Permanent Income Model Notes*

June 7, 2021

We construct a stochastic growth model capable of representing a rational expectations version of Milton Friedman's permanent income model of consumption. By building on Hansen et al. (1999), we generalize a consumer-investor's preferences to allow recursive utility, a preference for robustness, and habit persistence. We assume that a key exogenous driving process – nonfinancial income – is a multiplicative functional instead of a covariance stationary process. We incorporate intertemporal preferences like those posited by Ryder and Heal (1973) and Becker and Murphy (1988). The model of Hansen et al. (1999) fits within a linear-quadratic framework, but our model here does not. A consequence is that, except in some special cases, our model cannot be solved quasi-analytically as linear-quadratic models can.[1] That prompts us to approximate an optimal plan by deploying a small-noise expansion.

1 Exogenous income process

The model takes as exogenous a two-component nonfinancial income.[2] Let $Y \doteq \{Y_t : t \geqslant 0\}$ be an exogenous nonfinancial income process with logarithm $\widehat{Y} \doteq \{\widehat{Y}_t : t \geqslant 0\}$ governed by

$$\widehat{Y}_{t+1} - \widehat{Y}_t = \mathbb{D}X_t + \mathbb{F}W_{t+1} + \mathsf{g}$$

*We thank Dongchen Zou and John Wilson for helping with computations and Balint Szoke for comments.

[1]By "quasi-analytically" here we mean, "up to solving a matrix Riccati equation like those appearing in linear-quadratic dynamic programming".

[2]Nonfinancial income in the single-agent version of the model becomes an exogenous component of output in a "general equilibrium" version.

where $\mathbb{D}$ and $\mathbb{F}$ are row vectors,

$$X_{t+1} = \mathbb{A}X_t + \mathbb{B}W_{t+1},$$

$\mathbb{A}$ is a stable matrix, and $W_{t+1} \sim \mathcal{N}(0, I)$ be an i.i.d. vector process. Define

$$\begin{aligned}
\widehat{Y}_t &= \widehat{Y}_t^0 + \widehat{Y}_t^1 \\
\widehat{Y}_t^0 &= \widehat{Y}_0^0 + t\mathsf{g} \\
\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1 &= \mathbb{D}X_t + \mathbb{F}W_{t+1}
\end{aligned}$$

where $\{\widehat{Y}_t^0 : t \geqslant 0\}$ is a deterministic component of the nonfinancial income process and $\{\widehat{Y}_t^1 : t \geqslant 0\}$ is a stochastic component.

Example 1.1. *We can represent a model from Hansen et al. (1999) in terms X_t and W_{t+1} defined as*

$$X_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{2,t-1} \end{bmatrix}, \quad W_{t+1} = \begin{bmatrix} W_{1,t+1} \\ W_{2,t+1} \end{bmatrix}$$

where

$$\begin{bmatrix} X_{1,t+1} \\ X_{2,t+1} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} .704 & 0 & 0 \\ 0 & 1 & -.154 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} .144 & 0 \\ 0 & .206 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_{1,t+1} \\ W_{2,t+1} \end{bmatrix}$$

and

$$\begin{aligned}
\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1 &= .01(X_{1,t+1} + X_{2,t+1} - X_{2,t}) \\
&= .01\left(\begin{bmatrix} .704 & 0 & -.154 \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} .144 & .206 \end{bmatrix} \begin{bmatrix} W_{1,t+1} \\ W_{2,t+1} \end{bmatrix}\right)
\end{aligned}$$

Here $\{X_{1,t} : t \geqslant 0\}$ is a scalar first-order autoregression and $\{X_{2,t} : t \geqslant 0\}$ is a scalar second-order autoregression, each with its own shock process.

The process $\{\widehat{Y}_{t+1}^1 : t \geqslant 0\}$ has stationary increments to which $\{X_{1,t} : t \geqslant 0\}$ contributes a permanent component and $\{X_{2,t} : t \geqslant 0\}$ contributes a transient component, properties evident from impulse response functions in figure 1.

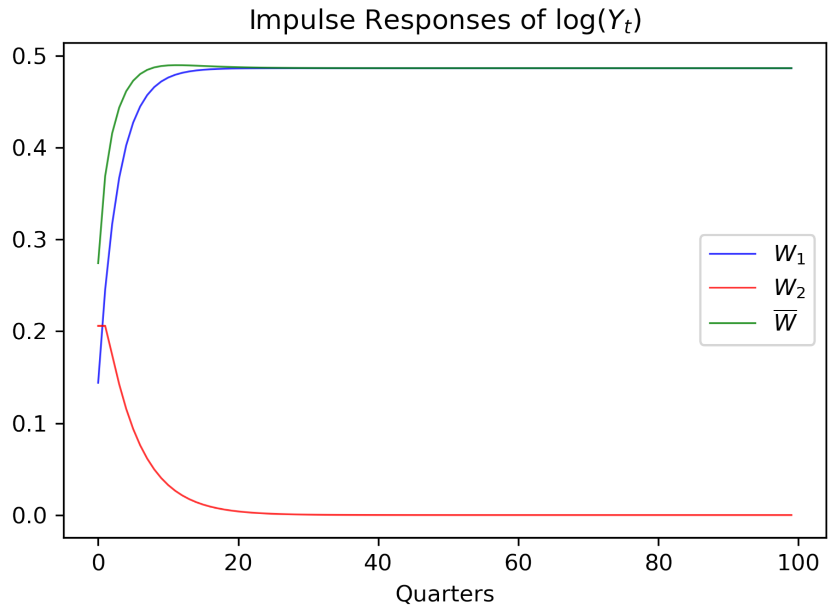

Figure 1: Impulse responses for two income shocks. Responses are multiplied by one hundred to convert them into percentages. Effects of W^1 are permanent while effects of W^2 transitory. LPH: I think we should hold off on presenting the $\overline{W}$ responses. The figure also reports the impulse response to an innovation $\overline{W}$ in the time-invariant innovations representation for Y^1 associated with a time-invariant Kalman filter.

That the limiting impulse response shocks W^1 converges to a nonzero constant indicates that its effects are permanent. That the response to the shock W^2 converges to zero indicates that its effects are transitory. That the response to the "mongrel shock" $\overline{W}$ in the innovations representation converges to a nonzero constant indicates that its effects are permanent. Notice that the permanent effect from the mongrel shock $\overline{W}$ in the time-invariant innovations representation equals the permanent effect from shock W^1.

2 Feasibility

Let C_t be consumption, Y_t be nonfinancial income, and K_t be the stock of a risk-free asset (or *liability* if it is negative) all at time t. The asset bears a fixed risk-free one-period gross return equal to $\exp(\mathsf{a})$. Feasibility at time t requires

$$K_{t+1} + C_t = \exp(\mathsf{a})K_t + Y_t \tag{1}$$

where we assume that the rate of return a exceeds the growth rate g. We will sometimes use the following rewriting of this equation:

$$K_{t+1} - K_t + C_t = [\exp(\mathsf{a} - 1]\, K_t + Y_t$$

where the left side includes both net investment and consumption and right side two sources of income, one from financial assets and the other from the exogenously specified income. Stochastic growth in the income process makes it convenient to divide by Y_t

$$\left[\left(\frac{K_{t+1}}{Y_{t+1}} \right) \left(\frac{Y_{t+1}}{Y_t} \right) - \frac{K_t}{Y_t} \right] + \frac{C_t}{Y_t} = [\exp(\mathsf{a}) - 1] \left(\frac{K_t}{Y_t} \right) + 1. \tag{2}$$

While nonlinear global and local numerical methods can in principle capture more of the underlying dynamics, a first-order approximation can still be enlightening and facilitate comparative dynamics. But "first-order approximation *to what?*" Our answer is "to stationary stochastic processes that solve first-order and other equilibrium conditions".

We use the following notation for capital, gross investment, and consumption to income ratios:

$$\begin{aligned}
\widetilde{K}_t &\doteq \frac{K_t}{Y_t}, \\
\widetilde{I}_{t+1} &\doteq \frac{K_{t+1} - K_t}{Y_t}, \\
\widetilde{C}_t &\doteq \frac{C_t}{Y_t} \\
\widehat{C}_t &\doteq \log C_t = \log \widetilde{C}_t + \widehat{Y}_t.
\end{aligned}$$

We allow the asset stock K_t to be negative, as can happen in so-called open economy models.

2.1 Order zero approximation:

We deduce an order-zero term by computing steady states. These are the same for both types of approximations that we shall describe. We seek steady states characterized by the following invariance relations:

$$\widetilde{K}_t^0 = \widetilde{K}_0^0$$

$$\widetilde{C}_t = \widetilde{C}_0$$

for all $t \geqslant 0$. In a deterministic steady state, gross investment, scaled by the exogenous income process, is

$$\widetilde{I}^0_{t+1} = \widetilde{K}^0_0[\exp(\mathsf{g}) - 1]$$

for all $t \geqslant 0$. Since

$$\widetilde{I}^0_{t+1} + \widetilde{C}^0_t = [\exp(\mathsf{a}) - 1]\widetilde{K}^0_0 + 1,$$

it follows that the consumption-income ratio in the steady state is:

$$\widetilde{C}^0_0 = [\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}^0_0 + 1.$$

Thus, we have a free initial condition that can be imposed on $\widetilde{K}^0_0$, which in turn determines $\widetilde{C}^0_0$. So there is a family of steady states indexed by $\widetilde{K}^0_0$ subject to the restriction $\widetilde{C}_0 > 0$.

We impose a constant consumption to income ratio now but will revisit this restriction after we describe consumer preferences.

2.2 Order one approximation

From equation (2) and the product rule for derivatives, we form the first-order approximation:

$$\widetilde{K}^1_{t+1}\exp(\mathsf{g}) + \widetilde{K}^0_0 \exp(\mathsf{g}) \left(\widehat{Y}^1_{t+1} - \widehat{Y}^1_t\right) + \widetilde{C}^0_0 \left(\widehat{C}^1_t - \widehat{Y}^1_t\right) = \exp(\mathsf{a})\widetilde{K}^1_t. \tag{3}$$

We approximate the asset to income ratio and accommodate borrowing by allowing for it to be negative. We approximate consumption and income both in logarithms. Processes with a superscript one are in effect first-derivative approximations. Subtracting $\widetilde{K}_t$ from both sides:

$$\left[\widetilde{K}^1_{t+1}\exp(\mathsf{g}) + \widetilde{K}^0_0 \exp(\mathsf{g}) \left(\widehat{Y}^1_{t+1} - \widehat{Y}^1_t\right) - \widetilde{K}^1_t\right] + \widetilde{C}^0_0 \left(\widehat{C}^1_t - \widehat{Y}^1_t\right) = [\exp(\mathsf{a}) - 1]\, \widetilde{K}^1_t.$$

where the term in the square brackets is the first-order approximation to net investment scaled by income.

Zhenhuan and Yad: Hi. Below is what LPH recommended today, June 5, 2021.

As observables for the ML estimation of the VAR from the first-order expansion, let's use $\log C_{t+1} - \log C_t$ and the investment consumption ratio. Start by writing:

$$K_{t+1} - K_t + C_t = [\exp(\mathsf{a}) - 1]K_t + Y_t.$$

Divide by C_t to get

$$\begin{aligned}
\frac{K_{t+1} - K_t}{C_t} &= [\exp(\mathsf{a}) - 1]\left(\frac{K_t}{C_t}\right) + \frac{Y_t}{C_t} - 1 \\
&= [\exp(\mathsf{a}) - 1]\widetilde{K}_t \exp\left(\log Y_t - \log C_t\right) + \exp\left(\log Y_t - \log C_t\right) - 1 \\
&= \left([\exp(\mathsf{a}) - 1]\,\widetilde{K}_t + 1\right) \exp\left(\log Y_t - \log C_t\right) - 1.
\end{aligned}$$

Compute the order-zero approximation from the steady state. We then take a first-order approximation of the right-hand side using the solution produced by the code. Let

$$\widetilde{I}_{t+1} = \frac{K_{t+1} - K_t}{C_t}$$

Then

$$\begin{aligned}
\widetilde{I}^1_{t+1} = &[\exp(\mathsf{a}) - 1]\,\widetilde{K}^1_t \exp\left(\log Y^0_t - \log C^0_t\right) \\
&+ \left([\exp(\mathsf{a}) - 1]\,\widetilde{K}^0_t + 1\right) \exp\left(\log Y^0_t - \log C^0_t\right)\left(\log Y^1_t - \log C^1_t\right)
\end{aligned}$$

This provides an affine approximation for the investment consumption ratio. Can we just update the data from HST, no removal of trend? Consumption real.

3 Planner's investment choice

Following Ryder and Heal (1973) and Hansen et al. (1999), we solve a planning problem. A representative consumer who has discounted time-separable preferences, subjective discount rate δ, and one-period utility of consumption

$$\frac{c^{1-\rho} - 1}{1 - \rho}$$

optimally chooses a consumption process by choosing one that satisfies the Euler equation:

$$MK_t = \exp\left(\mathsf{a} - \delta\right)\mathcal{E}\left(MK_{t+1} \mid \mathfrak{F}_t\right) \tag{4}$$

where $MK_t = MC_t = (C_t)^{-\rho}$ is a Lagrange multiplier on (1) that serves as a "co-state" variable on the stock of assets. Euler equation (4) is forward-looking because the asset stock at date t shows up in the resource constraint at two adjacent time periods. Rewriting equation (4) in terms of logarithms gives

$$\mathsf{a} - \delta + \log \mathcal{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \mid \mathfrak{F}_t\right] = 0. \tag{5}$$

In a deterministic steady state of the type described above,

$$\mathsf{a} - \delta - \rho \mathsf{g} = 0,$$

which restricts δ. In a first-order approximation

$$-\rho \mathcal{E}\left(\widehat{C}^1_{t+1} - \widehat{C}^1_t \mid \mathfrak{F}_t\right) = 0, \tag{6}$$

so the logarithm of $\widehat{C}^1$ is a random walk.

4 Approximate solution

We approximate a solution by solving the approximate state equation (3) and the approximate co-state equation (6) simultaneously. Multiply equation (3) by $\exp(-\mathsf{a})$ and deduce the forward-looking difference equation in $\tilde{K}^1_t$:

$$\widetilde{K}^1_t = \lambda \widetilde{K}^1_{t+1} + \tilde{\beta}_1\left(\widehat{Y}^1_{t+1} - \widehat{Y}^1_t\right) + \tilde{\beta}_2\left(\widehat{C}^1_t - \widehat{Y}^1_t\right) \tag{7}$$

where

$$\begin{aligned}
\lambda &= \exp(\mathsf{g} - \mathsf{a}) < 1 \\
\tilde{\beta}_1 &= \lambda \widetilde{K}^0_0 \\
\tilde{\beta}_2 &= \exp(-\mathsf{a})\widetilde{C}^0_0.
\end{aligned}$$

This difference equation can be solved forward to express $\widetilde{K}^1_t$ as a weighted sum of current and future values of the (approximate) growth rate $\widehat{Y}^1_{t+1} - \widehat{Y}^1_t$ and the logarithm of the (approximate) ratio of consumption to income$\left(\widehat{C}^1_t - \widehat{Y}^1_t\right)$ with weights that decline geometrically at rate λ. Thus, equation (7) implies:

$$\widetilde{K}_t^1 = \lambda\mathcal{E}\left(\widetilde{K}_{t+1}^1 \mid \mathfrak{F}_t\right) + \tilde{\beta}_1\mathcal{E}\left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1 \mid \mathfrak{F}_t\right) + \tilde{\beta}_2\left(\widehat{C}_t^1 - \widehat{Y}_t^1\right).$$

Solving this equation forward gives

$$\begin{aligned}\widetilde{K}_t^1 &= \tilde{\beta}_1\mathcal{E}\left[\sum_{j=0}^{\infty}\lambda^j\left(\widehat{Y}_{t+1+j}^1 - \widehat{Y}_{t+j}^1\right) \mid \mathfrak{F}_t\right] + \tilde{\beta}_2\mathcal{E}\left[\sum_{j=0}^{\infty}\lambda^j\left(\widehat{C}_{t+j}^1 - \widehat{Y}_{t+j}^1\right) \mid \mathfrak{F}_t\right] \\ &= \left[\frac{\tilde{\beta}_1(1-\lambda) - \lambda\tilde{\beta}_2}{\lambda}\right]\mathcal{E}\left(\sum_{j=1}^{\infty}\lambda^j\widehat{Y}_{t+j}^1 \mid \mathfrak{F}_t\right) + \left(\frac{\tilde{\beta}_2}{1-\lambda}\right)\widehat{C}_t^1 - \left(\tilde{\beta}_1 + \tilde{\beta}_2\right)\widehat{Y}_t^1\end{aligned}$$

where the second equality follows from (6). Solve for the approximate logarithm of consumption $\widehat{C}_t^1$ to get

$$\widehat{C}_t^1 = \left[\frac{1-\lambda}{\tilde{\beta}_2}\right]\widetilde{K}_t^1 + \left[\frac{(1-\lambda)\left(\tilde{\beta}_1 + \tilde{\beta}_2\right)}{\tilde{\beta}_2}\right]\widehat{Y}_t^1 - \left[\frac{\tilde{\beta}_1(1-\lambda)^2 - (1-\lambda)\lambda\tilde{\beta}_2}{\lambda\tilde{\beta}_2}\right]\mathcal{E}\left(\sum_{j=1}^{\infty}\lambda^j\widehat{Y}_{t+j}^1 \mid \mathfrak{F}_t\right)$$

Note that

$$\left[\frac{1-\lambda}{\tilde{\beta}_2}\right] = \frac{\exp(\mathsf{a}) - \exp(\mathsf{g})}{\left[\exp(\mathsf{a}) - \exp(\mathsf{g})\right]\widetilde{K}_0^0 + 1}.$$

Furthermore,

$$\frac{(1-\lambda)\tilde{\beta}_1 - \lambda\tilde{\beta}_2}{\exp(-\mathsf{a})\lambda} = -1,$$

and thus

$$\left[\frac{\tilde{\beta}_1(1-\lambda)^2 - (1-\lambda)\lambda\tilde{\beta}_2}{\lambda\tilde{\beta}_2}\right] = \frac{\lambda - 1}{\left[\exp(\mathsf{a}) - \exp(\mathsf{g})\right]\widetilde{K}_0^0 + 1}.$$

Finally,

$$\tilde{\beta}_1 + \tilde{\beta}_2 = \widetilde{K}_0^0 + \exp(-\mathsf{a}),$$

and consequently

$$\left[\frac{(1-\lambda)\left(\tilde{\beta}_1 + \tilde{\beta}_2\right)}{\tilde{\beta}_2}\right] = \frac{\left[\exp(\mathsf{a}) - \exp(\mathsf{h})\right]\widetilde{K}_0^0 + 1 - \lambda}{\left[\exp(\mathsf{a}) - \exp(\mathsf{g})\right]\widetilde{K}_0^0 + 1}.$$

Therefore,

$$
\begin{aligned}
\widehat{C}_t^1 = & \left[\frac{\exp(\mathsf{a}) - \exp(\mathsf{g})}{[\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}_0^0 + 1} \right] \widetilde{K}_t^1 \\
& + \left(\frac{[\exp(\mathsf{a}) - \exp(\mathsf{h})]\, \widetilde{K}_0^0 + 1 - \lambda}{[\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}_0^0 + 1} \right) \widehat{Y}_t^1 \\
& + \left(\frac{1 - \lambda}{[\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}_0^0 + 1} \right) \mathcal{E} \left(\sum_{j=1}^{\infty} \lambda^j \widehat{Y}_{t+j}^1 \mid \mathfrak{F}_t \right).
\end{aligned}
\tag{8}
$$

When $\widetilde{K}_0^0 = 0$, (8) simplifies to

$$
\widehat{C}_t^1 = [\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}_t^1 + (1 - \lambda) \mathcal{E} \left(\sum_{j=0}^{\infty} \lambda^j \widehat{Y}_{t+j}^1 \mid \mathfrak{F}_t \right),
$$

which is a logarithmic counterpart to the permanent income model of Flavin (1981). To deduce an optimal decision rule for consumption, we must evaluate the conditional expectations in the above formula. It can be verified that (8) also satisfies a balanced growth version:

$$
\begin{aligned}
\widehat{C}_t^1 - \widehat{Y}_t^1 = & \left(\frac{\exp(\mathsf{a}) - \exp(\mathsf{g})}{[\exp(\mathsf{a}) - \exp(\mathsf{g})]\, \widetilde{K}_0^0 + 1} \right) \widetilde{K}_t^1 \\
& + \left(\frac{1}{[\exp(\mathsf{a}) - \exp(\mathsf{g})]\, K_0^0 + 1} \right) \mathcal{E} \left[(1 - \lambda) \sum_{j=1}^{\infty} \lambda^j \widehat{Y}_{t+j}^1 - \lambda \widehat{Y}_t^1 \mid \mathfrak{F}_t \right].
\end{aligned}
$$

The denominator terms on the right-hand side equal $\widetilde{C}_0^0$ and adjust for the first-order terms being logarithmic derivatives.

When we use the specification in example 1.1, the increment to the random walk is

$$
\widehat{C}_{t+1}^1 - \widehat{C}_t^1 = .01 \begin{bmatrix} .48 & .0038 \end{bmatrix} W_{t+1},
$$

so that $\widehat{C}^1$ takes a random walk and so confirms to the predicitons of the permanent-income model of consumption. A transitory shock has little immediate impact on consumption because it its effects are "amortized over future time periods" as the consumer devotes most of an increment to nonfinancial income immediately to adjust the stock of financial assets. A permanent shock cannot be "amortized over future periods" in this way. Lars: here we want eventually to add a corresponding representation for the investment output ratio to buttress the discussion of the "amortization" effects and to setup the impulse responses that Zhenhuan is working on adding.

5 Uncertainty averse investors

To model responses to uncertainty, use a recursive utility specification. This activates a precautionary motive for savings that is reflected in a first-order approximation. For simplicity we maintain a unitary intertemporal elasticity of consumption, ($\rho = 1$). We represent preferences over consumption plans recursively in terms of continuation values. We show how an approximating model can be solved with a convenient change in probability measure. We offer an interpretation for this change of measure in terms of investor's aversion to feared misspecifications of the probability model.

Preferences over consumption plans can be represented in terms of a value V_t that satisfies the recursion defined by the two equations

$$
\begin{aligned}
V_t &= (C_t)^{[1-\exp(-\delta)]} (R_t)^{\exp(-\delta)} \\
R_t &= \left(\mathcal{E}\left[(V_{t+1})^{1-\gamma} \mid \mathfrak{F}_t\right]\right)^{\frac{1}{1-\gamma}}
\end{aligned}
\tag{9}
$$

for $\gamma \geqslant 1$. The random variable R_t is a risk-adjusted certainty equivalent of continuation value V_{t+1}; γ determines the magnitude of the risk adjustment. The first equation in (9) provides the continuation value V_t as a Cobb-Douglas function and the certainty equivalent of next period's continuation value. We require a terminal condition for the continuation value. For infinite-horizon versions, at least implicitly, we take limits as we push the terminal date arbitrarily far into the future. Including the parameter γ provides a way to distinguish an attitude about uncertainty from an attitude about intertemporal substitution as captured by an intertemporal elasticity of consumption.

5.1 Investor first-order conditions

To deduce investor first-order conditions, we recall some marginal computations familiar from CES utility functions, for which Cobb-Douglas is a special case. Let c, r, v be scalar variables that are potential realizations of C_t, R_t, and V_t, respectively. We let c^+ and v^+

be next period counterparts. From the Cobb-Douglas specification in the first equation of (9), we construct three marginal utilities:

$$
\begin{aligned}
mc &= [1 - \exp(-\delta)] \left(\frac{c}{v}\right)^{-1} \\
mr &= \exp(-\delta) \left(\frac{r}{v}\right)^{-1} \\
mc^{+} &= [1 - \exp(-\delta)] \left(\frac{c^{+}}{v^{+}}\right)^{-1}
\end{aligned}
$$

where the third is the next period counterpart of the first. From the second recursion, we form

$$
mv^{+} = \left(\frac{v^{+}}{r}\right)^{-\gamma}.
$$

When we use these calculations, we shall include a role for probabilities encoded in the conditional expectation used to construct the certainty equivalent of next period's continuation value.

To obtain the marginal rate of substitution for consumption tomorrow relative to today, we apply the "chain rule" for differentiation to obtain:

$$
mrs = \frac{mr \times mv^{+} \times mc^{+}}{mc} = \exp(-\delta) \left(\frac{c^{+}}{c}\right)^{-1} \left(\frac{v^{+}}{r}\right)^{1-\gamma}.
$$

When $\gamma = 1$, the formula collapses to the familiar intertermporal marginal rate of substitution for discounted logarithmic utility. More generally, when $\gamma \neq 1$, mrs is the ratio of next period's continuation value to a forward-looking certainty equivalent that captures uncertainty about future consumption.

Armed with these calculations, we modify the investor's first-order conditions (5) to

$$
\mathsf{a} - \delta + \log \mathcal{E} \left[\left(\frac{C_t}{C_{t+1}}\right) \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} \mid \mathfrak{F}_t \right] = 0.
$$

Notice how we include the probability weighting for the intertemporal marginal rate of substitution when we take conditional expectations.

The order zero steady state remains unchanged, but the order one approximation is altered. To prepare the way for this change, note that

$$N_{t+1} = \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} = \frac{(V_{t+1})^{1-\gamma}}{\mathcal{E}\left[(V_{t+1})^{1-\gamma} \mid \mathfrak{F}_t\right]} \tag{10}$$

where the right-hand side variable is positive and has conditional mean one, so that M_{t+1} has all of the properties of a likelihood ratio between an alternative probability model and the conditional probability model that we use to form expectations. This leads us to an alternative interpretation of the utility recursion to be described soon.

5.2 Logarithmic transformation

Continuation values are well defined only up to monotone increasing transformations. For purposes of approximation and interpretation, it is convenient to work with $\widehat{V}_t = \log V_t$. This leads us to a logarithmic version of recursion (9):

$$\begin{aligned} \widehat{V}_t &= [1 - \exp(-\delta)]\widehat{C}_t + \exp(-\delta)\widehat{R}_t \\ \widehat{R}_t &= \left(\frac{1}{1-\gamma}\right) \log \mathcal{E}\left(\exp\left[(1-\gamma)\widehat{V}_{t+1}\right] \mid \mathfrak{F}_t\right). \end{aligned} \tag{11}$$

Rather than interpreting $\widehat{R}_t$ in terms of sensitivity to risk, we now reinterpret it as a reflection of an investor's worries that the specification of the transition probabilities is mistaken. To do this, we begin by using a set of likelihood ratios

$$\mathfrak{N}_{t+1} \doteq \{N_{t+1} : N_{t+1} \geqslant 0, \mathcal{E}\left(N_{t+1} \mid \mathfrak{F}_t\right) = 1\}$$

to represent a set of possible changes in probability measures that concern the investor. Let

$$\widehat{V}_t = \min_{N_{t+1} \in \mathfrak{N}_{t+1}} [1 - \exp(-\delta)]\widehat{C}_t + \exp(-\delta)\left[\mathcal{E}\left(N_{t+1}\widehat{V}_{t+1} \mid \mathfrak{F}_t\right) + \xi\mathcal{E}\left(N_{t+1} \log N_{t+1} \mid \mathfrak{F}_t\right)\right]$$

where $\mathcal{E}\left(N_{t+1} \log N_{t+1} \mid \mathfrak{F}_t\right)$ is a measure of divergence between two probabilities called *relative entropy.* The positive scalar ξ is a penalty parameter that restrains the minimizer's search for alternative transition probability models within $\mathfrak{N}_{t+1}$. The minimizing N_{t+1} in the above problem is

$$N_{t+1} = \frac{\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right)}{\mathcal{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{F}_t\right]},$$

which equals the likelihood ratio that appears in (10) when $\gamma - 1 = \frac{1}{\xi}$. Furthermore, when $1 - \gamma = \frac{1}{\xi}$. the minimized objective function coincides with recursion (11).

In elsewhere, we justify the following first-order approximation to recursion (11)

$$\begin{aligned}
\widehat{V}_t^1 - \widehat{C}_t^1 &= \exp(-\delta)\left(\widehat{R}_t^1 - \widehat{C}_t^1\right) \\
\widehat{R}_t^1 - \widehat{C}_t^1 &= \frac{1}{1-\gamma}\log\mathcal{E}\left(\exp\left[(1-\gamma)\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + (1-\gamma)\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right] \mid \mathfrak{F}_t\right) \\
&= \mathcal{E}\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right) \mid \mathfrak{F}_t\right] + (1-\gamma)\frac{|\sigma_v^1|^2}{2}
\end{aligned} \tag{12}$$

where σ_v^1 satisfies:

$$\widehat{V}_{t+1}^1 - \mathcal{E}\left(W_{t+1} \mid \mathfrak{F}_t\right) = \sigma_v^1 \cdot W_{t+1}.$$

Conveniently, σ_v^1 can be solved assuming that $\gamma = 1$. This is a special case of the model that we solved earlier with $\rho = 1$.

Approximate first-order conditions are:

$$\mathsf{a} - \delta - \mathsf{g} = \log\mathcal{E}\left[N_{t+1}^1\left(\widehat{C}_t^1 - \widehat{C}_{t+1}^1\right) \mid \mathfrak{F}_t\right]$$

where

$$N_{t+1}^1 = \frac{\exp\left[(1-\gamma)\widehat{V}_{t+1}^1\right]}{\mathcal{E}\left(\exp\left[(1-\gamma)\widehat{V}_{t+1}^1\right] \mid \mathfrak{F}_t\right)}.$$

As we argue elsewhere, the change in the conditional probability measure induced by M_{t+1}^1 has the following particularly simple characterization: the mean of W_{t+1} conditioned on $\mathfrak{F}_t$ is the constant $(1-\gamma)\sigma_v^1$.

These observations can guide is in modifying our earlier solution to capture this change in preferences. From the first-order conditions

$$\mathsf{a} - \delta - \mathsf{g} = (1-\gamma)\sigma_v^1 \cdot \sigma_c^1$$

where σ_c^1 satisfies

$$\widehat{C}_{t+1}^1 - \widehat{C}_t^1 = \sigma_c^1 \cdot W_{t+1} + (1-\gamma)\sigma_v^1 \cdot \sigma_c^1$$

where the constant term on the right side is a first order adjustment to the growth rate of consumption. This term, which we will show to be negative, is an adjustment for precaution that scales with $(1-\gamma)$.

Now turn to the approximate recursion (12). We deploy a "guess and verify" tactic by assuming

$$\widehat{V}_t^1 = \widehat{C}_t^1 + \frac{\exp(-\delta)(1-\gamma)|\sigma_v^1|^2}{2[1-\exp(-\delta)]},$$

from which it follows that $\sigma_v^1 = \sigma_c^1$ where σ_c^1 can be computed with the logarithmic utility model for which $\rho = 1$.

Alternatively, we can impose a subjective discount factor that ensures that consumption and income grow at the same rates, the assumption that we have been maintaining in our approximation. To accomplish this, we set

$$\delta = \delta_0 + \delta_1$$

where

$$\begin{aligned} \delta_0 &= \mathsf{a} - \mathsf{g} \\ \delta_1 &= (\gamma - 1)\sigma_c^1 \cdot \sigma_c^1 = \frac{\sigma_c^1 \cdot \sigma_c^1}{\xi}. \end{aligned}$$

These adjustments assure that the original decision rule prevails after we activate risk-sensitivity or robustness adjustment via the preceding recursions. The term δ_1 is an adjustment for precaution that is positive because discounting the future more reduces incentives to save. The addition of δ_1 to the discount factor provides exactly the increase in impatience required to offset that precautionary saving motive and thereby make consumption and income continue to grow at the same rates.

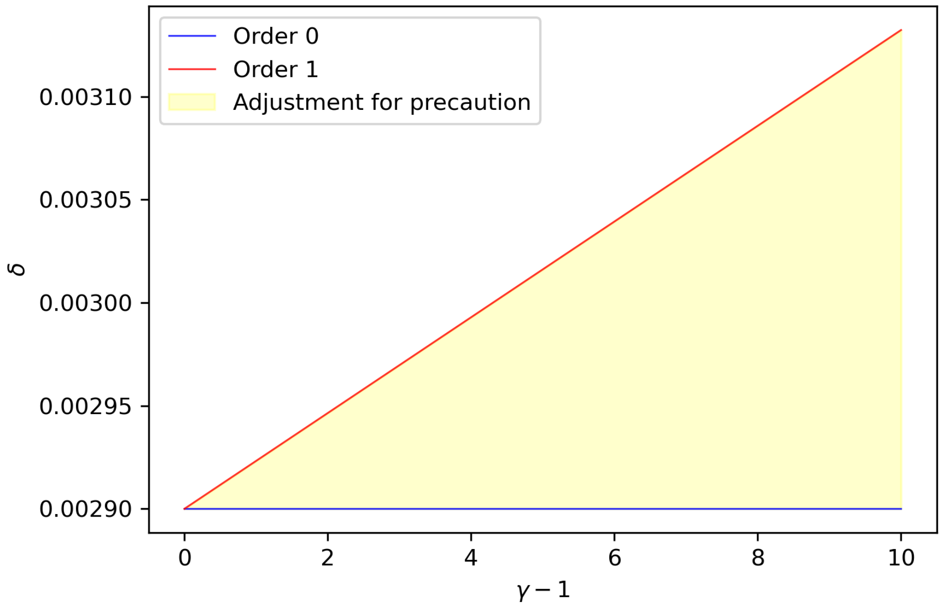

Figure 2: Changes in δ needed to offset precautionary saving induced by $\gamma > 1$.

Zhenhuan and Julien – could we add the net investment to income response along with the logarithm of the consumption/income response.

6 Habits and durable consumption

We now modify the above specification by adding a nested CES function that includes a household capital stock that can be interpreted either as a habit or as a durable good and an evolution equation for that capital stock. Specifically,

$$\begin{aligned} U_t &= \left[(1-\alpha)\left(C_t\right)^{1-\epsilon} + \alpha\left(H_t\right)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \\ V_t &= \left(U_t\right)^{1-\exp(-\delta)}\left(R_t\right)^{\exp(-\delta)} \\ R_t &= \left(\mathcal{E}\left[\left(V_{t+1}\right)^{1-\gamma} \mid \mathfrak{F}_t\right]\right)^{\frac{1}{1-\gamma}} \end{aligned} \tag{13}$$

where H_t is a household capital stock that can be used to represent habits or durable goods and that evolves as

$$H_{t+1} = \chi H_t + (1-\chi)C_t$$

where $0 \leqslant \chi < 1, 0 \leqslant \alpha < 1$ and $\eta > 0$. By setting $\eta = 1$, we get a Cobb-Douglas version of the first equation in (13)

$$U_t = (C_t)^{1-\alpha} (H_t)^{\alpha}.$$

This formulation thus introduces an additional state variable and a corresponding co-state variable.

Two static marginal utilities associated with the CES utility function are

$$\begin{aligned} mc &= [1 - \exp(-\delta)](1-\alpha)(u)^{\epsilon-1}(c)^{-\epsilon} \\ ms &= [1 - \exp(-\delta)]\alpha (u)^{\epsilon-1}(h)^{-\epsilon}. \end{aligned} \tag{14}$$

The intertemporal recursion induces us to make the multiplicative adjustment by

$$[1 - \exp(-\delta)](u)^{-1}.$$

As we specified previously,

$$N_{t+1} = \frac{V_{t+1}}{\mathcal{E}\left(\exp\left[(1-\gamma)V_{t+1}\right] \mid \mathfrak{F}_t\right)}$$

Consumption today alters future values of the stock H. In particular, a unit of consumption at date t increases H_{t+j} by $(1-\chi)\chi^{j-1}$ in period $t+j$. We account for this in the formula

$$MH_t - \exp(-\delta)\chi\mathcal{E}\left(M_{t+1}MH_{t+1} \mid \mathfrak{F}_t\right) = \exp(-\delta)\mathcal{E}\left(N_{t+1}MS_{t+1} \mid \mathfrak{F}_t\right) \tag{15}$$

where MS_{t+1} is the date $t+1$ static marginal utility obtained using the second formula in (14)

$$MS_{t+1} = [1 - \exp(-\delta)]\alpha (U_{t+1})^{\epsilon-1}(H_{t+1})^{-\epsilon}.$$

The forward-looking solution to this difference equation is

$$MH_t = \exp(-\delta)\sum_{j=1}^{\infty}[\exp(-\delta)\chi]^{j-1}\mathcal{E}\left(N_{t+j}MS_{t+j} \mid \mathfrak{F}_t\right) \tag{16}$$

This leads us to

$$MC_t = [1 - \exp(-\delta)](1-\alpha)(U_t)^{\epsilon-1}(C_t)^{-\epsilon} + (1-\chi)MH_t \tag{17}$$

where the first term uses the first formula in (14), which captures the current-period contribution to the marginal utility of consumption. Formula (17) augments the current-period contribution with a forward-looking component captured by $(1-\chi)MH_t$. This term appears because a change in consumption today alters the the stock of household capital in future periods with adjustments for depreciation.

In addition, we have the prospective co-state evolution:

$$\begin{aligned} MK_t &= \exp\left(\mathsf{a}-\delta\right)\mathcal{E}\left(N_{t+1}MK_{t+1} \mid \mathfrak{F}_t\right) \\ MK_t &= MC_t \end{aligned}$$

introduced previously.

To approximate a solution, we proceed as before but with additional state and co-state variables. We approximate the evolution of $\log H_t$ and $\log MH_t$.

6.1 Approximate state dynamics

Let

$$\begin{aligned} \widetilde{H}_t &\doteq \frac{H_t}{Y_t} \\ \widehat{H}_t &\doteq \log H_t = \log \widetilde{H}_t + \widehat{Y}_t. \end{aligned}$$

The zero order steady state is

$$\widetilde{H}_0^0 = \left[\frac{1-\chi}{\exp(\mathsf{g})-\chi}\right]\tilde{C}_0^0.$$

As derived in Appendix A

$$\left(\widehat{H}_{t+1}^1 - \widehat{Y}_{t+1}^1\right) + \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) = \hat{\beta}_3\left(\widehat{H}_t^1 - \widehat{Y}_t^1\right) + \hat{\beta}_4\left(\widehat{C}_t^1 - \widehat{Y}_t^1\right).$$

Appendix A also give formulas for the coefficients $\hat{\beta}_3$ and $\hat{\beta}_4$ used in this approximation.

6.2 Approximate co-state dynamics

We again look to the steady state for the date zero approximation. Define

$$\begin{aligned}
\widetilde{MH}_t &= Y_t \left(MH_t \right) \\
\widetilde{MC}_t &= Y_t \left(MC_t \right) \\
\widetilde{U}_t &= \frac{U_t}{Y_t} = \left[(1-\alpha) \left(\frac{C_t}{Y_t} \right)^{1-\epsilon} + \alpha \left(\frac{H_t}{Y_t} \right)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \\
\widetilde{MS}_t &= Y_t \left(MS_t \right) = [1 - \exp(-\delta)] \alpha \left(\widetilde{U}_t \right)^{\epsilon - 1} \left(\widetilde{H}_t \right)^{-\epsilon} \\
\widehat{MH}_t &= \log MH_t = \log \widetilde{MH}_t - \widehat{Y}_t \\
\widehat{MC}_t &= \log MC_t = \log \widetilde{MC}_t - \widehat{Y}_t \\
\widehat{U}_t &= \log U_t = \log \widetilde{U}_t - \widehat{Y}_t \\
\widehat{MS}_t &= \log MS_t = \log \widetilde{MS}_t - \widehat{Y}_t
\end{aligned}$$

Multiplying (15) by Y_t gives the steady state relation

$$[1 - \exp(-\mathsf{g} - \delta)\chi] \widetilde{MH}_0^0 = \exp(-\delta) \widetilde{MS}_0^0,$$

which we solve for the steady state $\widetilde{MH}_0$.

For the first-order approximation, we first construct

$$N_{t+1}^1 = \frac{\exp\left[(1-\gamma) \widehat{V}_{t+1}^1 \right]}{\mathcal{E}\left(\exp\left[(1-\gamma) \widehat{V}_{t+1}^1 \right] \mid \mathfrak{F}_t \right)}.$$

Appendix B derives the first-order approximating co-state evolution including formulas for the coefficients:

$$\begin{aligned}
&\left(\widehat{MH}_t + \widehat{Y}_t^1 \right) \\
&= \mathcal{E}\left(N_{t+1}^1 \left[\hat{\beta}_5 \left(\widehat{MH}_{t+1}^1 + \widehat{Y}_{t+1}^1 \right) + \hat{\beta}_6 \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1 \right) + \hat{\beta}_7 \left(\widehat{MS}_{t+1}^1 + \widehat{Y}_{t+1}^1 \right) \right] \mid \mathfrak{F}_t \right)
\end{aligned}$$

Finally, we form the first-order approximation for $\widehat{MC}_t + \widehat{Y}_t$:

$$\left[\widehat{MC}_t^1 + \widehat{Y}_t^1\right] = \hat{\beta}_8 \left[(\epsilon - 1)\left(\widehat{U}_t^1 - \widehat{Y}_t^1\right) - \epsilon\left(\widehat{C}_t^1 - \widehat{Y}_t^1\right)\right] + \hat{\beta}_9 \left[\widehat{MH}_t^1 + \widehat{Y}_t^1\right].$$

where coefficients are provide in Appendix B.

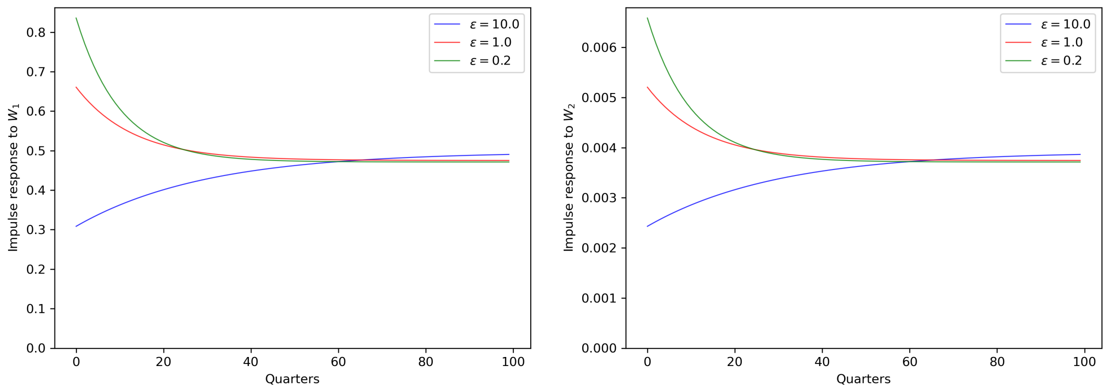

Figure 3: Impact of changing the substitution parameter ϵ. Parameter $\alpha = .5$. Zhenhuan and Julien: lets redo this figure replacing$\epsilon = 10$ with $\epsilon = 20$ and only with shock one. Then add a single plot with $\epsilon = 1$ and $\alpha = .5, .25, .10, .05$. Make sure the vertical axes remain the same.

Please produce a table with limiting responses for the two different shocks and the different α and ϵ configurations.

A Approximate state dynamics

Rewrite the evolution equation for H as:

$$\widetilde{H}_{t+1}\left(\frac{Y_{t+1}}{Y_t}\right) = \chi \widetilde{H}_t + (1-\chi)\widetilde{C}_t.$$

The first-order approximation is

$$\exp(\mathtt{g})\widetilde{H}_0^0\left(\widehat{H}_{t+1}^1 - \widehat{Y}_{t+1}^1\right) + \widetilde{H}_0^0 \exp(\mathtt{g})\left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) = \chi \widetilde{H}_0^0\left(\widehat{H}_t^1 - \widehat{Y}_t^1\right) + (1-\chi)\widetilde{C}_0^0\left(\widehat{C}_t^1 - \widehat{Y}_t^1\right)$$

Dividing by $\widetilde{H}_0^0 \exp(\mathsf{g})$ gives

$$\left(\widehat{H}_{t+1}^1 - \widehat{Y}_{t+1}^1\right) + \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) = \hat{\beta}_3 \left(\widehat{H}_t^1 - \widehat{Y}_t^1\right) + \hat{\beta}_4 \left(\widehat{C}_t^1 - \widehat{Y}_t^1\right)$$

where

$$\begin{aligned}
\hat{\beta}_3 &= \exp(-\mathsf{g})\chi \\
\hat{\beta}_4 &= \exp(-\mathsf{g})(1-\chi)\left(\frac{\widetilde{C}_0^0}{\widetilde{H}_0^0}\right).
\end{aligned}$$

B Approximate co-state dynamics

For the first-order approximation, we first construct

$$N_{t+1}^1 = \frac{\exp\left[(1-\gamma)\widehat{V}_{t+1}^1\right]}{\mathcal{E}\left(\exp\left[(1-\gamma)\widehat{V}_{t+1}^1\right] \mid \mathfrak{F}_t\right)}.$$

Since this formula is invariant to translations in $\widehat{V}_{t+1}$, it suffices to pre-compute $\widehat{V}_{t+1}$ for $\gamma = 1$. Armed with this object, the first-order approximation for the co-state dynamics are:

$$\begin{aligned}
\widetilde{MH}_0^0 \left(\widehat{MH}_t^1 + \widehat{Y}_t^1\right) = & \exp(-\delta - \mathsf{g})\chi\widetilde{MH}_0^0 \mathcal{E}\left[N_{t+1}^1 \left(\widehat{MH}_{t+1}^1 + \widehat{Y}_{t+1}^1\right) \mid \mathfrak{F}_t\right] \\
& - \exp(-\delta - \mathsf{g})\chi\widetilde{MH}_0^0 \mathcal{E}\left[N_{t+1}^1 \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) \mid \mathfrak{F}_t\right] \\
& + \exp(-\delta - \mathsf{g})\widetilde{MS}_0^0 \mathcal{E}\left[N_{t+1}^1 \left(\widehat{MS}_{t+1}^1 + \widehat{Y}_{t+1}^1\right) \mid \mathfrak{F}_t\right] \\
& - \exp(-\delta - \mathsf{g})\widetilde{MS}_0^0 \mathcal{E}\left[N_{t+1}^1 \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) \mid \mathfrak{F}_t\right]
\end{aligned}$$

Dividing by $\widetilde{MH}_0^0$ gives

$$\begin{aligned}
&\left(\widehat{MH}_t + \widehat{Y}_t^1\right) \\
&= \mathcal{E}\left(N_{t+1}^1 \left[\hat{\beta}_5 \left(\widehat{MH}_{t+1}^1 + \widehat{Y}_{t+1}^1\right) + \hat{\beta}_6 \left(\widehat{Y}_{t+1}^1 - \widehat{Y}_t^1\right) + \hat{\beta}_7 \left(\widehat{MS}_{t+1}^1 + \widehat{Y}_{t+1}^1\right)\right] \mid \mathfrak{F}_t\right)
\end{aligned}$$

where

$$
\begin{aligned}
\hat{\beta}_5 &= \exp(-\delta - \mathsf{g})\chi \\
\hat{\beta}_6 &= -\exp(-\delta - \mathsf{g}) \left(\chi + \frac{\widetilde{MS}_0^0}{\widetilde{MH}_0^0} \right) \\
\hat{\beta}_7 &= \exp(-\delta - \mathsf{g}) \left(\frac{\widetilde{MS}_0^0}{\widetilde{MH}_0^0} \right)
\end{aligned}
$$

which gives the first-order approximation to the prospective co-state evolution.

Finally, we form the first-order approximation for $\widehat{MC}_t + \widehat{Y}_t$:

$$
\left[\widehat{MC}_t^1 + \widehat{Y}_t^1 \right] = \hat{\beta}_8 \left[(\epsilon - 1) \left(\widehat{U}_t^1 - \widehat{Y}_t^1 \right) - \epsilon \left(\widehat{C}_t^1 - \widehat{Y}_t^1 \right) \right] + \hat{\beta}_9 \left[\widehat{MH}_t^1 + \widehat{Y}_t^1 \right].
$$

where

$$
\begin{aligned}
\hat{\beta}_8 &= \frac{[1 - \exp(-\delta)](1 - \alpha) \left(\widetilde{U}_0^0 \right)^{\epsilon - 1} \left(\widetilde{C}_0^0 \right)^{-\epsilon}}{\widetilde{MC}_0^0} \\
\hat{\beta}_9 &= \left(\frac{(1 - \chi) \widetilde{MH}_0^0}{\widetilde{MC}_0^0} \right)
\end{aligned}
$$

References

Becker, Gary S. and Kevin M. Murphy. 1988. A Theory of Rational Addiction. *Journal of Political Economy* 96 (4):675–700.

Flavin, Majorie A. 1981. The Adjustment of Consumption to Changing Expectations about Future Income. *Journal of Political Economy* 89 (5):974–1009.

Hansen, Lars Peter, Thomas Sargent, and Thomas Tallarini. 1999. Robust Permanent Income and Pricing. *Review of Economic Studies* 66 (4):873–907.

Ryder, Harl E. and Geoffrey M. Heal. 1973. Optimal Growth with Intertemporally Dependent Preferences. *The Review of Economic Studies* 40 (1):1–31.