Adjustment Cost Model

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1 The Model

1.1 Preference

1.1.1 Recursive utility without habit persistence

This is a special case of more general recursive preference, where there's no habits or durable goods. We take a consumption process $\{C_t\}$ as an input into $\{R_t, V_t\}$ processes that we define via backward recursion:

$$V_t = \left[(1-\beta)C_t^{1-\rho} + \beta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$R_t = \mathbb{E}\left[V_{t+1}^{1-\gamma} \mid \mathfrak{F}_t\right]^{\frac{1}{1-\gamma}}$$

$\{V_t\}$ is a continuation value process that ranks $\{C_t\}$ processes. The reciprocal of the parameter ρ describes the consumer's attitudes about intertemporal substitution, while the parameterγ describes the consumer's attitudes toward risk.

In practice, we are interested in finding a balanced growth path, where some variables grow at constant rates, while others are in a steady state. To this end, it's convenient to use a growing variable to scale others. In this specification, we can use C to scale V and R:

$$\frac{V_t}{C_t} = \left[(1-\beta) + \beta\left(\frac{R_t}{C_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$\frac{R_t}{C_t} = \mathbb{E}\left[\left(\frac{V_{t+1}}{C_t}\right)^{1-\gamma} |\mathfrak{F}_t\right]^{\frac{1}{1-\gamma}}$$

In the special case of $\rho = 1$,

$$\frac{V_t}{C_t} = \left(\frac{R_t}{C_t}\right)^{\beta}$$

More generally, we can include "preference shock" under this specification:

$$V_t = \left[(1-\beta)\left(C_t D_t\right)^{1-\rho} + \beta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where $\{D_t\}$ is an exogenous preference shifter process, whose dynamic will be introduced later. In this case, we can use CD to scale V and R to obtain a balanced growth version:

$$\frac{V_t}{C_t D_t} = \left[(1-\beta) + \beta\left(\frac{R_t}{C_t D_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$\frac{R_t}{C_t D_t} = \mathbb{E}\left[\left(\frac{V_{t+1}}{C_t D_t}\right)^{1-\gamma} |\mathfrak{F}_t\right]^{\frac{1}{1-\gamma}}$$

1.1.2 Recursive utility with habit persistence

This is the more general framework of recursive preference, where we introduce U_t via a CES aggregator of current consumption C_t and a household stock variable H_t. H_t can be interpreted either as habits or as durable goods. It will be clear that H_t is a geometrically weighted average of current and past consumptions, and the initial H_0.

Now, H_0 and $\{C_t\}$ are taken as inputs to form $\{U_t\}$ process, and $\{U_t\}$ is used as an input into $\{R_t, V_t\}$ processes via the recursion below:

$$V_t = \left[(1-\beta)U_t^{1-\rho} + \beta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$R_t = \mathbb{E}\left[V_{t+1}^{1-\gamma} \mid \mathfrak{F}_t\right]^{\frac{1}{1-\gamma}}$$

$$U_t = \left[(1-\alpha)C_t^{1-\epsilon} + \alpha H_t^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

$$H_{t+1} = \chi H_t + (1-\chi)C_t$$

Obviously, as $\alpha \to 0$, this preference specification degenerates to the no habit specification as described in section 1.1.1.

We are again interested in finding a balanced growth path. In this preference specification, we can use H to scale other preference variables V, R and U. Since H itself also grows, and C is also involved here, which also grows, we can scale them by K, whose dynamic will be introduced soon.

$$\frac{V_t}{H_t} = \left[(1-\beta)\left(\frac{U_t}{H_t}\right)^{1-\rho} + \beta\left(\frac{R_t}{H_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{1}$$

$$\frac{R_t}{H_t} = \mathbb{E}\left[\left(\frac{V_{t+1}}{H_t}\right)^{1-\gamma} |\mathfrak{F}_t\right]^{\frac{1}{1-\gamma}} \tag{2}$$

$$\frac{H_{t+1}}{K_t} = \chi\frac{H_t}{K_t} + (1-\chi)\frac{C_t}{K_t} \tag{3}$$

$$\frac{U_t}{H_t} = \left[(1-\alpha)\left(\frac{C_t}{H_t}\right)^{1-\epsilon} + \alpha\right]^{\frac{1}{1-\epsilon}} \tag{4}$$

In the special case of $\rho = 1$ and $\epsilon = 1$, (1) and (4) become respectively:

$$\frac{V_t}{H_t} = \left(\frac{U_t}{H_t}\right)^{1-\beta} \left(\frac{R_t}{H_t}\right)^{\beta}$$

$$\frac{U_t}{H_t} = \left(\frac{C_t}{H_t}\right)^{1-\alpha}$$

1.2 Technology: AK with adjustment cost

We consider an AK model with adjustment costs and state dependent growth G:

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = \mathbf{a} \tag{5}$$

$$\frac{K_{t+1}}{K_t} = \left[1 + \phi_2 \left(\frac{I_t}{K_t}\right)\right]^{\phi_1} G_{t+1} \tag{6}$$

$$G_{t+1} \equiv \exp\left(-\alpha_k + \mathbb{U}_k \cdot Z_t - \frac{1}{2} \mid \sigma_k \mid^2 + \sigma_k \cdot W_{t+1}\right) \tag{7}$$

$$Z_{t+1} = \mathbb{A} Z_t + \mathbb{B} W_{t+1} \tag{8}$$

Another version of (6) often of interest:

$$\log K_{t+1} - \log K_t = \phi_1 \log \left[1 + \phi_2 \left(\frac{I_t}{K_t}\right)\right] + \log G_{t+1}$$

where Z_{t+1} is a vector containing 2 entries:

$$Z_{t+1} = \left[Z_{1,t+1}, Z_{2,t+1}\right]'$$

with $Z_{1,t}$ and $Z_{2,t}$ being two components of capital growth;

W_{t+1} is a shock vector containing 3 entries:

$$W_{t+1} = \left[W_{1,t+1}, W_{2,t+1}, W_{3,t+1}\right]'$$

and they follow multivariate standard normal distribution.

$$\mathbb{A} = \begin{bmatrix} \exp(-\beta_1) & 0 \\ 0 & \exp(-\beta_2) \end{bmatrix}$$

$\mathbb{U}_k \cdot Z_t$ shifts the growth rate in technology, so it is a source of "long run risk".

When the preference shifter $\{D_t\}$ is of interest, it follows

$$\log D_{t+1} - \log D_t = \mathbb{U}_d \cdot Z_t + \sigma_d \cdot W_{t+1}$$

$\mathbb{U}_d \cdot Z_t$ shifts the growth rate in preference, so it is also a source of "long run risk".

1.3 Stochastic Discount Factor; FOC on investment

1.3.1 No household capital

The preferences described in Section 1.1.1 imply that the time $t+1$ multiplicative increment to the consumer's stochastic discount factor is (in units of C_t):

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}$$

The reason that the second term is written separately is that, it represents a change of probability measure to the shocks and thus opens the door to an expansion that we find to be revealing. The code has special treatment regarding this term.

The equilibrium $\{V_t\}$ process for a planner solves the Bellman equation

$$V_t = \max_{C_t, I_t} [(1-\beta) C_t^{1-\rho} + \beta((\mathbb{E}[V_{t+1}^{1-\gamma}|\mathfrak{F}_t])^{\frac{1}{1-\gamma}})^{1-\rho}]^{\frac{1}{1-\rho}}$$

where maximization is subject to equations in Section 1.2. The associated FOC (Euler equation) is:

$$\log \mathbb{E}\left[\frac{S_{t+1}}{S_t}\frac{MK_{t+1}}{MC_{t+1}}\psi\left(I_t, K_t, Z_t\right)\middle|\mathfrak{F}_t\right] = 0$$

where $MC_{t+1} = (1-\beta)C_{t+1}^{-\rho}V_{t+1}^{\rho}$ and $MK_{t+1} = \frac{V_{t+1}}{K_{t+1}}$ are the date $t+1$ marginal value of consumption and marginal value of capital, respectively; $\psi = -\frac{dK_{t+1}/K_t}{dC_t} = \phi_1\phi_2[1+\phi_2(\frac{I_t}{K_t})^{\phi_1 - 1}]G_{t+1}$.

If we include preference shifter, we need to modify two terms that appear in FOC: $MC_{t+1} = (1-\beta)C_{t+1}^{-\rho}V_{t+1}^{\rho}D_t^{1-\rho}$ and

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{D_{t+1}}{D_t}\right)^{1-\rho}$$

1.3.2 With household capital

Now we follow the preferences described in Section 1.1.2 instead.

SDF increment in units of U_t:

$$\frac{\widetilde{S_{t+1}}}{S_t} = \beta \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{U_t}\right)^{-\rho}$$

But we are more interested in viewing C_t as the numeraire. This leads us to introduce two additional equations in which enduring effects of consumption at t come into play. These equations in effect pin down two marginal rates of substitution, $\frac{MC_t}{MU_t}$ and $\frac{MH_t}{MU_t}$. They satisfy:

$$\frac{MC_t}{MU_t} = (1-\alpha)\left(\frac{U_t}{C_t}\right)^{\epsilon} + (1-\chi)\mathbb{E}\left[\frac{\widetilde{S_{t+1}}}{S_t}\frac{MH_{t+1}}{MU_{t+1}}\middle|\mathfrak{F}_t\right]$$

where $\frac{MH_t}{MU_t}$ satisfies:

$$\frac{MH_t}{MU_t} = \alpha\left(\frac{U_t}{H_t}\right)^{\epsilon} + \chi\mathbb{E}\left[\frac{\widetilde{S_{t+1}}}{S_t}\frac{MH_{t+1}}{MU_{t+1}}\middle|\mathfrak{F}_t\right]$$

Then we have SDF increment in units of C_t:

$$\frac{S_{t+1}}{S_t} = \left(\widetilde{\frac{S_{t+1}}{S_t}}\right)\left(\frac{MC_{t+1}/MU_{t+1}}{MC_t/MU_t}\right)$$

The FOC again takes the same form:

$$\log \mathbb{E}\left[\frac{S_{t+1}}{S_t}\frac{MK_{t+1}}{MC_{t+1}}\psi\left(I_t, K_t, Z_t\right)\middle|\mathfrak{F}_t\right] = 0 \tag{9}$$

where $MC_{t+1} = (1-\beta)C_{t+1}^{-\rho}V_{t+1}^{\rho}$ and $MK_{t+1} = \frac{V_{t+1}}{K_{t+1}} - MH_{t+1}\frac{H_{t+1}}{K_{t+1}}$ are the date $t+1$ marginal value of consumption and marginal value of capital, respectively; $\psi = -\frac{dK_{t+1}/K_t}{dC_t} = \phi_1\phi_2[1+\phi_2(\frac{I_t}{K_t})^{\phi_1-1}]G_{t+1}$. NB: MK_{t+1} here is different from that in Section 1.3.1.