The Permanent Income Model

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1 Model Construction

1.1 Preference: Recursive Utility with Habit Persistence

We introduce U_t via a CES aggregator of current consumption C_t and a household stock variable H_t. H_t, which is a geometrically weighted average of current and past consumptions and the initial H_0, can be interpreted either as habits or as durable goods.

A representative household ranks $\{U_t : t \geq 0\}$ processes with a utility functional $\{V_t : t \geq 0\}$ processes. We also introduce an $\{R_t : t \geq 0\}$ process as a risk adjusted version of $\{V_t : t \geq 0\}$, called a certainty equivalent. V_t, R_t, U_t and H_t are defined via the following recursion:

$$V_t = \left[(1-\beta)U_t^{1-\rho} + \beta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{1}$$

Stochastic growth in the income process (will be introduced shortly) makes it natural to divide every variable in the above equations by Y_t to form a balanced growth version:

$$\frac{V_t}{Y_t} = \left[(1-\beta)\left(\frac{U_t}{Y_t}\right)^{1-\rho} + \beta\left(\frac{R_t}{Y_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{2}$$

The reciprocal of the parameter ρ describes the consumer's attitudes toward intertemporal substitution, while the parameter γ describes the consumer's attitudes toward risk.

1.2 Technology: AK with Non-Financial Income

We construct a nonlinear version of a permanent income technology in the spirit of hansen1999robust and hansen2013recursive, Hansen and Sargent (2013, ch. 11) that assumes the consumer's non-financial income process $\{Y_t\}$ is an exogenous multiplicative functional.

$$K_{t+1} - K_t + C_t = \mathsf{a}K_t + Y_t \tag{3}$$

balanced growth version:

$$\frac{K_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} - \frac{K_t}{Y_t} + \frac{C_t}{Y_t} = \mathsf{a}\frac{K_t}{Y_t} + 1 \tag{4}$$

We use this to define a model variable "scaled gross investment":

$$\log Y_{t+1} - \log Y_t = DZ_t + FW_{t+1} + \mathsf{g} \tag{5}$$

$$Z_{t+1} = AZ_t + BW_{t+1} \tag{6}$$

Z_t is a 3×1 vector

$$Z_t = \left[Z_{1,t}, Z_{2,t}, Z_{2,t-1}\right]' \tag{7}$$

and W_{t+1} is a 2×1 vector

$$W_{t+1} = \left[W_{1,t+1}, W_{2,t+1}\right]' \tag{8}$$

where $W_{1,t+1}$ and $W_{2,t+1}$ are shocks to $Z_{1,t+1}$ and $Z_{2,t+1}$, respectively, and W_{t+1} follows a standardized multivariate normal distribution. We assume the following parameter values originally estimated by hansen1999robust:

$$\log\left(\frac{Y_{t+1}}{Y_t}\right) = .01(Z_{1,t+1}+Z_{2,t+1}-Z_{2,t}) = .01\left(\begin{bmatrix}.704 & 0 & -.154\end{bmatrix}\begin{bmatrix}Z_{1,t}\\ Z_{2,t}\\ Z_{2,t-1}\end{bmatrix} + \begin{bmatrix}.144 & .206\end{bmatrix}\begin{bmatrix}W_{1,t+1}\\ W_{2,t+1}\end{bmatrix}\right)+.00373 \tag{9}$$

$$\begin{bmatrix}Z_{1,t+1}\\ Z_{2,t+1}\\ Z_{2,t}\end{bmatrix} = \begin{bmatrix}.704 & 0 & 0\\ 0 & 1 & -.154\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}Z_{1,t}\\ Z_{2,t}\\ Z_{2,t-1}\end{bmatrix} + \begin{bmatrix}.144 & 0\\ 0 & .206\\ 0 & 0\end{bmatrix}\begin{bmatrix}W_{1,t+1}\\ W_{2,t+1}\end{bmatrix} \tag{10}$$

$W_{1,t+1}$ and $W_{2,t+1}$ play different roles in the non-financial income process. $W_{1,t+1}$ has a permanent effect on income, while $W_{2,t+1}$ only has a transient effect. This can be seen by computing and plotting the impulse response function of $\log Y_t$ to each shock. Note that the responses are multiplied by 100 to reflect percentage responses.

1.3 Stochastic Discount Factor; FOC on investment

SDF increment in units of U_t:

$$\frac{\widetilde{S_{t+1}}}{S_t} = \beta\left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}\left(\frac{V_{t+1}}{R_t}\right)^{\rho-1}\left(\frac{U_{t+1}}{U_t}\right)^{-\rho} \tag{11}$$

Note that the $\left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}$ term is written separately. This is because it is a random variable with mean 1 conditioned on time t information. Therefore, it represents a change of probability measure whenever we take a mathematical expectation. Specifically, W_{t+1} follows a standardized multivariate normal distribution under the "original" probability measure, but after taking into account of the change of measure, the distribution of W_{t+1} is different. What differences? How many details shall we provide. Therefore, the change of measure alters the structure of terms in which W_{t+1} is involved.

We are more interested in viewing C_t rather than U_t as the numeraire. This leads us to introduce two additional equations in which enduring effects of consumption at t come into play. These equations in effect pin down two marginal rates of substitution, $\frac{MC_t}{MU_t}$ and $\frac{MH_t}{MU_t}$.

$\frac{MC_t}{MU_t}$ has different specificiations depending on whether habit is "external (externality is ignored by the consumer)" or "internal (externality is internalized by the consumer)".

- External:

$$\frac{MC_t}{MU_t} = (1-\alpha)\left(\frac{U_t}{C_t}\right)^{\epsilon} \tag{12}$$

- Internal:

$$\frac{MC_t}{MU_t} = (1-\alpha)\left(\frac{U_t}{C_t}\right)^{\epsilon} + (1-\chi)\mathbb{E}\left[\frac{\widetilde{S_{t+1}}}{S_t}\frac{MH_{t+1}}{MU_{t+1}}\middle|\mathfrak{F}_t\right] \tag{13}$$

where $\frac{MH_t}{MU_t}$ satisfies:

$$\frac{MH_t}{MU_t} = \alpha\left(\frac{U_t}{H_t}\right)^{\epsilon} + \chi\mathbb{E}\left[\frac{\widetilde{S_{t+1}}}{S_t}\frac{MH_{t+1}}{MU_{t+1}}\middle|\mathfrak{F}_t\right] \tag{14}$$

Then we have SDF increment in units of C_t:

$$\frac{S_{t+1}}{S_t} = \widetilde{\left(\frac{S_{t+1}}{S_t}\right)} \left(\frac{MC_{t+1}/MU_{t+1}}{MC_t/MU_t}\right) \tag{15}$$

FOC on investment:

$$\log \mathbb{E}\left[(1+\mathsf{a})\,\frac{S_{t+1}}{S_t}\middle|\mathfrak{F}_t\right] = 0 \tag{16}$$