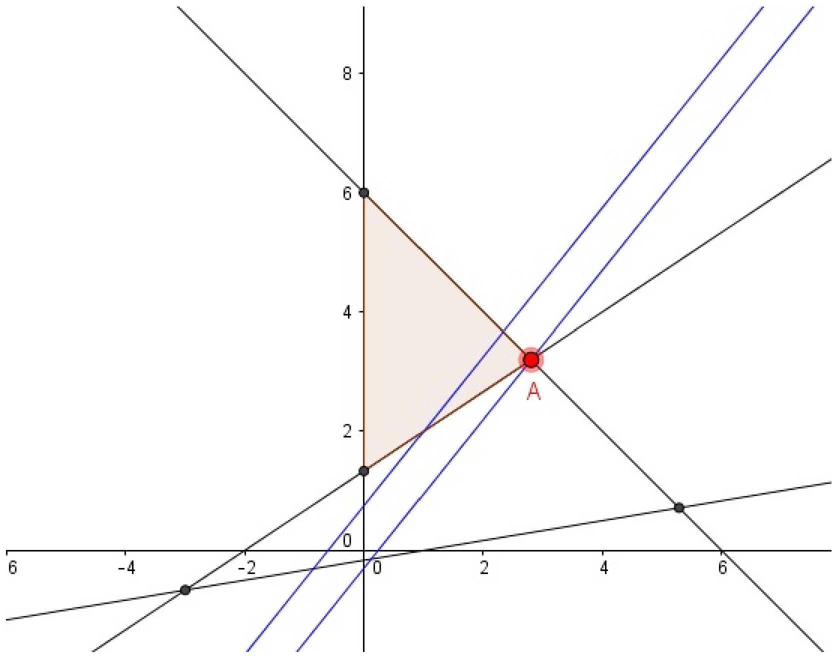
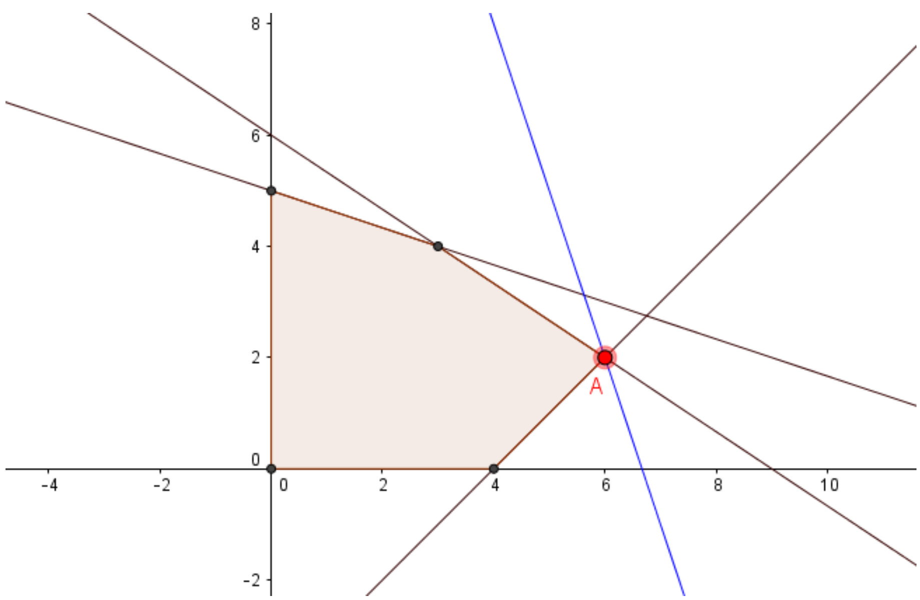
8.1.



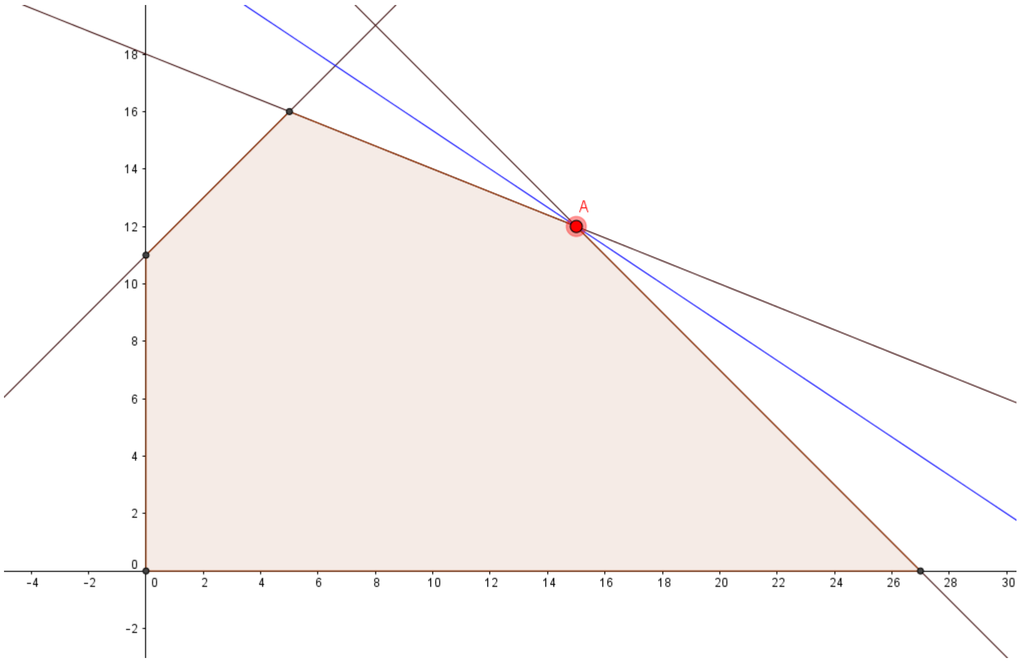
The optimal point is $A = (2.8, 3.2)$.

8.2.

(i)



The optimal point is $A = (6, 2)$ and the optimal value is 20.



(ii)

The optimal point is $A = (15, 12)$ and the optimal value is 132.

8.5.

(i)

$$\begin{array}{rcrcrcr}
\xi & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2
\end{array}$$

$$\begin{array}{rcrcrcr}
\xi & = & 12 & - & 3w_3 & + & 4x_2 \\
\hline
w_1 & = & 11 & + & w_3 & - & 4x_2 \\
w_2 & = & 10 & + & 2w_3 & - & 5x_2 \\
x_1 & = & 4 & - & w_3 & + & x_2
\end{array}$$

$$\begin{array}{rcrcrcr}
\xi & = & 20 & - & \frac{7}{5}w_3 & - & \frac{4}{5}w_2 \\
\hline
w_1 & = & 3 & - & \frac{3}{5}w_3 & + & \frac{4}{5}x_2 \\
x_2 & = & 2 & + & \frac{2}{5}w_3 & - & \frac{1}{5}w_2 \\
x_1 & = & 6 & - & \frac{3}{5}w_3 & - & \frac{1}{5}w_2
\end{array}$$

The optimal point is $x_1 = 6, x_2 = 2$, the optimum value is 20.

(ii)

$$\begin{array}{rcrcrcr}
\xi & = & & & 4x & + & 6y \\
\hline
z & = & 11 & + & x & - & y \\
t & = & 27 & - & x & - & y \\
w & = & 90 & - & 2x & - & 5y
\end{array}$$

$$\begin{array}{rcrcrcr}
\xi & = & 108 & - & 4t & + & 2y \\
\hline
z & = & 38 & - & t & - & 2y \\
x & = & 27 & - & t & - & y \\
w & = & 36 & + & 2t & - & 3y
\end{array}$$

$$\begin{array}{rcrcrcr}
\xi & = & 132 & - & \frac{8}{3}t & - & \frac{2}{3}w \\
\hline
z & = & 14 & - & \frac{7}{3}x & + & \frac{2}{3}w \\
x & = & 15 & - & \frac{5}{3}t & + & \frac{1}{3}w \\
y & = & 12 & + & \frac{2}{3}t & - & \frac{1}{3}w
\end{array}$$

The optimal point is $x = 15, y = 12$ and the optimum value is 132.

8.7.

(i) Solve the auxiliary problem

$$\begin{array}{rclcccccc}
\xi & = & & & & & & - & x_0 \\
\hline
w_1 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
w_2 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
w_3 & = & 3 & - & x_1 & & & + & x_0
\end{array}$$

$$\begin{array}{rclcccccc}
\xi & = & -8 & + & 4x_1 & + & 2x_2 & - & w_1 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & w_1 \\
w_2 & = & 14 & - & 2x_1 & - & 5x_2 & + & w_1 \\
w_3 & = & 11 & - & 5x_1 & - & 2x_2 & + & w_1
\end{array}$$

$$\begin{array}{rclcccccc}
\xi & = & & - & x_0 & & & & \\
\hline
x_1 & = & 2 & - & \frac{1}{4}x_0 & - & \frac{1}{2}x_2 & + & \frac{1}{4}w_1 \\
w_2 & = & 10 & + & \frac{1}{2}x_0 & - & 4x_2 & - & \frac{3}{2}w_1 \\
w_3 & = & 1 & + & \frac{5}{4}x_0 & + & \frac{1}{2}x_2 & - & \frac{9}{4}w_1
\end{array}$$

Hence we have found a feasible point $x_1 = 2, x_2 = 0$. Now we solve for the optimal value

$$\begin{array}{rclcccc}
\xi & = & 2 & + & \frac{1}{4}w_1 & + & \frac{3}{2}x_2 \\
\hline
x_1 & = & 2 & + & \frac{1}{4}w_1 & - & \frac{1}{2}x_2 \\
w_2 & = & 10 & - & \frac{3}{2}w_1 & - & 4x_2 \\
w_3 & = & 1 & - & \frac{9}{4}w_1 & + & \frac{1}{2}x_2
\end{array}$$

8.13. Suppose we have n decision variables. We start at the origin since it is feasible. Suppose $\mathbf{x} = 0$ is not an optimum point, then there exists a nonbasic variable x_i whose coefficient c_i is strictly positive. Let $w_j = -\sum_{k=1}^{n} a_{jk}x_k$ be the most binding basic variable. Since we know that we can improve x_i, it follows that $a_{ji} < 0$, otherwise improving x_i will make w_j negative or unchanged. However, then it follows that w_j is not a binding variable anymore, because improve x_i as large as possible won't make w_j negative. This shows that either there is not binding variable, which means the problem is unbounded, or $\mathbf{x} = 0$ is an optimum point.

8.17. Suppose we have a primal problem

$$\begin{array}{rl}
\text{maximize} & \mathbf{c}^T\mathbf{x} \\
\text{subject to} & A\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \succeq \mathbf{0}
\end{array}$$

Its corresponding dual is

$$\begin{array}{rl}
\text{minimize} & \mathbf{b}^T\mathbf{y} \\
\text{subject to} & A^T\mathbf{y} \succeq \mathbf{c} \\
& \mathbf{y} \succeq \mathbf{0}
\end{array}$$

which is equivalent to

$$\begin{array}{rl}
\text{maximize} & -\mathbf{b}^T\mathbf{y} \\
\text{subject to} & -A^T\mathbf{y} \preceq -\mathbf{c} \\
& \mathbf{y} \succeq \mathbf{0}
\end{array}$$

The dual of the dual is therefore

$$\begin{array}{rl}
\text{minimize} & -\mathbf{c}^T\mathbf{x} \\
\text{subject to} & -A\mathbf{x} \succeq -\mathbf{b} \\
& \mathbf{x} \succeq \mathbf{0}
\end{array}$$

or equivalently,

$$\begin{array}{rl} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{array}$$

which is exactly the primal problem.

9.10. We have

$$Df(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(Q^T + Q) - \mathbf{b}^T = \mathbf{x}^T Q - \mathbf{b}^T = (Q\mathbf{x} - \mathbf{b})^T.$$

and

$$D^2 f(\mathbf{x}) = Q.$$

The unique minimizer of f is the root of $Q\mathbf{x} = \mathbf{b}$. Since Q is positive definite, this means that the unique minimizer is $\mathbf{x} = Q^{-1}\mathbf{b}$. Suppose we start at any initial point $\mathbf{x}_0$, then one iteration of Newton's method yields

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 - D^2 f(\mathbf{x}_0)^{-1} Df(\mathbf{x}_0)^T \\ &= \mathbf{x}_0 - Q^{-1}(Q\mathbf{x}_0 - \mathbf{b}) \\ &= Q^{-1}\mathbf{b} + \mathbf{x}_0 - Q^{-1}Q\mathbf{x}_0 \\ &= Q^{-1}\mathbf{b}. \end{aligned}$$

This shows that $\mathbf{x}_1$ is indeed the unique minimizer of f.