

## Endogenously Determined Subjective Discount Rate

Before moving forward, let's have another look at the FOC [\(13\)](#). For compactness, we define  $\delta = -\log \beta$ . Essentially, FOC says

$$\log(1 + a) - \delta + \log \tilde{\mathbb{E}} \left[ \left( \frac{V_{t+1}}{R_t} \right)^{\rho-1} \left( \frac{U_{t+1}}{U_t} \right)^{-\rho} \left( \frac{MC_{t+1}/MU_{t+1}}{MC_t/MU_t} \right) \mathfrak{F}_t \right] = 0 \quad (14)$$

where  $\tilde{\mathbb{E}}[\cdot | \mathfrak{F}_t]$  is the conditional expectation operator under the **altered probability measure**, i.e.

taking into account the  $\left( \frac{V_{t+1}}{R_t} \right)^{1-\gamma}$  term. Here  $a$  is an exogenous one period risk-free capital growth

rate, thus the subjective discount rate  $\delta$  is not exogenously given, but rather endogenously pinned

down by [\(14\)](#). Our small noise expansion method gives the law of motion of model variables

$X_{t+1}(\mathbf{q}) = \psi[X_t(\mathbf{q}), \mathbf{q}W_{t+1}, \mathbf{q}]$ , and if we plug it into [\(14\)](#), there will surely be a bunch of terms that involve  $W_{t+1}$ . Since the structure of  $W_{t+1}$  under the altered probability measure differs as the order of expansion differs, it's clear that  $\beta$  pinned down by [\(14\)](#) in order 0, 1, 2 differs. Let's write them as  $\delta^0$ ,  $\delta^1$  and  $\delta^2$ .

In the deterministic steady state (order 0),  $\delta^0$  can be easily solved from [\(14\)](#) because change of measure and expectation operator don't matter (no  $W_{t+1}$  terms).

In order 1, an **additional constant term** needs to be added to [\(14\)](#), which equals  $\delta^0 - \delta^1$ . Without this constant term, the LHS of [\(14\)](#) is obviously non-zero, and the approximated law of motion coming from expansion that can make the wrong FOC "hold" will be wrong (not the equilibria that we want).

Similarly, in order 2, an **additional constant term** needs to be added to [\(14\)](#), which equals  $\delta^1 - \delta^2$ .

**Technical details:** When introducing the additional free constant term, we also impose an additional restriction, to set the constant component that applies to  $\frac{K_t}{Y_t}$  to zero. We make this latter adjustment because we want to set  $\frac{K_0}{Y_0}$  as an initial condition and don't want to adjust it later when we include higher order terms in an expansion.

**As in LPH's *PI Model Notes***, we provide a simplified example where there's no habit/durable goods. It can be viewed as a special case of the preference described in section 1.1, by setting  $\alpha = 0$ ,  $\rho = 1$ . In this special case, **LPH showed that**

In order 0 approximation,

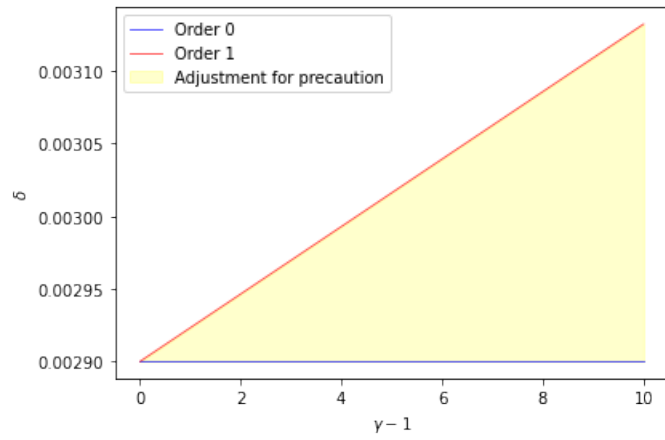
$$\delta^0 = a - g$$

In order 1 approximation,

$$\delta^1 = a - g - (1 - \gamma)\sigma_c^1 \cdot \sigma_c^1$$

The last term is an adjustment for precaution, which is exactly the "additional constant term" that should be included when we move from order 0 to order 1 expansion in this special case.

```
# discount rate adjustment graph
from demonstration import plot_figure_2
plot_figure_2()
```



By Lars P Hansen & Thomas Sargent

© Copyright 2021.