Notes on small noise expansion*

Zhenhuan Xie†

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1 Big picture

A nonlinear DSGE model usually cannot be solved quasi-analytically, so we resort to a small-noise expansion method to find an approximate solution. Let's say the model's equilibrium conditions are described by a system of possibly nonlinear, expectational difference equations

$$\mathbb{E}\left[f(X_t, X_{t+1}, \mathsf{q}W_{t+1}, \mathsf{q}) \mid \mathcal{F}_t\right] = 0 \tag{1}$$

the scalar q is a perturbation parameter that scales shock volatilities; f is a possibly nonlinear functions; W_{t+1} is an i.i.d. random vector that follows standard normal distribution; X_t is a vector that stacks together all uniquely determined[1] time t variables, and the components of X_t can be among the following categories:

- Endogenous non-predetermined variables[2];
- Endogenous predetermined variables[3];
- Exogenous forcing variables[4].

We follow Lombardo and Uhlig (2018) in assuming the solution of (1) forms a class of n dimensional stochastic processes $\{X_t(\mathsf{q})\}$ indexed by the perturbation parameter q, satisfying the equilibrium law of motion:

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†Please send corrections to: zhenhuanxie@uchicago.edu.

[1]This is usually referred as a "system reduction" before writing a DSGE model. For example, if investment I_t can be expressed using the same period consumption C_t and capital K_t, then I_t should not enter X_t when writing your model.

[2]For example, consumption at time t is a decision to be made at t, so it's not predetermined, and this decision is endogenous.

[3]For example, capital at time t is determined when the consumption/investment decision at time $t-1$ has been made, so it is an endogenous predetermined variable at t.

[4]For example, see how Bansal and Yaron (2004) models consumption growth and dividend growth.

$$X_{t+1}(\mathsf{q}) = \psi[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}] \tag{2}$$

Although we typically cannot obtain quasi-analytical formula for (2), we can approximate it with first- and second-order small noise expansions:

A first-order expansion of $X_t(\mathsf{q})$ around $\mathsf{q} = 0$ takes the form

$$X_t \approx X_t^0 + \mathsf{q}X_t^1$$

and a second-order expansion of $X_t(\mathsf{q})$ around $\mathsf{q} = 0$ takes the form

$$X_t \approx X_t^0 + \mathsf{q}X_t^1 + \frac{\mathsf{q}^2}{2}X_t^2$$

where we have denoted X_t^0 as the limiting process as $\mathsf{q} \downarrow 0$ that can be solved from the following fixed point problem[5]

$$X_{t+1}^0 = \psi(X_t^0, 0, 0)$$

and X_t^j, $j = 1, 2$ as the jth derivative of $X_t(\mathsf{q})$ with respect to the perturbation parameter q, evaluated at $\mathsf{q} = 0$.

2 Recursive utility, reconsidered

We emphasize the Epstein and Zin (1989) recursive utility here because **our Expansion-Suite code is specific to this preference**. This preference is extensively used in asset pricing literature, because intertemporal elasticity of substitution and relative risk aversion are two separate parameters[6]:

$$R_t = \mathbb{E}\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t\right]^{\frac{1}{1-\gamma}}$$
$$V_t = \left[(1-\beta)(C_t)^{1-\rho} + \beta (R_t)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

or written in terms of logarithm,

[5] X_t^0 is usually called a "deterministic steady state".
[6] One can prove that $IES = \frac{1}{\rho}$ and $RRA = \gamma$.

$$r_t = \frac{1}{1-\gamma} \log \mathbb{E}\left[\exp\left[(1-\gamma)\, v_{t+1}\right] \mid \mathcal{F}_t\right] \tag{3}$$

$$v_t = \frac{1}{1-\rho} \log \left[(1-\beta)\exp\left[(1-\rho)\, c_t\right] + \beta \exp\left[(1-\rho)\, r_t\right]\right] \tag{4}$$

Aside from **risk aversion**, γ can be equivalently reinterpreted from the **robustness (ambiguity aversion)** perspective. First note that if the agent has $\gamma = 1$, there is no probability distortion, i.e. $r_t = \mathbb{E}\left[v_{t+1} \mid \mathcal{F}_t\right]$:

$$\lim_{\gamma \to 1} \log R_t = \lim_{\gamma \to 1} \frac{\log \mathbb{E}\left[\exp\left[(1-\gamma)\log V_{t+1}\right] \mid \mathcal{F}_t\right]}{1-\gamma} = \mathbb{E}\left[\log V_{t+1} \mid \mathcal{F}_t\right]$$

where we suppose there is a "baseline probability/baseline model" that the agent uses to compute mathematical expectations like this.

Now instead we want to replace $\mathbb{E}\left[v_{t+1} \mid \mathcal{F}_t\right]$ by considering the following problem

$$r_t := \min_{M_{t+1} \geq 0, \mathbb{E}\left[M_{t+1} \mid \mathcal{F}_t\right] = 1} \mathbb{E}\left[M_{t+1} v_{t+1} \mid \mathcal{F}_t\right] + \xi \mathbb{E}\left[M_{t+1} \log M_{t+1} \mid \mathcal{F}_t\right] \tag{5}$$

where the random variable M_{t+1} represents an "alternative probability/alternative model" through a change of probability measure, and $M_{t+1} \log M_{t+1}$ is called "relative entropy", which measures the statistical divergence between the baseline model and the alternative model, and ξ penalizes this divergence so that the alternative model is statistically close to the baseline model. The minimizer of (5) is

$$M^*_{t+1} = \frac{\exp\left(-\frac{1}{\xi} v_{t+1}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{\xi} v_{t+1}\right) \mid \mathcal{F}_t\right]}$$

and the minimized objective is

$$-\xi \log \mathbb{E}\left[\exp\left(-\frac{1}{\xi} v_{t+1}\right) \mid \mathcal{F}_t\right]$$

The change of measure M^*_{t+1} evidently raises conditional probabilities of one-period ahead events that bring low continuation values. Note that if we set $\xi = \frac{1}{1-\gamma}$, the minimized objective will coincide with r_t defined in (3), and $\gamma = 1$ corresponds to $\xi = \frac{1}{1-\gamma} = \infty$, meaning that $M_{t+1} \equiv 1$, i.e. no probability distortion[7].

[7]Intuitively, the penalty is too large to deviate from the baseline probability.

3 Expansion

The endogenous law of motion ψ in (2) is unknown and needs to be solved for from the set of equilibrium conditions (1). We rewrite (1) by distinguishing two types of equilibrium conditions:

$$\mathbb{E}\left[M_{t+1} f(X_t, X_{t+1}, \mathsf{q} W_{t+1}, \mathsf{q}) \mid \mathcal{F}_t\right] = 0 \tag{6}$$

$$f_2\left(X_t, X_{t+1}, \mathsf{q} W_{t+1}, \mathsf{q}\right) = 0 \tag{7}$$

where f_1 are those equations in (1) that entertain "jump variables" so they (potentially) involve expectations[8], while f_2 are equations that represents the evolution of "state variables", which don't involve expectations because state variables are predetermined[9].

3.1 Special treatment for γ

You may remember LPH saying his expansion approach is not exactly the "standard" approach, and here is why.

The "standard" approach holds γ fixed independent of q, while our approach here makes γ vary with q according to

$$1 - \gamma = \frac{1 - \gamma_o}{\mathsf{q}}$$

where γ_o is fixed. Such an approximation has been used in the control theory literature. It has the virtue that risk adjustments occur more prominently at lower order terms in an expansion than they would were γ to be treated as fixed. Note that γ only shows up in the equilibrium condition $R_t = \mathbb{E}\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t\right]^{\frac{1}{1-\gamma}}$, or equivalently $r_t = \frac{1}{1-\gamma} \log \mathbb{E}\left[\exp\left[(1-\gamma)\, v_{t+1}\right] \mid \mathcal{F}_t\right]$, so it's worth us looking at how our expansion approach approxiamtes r_t.

In order 0, there's no uncertainty, so we can directly drop the expectation

$$r_t^0 = v_{t+1}^0$$

In order to derive the higher order approximation, we first construct

[8] Why do I say "potentially"? Because in our specific setting of a representative agent with E-Z preference, we have only one equilibrium condition that involves expectation. What is it?

[9] The terminology "jump variable" corresponds to endogenous non-predetermined variables, while "state variable" corresponds to both endogenous predetermined variables and exogenous forcing variables.

$$\tilde{v}_t = \frac{v_t - v_t^0}{\mathsf{q}}$$
$$\tilde{r}_t = \frac{r_t - v_{t+1}^0}{\mathsf{q}}$$

and we assume that limits of these two quantities remain well defined as $\mathsf{q} \to 0$, and denote these limites by $\tilde{v}_t^0$ and $\tilde{r}_t^0$. Mutiplying $\tilde{v}_t$ by q and then differentiate with respect to q:

$$\mathsf{q}\frac{\mathrm{d}\tilde{v}_t}{\mathrm{d}\mathsf{q}} + \tilde{v}_t = \frac{\mathrm{d}v_t}{\mathrm{d}\mathsf{q}}$$
$$\mathsf{q}\frac{\mathrm{d}^2\tilde{v}_t}{\mathrm{d}\mathsf{q}^2} + 2\frac{\mathrm{d}\tilde{v}_t}{\mathrm{d}\mathsf{q}} = \frac{\mathrm{d}^2 v_t}{\mathrm{d}\mathsf{q}^2}$$

Thus,

$$\tilde{v}_t^0 = v_t^1$$
$$2\tilde{v}_t^1 = v_t^2$$

Similarly for r_t,

$$\tilde{r}_t^0 = r_t^1 = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left[\exp\left[(1-\gamma_o)\, v_{t+1}^1\right] \mid \mathcal{F}_t\right]$$
$$2\tilde{r}_t^1 = r_t^2 = \mathbb{E}\left[\frac{\exp\left((1-\gamma_o)\, v_{t+1}^1\right)}{\mathbb{E}\left[\exp\left(\left((1-\gamma_o)\, v_{t+1}^1\right) \mid \mathcal{F}_t\right]} v_{t+1}^2 \mid \mathcal{F}_t\right]$$

3.2 Order 0

The zeroth-order approximation of the system (6)-(7) is:

$$\mathbb{E}\left[M_{t+1}^0 f_1^0 \mid \mathcal{F}_t\right] = 0$$
$$f_2^0 = 0$$

Since f^0 is, by definition of $\mathsf{q} = 0$, deterministic, and $\mathbb{E}\left[M^0_{t+1} \mid \mathcal{F}_t\right] = 1$, so the whole system simplifies to $f^0 = 0$. This amounts to solving for the deterministic path (typically the deterministic steady state) of the model[10]

$$0 = f\left(X^0_t, X^0_{t+1}, 0, 0\right)$$

3.3 Order 1

The first-order expansion of the system (6)-(7) is

$$\begin{aligned} \mathbb{E}\left[M^0_{t+1} f^1_1 \mid \mathcal{F}_t\right] &= 0 \\ f^1_2 &= 0 \end{aligned}$$

We will use them to solve for the coefficient matrices in the law of motion

$$X^1_{t+1} = \psi_x X^1_t + \psi_w W_{t+1} + \psi_q \tag{8}$$

To facilitate computation, we partition the vector X_t into two parts

$$X_t := \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix}$$

where $X_{1,t}$ includes all jump variables and $X_{2,t}$ includes all state variables. Note that the number in the subscript, indicating the type of variables, is not to be confused with the number in the superscript, indicating the order of differentiation.

As an intermediate step, we will first solve for

$$X^1_{1,t} = \mathbb{N} X^1_{2,t} + C \tag{9}$$

and

$$X^1_{2,t+1} = \tilde{\psi}_x X^1_{2,t} + \tilde{\psi}_w W_{t+1} + \tilde{\psi}_q \tag{10}$$

[10] Common senses in solving a fixed point problem apply. For example, in the AK adjustment cost model, we use a numerical root solver to solve the fixed point of one variable $\log \frac{C}{K}$ and then compute the others, and in the permanent income model, we also first determine the fixed point of one variable $\frac{K}{Y}$ and then find the others'. Multivariate root solver may also be used but the performance may not be very promising.

3.3.1 Details

Since we are assuming the underlying shocks are normally distributed, the first-order approximation for the logarithm of consumption is Gaussian. In this case, the parameter γ_o contributes only to a constant term that is added to the first-order approximation for the logarithm of the continuation value process relative to the case in which $\gamma_o = 1$. We will first solve the system (17) for the $\gamma_o = 1$ case, where we can express the variables in X_t^1 that are not predetermined using the predetermined ones.

Step 1 Set $\gamma_o = 1$, then $M_{t+1}^0 = 1$. Take expectation to f_2, and then system (6)-(7) becomes

$$\mathbb{E}\left[f^1 \mid \mathcal{F}_t\right] = 0$$

Expand f^1 and we have

$$-f_{x^+}\mathbb{E}\left[X_{t+1}^1 \mid \mathcal{F}_t\right] = f_x X_t^1 \tag{11}$$

where f_{x^+} and f_x are partial derivatives of f (we assume $f_q = 0$; f_w doesn't show up here because of the undistorted expectation). Apply the generalized Schur decomposition[11] to $-f_{x^+}$ and f_x and transform the first-order system to be lower triangular:

$$-\mathbb{Q}^* f_{x^+}\mathbb{Z}\mathbb{Z}^*\mathbb{E}\left[X_{t+1}^1 \mid \mathcal{F}_t\right] = \mathbb{Q}^* f_x \mathbb{Z}\mathbb{Z}^* X_t^1$$

Rewrite this as:

$$\begin{bmatrix} \Lambda_{11}^+ & \Lambda_{12}^+ \\ 0 & \Lambda_{22}^+ \end{bmatrix} \mathbb{Z}^*\mathbb{E}\left[X_{t+1}^1 \mid \mathcal{F}_t\right] = \begin{bmatrix} \Lambda_{11}^0 & \Lambda_{12}^0 \\ 0 & \Lambda_{22}^0 \end{bmatrix} \mathbb{Z}^* X_t^1$$

Write the second block of this equation as:

$$\Lambda_{22}^+ \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^*\mathbb{E}\left[X_{t+1}^1 \mid \mathcal{F}_t\right] = \Lambda_{22}^0 \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_t^1$$

Since $\mathbb{Z}^*$ is nonsingular and there exists a (stable) solution to the dynamic system, we must have

$$\begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_t^1 = 0$$

which we use to solve for the non-predetermined variables in X_t^1, i.e. we can obtain

[11] We push the explosive eigenvalues to the right bottom, which is a standard treatment in solving DSGE models. See, for example, Blanchard and Kahn (1980), Klein (2000). We assume *saddle-path stability*, i.e. the number of explosive eigenvalues equals the number of non-predetermined variables.

$$X^1_{1,t} = \mathbb{N} X^1_{2,t}$$

Step 2 Set $\gamma_o \neq 1$. Under the M^0_{t+1} change of measure, one can prove that the shock distribution is

$$W_{t+1} \sim \mathcal{N}\left((1-\gamma_o)\,\sigma^1_v, I\right)$$

where σ^1_v is the loading of $v_{t+1} - r_t$ on W_{t+1}.

We seek a solution of the form

$$\begin{aligned} X^1_{1,t+1} - D_1 &= \mathbb{N}(X^1_{2,t+1} - D_2) \\ X^1_{2,t+1} - D_2 &= \tilde{\psi}_x (X^1_{2,t} - D_2) + \tilde{\psi}_w W_{t+1} \end{aligned}$$

where we can first obtain $\tilde{\psi}_x$ and $\tilde{\psi}_w$ by substituting $X^1_{1,\cdot} = \mathbb{N} X^1_{2,\cdot}$ into (11) and solve for $X_{2,t+1}$:

$$\tilde{\psi}_x = \left(-f_{2,x^+}\begin{bmatrix}\mathbb{N}\\ I\end{bmatrix}\right)^{-1}\left(f_{2,x}\begin{bmatrix}\mathbb{N}\\ I\end{bmatrix}\right) \tag{12}$$

and

$$\tilde{\psi}_w = \left(-f_{2,x^+}\begin{bmatrix}\mathbb{N}\\ I\end{bmatrix}\right)^{-1} f_{2,w} \tag{13}$$

Then let $D := \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, and we add and subtract D in (11), and substitute $X^1_{1,t+1} - D_1$ and $X^1_{2,t+1} - D_2$ into it, meaning that we can compute D by solving

$$\begin{aligned} (f_{1,x^+} + f_{1,x})D + \left(f_{1,x^+}\begin{bmatrix}\mathbb{N}\\ I\end{bmatrix}\tilde{\psi}_w + f_{1,w}\right)(1-\gamma_o)\,\sigma^1_v + \begin{bmatrix} -\frac{1}{2}(1-\gamma_o)|\sigma_v|^2 \\ 0 \end{bmatrix} &= 0 \\ (f_{2,x^+} + f_{2,x})D &= 0 \end{aligned}$$

Step 3 Write the solved $X^1_{1,t}$ as

$$X^1_{1,t} = \mathbb{N} X^1_{2,t} + C \tag{14}$$

where $C = D_1 - \mathbb{N} D_2$. Also we have $\tilde{\psi}_q = D_2 - \tilde{\psi}_x D_2$.

3.4 Order 2 - I haven't carefully proofread this

Take expectation to equation (7) under the distorted measure M. The second-order expansion of the system (6)-(7) is

$$E_t\left[M^0_{t+1}f^2\right] + 2E_t\left[M^1_{t+1}f^1\right] = 0 \tag{15}$$

The first term on the left-hand side is a standard second-order expansion with constant mean distortions of the shocks in f^2_{t+1} imposed by the known (from the solution of the first-order approximation) worst-case distortion M^0_{t+1}.

The second term has an different structure. It contains the term f^1, which will only depend on the known first-order dynamics of the law of motion ψ, and the unknown M^1_{t+1} constructed from distortions.

In the following computational details, we will first solve $X^2_{1,t}$ as a linear function of $X^2_{2,t}$ and contributions from first-order expansion, and then solve for $\tilde{\psi}$ derivatives.

3.4.1 Details

Step 1 Start with an initial guess $M^1_{t+1} = 0$.

Step 2 With known M^0_{t+1} and M^1_{t+1}, calculate the first-order contributions from (15) and write the stacked equation for the second order computation as:

$$-f_{x^+}E_t\left[M^0_{t+1}X^2_{t+1}\right] = f_x X^2_t + D_t \tag{16}$$

where D_t includes all of the first-order contributions. Let D^1_{t+1} the contributions from first-order expansion and D^2_{t+1} from second-order expansion. And $D_t = E[M^0_{t+1}D^2_{t+1}] + 2E[M^1_{t+1}D^1_{t+1}]$. Notice that f_{x^+} and f_x matrices are the first-order derivatives, and thus the generalized Schur decomposition remains the same as in the first-order expansion.

Transform the second-order system to be lower triangular:

$$-\mathbb{Q}^* f_{x^+}\mathbb{Z}\mathbb{Z}^* E_t\left[M^0_{t+1}X^2_{t+1}\right] = \mathbb{Q}^* f_x \mathbb{Z}\mathbb{Z}^* X^2_t + \mathbb{Q}^* D_t \tag{17}$$

where $\mathbb{Q}^*$ is the conjugate transpose of $\mathbb{Q}$ and $\mathbb{Q}$ is unitary. Rewrite this as

$$\begin{bmatrix} \Lambda^+_{11} & \Lambda^+_{12} \\ 0 & \Lambda^+_{22} \end{bmatrix} \mathbb{Z}^* E_t\left[M^0_{t+1}X^2_{t+1}\right] = \begin{bmatrix} \Lambda^0_{11} & \Lambda^0_{12} \\ 0 & \Lambda^0_{22} \end{bmatrix} \mathbb{Z}^* X^2_t + \mathbb{Q}^* D_t \tag{18}$$

Write the second block of this equation as:

$$\Lambda_{22}^{+} E_t \left[M_{t+1}^0 \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_{t+1}^2 \right] = \Lambda_{22}^0 \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_t^2 + \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Q}^* D_t \tag{20}$$

Suppose there exists a $\tilde{D}$ that satisfies:

$$\Lambda_{22}^{+} E_t \left[M_{t+1}^0 \left(\begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_{t+1}^2 - \tilde{D}_{t+1} \right) \right] = \Lambda_{22}^0 \left(\begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_t^2 - \tilde{D}_t \right) \tag{21}$$

Combining equation (20) and (21) gives

$$\tilde{D}_t = \left(\Lambda_{22}^0\right)^{-1} \Lambda_{22}^{+} E_t \left[M_{t+1}^0 \tilde{D}_{t+1} \right] - \left(\Lambda_{22}^0\right)^{-1} \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Q}^* D_t \tag{22}$$

Write $D_t = D \begin{bmatrix} 1 \\ X_t \\ X_t \otimes X_t \end{bmatrix}$ and $\tilde{D}_t = \tilde{D} \begin{bmatrix} 1 \\ X_t \\ X_t \otimes X_t \end{bmatrix}$

Then we have

$$\text{vec}(\tilde{D}) = (A_2^T \otimes A_1)\text{vec}(\tilde{D}) + \text{vec}(A_3) \tag{19}$$

where $A_1 = \left(\Lambda_{22}^0\right)^{-1} \Lambda_{22}^{+}$, $A_2 = M$, $A_3 = -\left(\Lambda_{22}^0\right)^{-1} \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Q}^* D$. M is implied by

$$E \left[M_{t+1}^0 A \begin{bmatrix} 1 \\ X_{t+1} \\ X_{t+1} \otimes X_{t+1} \end{bmatrix} \right] = AM \begin{bmatrix} 1 \\ X_t \\ X_t \otimes X_t \end{bmatrix}$$

Solve the linear equation for $\tilde{D}_t$. The the solution of interest is:

$$\begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \mathbb{Z}^* X_t^2 = \tilde{D}_t \tag{20}$$

which we use to solve for the variables in X_t^2 that are not predetermined.

Step 3 Write the solved X_t^2 as

$$X_{1,t}^2 = \mathbb{N} X_{2,t}^2 + G_t \tag{23}$$

where G_t includes all of the first-order contributions. Take second-order derivatives to $f_{2,t+1}$ and substitue $X_{1,t}^1$, $X_{1,t}^2$ using (18), (23). We have

$$-f_{2,x^+} \left(\begin{bmatrix} \mathbb{N} \\ I \end{bmatrix} X_{2,t+1}^2 + \begin{bmatrix} G_{t+1} \\ 0 \end{bmatrix} \right) = f_{2,x} \left(\begin{bmatrix} \mathbb{N} \\ I \end{bmatrix} X_{2,t}^2 + \begin{bmatrix} G_t \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 & I \end{bmatrix} D_{t+1}^2 \tag{21}$$

Invert the coefficient of $X^2_{2,t+1}$ and rearrange to get the $\tilde{\psi}$ derivatives.

Step 4 Use the solved ψ derivatives to calculate the first- and second-order derivatives of v_{t+1} and then update M^1_{t+1}. Repeat step 2 - step 4 until M^1_{t+1} converges.

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