m|^2 = r^2 + d^2 - 2dr\cos\theta.$ (2)

$$\vec{r}\times\vec{r}_m = \vec{r}\times(\vec{r} - d\hat{z}) = \cancelto{0}{\vec{r}\times\vec{r}} - \vec{r}\times d\hat{z}$$

$$= -\vec{r}\times d\hat{z} = -\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & 0 & d \end{pmatrix} \quad (2)$$

(2)

$$= -\left[\hat{i}(yd) - \hat{j}\,xd\right] = -(r\sin\theta\sin\varphi\, d)\,\hat{i} + (r\sin\theta\cos\varphi)\,d\,\hat{j}.$$

Hence, $\vec{p} = \dfrac{\mu_0 q_e q_m d}{(4\pi)^2}\,\dfrac{-(\cancel{r}\sin\theta\sin\varphi)\hat{i} + \cancel{r}(\sin\theta\cos\varphi)\hat{j}}{r^2(r^2+d^2-2rd\cos\theta)^{3/2}}$ [2]

b) $\vec{L} = \vec{r}\times\vec{p} = \dfrac{\mu_0 q_e q_m d}{(4\pi)^2}\,\dfrac{1}{r^2(r^2+d^2-2dr\cos\theta)^{3/2}}\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}$ (2)

$$\Rightarrow \vec{L} = \frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\frac{1}{r^2(r^2+d^2-2dr\cos\theta)^{3/2}}\begin{pmatrix} -z\sin\theta\cos\varphi \\ -z\sin\theta\sin\varphi \\ x\sin\theta\cos\varphi + y\sin\theta\sin\varphi \end{pmatrix} \quad (2)$$

$$\vec{L} = \int \vec{\mathcal{L}}\, dv = \int_0^\infty \int_0^\pi \int_0^{2\pi} \vec{\mathcal{L}} \quad r^2 \sin\theta \quad dr\, d\theta\, d\varphi \qquad (2)$$

$$= \frac{\mu_0 q_e q_m d}{(4\pi)^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{(r^2+d^2-2dr\cos\theta)^{3/2}} \begin{pmatrix} -r\cos\theta\sin\theta\cos\varphi \\ -r\cos\theta\sin\theta\sin\varphi \\ r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi \end{pmatrix}$$

$$r\sin^2\theta \quad dr\, d\theta\, d\varphi$$

$$= \frac{\mu_0 q_e q_m d}{(4\pi)^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{r\sin^3\theta}{(r^2+d^2-2dr\cos\theta)^{3/2}} \, dr\, d\theta\, d\varphi \qquad (2)$$

$$= \frac{\mu_0 q_e q_m \cancel{d}}{(4\pi)^{\cancel{2}}} \hat{z}\, 2\cancel{\pi} \cdot \frac{\cancel{2}}{\cancel{d}} = \frac{\mu_0 q_e q_m}{4\pi} \hat{z} \qquad (2)$$

$$\int_0^\pi \int_0^\infty \frac{r\sin^3\theta}{(d^2+r^2-2dr\cos\theta)^{3/2}}\, d\theta\, dr$$

$$\rightarrow \frac{r^2+d^2-u^2}{2dr} = \cos\theta$$

Let $u = \sqrt{r^2+d^2-2dr\cos\theta} \Rightarrow du = \dfrac{dr\sin\theta}{\sqrt{r^2+d^2-2dr\cos\theta}}\, d\theta$

$$= \int_0^\infty \int_{|r-d|}^{(r+d)} \frac{\cancel{r}\sin^2\theta}{u^2}\,\frac{du}{d}\, dr \qquad \sin^2\theta = 1-\cos^2\theta$$

$$= 1-\left(\frac{r^2+d^2-u^2}{2dr}\right)^2$$

$$= \frac{1}{d}\int_0^\infty \int_{|r-d|}^{(r+d)} \frac{1}{u^2} - \frac{1}{u^2}\left(\frac{r^2+d^2-u^2}{2dr}\right)^2 du\, dr$$

$$= \frac{1}{d}\int_0^\infty \int_{|r-d|}^{r+d} \frac{1}{u^2}\left(1-\frac{(r^2+d^2)^2}{4d^2r^2}\right) - \frac{1}{\cancel{u^2}}\,\frac{(-2)\cancel{u^2}(r^2+d^2)}{4d^2r^2} - \frac{1}{\cancel{u^2}}\,\frac{u^{\cancel{4}}}{4d^2r^2}\; du\, dr$$

$$= \frac{1}{d}\int_0^\infty \left(1-\frac{(r^2+d^2)^2}{4d^2r^2}\right)\left(-\frac{1}{u}\right)\Bigg|_{|r-d|}^{|r+d|} + \frac{2(r^2+d^2)}{4d^2r^2}\, u\Bigg|_{|r-d|}^{r+d} - \frac{1}{3}\,\frac{u^3}{4d^2r^2}\Bigg|_{|r-d|}^{r+d} dr$$

⊛ $r > d$.

$$\left(1-\frac{(r^2+d^2)^2}{4d^2r^2}\right)\left(-\frac{1}{r+d}+\frac{1}{r-d}\right) + \frac{\cancel{8}(r^2+d^2)}{\cancel{4}d^{\cancel{2}}r^2}(\cancel{r}+d-\cancel{r}+d) - \frac{1}{3}\,\frac{(r+d)^3-(r-d)^3}{4d^2r^2}$$

$$= \left(1-\frac{(r^2+d^2)^2}{4d^2r^2}\right)\frac{\cancel{r}+d+d-\cancel{r}}{r^2-d^2} + \frac{(r^2+d^2)}{dr^2} - \frac{1}{3}\,\underbrace{\frac{d^3+\cancel{3d^2r}+3dr^2+\cancel{r^3}-(\cancel{r^3}-3r^2d+\cancel{3rd^2}-d^3)}{4d^2r^2}}$$

$$= -\frac{6r^2d+2d^3}{3\times 4d^2r^2} = -\frac{3r^2+d^2}{6dr^2}$$

$$= \left(\frac{4d^2r^2-(r^2+d^2)^2}{4d^2r^2}\right)\frac{2d}{r^2-d^2} + \frac{(r^2+d^2)}{dr^2} +$$

$$= \frac{-(r^2-d^2)^{\cancel{2}}}{4d^2r^2}\,\frac{2d}{\cancel{r^2-d^2}} + \frac{r^2+d^2}{dr^2} - \frac{3r^2+d^2}{6dr^2}$$

$$= -\frac{(r^2-d^2)}{2dr^2} + \frac{r^2+d^2}{dr^2} - \frac{3r^2+d^2}{6dr^2} = \frac{\cancel{-3r^2}+3d^2\cancel{+6r^2}+6d^2\cancel{-3r^2}\cancel{-d^2}}{6dr^2}$$

$$\overbrace{}^{8d^2}$$

$$= \frac{8d^2}{6dr^2} = \frac{4}{3}\frac{d}{r^2}$$

② $r<d$:

$$\left(1-\frac{(r^2+d^2)^2}{4d^2r^2}\right)\underbrace{\left(-\frac{1}{r+d}+\frac{1}{d-r}\right)}_{\frac{r\cancel{+d}+r\cancel{-d}}{d^2-r^2}} + \frac{2(r^2+d^2)}{4d^2r^2}(r\cancel{+d}\cancel{-d}+r) - \frac{1}{3}\frac{\cancel{d^3}}{4d^2r^2}\underbrace{\left((r+d)^3-(d-r)^3\right)}$$

$$= \frac{-(d^2-r^2)^{\cancel{2}}}{4d^2r^2}\frac{2r}{\cancel{d^2-r^2}} + \frac{r^2+d^2}{d^2r} +$$

$$-\frac{1}{3\times 4d^2r^2}\times\left[r^3+\cancel{3r^2d}+3rd^2\cancel{+d^3} - (\cancel{d^3}-3d^2r+\cancel{3dr^2}-r^3)\right]$$

$$= -\frac{1}{12d^2r^2}(2r^3+6rd^2)$$

$$= -\frac{r^2+3d^2}{6d^2r}$$

$$= -\frac{(d^2-r^2)}{2d^2r} + \frac{r^2+d^2}{d^2r} - \frac{r^2+3d^2}{6d^2r} = \frac{\cancel{-3d^2}+3r^2+6r^2\cancel{+6d^2}-r^2\cancel{-3d^2}}{6d^2r}$$

$$= \frac{8r^2}{6d^2r} = \frac{4r}{3d^2}$$

=

$$\Rightarrow dI = \int_0^d \frac{4r}{3d^2}\,dr + \int_d^\infty \frac{4d}{3r^2}\,dr$$

$$= \frac{2}{3d^2}d^2 + \frac{4d}{3}\left(-\frac{1}{r}\right)\Big|_d^\infty = \frac{2}{3}+\frac{4}{3} = \frac{6}{3} = 2$$

$$\Rightarrow I = \frac{2}{d}$$