Helmholtz Coils Theory

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Abstract

These notes summarize the theory of Helmholtz coils.

1 Magnetic Field of a Current Loop

Let's consider a circular current with radius R. The wire is assumed to be infinitesimally thin and carries a current I. Due to the symmetry of the problem, one can conclude that there is only a magnetic field component in the direction of the axis B_z and radially B_ρ. The tangential component vanishes $\vec{B}_\theta = 0$.

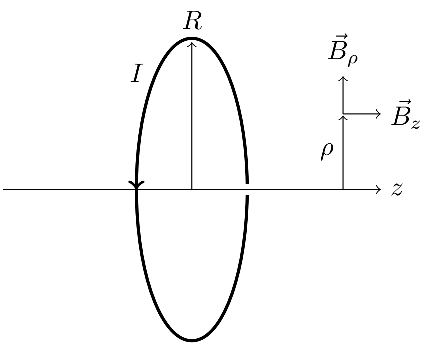

Figure 1: Current loop.

1.1 Field on the Axis

Using the Biot-Savart law 6, one can derive the magnetic field on the axis by a circular wire.

$$\vec{B}(z) = \frac{\mu_0 I}{2R} \frac{R^3}{(R^2 + z^2)^{3/2}} \hat{e}_z \tag{1}$$

The field in the center is

$$\vec{B}(0) = \frac{\mu_0 I}{2R} \hat{e}_z = B_0 \hat{e}_z \tag{2}$$

Note that $B_0 = \mu_0 I / 2R$ is a natural unit for the current loop and the problems in the following chapters. The second part in equation (1) is dimensionless and ... Figure 2

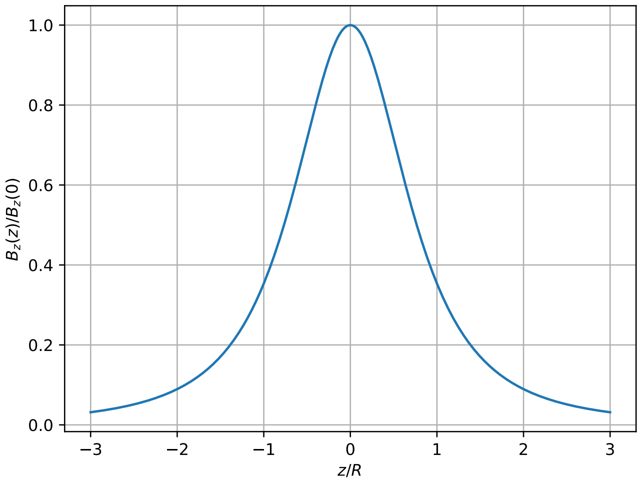

Figure 2: Magnetic field on axis of a current loop . ... equation (1)

1.2 General Solution

According to Bergeman et al. ([1]) the solution of the Biot-Svart integral (eq. 7) is

$$\vec{B}(z,\rho) = B_z(z,\rho)\hat{e}_z + B_\rho(z,\rho)\hat{e}_\rho$$

$$B_z(z,\rho) = \frac{\mu_0 I}{2R}\frac{R}{\pi\sqrt{(R+\rho)^2+z^2}}\left(E(k^2)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}+K(k^2)\right) \tag{3}$$

$$B_\rho(z,\rho) = \frac{\mu_0 I}{2R}\frac{z}{\pi\rho\sqrt{(R+\rho)^2+z^2}}\left(E(k^2)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}-K(k^2)\right) \tag{4}$$

$$k^2 = \frac{4R\rho}{(R+\rho)^2+z^2}$$

$K(k^2)$ is the complete elliptic integral of the first kind and $E(k^2)$ the complete elliptic integral of the second kind.

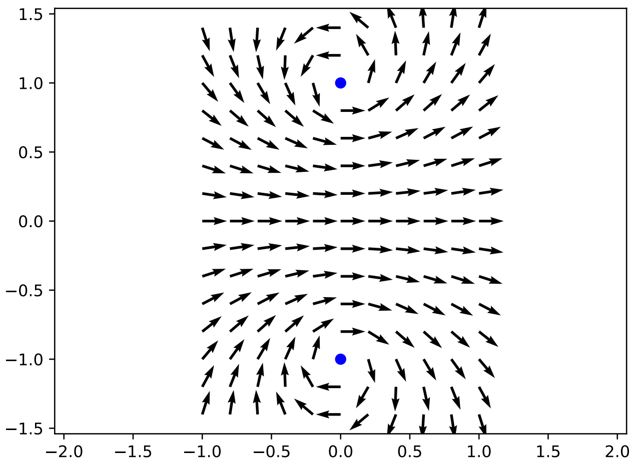

Figure 3: Magnetic field of a current loop .

2 Helmholtz Coils

A Helmholtz setup consists of two identical current loops with radius R which are placed at a distance that is exactly the radius R of the loops. The current I in both loops is exactly the same and is running in the same direction.

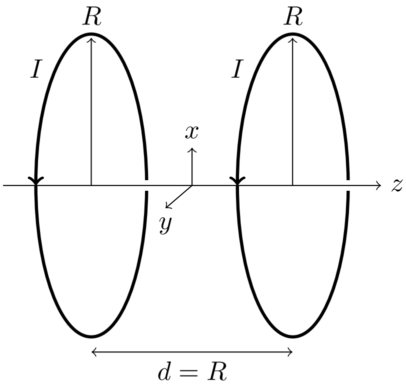

Figure 4: Helmholtz coil.

The field of both coils on the axis ($\rho = 0$) is then

$$\vec{B}_{\mathrm{Hh}}(z) = \frac{\mu_0 I}{2R}\left(\frac{R^3}{\left(R^2+\left(z-\frac{R}{2}\right)^2\right)^{3/2}} + \frac{R^3}{\left(R^2+\left(z+\frac{R}{2}\right)^2\right)^{3/2}}\right)\vec{e}_z \tag{5}$$

The field at the center is

$$B_{\mathrm{Hh}}(z=0) = B(R/2) + B(-R/2) = \frac{\mu_0 I}{2R}\frac{2R^3}{\left(R^2+\frac{R^2}{4}\right)^{3/2}} = \frac{16\sqrt{5}}{25}\frac{\mu_0 I}{2R} \approx 1.431\frac{\mu_0 I}{2R} \tag{6}$$

Since $B_{\mathrm{H}}(z)$ is a symmetric function, all odd gradients are zero

$$\frac{\partial^n B_{\mathrm{H}}}{\partial z^n}(0) = 0 \qquad \text{for } n \text{ odd}$$

The condition that the distance between the coils is equal to the radius leads to the fact that the second order gradient is zero

$$\frac{\partial^2 B_{\mathrm{H}}}{\partial z^2}(0) = 0$$

Hence, the field in the center of the coils is very constant. The first non-zero gradient is the fourth order.

$$\begin{aligned}\vec{B}_{\mathrm{H}}(z) &= \frac{\mu_0 I}{2R}\vec{e}_z\left(\frac{16\sqrt{5}}{25} - \frac{512\sqrt{5}}{625}(z/R)^4 + \frac{315392\sqrt{5}}{390625}(z/R)^6 + \mathcal{O}((z/R)^8)\right)\\ &= \frac{\mu_0 I}{2R}\vec{e}_z\left(1.431 - 1.649\,(z/R)^4 + 1.805\,(z/R)^6 + \mathcal{O}((z/R)^8)\right)\end{aligned}$$

Another aspect of the Helmholtz configuration is that the field in the symmetry plane ($z = 0$) has only a component in the z direction

$$\vec{B}(x, y, z=0) = B(x, y, 0)\vec{e}_z$$

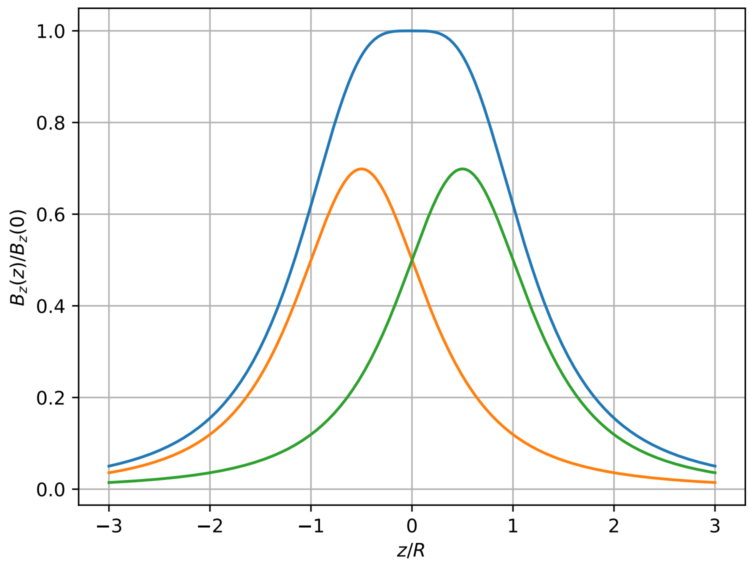

Figure 5: Magnetic field on axis of a Helmholtz coil. Green shows the contribution of the current loop at position $R/2$ and orange is the contribution of the current loop at position $-R/2$.

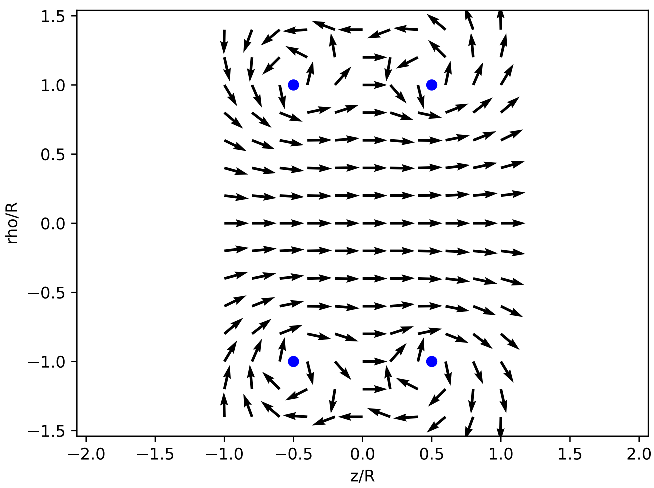

Figure 6: Magnetic field of a Helmholtz coil.

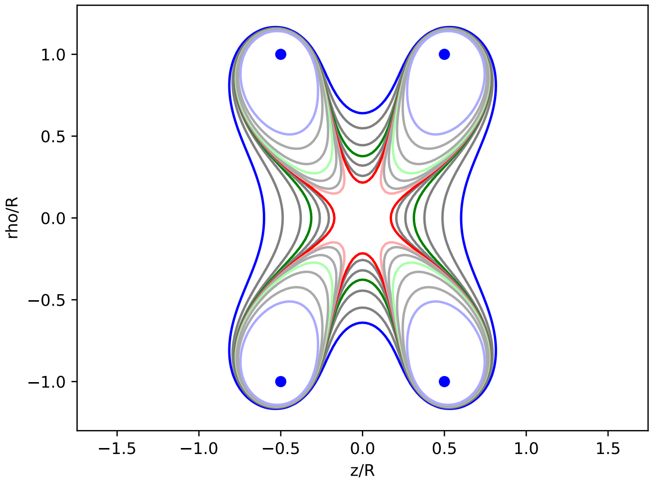

Figure 7: Magnetic field strenth of a Helmholtz coil $\left|\vec{B}_{\mathrm{Hh}}(z,\rho)\right|$. The field in the center has the strength of $|\vec{B}_{\mathrm{Hh}}(0,0)| = \frac{16\sqrt{5}}{25}\frac{\mu_0 I}{2R}$. The contourlines indicate relative deviation from the center value, red: $\pm 10\%$, green: $\pm 1\%$, blue: $\pm 0.1\%$. The field decreases on the z and the ρ axes. It increases towards the current loops (blue dots).

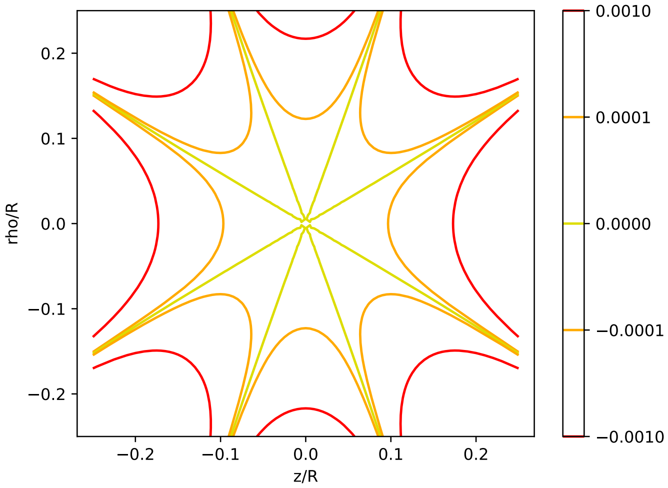

Figure 8: Magnetic field strength, zoom in on figure 7. yellow: $|\vec{B}_{\mathrm{Hh}}| = \frac{16\sqrt{5}}{25}\frac{\mu_0 I}{2R}$., orange: $\pm 10^{-4}$, red: $\pm 10^{-3}$

3 Anti-Helmholtz Coil

The anti-Helmholtz coil consists of exactly the same geometry as the Helmholtz coil. However, for an anti-Helmholtz the currents are running in opposite directions.

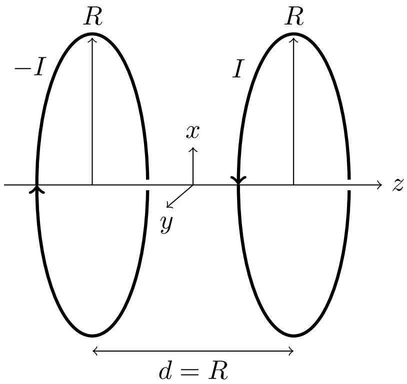

Figure 9: Anit-Helmholtz coil.

$$\vec{B}_{\mathrm{aHh}}(z) = \frac{\mu_0 I}{2R}\left(\frac{R^3}{\left(R^2+\left(z-\frac{R}{2}\right)^2\right)^{3/2}} - \frac{R^3}{\left(R^2+\left(z+\frac{R}{2}\right)^2\right)^{3/2}}\right)\vec{e}_z \tag{7}$$

Figure

$$\vec{B}_{\mathrm{aHh}}(0) = 0$$

$$\frac{\partial^n B_{\mathrm{aHh}}}{\partial z^n}(0) = 0 \qquad \text{for } n \text{ even}$$

$$\begin{aligned}\vec{B}_{\mathrm{aHh}}(z) &= \frac{\mu_0 I}{2R}\vec{e}_z\left(\frac{96\sqrt{5}}{125}(z/R) - \frac{512\sqrt{5}}{625}(z/R)^3 + \mathcal{O}((z/R)^5)\right) \\ &= \frac{\mu_0 I}{2R}\vec{e}_z\left(1.717\,(z/R) - 1.832\,(z/R)^3 + \mathcal{O}((z/R)^5)\right)\end{aligned}$$

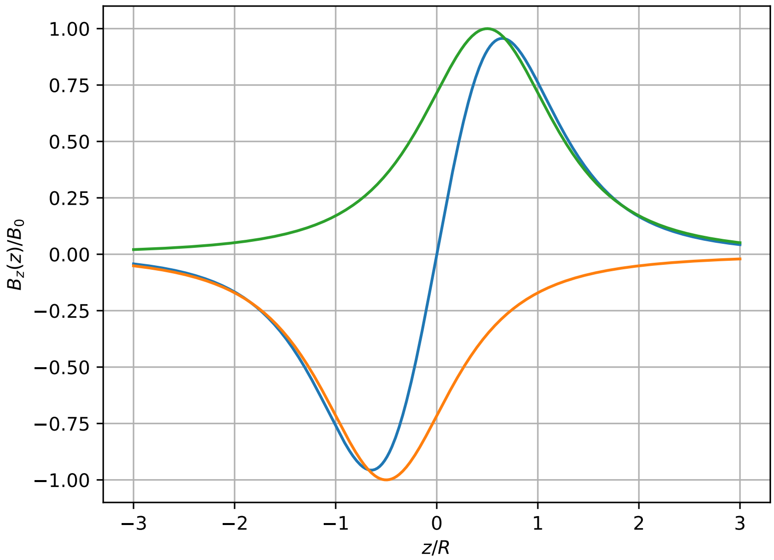

Figure 10: Anti-Helmholtz coil provides zero field in the center and a very constant gradient. The blue curve is given by equation 5. The green and the orange curve are the contributions of the right and the left current loop. All curves are normalized by $B_0 = \mu_0 I/2R$.

A General Information and Derivations

A.1 Biot-Savart Law

Biot-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{\mathrm{wire}} \frac{I d\vec{l} \times \vec{s}}{s^3} \tag{8}$$

Note that vacuum permeability μ_0 has the value of

$$\mu_0 = 4\pi 10^{-7} \mathrm{H/m} = 4\pi 10^{-7} \mathrm{Tm/A}$$

A.2 Cylindrical Coordinates

cylindrical coordinates (ρ, θ, z). $\rho = \sqrt{x^2 + y^2}$ $\theta = \arg(x + iy)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos(\theta) \\ \rho \sin(\theta) \\ z \end{pmatrix}$$

A.3 Current Loop

Biot-Savart law 6

$$\vec{s} = \begin{pmatrix} \rho - R\cos(\theta) \\ -R\sin(\theta) \\ z \end{pmatrix}$$

$$d\vec{l} = R \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} d\theta$$

$$|\vec{s}|^2 = R^2 - 2R\rho\cos(\theta) + \rho^2 + z^2$$

$$d\vec{l} \times \vec{s} = R \begin{pmatrix} z\cos(\theta) \\ z\sin(\theta) \\ R - \rho\cos(\theta) \end{pmatrix} d\theta$$

$$\vec{B}(\rho, z) = \frac{\mu_0 I}{4\pi} \int_{-\pi}^{\pi} \frac{R \begin{pmatrix} z\cos(\theta) \\ z\sin(\theta) \\ R - \rho\cos(\theta) \end{pmatrix} d\theta}{(R^2 - 2R\rho\cos(\theta) + \rho^2 + z^2)^{3/2}}$$

By introducing $\tilde{\rho} = \rho/R$ and $\tilde{z} = z/R$

$$\vec{B}(\rho, z) = \frac{\mu_0 I}{2R} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\begin{pmatrix} \tilde{z}\cos(\theta) \\ \tilde{z}\sin(\theta) \\ 1 - \tilde{\rho}\cos(\theta) \end{pmatrix} d\theta}{(1 - 2\tilde{\rho}\cos(\theta) + \tilde{\rho}^2 + \tilde{z}^2)^{3/2}} \tag{9}$$

B Document Information

References

[1] T. Bergeman, Gidon Erez, and Harold J. Metcalf. Magnetostatic trapping fields for neutral atoms. *Phys. Rev. A*, 35:1535–1546, Feb 1987.