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6.1.

$$\begin{aligned} &\text{minimize} && f(\mathbf{w}) = e^{\mathbf{w}^T\mathbf{x}} \\ &\text{subject to} && G(\mathbf{w}) = \mathbf{w}^T A\mathbf{w} - \mathbf{w}^T\mathbf{x} - \mathbf{w}^T A\mathbf{y} \leq a \\ & && H(\mathbf{w}) = \mathbf{y}^T\mathbf{w} - \mathbf{w}^T\mathbf{x} = b \end{aligned}$$

6.5. Let $\mathbf{x} = [m,k] \in \mathbb{R}^2$ denote the amount of milk cartons and knobs the company produce, respectively. The profit of the company by producing $\mathbf{x}$ is therefore $\mathbf{x}^T\mathbf{w}$, where $\mathbf{w} = [0.07, 0.05]$. The cost of production is $4m+3k$ grams of plastic and $2m+k$ minutes of labor. The company cannot exceed its resources of plastic and labor, hence we must have $4m+3k \leq 24\cdot 10^4$ and $2m+k \leq 100$. Finally, we must have $k,m \geq 0$. Let

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = [24\cdot 10^4, 100, 0, 0].$$

Then the resource constraints can be written as $A\mathbf{x} \preceq \mathbf{b}$. Hence, the problem in standard form is

$$\begin{aligned} &\text{minimize}_{\mathbf{x}} && -\mathbf{x}^T\mathbf{w} \\ &\text{subject to} && A\mathbf{x} \preceq \mathbf{b}. \end{aligned}$$

6.6. We have

$$Df(x,y) = [6xy + 4y^2 + y, 3x^2 + 8xy + x].$$

Solve for $Df(x,y) = 0$, we get

$$\begin{cases} y(6x+4y+1) = 0 \\ x(3x+8y+1) = 0 \end{cases}$$

The solutions to the above equations, which are also the critical points of f, are

$$(x,y) \in \left\{(0,0), (0, -\frac{1}{4}, (-\frac{1}{3}, 0), (-\frac{1}{9}, -\frac{1}{12})\right\}.$$

The Hessian matrix of f is

$$H(x,y) = \begin{pmatrix} 6y & 6x+8y+1 \\ 6x+8y+1 & 8x \end{pmatrix}$$

We have

$$H(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has mixed eigenvalues (± 1), so $(0,0)$ is a saddle point.

$$H(0, -\frac{1}{4}) = \begin{pmatrix} -1.5 & -1 \\ -1 & 0 \end{pmatrix}$$

also has mixed eigenvalues (-2 and 0.5), hence $(0, -\frac{1}{4})$ is also a saddle point.

$$H(-\frac{1}{3}, 0) = \begin{pmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{pmatrix}$$

also has mixed eigenvalues (-3 and $\frac{1}{3}$), hence $(-\frac{1}{3}, 0)$ is also a saddle point. Finally,

$$H(-\frac{1}{9}, -\frac{1}{12}) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{8}{9} \end{pmatrix}$$

has two negative eigenvalues ($\frac{-25 \pm \sqrt{193}}{36}$), hence is negative-definite and thus $(-\frac{1}{9}, -\frac{1}{12})$ is a local maximum.

6.11. The unique minimizer x^* of f is $x^* = \frac{-b}{2a}$. Now for any $x_0 \in \mathbb{R}$, apply one iteration of Newton's method yields

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = \frac{-b}{2a} = x^*.$$

Thus, one iteration of Newton's method lands at the unique minimizer of f.

6.14. Python Code.