Expansion Notes

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1 Introduction

We present explicit formulas for second-order approximations of stochastic processes that feature prominently in our asset pricing applications. We also describe how we induce these approximations to respect a collection of equilibrium conditions.

2 Small noise expansion

We follow Lombardo and Uhlig (2018) by considering the following class of stochastic processes indexed by a scalar perturbation parameter q:

$$X_{t+1}(\mathsf{q}) = \psi\left[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right]. \tag{1}$$

Here X is an n-dimensional stochastic process and $\{W_{t+1}\}$ is an i.i.d. normally distributed random vector with conditional mean vector 0 and conditional covariance matrix I.

We denote a zero-order expansion $\mathsf{q} = 0$ limit as:

$$X^0_{t+1} = \psi\left(X^0_t, 0, 0\right), \tag{2}$$

and assume that there exists a second-order expansion of X_t around $\mathsf{q} = 0$:

$$X_t \approx X^0_t + \mathsf{q}X^1_t + \frac{\mathsf{q}^2}{2}X^2_t \tag{3}$$

where X^1_t is a first-order expansion.

In the remainder of this chapter we shall construct instances of the second-order expansion (3) in which the generic random variable X_t is replaced, for example, by the logarithm of consumption, a value function, and so on. In approximation (3), the stochastic processes X^j, $j = 0, 1, 2$ are appropriate derivatives of X with respect to the perturbation parameter q.

Processes $X_t^j, j = 0, 1, 2.$ have a recursive structure: the stochastic process X_t^0 can be computed first, then the process X_t^1 next (it depends on X_t^0), and then the process X_t^2 (it depends on both X_t^0 and X_t^1).

We use a prime ($'$) to denote a transpose of a matrix or vector. When we include x' in a partial derivative of a scalar function it means that the partial derivative is a row vector. Consistent with this convention, let $\psi^i_{x'}$, the i^{th} entry of $\psi_{x'}$, denote the row vector of first derivatives with respect to the vector x, and similarly for $\psi^i_{w'}$. Since q is scalar, ψ^i_{q} is the scalar derivative with respect to q. Derivatives are evaluated at X_t^0, which in some examples is invariant over time, unless otherwise stated.

The first-derivative process obeys a recursion

$$X_{t+1}^1 = \begin{bmatrix} \psi^1_{x'} \\ \psi^2_{x'} \\ \vdots \\ \psi^n_{x'} \end{bmatrix} X_t^1 + \begin{bmatrix} \psi^1_{w'} \\ \psi^2_{w'} \\ \vdots \\ \psi^n_{w'} \end{bmatrix} W_{t+1} + \begin{bmatrix} \psi^1_{\mathsf{q}} \\ \psi^2_{\mathsf{q}} \\ \vdots \\ \psi^n_{\mathsf{q}} \end{bmatrix} \tag{4}$$

that we can write compactly as the following *a first-order vector autoregression*:

$$X_{t+1}^1 = \psi_{x'} X_t^1 + \psi_{w'} W_{t+1} + \psi_{\mathsf{q}}$$

We assume that the matrix ψ_x' is stable in the sense that all of its eigenvalues are strictly less than one in modulus.

It is natural for us to denote second derivative processes with double subscripts. For instance, for the double script used in conjunction with the second derivative matrix of ψ^i, the first subscript without a prime ($'$) reports the row location; second subscript with a prime ($'$) reports the column location. Differentiating recursion (4) gives:

$$X_{t+1}^2 = \psi_{x'} X_t^2 + \begin{bmatrix} {X_t^1}' \psi^1_{xx'} X_t^1 \\ {X_t^1}' \psi^2_{xx'} X_t^1 \\ \vdots \\ {X_t^1}' \psi^n_{xx'} X_t^1 \end{bmatrix} + 2 \begin{bmatrix} {X_t^1}' \psi^1_{xw'} W_{t+1} \\ {X_t^1}' \psi^2_{xw'} W_{t+1} \\ \vdots \\ {X_t^1}' \psi^n_{xw'} W_{t+1} \end{bmatrix} + \begin{bmatrix} {W_{t+1}}' \psi^1_{ww'} W_{t+1} \\ {W_{t+1}}' \psi^2_{ww'} W_{t+1} \\ \vdots \\ {W_{t+1}}' \psi^n_{ww'} W_{t+1} \end{bmatrix}$$

$$
+ 2 \begin{bmatrix} \psi^1_{\mathsf{q}x'} X^1_t \\ \psi^2_{\mathsf{q}x'} X^1_t \\ \vdots \\ \psi^n_{\mathsf{q}x'} X^1_t \end{bmatrix} + 2 \begin{bmatrix} \psi^1_{\mathsf{q}w'} W_{t+1} \\ \psi^2_{\mathsf{q}w'} W_{t+1} \\ \vdots \\ \psi^n_{\mathsf{q}w'} W_{t+1} \end{bmatrix} + \begin{bmatrix} \psi^1_{\mathsf{qq}} \\ \psi^2_{\mathsf{qq}} \\ \vdots \\ \psi^n_{\mathsf{qq}} \end{bmatrix} \tag{5}
$$

Recursions (4) and (5) have a linear structure with some notable properties. The law of motion for X^0 is deterministic and is time invariant if (1) comes from a stationary $\{X_t\}$ process. The dynamics of X^2 are nonlinear only in X^1 and W_{t+1}; so the stable dynamics for X^1 that prevail when ψ_x is a stable matrix imply stable dynamics for X^2. Stability carries over to higher-order terms too.

Perturbation methods have been applied to many rational expectations models in which partial derivatives of ψ with respect to q are often zero.[1] However, derivatives of ψ with respect to q aren't zero in models with the robust or recursive utility specifications that we shall study here.

3 Approximating a recursive utility value function

In this section, we obtain second order expansions for components of a continuation value process. This process along with its associated stochastic discount factor process are important constituents of models. Let C denote consumption and $\widehat{C}$ the logarithm of consumption. We start by assuming that we have in hand a second-order approximation for the process $\widehat{C}$ whose expansion takes the form (3) with $\widehat{C}$ replacing the generic variable X everywhere so that

$$
\widehat{C}_t \approx \widehat{C}^0_t + \mathsf{q}\widehat{C}^1_t + \frac{\mathsf{q}^2}{2}\widehat{C}^2_t. \tag{6}
$$

The processes $\widehat{C}^j$ for $j = 0, 1, 2$ are computed as described in section 2. These components of approximation (6) will be inputs into calculations for component of second-order approximation to a value function and associated processes.

The homogeneous of degree one representation of recursive utility is

$$
V_t = \left[(1-\beta)\left(C_t\right)^{1-\rho} + \beta \exp\left(R_t\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{7}
$$

[1]See, for instance, Schmitt-Grohé and Uribe (2004).

where

$$R_t = \left(\mathbb{E}\left[(V_{t+1})^{1-\gamma} \mid \mathfrak{F}_t\right]\right)^{\frac{1}{1-\gamma}}. \tag{8}$$

Notice that in equation (7), V_t is a homogeneous of degree one function of C_t and R_t. In equation (8), R_t is a homogeneous of degree one function of another function, namely, V_{t+1} as it varies over date $t+1$ information. In equation (7), $0 < \beta < 1$ is a subjective discount factor and ρ describes attitudes toward intertemporal substitution. Formally, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution. In equation (8), γ describes attitudes towards risk.

Continuation values are determined only up to an increasing transformation. For computational and conceptual reasons, we find it advantageous to work with the logarithm $\widehat{V}_t = \log V_t$. The corresponding recursions for $\widehat{V}_t$ expressed in terms of the logarithm of consumption $\widehat{C}_t$ are

$$\widehat{V}_t = \frac{1}{1-\rho} \log\left[(1-\beta)\exp[(1-\rho)\widehat{C}_t] + \beta \exp\left[(1-\rho)\widehat{R}_t\right]\right] \tag{9}$$

where

$$\widehat{R}_t = \frac{1}{1-\gamma} \log \mathbb{E}\left(\exp\left[(1-\gamma)\widehat{V}_{t+1}\right] \mid \mathfrak{F}_t\right). \tag{10}$$

The right side of recursion (9) is the logarithm of a constant elasticity of substitution (CES) function of $\exp(\widehat{C}_t)$ and $\exp(\widehat{R}_t)$.

Our approach will be to construct small noise expansions for both $\widehat{V}_t$ and $\widehat{R}_t$ and then to assemble them appropriately. Before doing so, we consider a reinterpretation of (10).

3.1 Robustness to Model Misspecification

A reinterpretation of the utility recursion and the two-small-noise expansion approach that we'll deploy comes from recognizing that when $\gamma > 1$, (10) emerges from an instance robust control theory in which $\frac{1}{\gamma-1}$ is a penalty parameter on entropy relative to alternatives that constrains the alternative probability models that a decision maker considers when evaluating consumption processes. This interpretation originated in work by Jacobson (1973) and Whittle (1981) that was extended and reformulated recursively by Hansen and Sargent (1995).

Let the random variable $N_{t+1} \geq 0$ satisfy $\mathbb{E}(N_{t+1} \mid \mathfrak{F}_t) = 1$ so that it is a likelihood ratio. Think of replacing the expected continuation value $\mathbb{E}\left(\widehat{V}_{t+1} \mid \mathfrak{F}_t\right)$ by the minimized value of the following problem:

$$\min_{N_{t+1}\geq 0, \mathbb{E}(N_{t+1}|\mathfrak{F}_t)=1} \mathbb{E}\left(N_{t+1}\widehat{V}_{t+1} \mid \mathfrak{F}_t\right) + \xi \mathbb{E}\left(N_{t+1} \log N_{t+1} \mid \mathfrak{F}_t\right) \tag{11}$$

where ξ is a parameter that penalizes departures of N_{t+1} from unity as measured by relative entropy. Conditional relative entropy for an altered conditional probability induced by applying change of measure N_{t+1} is

$$\mathbb{E}\left(N_{t+1} \log N_{t+1} \mid \mathfrak{F}_t\right) \geq 0$$

where, because the function $y \log y$ is convex, the inequality follows from Jensen's inequality. Relative entropy is zero when $N_{t+1} = 1$. The minimizer of problem (11), namely,

$$N_{t+1}^* = \frac{\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{F}_t\right]}$$

"tilts" probabilities towards low continuation values, a version of what Bucklew (2004) calls a stochastic version of Murphy's law. The minimized objective

$$-\xi \log \mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{F}_t\right] = \widehat{R}_t$$

where $\widehat{R}_t$ was given previously by equation (10) if we set $\xi = \frac{1}{\gamma - 1}$.

3.2 Two expansion protocols

The two alternative expansion protocols have significant and enlightening consequences for continuation value processes and for the minimizing N process used to alter expectations. Under the first protocol, we fix $\xi = \xi_o$ while under the second we set

$$\xi = \mathsf{q}\xi_o$$

for a fixed ξ_o. The is first is essentially what is typically used in macro-finance research, while we will show that the second one has the advantage of letting uncertainty matter for lower order terms in the approximation. The second protocol has antecedents in the control theory literature and has the virtue that implied uncertainty adjustments occur more prominently at lower order terms in the approximation. Under the "risk" interpretation,

$$\frac{1}{\gamma - 1} = \xi = \mathsf{q}\xi_o = \frac{\mathsf{q}}{\gamma_o - 1}$$

for $\gamma_o > 1$. Thus the $\gamma - 1$ moves inversely with q.

3.2.1 Approximating the change in probability measure

Recall that the change of measure deduced by solving Problem (11) is:

$$N^*_{t+1} = \frac{\exp\left[-\left(\frac{1}{\xi}\right)\widehat{V}_{t+1}\right]}{\mathbb{E}\left(\exp\left[-\left(\frac{1}{\xi}\right)\widehat{V}_{t+1}\right] \mid \mathfrak{F}_t\right)}.$$

To illustrate the difference in the implications, we compute zero and first-order contributions under the two protocols. Under the first protocol, the $\mathsf{q} = 0$ limit is:

$$\lim_{\mathsf{q}\downarrow 0} N^*_{t+1} = 1,$$

and

$$\frac{d}{d\mathsf{q}} N^*_{t+1}\bigg|_{\mathsf{q}=0} = -\frac{1}{\xi_o}\left[\widehat{V}^1_{t+1} - \mathbb{E}\left(\widehat{V}^1_{t+1} \mid \mathfrak{F}_t\right)\right] \doteq N^1_{t+1}$$

Recall that under the second protocol $\xi = \mathsf{q}\xi_o$. In this case we rewrite the formula for N^*_{t+1} as:

$$N^*_{t+1} = \frac{\exp\left[-\left(\frac{1}{\xi_o}\right)\widetilde{V}_{t+1}\right]}{\mathbb{E}\left(\exp\left[-\left(\frac{1}{\xi_o}\right)\widetilde{V}_{t+1}\right] \mid \mathfrak{F}_t\right)}$$

with a $\mathsf{q} = 0$ limit of N^*_{t+1} for the second protocol given by:

$$\begin{aligned}
\lim_{\mathsf{q}\downarrow 0} N^*_{t+1} =& \frac{\exp\left[-\frac{1}{\xi_o}\left(\widetilde{V}^0_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[-\frac{1}{\xi_o}\left(\widetilde{V}^0_{t+1}\right)\right] \mid \mathfrak{F}_t\right)} \\
=& \frac{\exp\left[-\frac{1}{\xi_o}\left(\widehat{V}^1_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[-\frac{1}{\xi_o}\left(\widehat{V}^1_{t+1}\right)\right] \mid \mathfrak{F}_t\right)} \\
\doteq& N^0_{t+1}
\end{aligned} \tag{12}$$

Consider next,

$$\frac{d}{d\mathsf{q}} N^*_{t+1}\bigg|_{\mathsf{q}=0} =(1-\gamma_0)\frac{\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]\widetilde{V}^1_{t+1}}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{F}_t\right)}$$

$$
\begin{aligned}
&- (1-\gamma_o)\left(\frac{\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{F}_t\right)}\right)\left(\frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]\widetilde{V}^1_{t+1} \mid \mathfrak{F}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{F}_t\right)}\right) \\
=& \left(\frac{1-\gamma_o}{2}\right)\frac{\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right]\widehat{V}^2_{t+1}}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right] \mid \mathfrak{F}_t\right)} \\
&- \left(\frac{1-\gamma_o}{2}\right)\left(\frac{\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right] \mid \mathfrak{F}_t\right)}\right)\left(\frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right]\widehat{V}^2_{t+1} \mid \mathfrak{F}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right] \mid \mathfrak{F}_t\right)}\right) \\
\doteq& N^1_{t+1}.
\end{aligned}
$$

Notice that the zeroth and first-order adjustments involve higher order approximations of the continuation value process under the second protocol. Neither of the first-order approximations to N^*_{t+1} preserve positivity, however. In both cases, the first-order adjustment terms have conditional mean zero. As an alternative, we also consider using:

$$
\begin{aligned}
\log N^*_{t+1} \approx & . \frac{\exp\left[(1-\gamma_o)\left(\mathsf{q}\widehat{V}^1_{t+1} + \frac{\mathsf{q}^2}{2}\widehat{V}^2_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\mathsf{q}\widehat{V}^1_{t+1} + \frac{\mathsf{q}^2}{2}\widehat{V}^2_{t+1}\right)\right] \mid \mathfrak{F}_t\right)} \\
\doteq & \widetilde{N}^2_{t+1}.
\end{aligned}
$$

We now explore the approximation of the continuation values.

3.3 Continuation values

3.3.1 Order zero

We write the order-zero expansion of (9) as

$$
\begin{aligned}
\widehat{V}^0_t &= \frac{1}{1-\rho}\log\left[(1-\beta)\exp[(1-\rho)\widehat{C}^0_t] + \beta\exp\left[(1-\rho)\widehat{R}^0_t\right]\right] \\
\widehat{R}^0_t &= \widehat{V}^0_{t+1},
\end{aligned}
$$

where the second equation follows from noting that randomness vanishes in the limit as q approaches 0.

We assume that the logarithm of consumption is an additive process so that its first-difference is stationary. For models with production, this representation will become part of the derived expansion. For order zero, we presume a constant growth rate:

$$\widehat{C}^0_{t+1} - \widehat{C}^0_t = \eta_c.$$

The order-zero approximation of (9) is:

$$\widehat{V}^0_t - \widehat{C}^0_t = \frac{1}{1-\rho} \log \left[(1-\beta) + \beta \exp \left[(1-\rho) \left(\widehat{V}^0_{t+1} - \widehat{C}^0_{t+1} + \eta_c \right) \right] \right]$$

We guess that $\widehat{V}^0_t - \widehat{C}^0_t = \eta_{v-c}$ and will have verified the guess if the following equation is satisfied

$$\exp\left[(1-\rho)(\eta_{v-c})\right] = (1-\beta) + \beta \exp\left[(1-\rho)(\eta_{v-c})\right] \exp\left[(1-\rho)\eta_c\right],$$

which implies

$$\exp\left[(1-\rho)(\eta_{v-c})\right] = \frac{1-\beta}{1-\beta\exp\left[(1-\rho)\eta_c\right]}. \tag{13}$$

Equation (13) determines η_{v-c} as a function of η_c and the preference parameters ρ, β, but not the risk aversion parameter γ.

3.3.2 Order one

We temporarily take $\widehat{R}^1_t$ as given (we'll compute it in subsection (3.3.4)). We construct a recursion by taking a first-order approximation to the nonlinear utility recursion (9)

$$\begin{aligned} \widehat{V}^1_t = & \left[\frac{(1-\beta)}{(1-\beta) + \beta\exp\left[(1-\rho)\left(\eta_{v-c} + \eta_c\right)\right]} \right] \widehat{C}^1_t \\ & + \left[\frac{\beta\exp\left[(1-\rho)\left(\eta_{v-c} + \eta^0_c\right)\right]}{(1-\beta) + \beta\exp\left[(1-\rho)\left(\eta_{v-c} + \eta_c\right)\right]} \right] \widehat{R}^1_t, \end{aligned}$$

and represent it as a weighted average of $\widehat{C}^1_t$ and $\widehat{R}^1_t$

$$\widehat{V}^1_t = (1-\lambda)\widehat{C}^1_t + \lambda \widehat{R}^1_t \tag{14}$$

where $0 < \lambda < 1$ satisfies

$$\begin{aligned} \lambda = & \left[\frac{\beta\exp\left[(1-\rho)\left(\eta_{v-c} + \eta^0_c\right)\right]}{(1-\beta) + \beta\exp\left[(1-\rho)\left(\eta_{v-c} + \eta_c\right)\right]} \right] \\ = & \left[\frac{\beta\exp\left[(1-\rho)\eta_c\right]}{(1-\beta)\exp\left[-(1-\rho)\eta_{v-c}\right] + \beta\exp\left[(1-\rho)\eta_c\right]} \right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\beta \exp\left[(1-\rho)\eta_c\right]}{1-\beta \exp\left[(1-\rho)\eta_c\right] + \beta \exp\left[(1-\rho)\eta_c\right]}\right] \\
&= \beta \exp\left[(1-\rho)\eta_c\right] \qquad (15)
\end{aligned}
$$

Notice how parameter ρ influences the weight λ when $\eta_c \neq 0$, in which case the log consumption process displays growth or decay. When $0 < \rho < 1$, the condition $\lambda < 1$ restricts the parameter ρ relative to the consumption growth rate η_c since

$$
(1-\rho)\eta_c < -\log \beta
$$

To anticipate consequences of growth in log consumption, we express the first-order approximation as:

$$
\widehat{V}_t^1 - \widehat{C}_t^1 = \lambda \left(\widehat{R}_t^1 - \widehat{C}_t^1\right) \qquad (16)
$$

3.3.3 Order two

Temporarily take both $\widehat{R}_t^1$ and $\widehat{R}_t^2$ as given. (We'll construct both in subsection (3.3.4).) Differentiating equation (9) a second time gives:

$$
\begin{aligned}
\widehat{V}_t^2 =\ & (1-\lambda)\widehat{C}_t^2 + \lambda \widehat{R}_t^2 \\
& + (1-\rho)\left[(1-\lambda)\left(\widehat{C}_t^1\right)^2 + \lambda\left(\widehat{R}_t^1\right)^2 - \left[(1-\lambda)\widehat{C}_t^1 + \lambda \widehat{R}_t^1\right]^2\right] \\
=\ & (1-\lambda)\widehat{C}_t^2 + \lambda \widehat{R}_t^2 + (1-\rho)(1-\lambda)\lambda\left(\widehat{C}_t^1 - \widehat{R}_t^1\right)^2. \qquad (17)
\end{aligned}
$$

Equivalently,

$$
\widehat{V}_t^2 - \widehat{C}_t^2 = \lambda\left(\widehat{R}_t^2 - \widehat{C}_t^2\right) + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2.
$$

3.3.4 Approximating $\widehat{R}_t$

We now turn to computing the $\widehat{R}_t^1$ and $\widehat{R}_t^2$ that appear in the second-order approximation to $\widehat{R}_t$. For both protocols,

$$
\widehat{R}_t^0 - \widehat{C}_t^0 = \left(\widehat{V}_{t+1}^0 - \widehat{C}_{t+1}^0\right) + \left(\widehat{C}_{t+1}^0 - \widehat{C}_t^0\right)
$$

For the first protocol where $\xi = \xi_o$ and hence $\gamma - 1 = \gamma_o - 1$ are held fixed for all q,

$$
\begin{aligned}
\widehat{R}_t^1 - \widehat{C}_t^1 =& \mathbb{E}\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1 \mid \mathfrak{F}_t\right) + \mathbb{E}\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1 \mid \mathfrak{F}_t\right) \\
\widehat{R}_t^2 - \widehat{C}_t^2 =& \mathbb{E}\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2 \mid \mathfrak{F}_t\right) + \mathbb{E}\left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2 \mid \mathfrak{F}_t\right) \\
&+ (1-\gamma)\left(\mathbb{E}\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1\right)^2 \mid \mathfrak{F}_t\right] - \left[\mathbb{E}\left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1 \mid \mathfrak{F}_t\right)\right]^2\right).
\end{aligned}
$$

Notice that γ does not contribute to the first-order approximation $\widehat{R}_t^1$ and that the term scaled by $1-\gamma$ in the second-order approximation $\widehat{R}_t^2$ is the conditional variance of the first-order approximation for the continuation value, a term that is typically constant.

The implied continuation value recursions are:

$$
\widehat{V}_t^1 - \widehat{C}_t^1 = \lambda \mathbb{E}\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1 \mid \mathfrak{F}_t\right) + \lambda \mathbb{E}\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1 \mid \mathfrak{F}_t\right) \tag{18}
$$

and

$$
\begin{aligned}
\widehat{V}_t^2 - \widehat{C}_t^2 =& \lambda \mathbb{E}\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2 \mid \mathfrak{F}_t\right) + \lambda \mathbb{E}\left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2 \mid \mathfrak{F}_t\right) \\
&+ \lambda(1-\gamma)\left(\mathbb{E}\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1\right)^2 \mid \mathfrak{F}_t\right] - \left[\mathbb{E}\left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1 \mid \mathfrak{F}_t\right)\right]^2\right) \\
&+ (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2
\end{aligned} \tag{19}
$$

We now consider the second protocol with

$$
\frac{1}{\gamma - 1} = \xi = \xi_o \mathsf{q} = \frac{\mathsf{q}}{\gamma_o - 1}.
$$

To facilitate computing some useful limits we construct:

$$
\begin{aligned}
\widetilde{V}_t &= \frac{\widehat{V}_t - \widehat{V}_t^0}{\mathsf{q}} \\
\widetilde{R}_t &= \frac{\widehat{R}_t - \widehat{V}_{t+1}^0}{\mathsf{q}}
\end{aligned}
$$

which we assume remain well defined as q declines to zero, with limits denoted by $\widetilde{V}_t^0, \widetilde{R}_t^0$.

Multiplying through by q and differentiating, we deduce that

$$\mathsf{q}\frac{d}{d\mathsf{q}}\widetilde{V}_t + \widetilde{V}_t = \frac{d}{d\mathsf{q}}\widehat{V}_t$$
$$\mathsf{q}\frac{d^2}{d\mathsf{q}^2}\widetilde{V}_t + 2\frac{d}{d\mathsf{q}}\widetilde{V}_t = \frac{d^2}{d^2\mathsf{q}}\widehat{V}_t$$

Thus,

$$\widetilde{V}_t^0 = \widehat{V}_t^1$$
$$2\widetilde{V}_t^1 = \widehat{V}_t^2.$$

Analogous formulas relate $\widetilde{R}_t$ and its derivatives to the derivatives of $\widehat{R}_t$. Thus, we write:

$$\widetilde{R}_t = \left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{F}_t\right),$$

and note that

$$\widetilde{R}_t^0 = \left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right] \mid \mathfrak{F}_t\right).$$

Equivalently,

$$\widehat{R}_t^1 = \left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right] \mid \mathfrak{F}_t\right),$$

and thus

$$\widehat{R}_t^1 - \widehat{C}_t^1 = \left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right]\right] \mid \mathfrak{F}_t\right) \tag{20}$$

Combining (20) with (16) gives:

$$\begin{aligned}\widehat{V}_t^1 - \widehat{C}_t^1 =& \lambda\left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1\right)\right] \mid \mathfrak{F}_t\right)\\ =& \lambda\left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right]\right] \mid \mathfrak{F}_t\right)\end{aligned} \tag{21}$$

Equation (21) is a standard risk-sensitive recursion, a starting point of several papers in economics, for instance, Tallarini (2000)'s paper on risk-sensitive business cycles and Hansen et al. (2008)'s paper on measurement and inference challenges created by the presence of long-term risk. Both of those papers assumed a logarithmic one-period utility function, so that for them $\rho = 1$. Here we have instead obtained the recursion as a first-order approximation without necessarily assuming log utility. In equation (**??**), the impact of $\rho \neq 1$ shows up via the effects of (14) and (15) on the weight λ.

Consider components of the second-order approximation next. Note that

$$\frac{d\widetilde{R}_t}{d\mathsf{q}} = \frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \frac{d\widetilde{V}_{t+1}}{d\mathsf{q}} \mid \mathfrak{F}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{F}_t\right)},$$

and thus

$$\begin{aligned} \widehat{R}_t^2 = 2\widetilde{R}_t^1 &= 2E\left(N_{t+1}^0 \widetilde{V}_{t+1}^1 \mid \mathfrak{F}_t\right) \\ &= E\left(N_{t+1}^0 \widehat{V}_{t+1}^2 \mid \mathfrak{F}_t\right), \end{aligned} \tag{22}$$

where N_{t+1}^0 is given by (12)

$$\begin{aligned} N_{t+1}^0 = &\frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right] \mid \mathfrak{F}_t\right)} \\ = &\frac{\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right] \mid \mathfrak{F}_t\right)}. \end{aligned}$$

Combining this adjustment with (22) gives:

$$\begin{aligned} \widehat{V}_t^2 - \widehat{C}_t^2 = \ &\lambda \mathbb{E}\left(N_{t+1}^0 \widehat{V}_{t+1}^2 - \widehat{C}_t^2 \mid \mathfrak{F}_t\right) \\ &+ (1-\rho)(1-\lambda)\lambda\left[\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2\right] \end{aligned}$$

The form of equations (19) and (22) for $\widehat{V}_t^2 - \widehat{C}_t^2$ is very similar for the two protocols. There is an extra constant term for the first protocol and a distorted expectation for the second protocol.

For computational purposes, it is advantageous to compute $\widehat{V}_t^2 - \widehat{C}_t^2$ for the second protocol using:

$$\widehat{V}_t^2 - \widehat{C}_t^2 = \lambda \mathbb{E}\left[N_{t+1}^0 \left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2\right) \mid \mathfrak{F}_t\right] \\ + \lambda \mathbb{E}\left[N_{t+1}^0 \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2\right) \mid \mathfrak{F}_t\right] + (1-\rho)(1-\lambda)\lambda \left[\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2\right]. \qquad (23)$$

This is a Lyapunov equation to be solved forward for $\widehat{V}_t^2 - \widehat{C}_t^2$.

4 Example: Long run risk

We should convert much of this section into a Jupyter notebook.

Consider the evolution equation adapted from Bansal and Yaron.

$$\begin{aligned} Y_{t+1} &= 1 + \alpha_z(Y_t - 1) + \sigma_y \cdot W_{t+1} \\ Z_{t+1} &= \alpha_z Z_t + Y_t \sigma_z \cdot W_{t+1} \\ \widehat{C}_{t+1} - \widehat{C}_t &= \eta_c + Z_t + Y_t \sigma_c \cdot W_{t+1} \end{aligned}$$

Let

$$X_{t+1} = \begin{bmatrix} Y_{t+1} \\ Z_{t+1} \end{bmatrix}$$

For this example, the second order approximation will be exact, but it remains interesting to compute the terms of the expansion. Order zero:

$$\begin{aligned} Y_t^0 &= 1 \\ Z_t^0 &= 0 \\ \widehat{C}_{t+1}^0 - \widehat{C}_t^0 &= \eta_c \end{aligned}$$

Order one:

$$\begin{aligned} Y_{t+1}^1 &= \alpha_y Y_t^1 + \sigma_y \cdot W_{t+1} \\ Z_{t+1}^1 &= \alpha_z Z_t^1 + \sigma_z \cdot W_{t+1} \\ \widehat{C}_{t+1}^1 - \widehat{C}_t^1 &= Z_t^1 + \sigma_c \cdot W_{t+1} \end{aligned}$$

Order two:

$$
\begin{aligned}
Y_t^2 &= 0 \\
Z_{t+1}^2 &= \alpha_z Z_t^2 + 2 Y_t^1 \sigma_z \cdot W_{t+1} \\
\widehat{C}_{t+1}^2 - \widehat{C}_t^2 &= Z_t^2 + 2 Y_t^1 \sigma_c \cdot W_{t+1}
\end{aligned}
$$

Haomin: please produce a table of the parameter values. You can pull these from your approximation.

4.1 Continuation value approximation

Order zero follows directly from (13).

Order one, protocol one: Solving (18) gives:

$$
\widehat{V}_t^1 - \widehat{C}_t^1 = \left(\frac{\lambda}{1 - \lambda \alpha_z} \right) Z_t^1,
$$

Thus

$$
\widehat{R}_t^1 - \widehat{C}_t^1 = \left(\frac{1}{1 - \lambda \alpha_z} \right) Z_t^1.
$$

Order one, protocol two: From (20)

$$
\widehat{R}_t^1 - \widehat{C}_t^1 = \left(\frac{1}{1 - \lambda \alpha_z} \right) Z_t^1 + \left[\frac{(1-\gamma_o)}{2(1-\lambda)} \right] \left| \left(\frac{\lambda}{1 - \lambda \alpha_z} \right) \sigma_z + \sigma_c \right|^2 \tag{24}
$$

Haomin and Hanson, please check the constant term

Notice that the difference between the two protocols shows up in order one captured by constant terms.

4.2 Order zero approximation of N_{t+1}^*

For protocol one:

$$
N_{t+1}^0 = 1
$$

For protocol two:

$$
N_{t+1}^0 = \frac{\exp\left[(1-\gamma_o) \left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1 \right) \right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o) \left(\widehat{V}_{t+1} - \widehat{C}_t^1 \right) \right] \mid \mathfrak{F}_t \right)}.
$$

which changes the distribution of W_{t+1} from being $\mathcal{N}(0, \mathbb{I})$ to being: $\mathcal{N}\left(\tilde{\mu}, \mathbb{I}\right)$. where

$$\tilde{\mu} \doteq (1-\gamma_o)\left[\left(\frac{\lambda}{1-\lambda\alpha_z}\right)\sigma_z + \sigma_c\right]$$

By a complete-the-square argument, the mean should be the coefficient in the numerator on W_{t+1} inside the exponential. Haomin and Hanson, make sure this the correct mean.

The implied stochastic dynamics under the distorted evolution are: Order one:

$$\begin{aligned}
Y^1_{t+1} &= \alpha_y Y^1_t + \sigma_y \cdot \tilde{\mu} + \sigma_y \cdot W_{t+1} \\
Z^1_{t+1} &= \alpha_z Z^1_t + \sigma_z \cdot \tilde{\mu} + \sigma_z \cdot W_{t+1} \\
\widehat{C}^1_{t+1} - \widehat{C}^1_t &= Z^1_t + \sigma_c \cdot \tilde{\mu} + \sigma_c \cdot W_{t+1}
\end{aligned}$$

Order two:

$$\begin{aligned}
Y^2_t &= 0 \\
Z^2_{t+1} &= \alpha_z Z^2_t + 2Y^1_t \sigma_z \cdot \tilde{\mu} + 2Y^1_t \sigma_z \cdot W_{t+1} \\
\widehat{C}^2_{t+1} - \widehat{C}^2_t &= Z^2_t + 2Y^1_t \sigma_c \cdot \tilde{\mu} + 2Y^1_t \sigma_c \cdot W_{t+1}
\end{aligned}$$

Next consider the second-order approximation of the value function. Under protocol one,

$$\begin{aligned}
\widehat{V}^2_t - \widehat{C}^2_t =& \lambda\mathbb{E}\left(\widehat{V}^2_{t+1} - \widehat{C}^2_{t+1} \mid \mathfrak{F}_t\right) + \lambda\mathbb{E}\left(\widehat{C}^2_{t+1} - \widehat{C}^2_t \mid \mathfrak{F}_t\right) \\
&+ \lambda(1-\gamma)\left(\mathbb{E}\left[\left(\widehat{V}^1_{t+1} - \widehat{C}^1_t\right)^2 \mid \mathfrak{F}_t\right] - \left[\mathbb{E}\left(\widehat{V}^1_{t+1} - \widehat{C}^1_t \mid \mathfrak{F}_t\right)\right]^2\right) \\
&+ (1-\rho)(1-\lambda)\lambda\left(\widehat{R}^1_t - \widehat{C}^1_t\right)^2
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{E}\left(\widehat{C}^2_{t+1} - \widehat{C}^2_t \mid \mathfrak{F}_t\right) &= Z^2_t \\
\mathbb{E}\left[\left(\widehat{V}^1_{t+1} - \widehat{C}^1_t\right)^2 \mid \mathfrak{F}_t\right] - \left[\mathbb{E}\left(\widehat{V}^1_{t+1} - \widehat{C}^1_t \mid \mathfrak{F}_t\right)\right]^2 &= \left|\left(\frac{\lambda}{1-\lambda\alpha_z}\right)\sigma_z + \sigma_c\right|^2 \\
\left(\widehat{R}^1_t - \widehat{C}^1_t\right)^2 &= \left(\frac{1}{1-\lambda\alpha_z}\right)^2 \left(Z^1_t\right)^2
\end{aligned}$$

The equation of interest can be solved as a Lyapunov function using a standard Python package. The state variable processes Y^1_t and Y^2_t drop out of this computation.

Consider next protocol two.

$$
\begin{aligned}
\widehat{V}_t^2 - \widehat{C}_t^2 = \; & \lambda \mathbb{E}\left[N_{t+1}^0 \left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2 \right) \mid \mathfrak{F}_t \right] \\
& + \lambda \mathbb{E}\left[N_{t+1}^0 \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2 \right) \mid \mathfrak{F}_t \right] + (1-\rho)(1-\lambda)\lambda \left[\left(\widehat{R}_t^1 - \widehat{C}_t^1 \right)^2 \right].
\end{aligned}
$$

where $\widetilde{\mathbb{E}}$ uses the change of measure implied by N_{t+1}^0. Since

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[N_{t+1}^0 \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2 \right) \mid \mathfrak{F}_t \right] &= Z_t^2 + 2Y_t^1 \sigma_c \cdot \tilde{\mu} \\
\widetilde{\mathbb{E}}\left(N_{t+1}^0 Z_{t+1}^2 \mid \mathfrak{F}_t \right) &= \alpha_z Z_t^2 + 2Y_t^1 \sigma_z \cdot \tilde{\mu}
\end{aligned}
$$

Y_t^1 will show up in the value function approximation for the second protocol.

4.3 Results to report in the Jupyter notebook

i) approximation of $\widehat{V}_t - \widehat{C}_t$ for both protocols, two state variables

ii) approximation of $\widehat{R}_t - \widehat{C}_t$ for both protocols, two state variables

iii) approximation of $\left(\widehat{V}_{t+1} - \widehat{C}_{1+1} \right) - \left(\widehat{R}_t - \widehat{C}_t \right)$ for both protocols, two state variables

iv) approximation of $\widehat{C}_{t+1} - \widehat{C}_t$ both protocols

v) three shock prices, which are the coefficients on W_{t+1} in the log stochastic discount factor approximations

vi) dominate eigenvalue and eigenfunction approximations when growth term set to one

5 Stochastic discount factor approximation

Next we construct a second-order expansion of the stochastic discount factor process S where

$$
\log S_{t+1} - \log S_t = \log \beta - \rho \left(\widehat{C}_{t+1} - \widehat{C}_t \right) + (1-\gamma)(\widehat{V}_{t+1} - \widehat{R}_t) + (\rho - 1)(\widehat{V}_{t+1} - \widehat{R}_t).
$$

Recall that

$$
(1-\gamma)(\widehat{V}_{t+1} - \widehat{R}_t) = \log N_{t+1}^*.
$$

where N^*_{t+1} has conditional mean one. Write:

$$\frac{S_{t+1}}{S_t} = N^*_{t+1} \exp\left[(\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \exp\left(\widehat{S}_{t+1} - \widehat{S}_t\right) \tag{25}$$

where

$$\widehat{S}_{t+1} - \widehat{S}_t = \log\beta - \rho\left(\widehat{C}_{t+1} - \widehat{C}_t\right)$$

We discussed previously the approximation of N^*_{t+1}.

5.1 Approximating values

We now deduce some approximation formulas to support recursive valuation. Let G denote a stochastic growth process. We are interested in repeated applications of the following operator:

$$\begin{aligned}
\mathbb{M}\left[\phi(X_{t+1}) \mid \mathfrak{F}_t\right] &= \mathbb{E}\left[\left(\frac{S_{t+1}G_{t+1}}{S_tG_t}\right)\phi(X_{t+1}) \mid \mathfrak{F}_t\right] \\
&= \mathbb{E}\left[N^*_{t+1}\exp\left[(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right]\left(\frac{M_{t+1}}{M_t}\right)\phi(X_{t+1}) \mid \mathfrak{F}_t\right]
\end{aligned}$$

where

$$\frac{M_{t+1}}{M_t} \doteq \exp\left(\widehat{S}_{t+1} - \widehat{S}_t + \widehat{G}_{t+1} - \widehat{G}_t\right). \tag{26}$$

We have already discussed approximations of N^*_{t+1}.

5.1.1 Approximating equity prices

We wish to approximate the solution the forward equation:

$$\psi_t = \log\mathbb{E}\left[N^*_{t+1}\exp\left[(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right]\ \left(\frac{M_{t+1}}{M_t}\right)\left(\exp\left[\psi_{t+1}\right)\right] + 1\right) \mid \mathfrak{F}_t\right].$$

We take and approximation for N^*_{t+1} as given in what follows. In this calculation, ψ_t is a so-called jump variable.

In what follows we use the approximations

$$\begin{aligned}
\widehat{M}_{t+1} - \widehat{M}_t &\approx\ \eta_m + \mathsf{q}\left(\widehat{M}^1_{t+1} - \widehat{M}^1_t\right) + \frac{\mathsf{q}^2}{2}\left(\widehat{M}^2_{t+1} - \widehat{M}^2_t\right) \\
(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right) &\approx\ (\rho-1)\left[\mathsf{q}\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{\mathsf{q}^2}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]
\end{aligned}$$

and solve for the approximation:

$$\psi_t \approx \psi^0 + \mathsf{q}\psi_t^1 + \frac{\mathsf{q}^2}{2}\psi_t^2$$

Order zero:

$$\exp\left(\psi^0\right) = \exp(\eta_m)\left[\exp\left[(\psi^0\right) + 1\right],$$

implying that

$$\exp\left(\psi^0\right) = \frac{\exp(\eta_m)}{1 - \exp(\eta_m)}.$$

Order one:

$$\psi_t^1 = \mathbb{E}\left(N_{t+1}^*\left[(\rho - 1)\left(\widehat{V}_{t+1}^1 - \widehat{R}_t^1\right)\right]\left[\left(\widehat{M}_{t+1}^1 - \widehat{M}_t^1\right) + \exp(\eta_m)\psi_{t+1}^1\right] \mid \mathfrak{F}_t\right)$$

Order two approximation :

$$\begin{aligned}
\psi_t^2 =& \mathbb{E}\left(N_{t+1}^0\left[\left(\widehat{M}_{t+1}^2 - \widehat{M}_t^2\right) + \exp(\eta_m)\psi_{t+1}^2\right] \mid \mathfrak{F}_t\right) - (\psi_t^1)^2 \\
&+ \mathbb{E}\left(N_{t+1}^0\left[\left(\widehat{M}_{t+1}^1 - \widehat{M}_t^1\right)^2 + 2\exp(\eta_m)\psi_{t+1}^1\left(\widehat{M}_{t+1}^1 - \widehat{M}_t^1\right)\right] \mid \mathfrak{F}_t\right) \\
&+ \mathbb{E}\left[N_{t+1}^0 \exp(\eta_m)\left(\psi_{t+1}^1\right)^2 \mid \mathfrak{F}_t\right] \\
&+ 2\mathbb{E}\left(N_{t+1}^1\left[\left(\widehat{M}_{t+1}^1 - \widehat{M}_t^1\right) + \exp(\eta_m)\psi_{t+1}^1\right] \mid \mathfrak{F}_t\right)
\end{aligned}$$

5.1.2 A multi-step approach

In what follows, we consider:

$$\widehat{S}_{t+1} - \widehat{S}_t = \log\beta - \rho\left(\widehat{C}_{t+1} - \widehat{C}_t\right) + (\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)$$

The third term requires special treatment.

Suppose we take as given first-order and second-order approximations to

$$\widehat{V}_{t+1} - \widehat{R}_t.$$

The following are some useful derivatives:

$$D\exp\left[(\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] = (\rho - 1)\exp\left[(\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] D\left(\widehat{V}_{t+1} - \widehat{R}_t\right)$$

$$D^2 \exp\left[(1-\rho)\left(\widehat{V}_{t+1}-\widehat{R}_t\right)\right] =(1-\rho)\exp\left[(\rho-1)\left(\widehat{V}_{t+1}-\widehat{R}_t\right)\right] D^2\left(\widehat{V}_{t+1}-\widehat{R}_t\right) + (\rho-1)^2\exp\left[(1-\rho)\left(\widehat{V}_{t+1}-\widehat{R}_t\right)\right]\left[D\left(\widehat{V}_{t+1}-\widehat{R}_t\right)\right]^2$$

i) Given an approximation N and the first and second-order approximations for $\widehat{V}_{t+1}-\widehat{R}_t$ compute first and second-order expansions:

$$\begin{aligned}
\psi_t^1 &= \mathbb{E}\left(N_{t+1}\left[\left(\widehat{M}_{t+1}^1-\widehat{M}_t^1\right)+\exp(\eta_m)\psi_{t+1}^1\right] \mid \mathfrak{F}_t\right)\\
\psi_t^2 &=\mathbb{E}\left(N_{t+1}\left[\left(\widehat{M}_{t+1}^2-\widehat{M}_t^2\right)+\exp(\eta_m)\psi_{t+1}^2\right] \mid \mathfrak{F}_t\right)-(\psi_t^1)^2\\
&\quad+\mathbb{E}\left(N_{t+1}\left[\left(\widehat{M}_{t+1}^1-\widehat{M}_t^1\right)^2+2\exp(\eta_m)\psi_{t+1}^1\left(\widehat{M}_{t+1}^1-\widehat{M}_t^1\right)\right] \mid \mathfrak{F}_t\right)\\
&\quad+\mathbb{E}\left[N_{t+1}\exp(\eta_m)\left(\psi_{t+1}^1\right)^2 \mid \mathfrak{F}_t\right]
\end{aligned}$$

ii) Compute first and second-order expansions for approximation of $\widehat{V}-\widehat{C}$ and $\widehat{R}-\widehat{C}$. where

$$\begin{aligned}
\widehat{V}_t^1-\widehat{C}_t^1 &=\lambda\left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1-\widehat{C}_{t+1}^1\right)+\left(\widehat{C}_{t+1}^1-\widehat{C}_t^1\right)\right]\right] \mid \mathfrak{F}_t\right),\\
\widehat{R}_t^1-\widehat{C}_t^1 &=\left(\frac{1}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1-\widehat{C}_{t+1}^1\right)+\left(\widehat{C}_{t+1}^1-\widehat{C}_t^1\right)\right]\right] \mid \mathfrak{F}_t\right)
\end{aligned}$$

$$\begin{aligned}
\widehat{V}_t^2-\widehat{C}_t^2 =&\ \lambda\mathbb{E}\left[N_{t+1}^0\left(\widehat{V}_{t+1}^2-\widehat{C}_{t+1}^2\right) \mid \mathfrak{F}_t\right]\\
&+\lambda\mathbb{E}\left[N_{t+1}^0\left(\widehat{C}_{t+1}^2-\widehat{C}_t^2\right) \mid \mathfrak{F}_t\right]+(1-\rho)(1-\lambda)\lambda\left[\left(\widehat{R}_t^1-\widehat{C}_t^1\right)^2\right],\\
\widehat{R}_t^2-\widehat{C}_t^2 =&\ \mathbb{E}\left(N_{t+1}^0\left[\left(\widehat{V}_{t+1}^2-\widehat{C}_{t+1}^2\right)+\left(\widehat{C}_{t+1}^2-\widehat{C}_t^2\right)\right] \mid \mathfrak{F}_t\right),
\end{aligned}$$

and

$$N_{t+1}^0=\frac{\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right] \mid \mathfrak{F}_t\right)}.$$

iii) Form:

$$\widetilde{N}^2_{t+1} = \frac{\exp\left[(1-\gamma_o)\left(\mathsf{q}\widehat{V}^1_{t+1} + \frac{\mathsf{q}^2}{2}\widehat{V}^2_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\mathsf{q}\widehat{V}^1_{t+1} + \frac{\mathsf{q}^2}{2}\widehat{V}^2_{t+1}\right)\right] \mid \mathfrak{F}_t\right)},$$

and construct a change of measure implied by $\widetilde{N}^2_{t+1}$. Write:

$$\widetilde{N}^2_{t+1} \propto \exp\left[W_{t+1}{}'\left(\mathbb{H}_0 + \mathbb{H}_1 X^1_t + \mathbb{H}_2 X^2_t\right) - \frac{1}{2}W'_{t+1}\Lambda W_{t+1}\right] \tag{27}$$

where the proportionality constant does not depend on W_{t+1} and only date t conditioning information. Let lower case variables denote potential realizations of random vectors. Since W_{t+1} is a multivariate standard normally distributed random vector, write it's density up to proportionality factor as:

$$\exp\left(-\frac{1}{2}w'w\right),$$

and express the product of two functions:

$$\begin{aligned} &\exp\left[w'\left(\mathbb{H}_0 + \mathbb{H}_1 x^1 + \mathbb{H}_2 x^2\right) - \frac{1}{2}w'\Lambda w\right]\exp\left(-\frac{1}{2}w'w\right) \\ &= \exp\left[w'\widetilde{\Lambda}\left(\widetilde{\mathbb{H}}_0 + \widetilde{\mathbb{H}}_1 x^1 + \widetilde{\mathbb{H}}_2 x^2\right) - \frac{1}{2}w'\widetilde{\Lambda}w\right] \end{aligned}$$

where

$$\begin{aligned} \widetilde{\Lambda} &= \mathbb{I} + \Lambda \\ \widetilde{\mathbb{H}}_0 &= \widetilde{\Lambda}^{-1}\mathbb{H}_0 \\ \widetilde{\mathbb{H}}_1 &= \widetilde{\Lambda}^{-1}\mathbb{H}_1 \\ \widetilde{\mathbb{H}}_2 &= \widetilde{\Lambda}^{-1}\mathbb{H}_2 \end{aligned}$$

Then under the change in probability measure induced by $\widetilde{N}_{t+1}$, W_{t+1} has mean:

$$\widetilde{\mathbb{H}}_0 + \widetilde{\mathbb{H}}_1 X^1_t + \widetilde{\mathbb{H}}_2 X^2_t$$

and covariance matrix $\widetilde{\Lambda}^{-1}$.

iv) Transform the state equation using this change of distribution for W_{t+1}. In particular, write

$$W_{t+1} = \tilde{\mu}_t + \widetilde{\Gamma}\widetilde{W}_{t+1}$$

where $\widetilde{W}_{t+1}$ is a multivariate standard normal and

$$\widetilde{\Gamma}\widetilde{\Gamma}' = \widetilde{\Lambda}^{-1}.$$

v) Let $N_{t+1} = \widetilde{N}_{t+1}$ and return to step i) and update first and second-order approximations for $\widehat{V} - \widehat{C}$ and $\widehat{R} - \widehat{C}$ and return to step i).

You may start the iterations by setting $N_{t+1} = 1$. This same method has a direct extension to problems with endogenous state variables.

5.1.3 Adjustment cost model

We consider an AK model with adjustment costs and state dependent growth G:

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = \alpha \tag{28a}$$

$$\frac{K_{t+1}}{K_t} = \left[1 + \phi_2\left(\frac{I_t}{K_t}\right)\right]^{\phi_1} G_{t+1} \tag{28b}$$

$$G_{t+1} \equiv \exp\left(-\mathsf{a}_k + Z_t - \frac{1}{2} \mid \sigma_k \mid^2 + \sigma_k \cdot W_{t+1}\right) \tag{28c}$$

$$Z_{t+1} = \mathsf{a}_z Z_t + \sigma_z W_{t+1} \tag{28d}$$

Taking logarithms of the capital evolution (28b):

$$\log K_{t+1} - \log K_t = \phi_1 \log\left[1 + \phi_2\left(\frac{I_t}{K_t}\right)\right] + \log G_{t+1}.$$

The shock vector W_{t+1} contains two entries:

$$W_{t+1} = \begin{bmatrix} W_{1,t+1} \\ W_{2,t+1} \end{bmatrix},$$

and it follows a multivariate standard normal distribution.

At date t, K_t is predetermined while $\frac{C_t}{K_t}$ is to be determined via optimization and hence is a "jump variable." For this reason, we will scale V by K and R by K in the computational algorithm. Recall the one-period stochastic discount factor given by (25)

$$\begin{aligned}\frac{S_{t+1}}{S_t} &= \beta N^*_{t+1}\left(\frac{V_{t+1}}{R_t}\right)^{\rho-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\\ &= \beta N^*_{t+1}\left(\frac{V_{t+1}}{K_{t+1}}\right)^{\rho-1}\left(\frac{R_t}{K_t}\right)^{1-\rho}\left(\frac{C_{t+1}}{K_{t+1}}\right)^{-\rho}\left(\frac{C_t}{K_t}\right)^{\rho}\left(\frac{K_{t+1}}{K_t}\right)^{-1}\end{aligned}$$

The associated investment FOC (Euler equation) is:

$$\log \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\psi\left(\frac{I_t}{K_t}, G_{t+1}\right)\middle|\mathfrak{F}_t\right] = 0$$

where

$$\frac{MK_{t+1}}{MC_{t+1}} = \left(\frac{1}{1-\beta}\right)\left(\frac{V_{t+1}}{K_{t+1}}\right)\left(\frac{V_{t+1}}{C_{t+1}}\right)^{-\rho} = \left(\frac{1}{1-\beta}\right)\left(\frac{V_{t+1}}{K_{t+1}}\right)^{1-\rho}\left(\frac{C_{t+1}}{K_{t+1}}\right)^{\rho}.$$

where $MC_{t+1} = (1-\beta)C_{t+1}^{-\rho}V_{t+1}^{\rho}$ and $MK_{t+1} = \frac{V_{t+1}}{K_{t+1}}$ are the date $t+1$ marginal values of consumption and marginal value of capital, respectively. Thus

$$\begin{aligned}\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right) &= \left(\frac{\beta}{1-\beta}\right)N^*_{t+1}\left(\frac{R_t}{K_t}\right)^{1-\rho}\left(\frac{C_t}{K_t}\right)^{\rho}\left(\frac{K_{t+1}}{K_t}\right)^{-1}\\ &= \left(\frac{\beta}{1-\beta}\right)N^*_{t+1}\left(\frac{R_t}{K_t}\right)^{1-\rho}\left(\frac{C_t}{K_t}\right)^{\rho}\left[1+\phi_2\left(\frac{I_t}{K_t}\right)\right]^{-1}(G_{t+1})^{-1}.\end{aligned}$$

Also,

$$\psi\left(\frac{I_t}{K_t}, G_{t+1}\right) = \phi_1\phi_2\left[1+\phi_2\left(\frac{I_t}{K_t}\right)\right]^{\phi_1-1}G_{t+1}$$

Combining expressions, we see that

$$\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\psi\left(\frac{I_t}{K_t}, G_{t+1}\right) = \left(\frac{\beta}{1-\beta}\right)N^*_{t+1}\left(\frac{R_t}{K_t}\right)^{1-\rho}\left(\frac{C_t}{K_t}\right)^{\rho}\phi_1\phi_2\left[1+\phi_2\left(\frac{I_t}{K_t}\right)\right]^{-1}.$$

Since N^*_{t+1} had conditional expectation one, the expectation is inconsequential. The equation of interest becomes:

$$\log\left(\frac{\beta}{1-\beta}\right) + (1-\rho)\left(\log R_t - \log K_t\right)$$

$$+\rho\left(\log C_t-\log K_t\right)+\log\phi_1+\log\phi_2-\log\left[1+\phi_2\left(\frac{I_t}{K_t}\right)\right]=0.$$

References

Bucklew, James A. 2004. *Large Deviation Theory*. Springer Series in. Statistics. New York: Springer Verlag.

Hansen, Lars Peter and Thomas J. Sargent. 1995. Discounted Linear Exponential Quadratic Gaussian Control. *IEEE Transactions on Automatic Control* .

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long-Run Risk. *Journal of Political Economy* 116:260–302.

Jacobson, David H. 1973. Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Differential Games. *IEEE Transactions for Automatic Control* AC-18:1124–1131.

Lombardo, Giovanni and Harald Uhlig. 2018. A Theory of Pruning. *International Economic Review* 59 (4):1825–1836.

Schmitt-Grohé, Stephanie and Martín Uribe. 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control* 28 (4):755–775.

Tallarini, Jr., Thomas D. 2000. Risk-sensitive Real Business cycles. *Journal of Monetary Economics* 45:507–532.

Whittle, Peter. 1981. Risk Sensitive Linear Quadratic Gaussian Control. *Advances in Applied Probability* 13 (4):764–777.